Equity Derivatives and Volatility Workshop

Marco Avellaneda

New York University

RiskMathics Trading and Risk Management

Mexico, June 19 & 20, 2019

Options

- A **call option** is a contract that allows the holder (long) to purchase an underlying asset from the writer (short) at a fixed price during a specified period of time.

- A **put option** is a contract that allows the holder (long) to sell an underlying asset from the writer (short) at a fixed price during a specified period of time.

- Options can be traded OTC between two counterparties (e.g. banks, or banks and their clients) or in exchanges (CBOE, ISE, NYSE, PCOST, PHLX, BOX).

- Main underlying assets: equity shares, equity indexes, swaps and bonds, futures on bonds, futures on equity indexes, foreign exchange (OTC mostly).

- Many OTC derivatives such as **swaps and strucutured notes** have ``embedded options in them, which makes their study very relevant.

- **Convertible bonds** also contain embedded options.

Specifying an Option Contract

- An option contract is specified by
  -- put or call
  -- underlying asset
  -- notional amount
  -- exercise price
  -- maturity date or expiration date
  -- style (American, European)
  -- settlement (cash or physical)

- An American option can be exercised anytime before the expiration date

- A European option can be exercised only at the maturity date

Example: Exchange Traded Equity Option

SPY December 120 Call
Underlying asset: SPY Notional Amount: 100 Shares Exercise Price: \$120 Expiration date: Friday, December 16 2011 Style: American Settlement: Physical

- This option trades in the six US options exchange
- Most US exchange-traded options are standardized to a notional of 100 shares
- Expiration is on the 3rd Friday of the expiration month
- Strikes are standardized as well, in increments of \$2.50, \$5 or \$1, depending on the underlying asset and the strike price.
- Regulated by Options Clearing Corporation,US laws, etc. Centrally cleared.

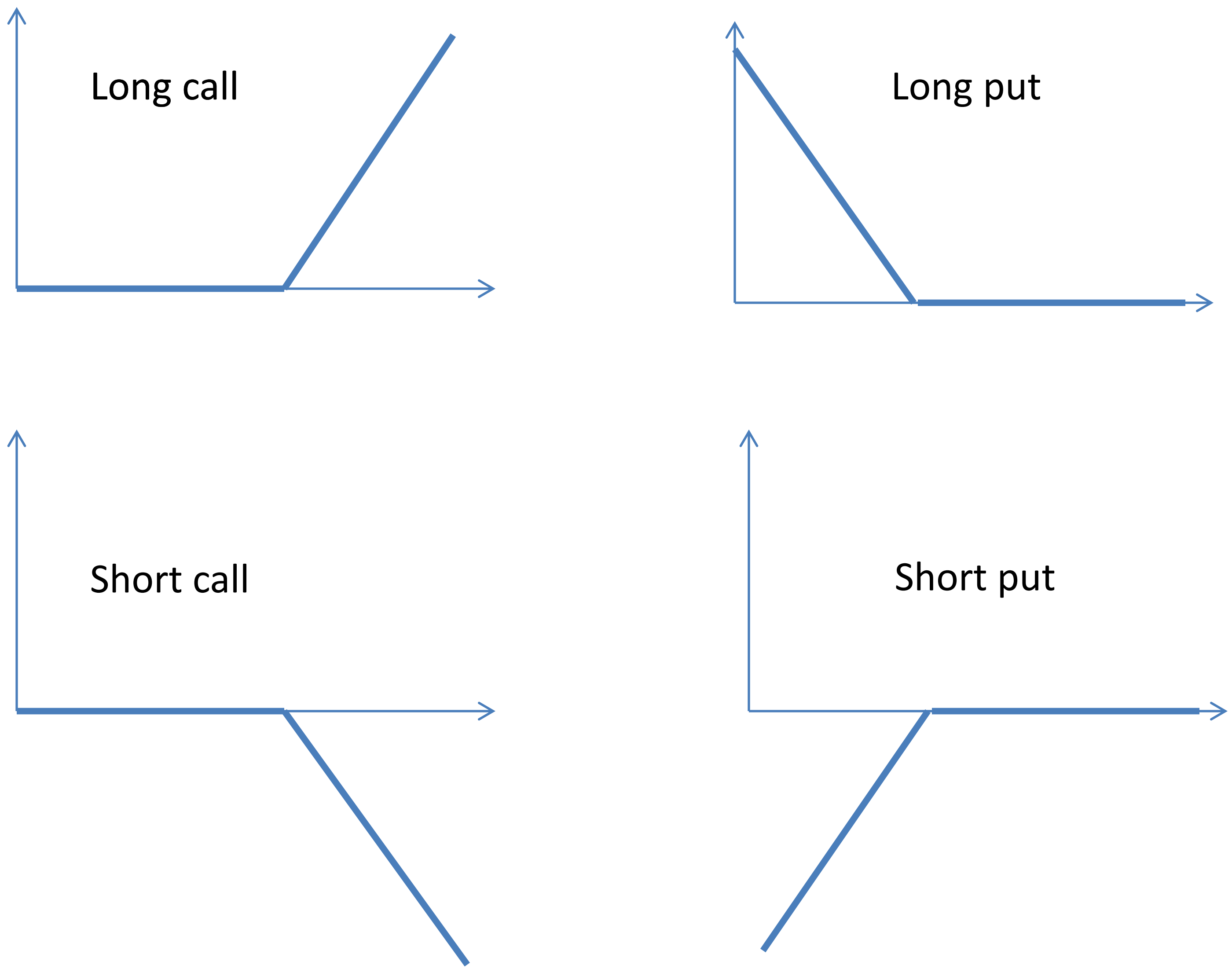
Example: an OTC currency option

120 day USD/JPY 85 Put

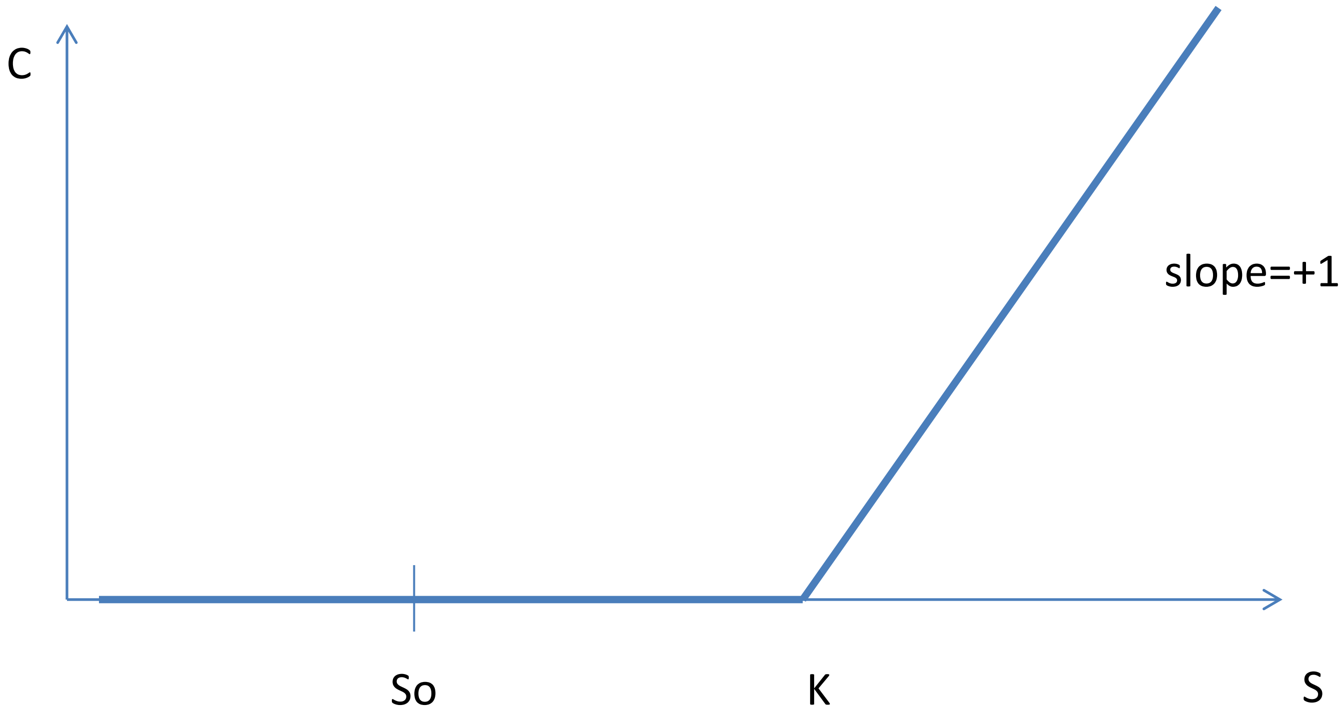
- Underlying asset: USD/JPY
- Notional amount: USD 40,000,000
- Trade date: Sep 19 2011
- Expiration date: Jan 17 2012
- Style: European
- Settlement: Cash

- OTC contract between banks or banks and clients
- Notional not standardized (minimum notional ~ 10 MM USD)
- Strikes are not necessarily standardized
- Governed by interbank agreements.

Basic positions & profit diagrams



A closer look at the call payoff



Payoff = max(S-K,0)

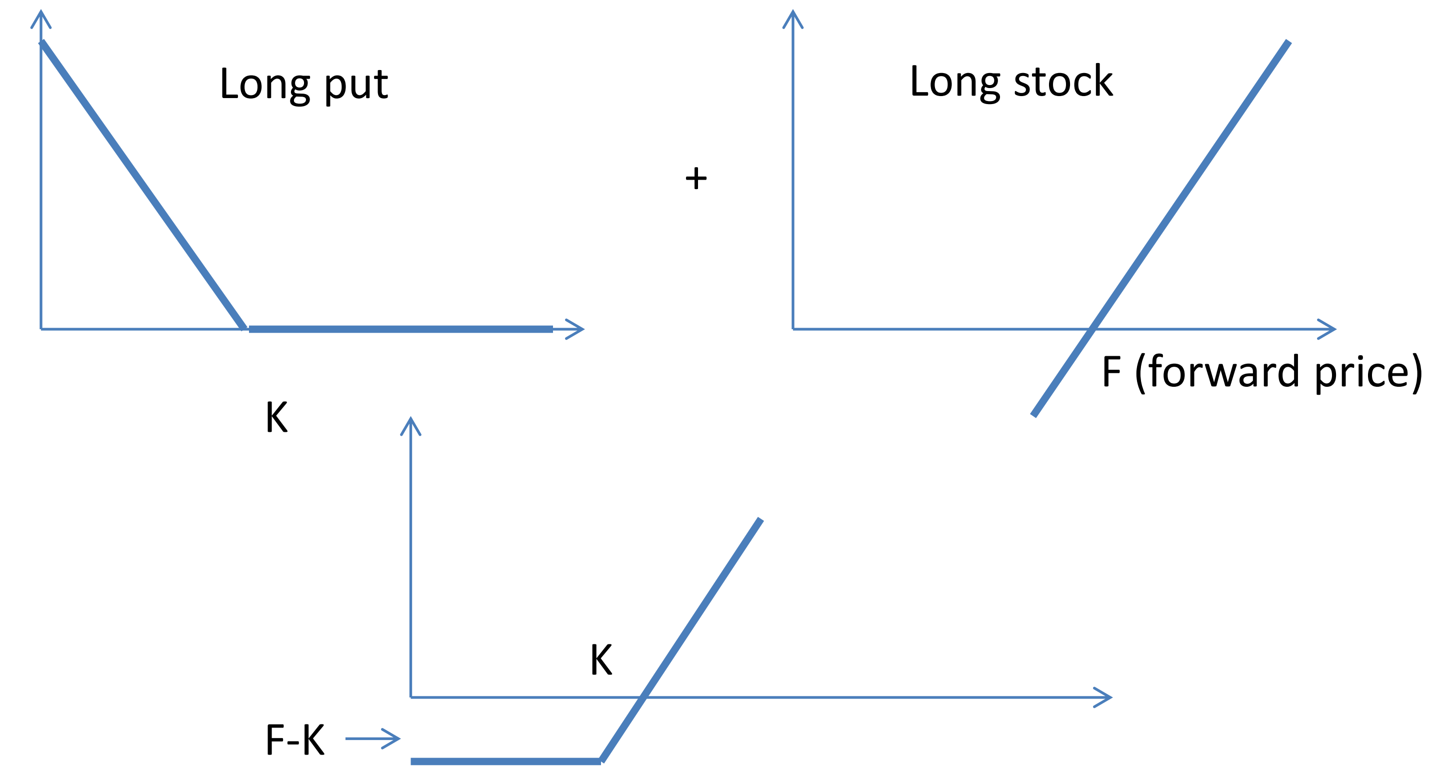
If So<K, the option is **out-of-the-money**

If So>K, the option is **in-the-money**

Put-Call Parity

- This principle applies to European options but is also widely use to analyze American-style options as well.

- A position long put + long forward is equivalent to long call (up to a cash position) as shown by the diagram below:



Put-Call Parity

Call payoff Put payoff

$$\max(S_T - K, 0) - \max\left(K - S_T 0\right) = S_T - K$$

$$= \underbrace{S_T - F_T}_{\text{Forward payoff}} + \underbrace{F_T - K}_{\text{Cash}}$$

- Since, by definition, the ATM forward contract has zero value, we have, in terms of the option premia,

$$\boxed{Call\left(K,T\right) - Put\left(K,T\right) = PV\left(F_T - K\right)}$$

- Arbitrage relation between the fair values of European-style puts and calls

Put-Call Parity in terms of forward & spot prices

- If the options are at-the-money forward,

$$K = F_T \quad \Rightarrow \quad Call(F_T, T) = Put(F_T, T)$$

- In general, we have

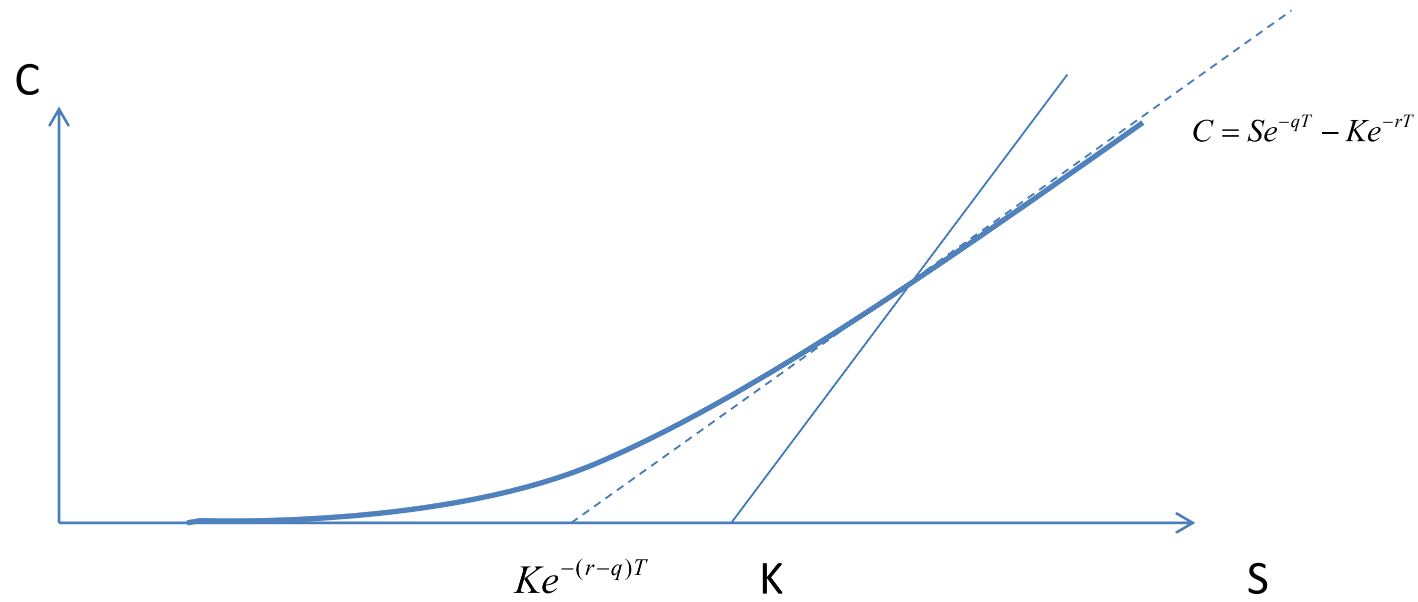
$$\begin{aligned} Call(K,T) - Put(K,T) &= PV(F_T - K) \\ &= e^{-rT}\left(e^{(r-q)T} S_0 - K\right) \\ &= e^{-qT} S_0 - e^{-rT} K \end{aligned}$$

$q =$ dividend yield for the stock over period $(0, T)$

$r =$ funding rate over the period $(0, T)$

Basic properties of options: calls

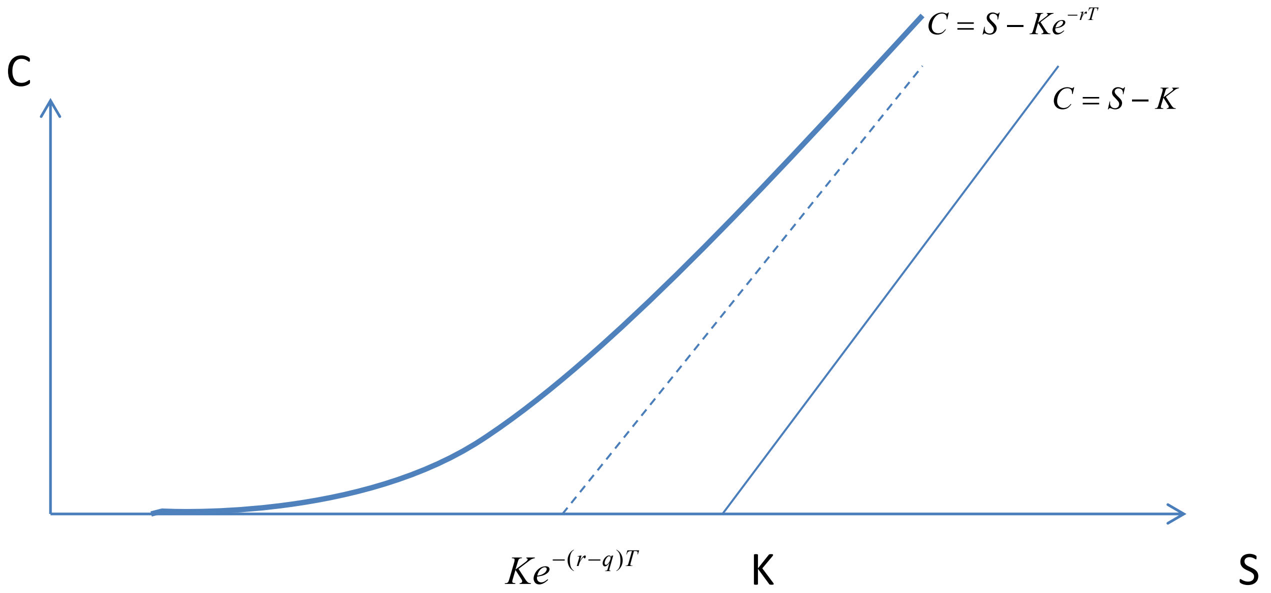
$$Call(S,K,T) > 0, \quad Call(S,K,T) > Se^{-qT} - Ke^{-rT}$$
$$Call(S,K,T) \approx Se^{-qT} - Ke^{-rT}, \; S/K >> 1$$



Call premium is increasing in S/K and asymptotic to PV(F-K).

If there are no dividend payments then C>Max(S-K,0)

$$Call(S,K,T) > 0, \quad Call(S,K,T) > S - Ke^{-rT}$$



Call premium is increasing in S/K and asymptotic to PV(F-K).

American-style vs. European-Style calls

$$Call_{am}(K,T) \geq Call_{eu}(K,T)$$

always

$$\text{If } \; q=0, \; Call_{eu}(K,T) \geq (S - Ke^{-rT})^{+} > (S-K)^{+}$$

$$\therefore \qquad Call_{am}(K,T) > (S-K)^{+}$$

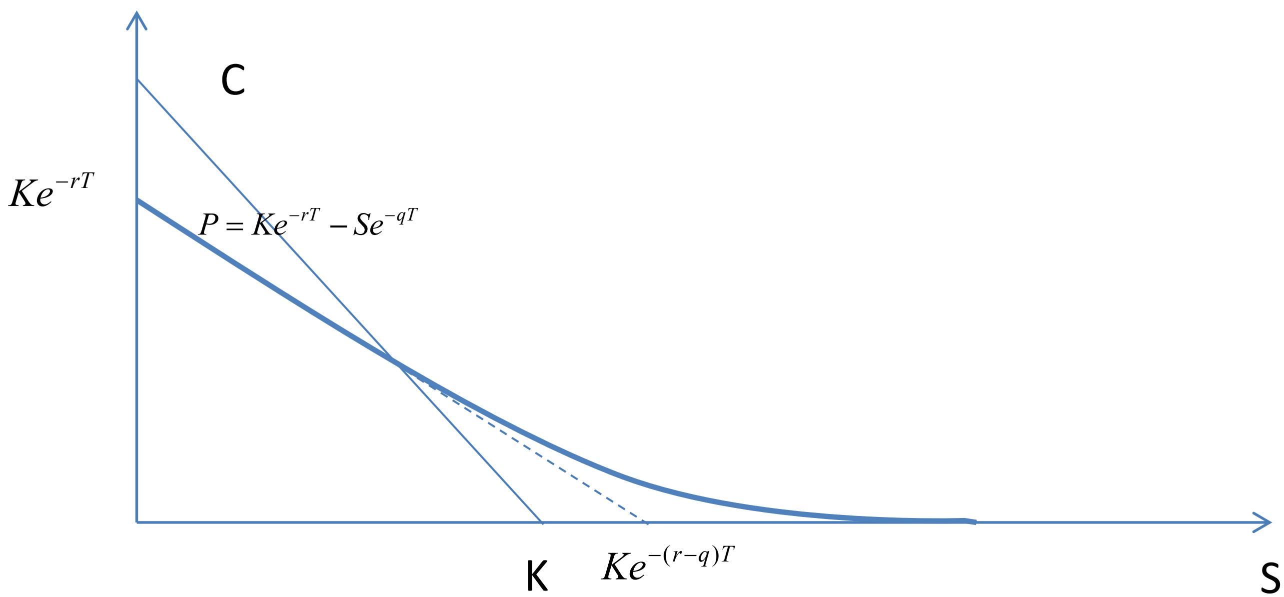
if S>K

$$\therefore \; \boxed{Call_{am}(K,T) = Call_{eu}(K,T)}$$

- If the stock does not pay dividends over the life of the option, there is no early-exercise premium.
- More generally, if a commodity does not have a positive convenience yield, then American-style and European-style options have the same premium.

Basic properties of puts

$$Put(S,K,T) > 0, \quad Put(S,K,T) > Ke^{-rT} - Se^{-qT}$$
$$Put(S,K,T) \approx Ke^{-rT} - Se^{-qT}, \quad S/K << 1$$



- Put premium is decreasing in S/K and asymptotic to PV(K-F).
- The asymptotic is below intrinsic value if r>0
- American puts have early exercise premium if r>0

Puts vs. Calls, philosophically

- There is complete symmetry between puts and calls in several respects.

- Put = ``Call on Cash'' using stock as the currency.

- Best context for this is FX

  N-day USD Call/ JPY put with strike 75 Y , notional 10,000,000 USD, is

  -- an option to buy 10 MM USD at 75 JPY per dollar N days from now (Dollar Call)

  -- an option to sell 750 MM JPY at 0.013333 USD per JPY N days from now (Yen Put)

- In FX, the foreign interest rate plays the role of dividend. $r = r_d,\ q = r_f$

SPY November 2011 Options (partial view)

SPY=$119.50, Expiration Date, Nov 18 2011, 43 trading days left

CALLS							PUTS					
Last	Change	Bid	Ask	Volume	Open Int	Strike	Last	Change	Bid	Ask	Volume	Open Int
12.5	0.83	12.29	12.35	39	5,185	110	2.88	0.24	2.85	2.87	1,452	79,332
11.74	1.54	11.47	11.59	46	4,000	111	3.15	0.29	3.06	3.11	236	21,256
10.97	0.73	10.69	10.81	43	4,229	112	3.33	0.23	3.31	3.35	224	49,388
10.1	0.4	10.01	10.04	617	7,244	113	3.62	0.29	3.58	3.62	1,460	34,591
9.3	0.5	9.24	9.26	800	5,480	114	3.86	0.24	3.83	3.87	2,903	45,415
8.6	0.46	8.55	8.57	1,480	12,266	115	4.13	0.22	4.09	4.11	1,546	95,122
7.87	0.54	7.86	7.87	1,091	12,448	116	4.44	0.26	4.45	4.47	2,117	36,139
7.22	0.5	7.19	7.21	1,505	6,763	117	4.76	0.26	4.78	4.8	1,953	48,040
6.63	0.4	6.54	6.56	2,143	22,235	118	5.15	0.35	5.11	5.14	3,500	42,238
5.95	0.39	5.95	5.97	3,929	12,458	119	5.54	0.35	5.51	5.55	4,684	13,423
5.3	0.47	5.34	5.36	4,200	27,501	120	5.87	0.24	5.91	5.93	4,669	63,099
4.74	0.45	4.77	4.78	3,034	26,806	121	6.36	0.36	6.33	6.34	2,835	24,034
4.3	0.31	4.26	4.27	2,791	26,410	122	6.82	0.35	6.8	6.81	1,558	17,818
3.74	0.34	3.71	3.73	2,269	16,416	123	7.31	0.45	7.26	7.28	1,970	5,564
3.27	0.29	3.23	3.24	1,703	40,432	124	7.88	0.44	7.86	7.88	1,433	33,682
2.79	0.37	2.81	2.82	1,525	57,405	125	8.38	0.43	8.4	8.42	635	30,826
2.46	0.25	2.42	2.43	1,626	26,484	126	9.01	0.48	8.97	8.99	865	8,792
2.09	0.22	2.03	2.05	1,797	41,838	127	9.56	0.39	9.56	9.59	419	14,686
1.76	0.16	1.71	1.72	1,625	11,681	128	10.23	0.44	10.3	10.3	503	6,495
1.41	0.2	1.42	1.45	1,530	7,092	129	10.97	0.6	10.9	11.1	109	1,668

Implied Dividend Yield

- The implied dividend yield is the yield that makes Put-Call parity true.

$$C_{eur}(K,T) - P_{eur}(K,T) = Se^{-qT} - Ke^{-rT}$$

$$q = q(K,T) = -\frac{1}{T}\ln\left(\frac{C_{eur}(K,T) - P_{eur}(K,T) + Ke^{-rT}}{S}\right)$$

- Option markets contain information about funding rates and dividends. If the options are European-style, q should be roughly independent of K.

- If the options are American-style, we can still use the market to estimate the dividend yield.

Implied Forward Calculation

- Implied Forward

$$i_0 = \arg\min_i \; |C(K_i, T) - P(K_i, T)|$$

$$F_{im} = K_{i_0} + e^{RT}\left(C\left(K_{i_0}, T\right) - P\left(K_{i_0}, T\right)\right)$$

- Implied Dividend

$$q_{im} = r - \frac{1}{T} ln(F/Spot)$$

CALLS				PUTS			
Bid	Ask	Mid	Strike	Bid	Ask	Mid	IDIV
12.29	12.35	12.32	110	2.85	2.87	2.86	0.33%
11.47	11.59	11.53	111	3.06	3.11	3.09	0.41%
10.69	10.81	10.75	112	3.31	3.35	3.33	0.53%
10.01	10.04	10.03	113	3.58	3.62	3.6	0.51%
9.24	9.26	9.25	114	3.83	3.87	3.85	0.63%
8.55	8.57	8.56	115	4.09	4.11	4.1	0.34%
7.86	7.87	7.865	116	4.45	4.47	4.46	0.61%
7.19	7.21	7.2	117	4.78	4.8	4.79	0.59%
6.54	6.56	6.55	118	5.11	5.14	5.13	0.52%
5.95	5.97	5.96	119	5.51	5.55	5.53	0.49%
5.34	5.36	5.35	120	5.91	5.93	5.92	0.49%
4.77	4.78	4.775	121	6.33	6.34	6.34	0.45%
4.26	4.27	4.265	122	6.8	6.81	6.81	0.35%
3.71	3.73	3.72	123	7.26	7.28	7.27	0.40%
3.23	3.24	3.235	124	7.86	7.88	7.87	0.82%
2.81	2.82	2.815	125	8.4	8.42	8.41	0.62%
2.42	2.43	2.425	126	8.97	8.99	8.98	0.43%
2.03	2.05	2.04	127	9.56	9.59	9.58	0.33%
1.71	1.72	1.715	128	10.3	10.3	10.3	0.53%
1.42	1.45	1.435	129	10.9	11.1	11	0.43%

SPY=119.50
FF=0.10%

Average IDIV around
The money=0.49%

Implied Dividend Yields from Option prices (American)

The effect of implying dividends from American-style options

- American in-the-money puts are higher than the European counterparts
- **IDIV is less than q for low strikes, IDIV is greater than q for high strikes**

$$S >> K \quad \Rightarrow \quad C_{am}(K,T) > C_{eur}(K,T) \;\; \& \;\; P_{am}(K,T) \approx P_{eur}(K,T)$$

$$\therefore IDIV = -\frac{1}{T}\ln\left(\frac{C_{am} - P_{am} + Ke^{-rT}}{S}\right) < -\frac{1}{T}\ln\left(\frac{C_{eur} - P_{eur} + Ke^{-rT}}{S}\right) \approx q$$

$$S >> K \quad \Rightarrow \quad P_{am}(K,T) > P_{eur}(K,T) \;\; \& \;\; C_{am}(K,T) \approx C_{eur}(K,T)$$

$$\therefore IDIV = -\frac{1}{T}\ln\left(\frac{C_{am} - P_{am} + Ke^{-rT}}{S}\right) > -\frac{1}{T}\ln\left(\frac{C_{eur} - P_{eur} + Ke^{-rT}}{S}\right) \approx q$$

XOM January 2013 options (near the money)

Calls			Strike	Puts				
Symbol	Bid	Ask		Symbol	Bid	Ask	IDIV	C-P
XOM1301	15.75	16.5	60	XOM1301	5.7	5.8	1.84%	10.4
XOM1301	12.45	12.7	65	XOM1301	7.4	7.6	2.16%	5.08
XOM1301	9.55	9.7	70	XOM1301	9.5	9.75	2.26%	0
XOM1301	8.3	8.45	72.5	XOM1301	10.9	11	2.32%	-2.55
XOM1301	7.1	7.3	75	XOM1301	12.1	12.4	2.30%	-5.03
XOM1301	6.05	6.25	77.5	XOM1301	13.6	13.8	2.31%	-7.53
XOM1301	5.15	5.3	80	XOM1301	15.1	15.4	2.27%	-9.98

XOM=71.97

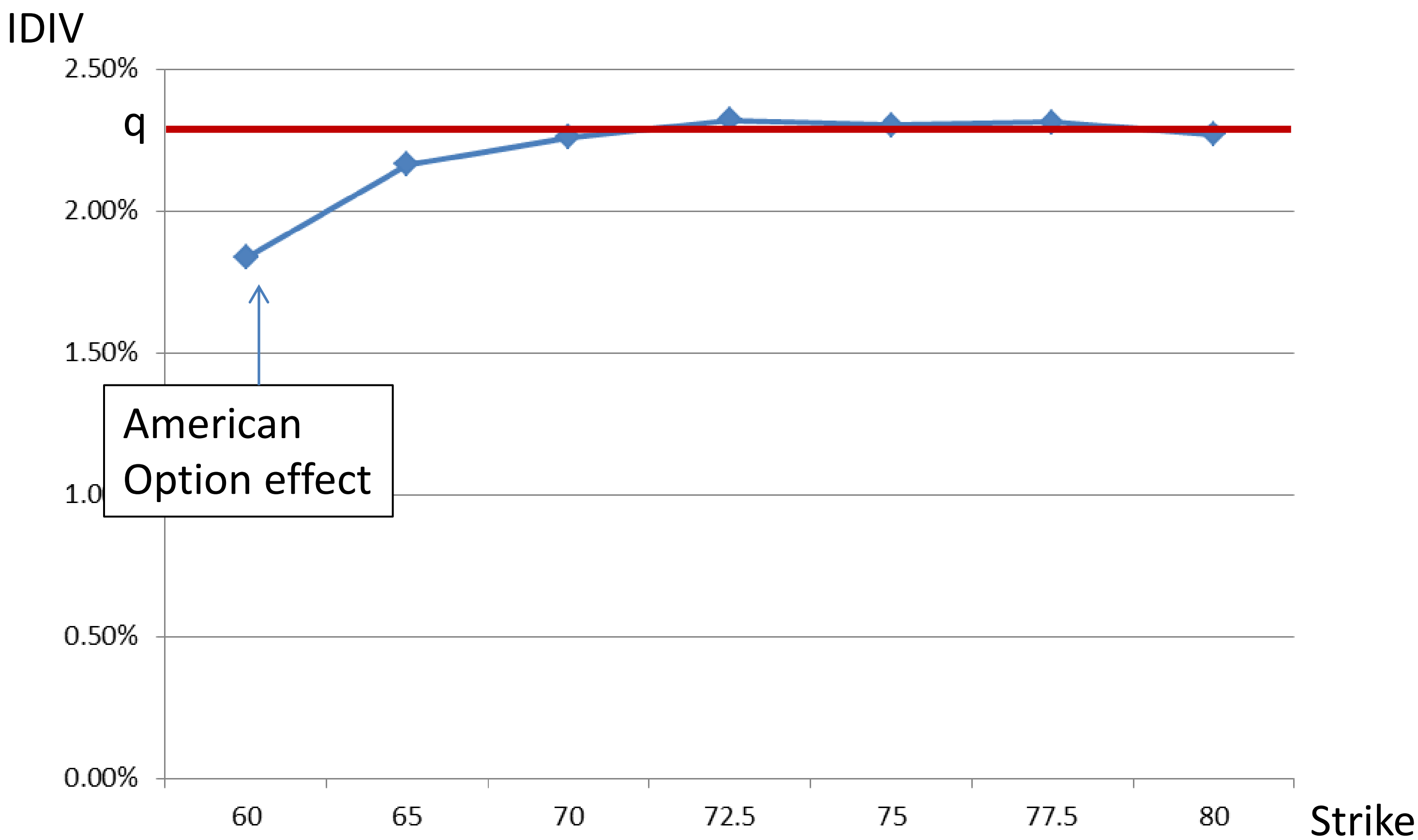
Implied Dividend

Call-Put

XOM January 2013 options, IDIV

Actual distributions

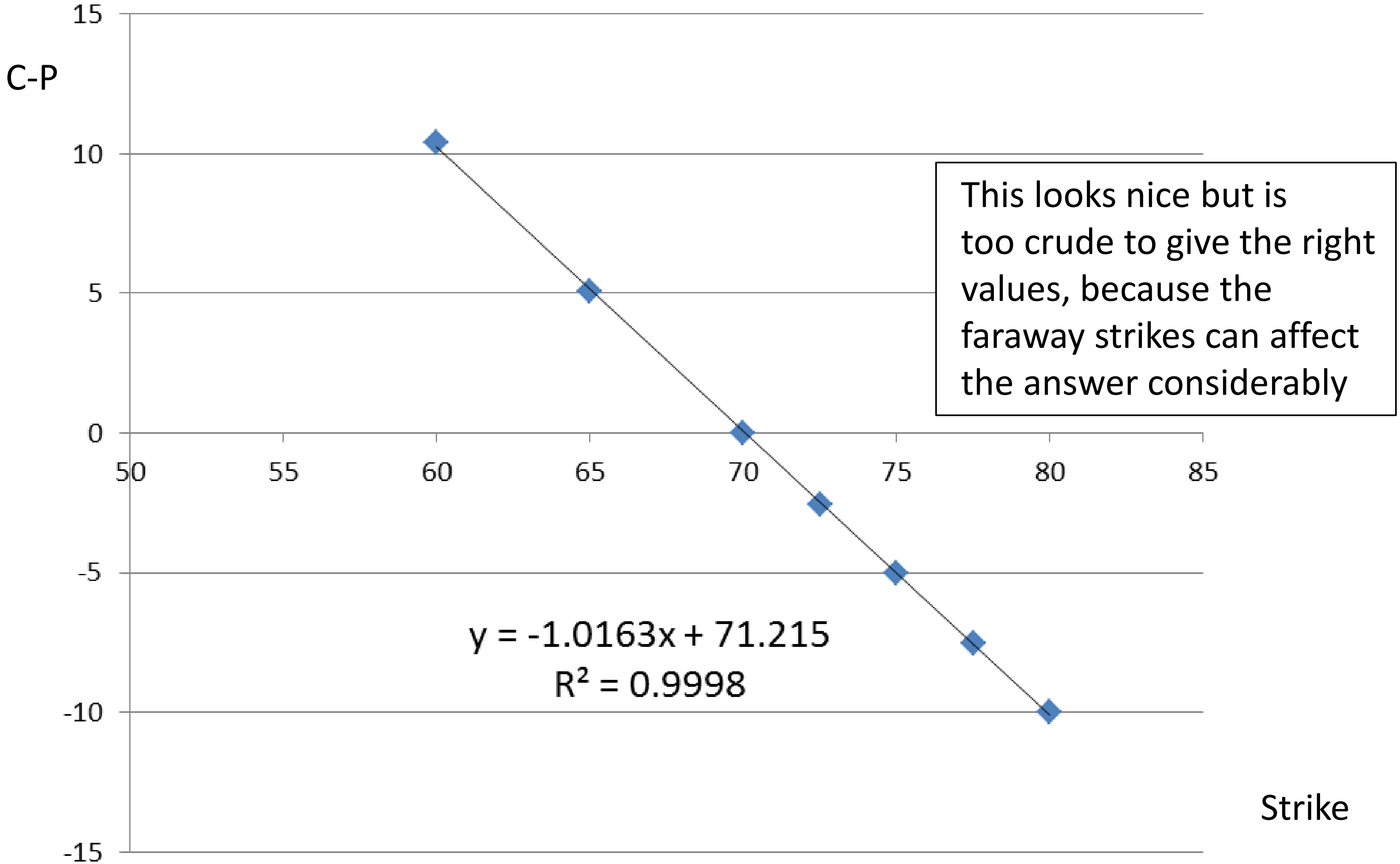
Date	Dividends
8/10/2011	0.47
5/11/2011	0.47
2/8/2011	0.44
11/9/2010	0.44
8/11/2010	0.44
5/11/2010	0.44
2/8/2010	0.42
11/9/2009	0.42
8/11/2009	0.42
5/11/2009	0.42
2/6/2009	0.4



Last calendar year's distributions = 1.82/71.97=2.53%

Options markets imply a slightly lower dividend yield (2.29%), but close.

Regressing C-P on Strike Price



Arbitrage Argument for Put-Call Parity

- Based on cash-and-carry
- If C-P > PV(F-K), sell call, buy put and buy stock (conversion)
- If C-P < PV(F-K), buy call, sell put and short stock (reversal)
- More precisely: if C-P > PV(F-K) then

-- sell 1 call

-- buy 1 put

This is a synthetic short forward with strike K.

Note: the proceeds are greater than the upfront fee for entering into a long forward with price K. [So it’s a profitable trade ☺].

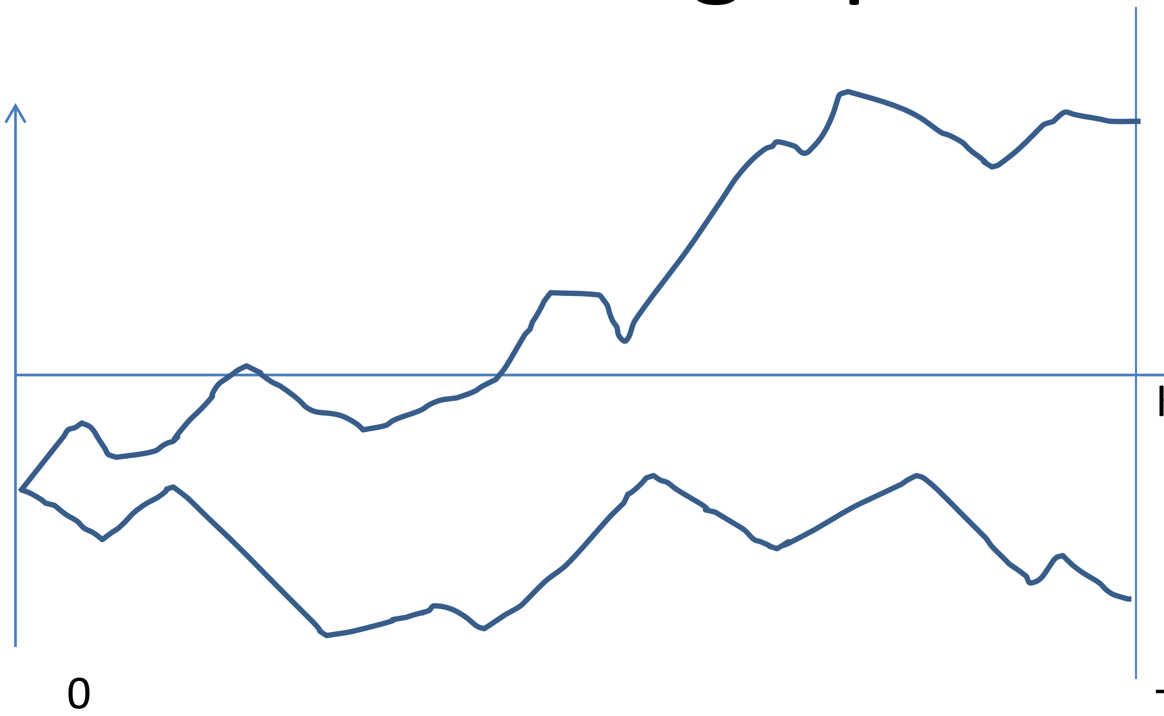
-- **Cash & carry:** borrow $$, buy 100 shares of stock, invest the proceeds of the option trade. Collect stock dividends if any and deliver the stock against the short forward. This pays K, which gives a total PNL = (F-K)+K=F, enough to pay back the loan with the dividends collected.

With a graph

At time 0

1. Sell the Conversion

2. Go long stock (financing)



At time T

You now have the stock by carrying it (PNL = -F).

Deliver the stock against the option position and collect K

Hedging PNL= -(loan + interest) + (dividends) = - F

Option PNL = K   (deliver stock and get K)

Net PNL = K-F + FV(C-P) = FV[C-P – PV(F-K)]>0

Conversion and reversals

- Conversions and reversals are simplest examples of **option spreads**

  Reversal: sell put, buy call, short stock, or (synthetic long + physical short)

  Conversion: buy put, short call, buy stock, or (synthetic short + physical long)

- One nice way to thinking about when to do a conversion or a reversal is to compare the implied dividend from the options market and the implied from the forward (or, equivalently, the cost of carry).

$$q_{options} > q_{carry} + \varepsilon \quad \Rightarrow \quad \text{do a reversal}$$
$$q_{options} < q_{carry} - \varepsilon \quad \Rightarrow \quad \text{do a conversion}$$

- In other words, ``collect the most dividends''!

Hard to borrow stocks

- When you finance a long stock, you usually pay interest: FF (plus fee). This is a debit to the cash account.

- When you finance a short stock, you usually receive interest: FF (minus fee). This is usually a credit to the cash account.

- A stock is said to be hard-to-borrow, or special, if it is not easily available for stock-loan and therefore costs more to short.

- Lenders of special stock require an increased rate of interest (like a ``rent''). This extra interest can be viewed as a dividend that is collected by traders who are long and loan the stock at more than FF.

- In this case, since conversions are substitutes for short stock, conversions are expensive or equivalently reversals are attractive.

LNKD December 2011

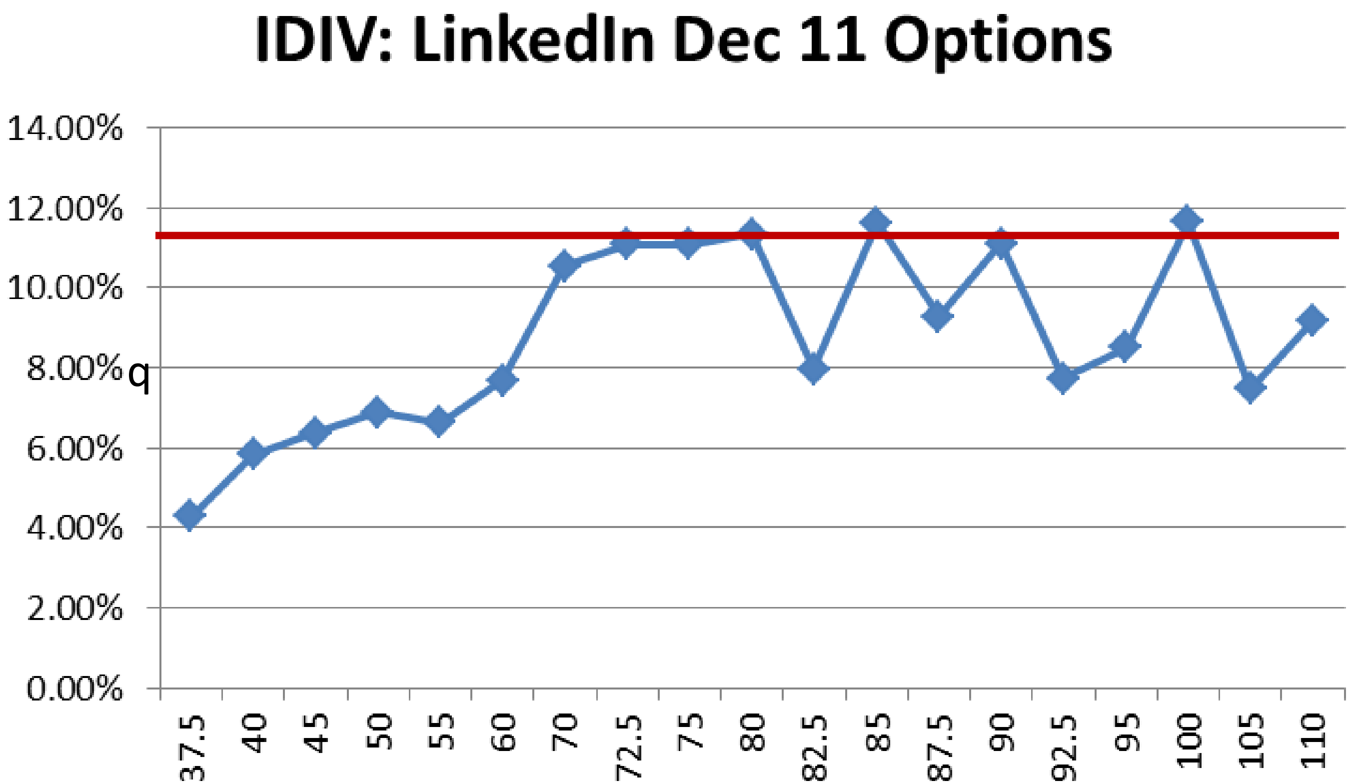
CALLS					PUTS				
Bid	Ask	Volume	Open Int	**STRIKE**	Bid	Ask	Volume	Open Int	IDIV
40	43.5	0	0	37.5	1.3	1.7	10	10	4.30%
37.4	41.2	0	0	40	1.55	2.15	1	4	5.85%
32.9	36.6	0	0	45	2.15	2.65	3	20	6.37%
28.8	32.3	1	1	50	3	3.6	20	77	6.89%
25.1	28.3	0	0	55	4.1	4.7	2	30	6.64%
21.2	24.5	0	0	60	5.5	6	10	88	7.68%
15.2	16.3	5	12	70	8.9	9.5	3	8	10.56%
13.8	14.7	5	52	72.5	9.9	10.7	8	8	11.09%
12.5	13.4	2	15	75	11.1	11.9	10	32	11.09%
10	10.9	10	43	80	13.5	14.6	10	65	11.36%
8.5	11.7	0	0	82.5	15	16.1	8	70	7.97%
8.1	8.9	11	22	85	16.7	17.6	1	96	11.63%
6.9	9.3	0	0	87.5	18.4	19.2	15	27	9.28%
6.3	7.7	32	337	90	20.1	21	3	9	11.11%
5.7	6.5	1	72	92.5	20	23	0	0	7.73%
5	5.5	6	38	95	21.8	24.8	0	0	8.51%
3.8	4.6	2	19	100	26.9	28.8	16	564	11.65%
2.9	4.1	12	110	105	29.9	32.8	0	0	7.48%
2.3	2.65	4	28	110	34.2	37.1	0	0	9.18%

IDIV is not zero although LinkedIn has never paid and has not announced a dividend.

This is due to the cost of shorting the stock:

$P-C > PV(F-K)$!

Plotting IDIV for LNKD



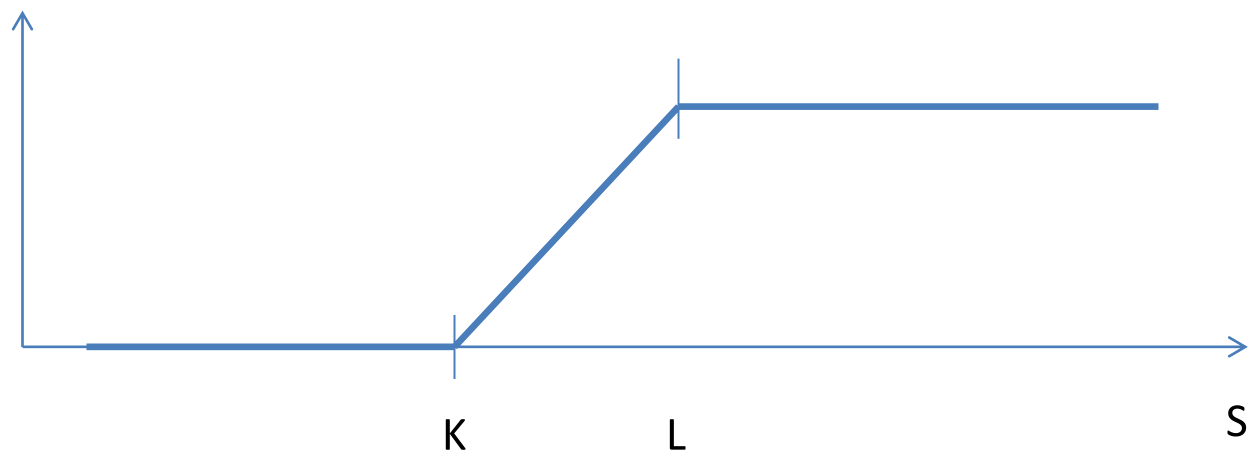
- Averaging the 75 and 80 strikes leads to $q_{\_option}$=11%, reflecting the difficulty of borrowing LNKD for short-selling.
- LNKD has traded with option-implied q's above 80% since its IPO last summer

Option Pricing

- In previous lectures, we covered forward pricing and the importance of cost-of carry
- We also covered Put-Call Parity, which can be viewed as relation that should hold between European-style puts and calls with the same expiration
- Put-call parity can be seen as pricing conversions relative to forwards on the same underlying asset
- What other relations exist between options and spreads on the same underlying asset?

Call Spread

Call Spread: Long a call with strike K, short a call with strike L (L>K)



Since the payoff is non-negative, the value of the spread must be positive

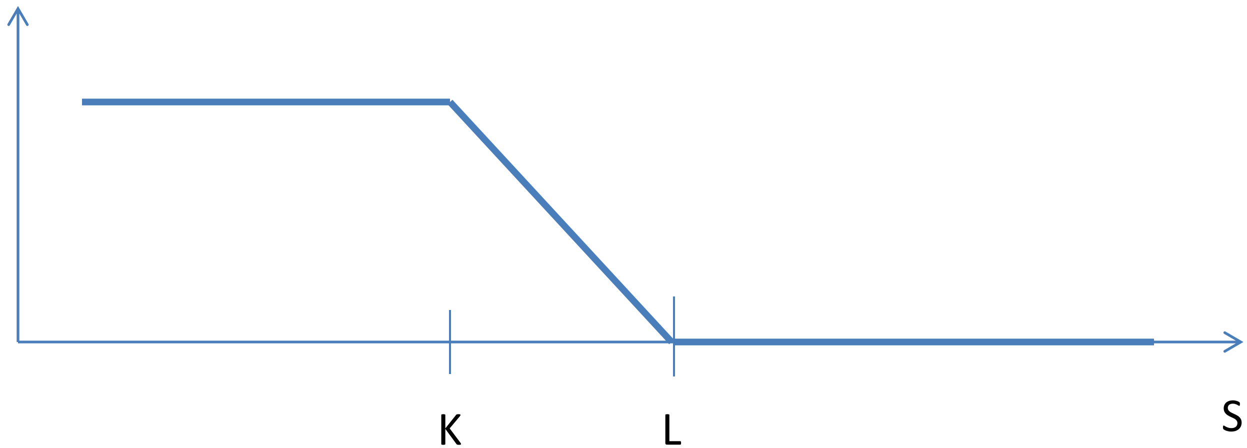
$$K < L \quad \Rightarrow \quad Call(K,T) > Call(L,T)$$

$$CS(K,L,T) = Call(K,T) - Call(L,T) > 0$$

Spread makes money if the price of the underlying goes up

Put Spread

Put Spread: Long a put with strike L, short a put with strike L (L>K)



Since the payoff is non-negative, the value of the spread must be positive

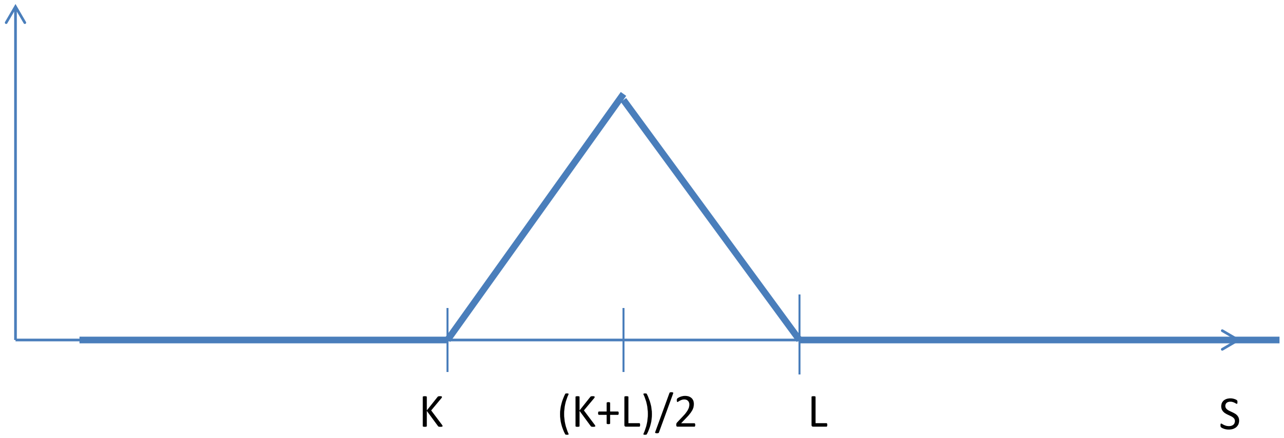
$$K < L \quad \Rightarrow \quad Put(K,T) < Put(L,T)$$

$$PS(K,L,T) = Put(L,T) - Put(K,T) > 0$$

Spread makes money if the price of the underlying goes down

Butterfly Spread

Butterfly spread: Long call with strike K, long call with strike L, short 2 calls with strike (K+L)/2



Since the spread has non-negative payoff, it must have positive value

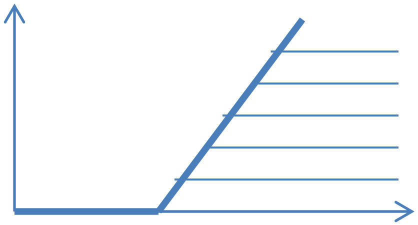
$$B\left(K,(K+L)/2,L,T\right) \equiv Call\left(K,T\right) + Call\left(L,T\right) - 2Call\left(\frac{K+L}{2},T\right) > 0$$

Butterflies make $ if the stock price is near (K+L)/2 at expiration.

Reconstructing Call prices from Butterfly Spreads

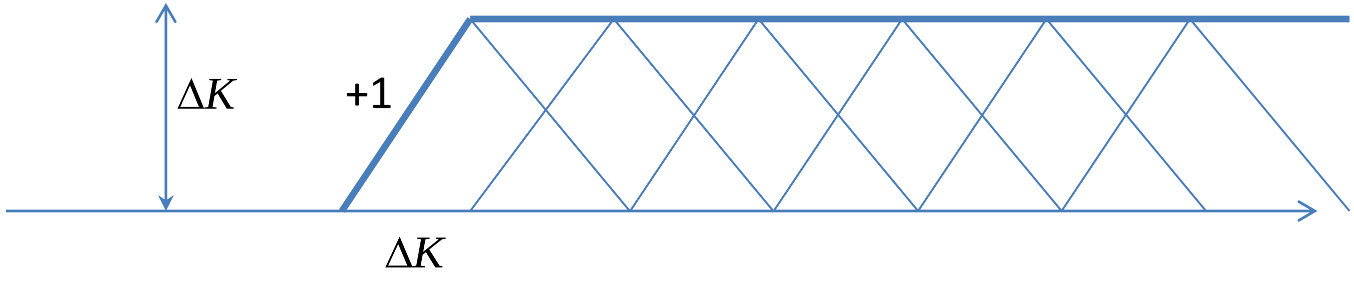
Assume for simplicity a countable number of strikes, $K_n = n\Delta K,\ n = 0,1,2,3,.....$
and that the stock price can only take values on the lattice $S_T = m\Delta K,\ m = 1,2,3,....$

$$Call(K_n, T) = \sum_{j \ge n} \left(Call(K_j, T) - Call(K_{j+1}, T) \right) = \sum_{j \ge n} CS(K_j, K_{j+1}, T)$$



- A call can be viewed as a portfolio of call spreads
- A call spread can be viewed as a portfolio of butterfly spreads

$$CS(K_j, K_{j+1}, T) = \sum_{i \ge j} B(K_i, K_{i+1}, K_{i+2}, T)$$



Calls as super-positions of butterfly spreads

$$Call(K_n,T) = \sum_{j=n}^{\infty} \sum_{i \geq j} B(K_i, K_{i+1}, K_{i+2}, T)$$

$$= \sum_{j=n}^{\infty} (j+1-n) B(K_j, K_{j+1}, K_{j+2}, T)$$

$$= \sum_{j=n}^{\infty} (j+1-n) \Delta K \cdot \left(\frac{B(K_j, K_{j+1}, K_{j+2}, T)}{\Delta K} \right)$$

$$= \sum_{j=n}^{\infty} (K_{j+1} - K_n) w(K_{j+1}, T) \qquad w(K_{j+1}, T) \equiv \frac{B(K_j, K_{j+1}, K_{j+2}, T)}{\Delta K}$$

$$= \sum_{j=0}^{\infty} (K_{j+1} - K_n)^+ w(K_{j+1}, T)$$

The weights correspond to values of Butterfly spreads centered at each Kj. In particular, they are positive

From weights to probabilities

$$w(K,T) > 0$$

$$\sum_{1}^{\infty} w(K_j,T) = \frac{1}{\Delta K}\sum_{1}^{\infty} B(K_{j-1},K_j,K_{j+1},T) = \frac{1}{\Delta K} CS(0,\Delta K,T)$$

$$= \mathrm{PV}(\$1) = e^{-rT}$$ (assuming that the stock can only take values $> \Delta K$)

$$w(K_j,T) = e^{-rT} p(K_j,T); \qquad \sum_j p(K_j,T) = 1, \quad p(K_j,T) > 0.$$

$$\boxed{\begin{aligned} Call(K,T) &= e^{-rT}\sum_j \max(K_j - K,0)\, p(K_j,T) \\ &= e^{-rT} E^{p}\{\max(S_T - K,0)\} \end{aligned}}$$

First moment of *p* is the forward price

$$
\begin{aligned}
E^{p}\{S\} &= e^{rT} Call(0,T) \\
&= e^{rT} e^{-qT} S_0 = S_0 e^{(r-q)T} \\
&= F_{0,T}
\end{aligned}
$$

- A call with strike 0 is the option to buy the stock at zero at time T
  Its value is therefore the present value of the forward price (pay now, get stock later).

- It follows that the first moment of p is the forward price.

- It also follows that put prices are given by a similar formula, namely

$$
\begin{aligned}
Put(K,T) &= Call(K,T) + Ke^{-rT} - Se^{-qT} \\
&= e^{-rT} E^{p}\left\{(S_T - K)^{+}\right\} + e^{-rT} K - e^{-rT} F \\
&= e^{-rT} E^{p}\left\{(S_T - K)^{+}\right\} + e^{-rT} E^{p}\{K - S_T\} \\
&= e^{-rT} E^{p}\left\{(K - S_T)^{+}\right\}
\end{aligned}
$$

General Payoffs

- Any twice differentiable function *f(S)* can be expressed as a combination
- of put and call payoffs, using the formula
- $$f(S) = f(0) + f'(0)(S-F) + \int_F^S (S-Y) f''(Y)\,dY$$ (this is just Taylor expansion)

$$= f(0) + f'(0)(S-F) + \int_0^F \underline{(Y-S)}^+ f''(Y)\,dY + \int_F^\infty \underline{(S-Y)}^+ f''(Y)\,dY$$

- Thus, a European-style payoff can be viewed as a spread of puts and calls. By linearity of pricing,

Fair value of a claim with payoff $f(S_T) =$

$$= e^{-rT} f(0) + 0 + \int_0^F Put(Y,T)\, f''(Y)\,dY + \int_F^\infty Call(Y,T)\, f''(Y)\,dY$$

$$= e^{-rT} f(0) + e^{-rT} f'(0) E^p\{S_T - F\} + e^{-rT} E^p \left\{ \int_0^F (Y - S_T)^+ f''(Y)\,dY + \int_F^\infty (S_T - Y)^+ f''(Y)\,dY \right\}$$

$$= e^{-rT} E^p\{f(S_T)\}$$

Fundamental theorem of pricing (one period model)

- An **arbitrage opportunity** is a portfolio of derivative securities and cash which has the following properties:

  - The payoff is non-negative in all future states of the market
  - The price of the portfolio is zero or negative (a credit)

Assume that each security has a unique price (i.e. assume bid-offer).

- If there are no arbitrage opportunities, then there exists a probability distribution of future states of the market such that, for any function *f(S)*, the price of a security with payoff $f(S_T)$ is

$$\boxed{P_f = e^{-rT} E^p \{ f(S_T) \}}$$

- Conversely, if such a probability exists there are no arbitrage opportunities

Practical Application to European Options

- A pricing measure is a probability of future prices of the underlying asset with the property that

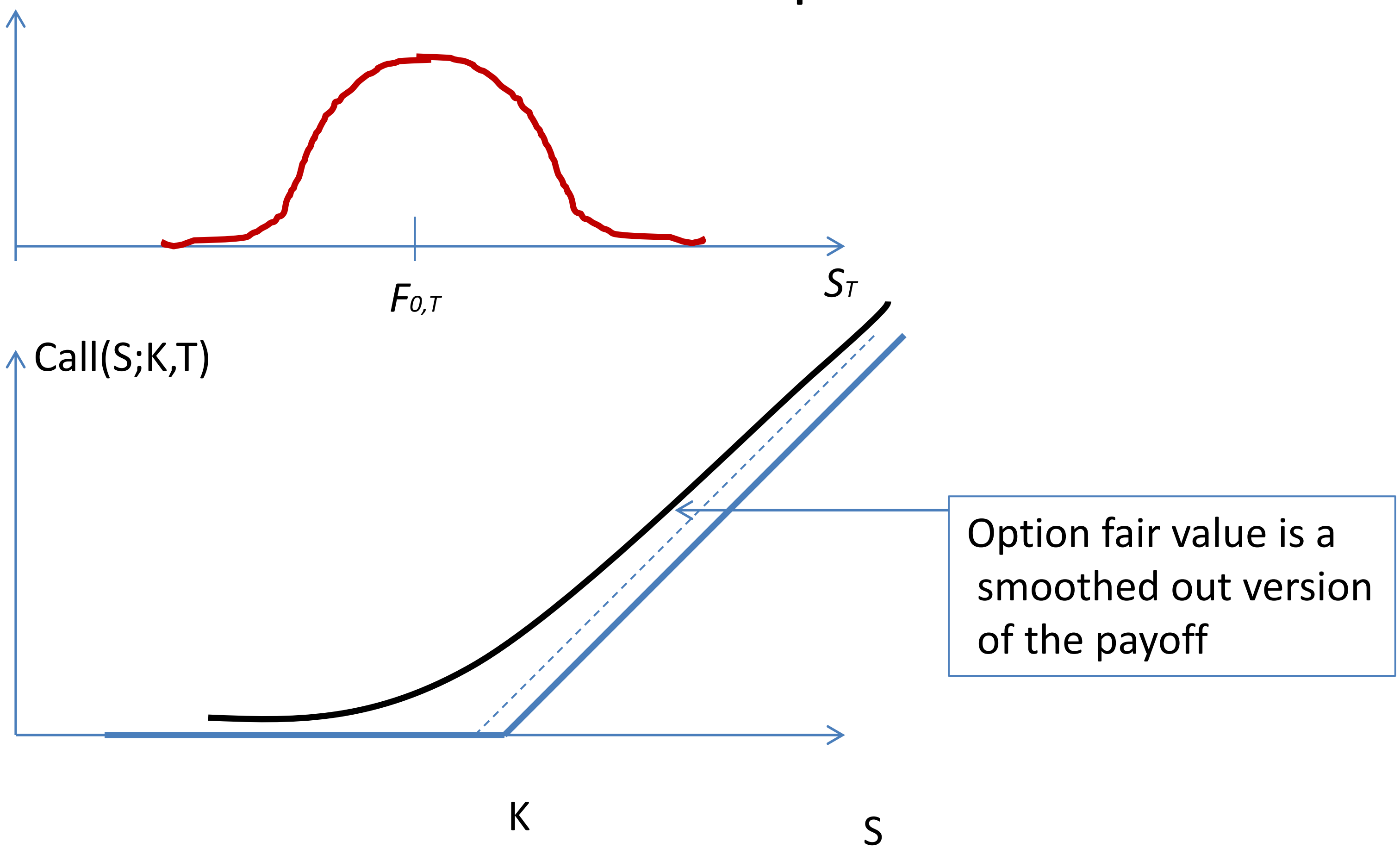
$$E^{p}\{S_T\} = F_{0,T}$$

- If we determine a suitable pricing measure, then all European options with expiration date *T* should have value given by

$$Call(K,T) = e^{-rT} E\{(S_T - K)^+\}, \quad Put(K,T) = e^{-rT} E\{(K - S_T)^+\}$$

- The main issue is then to determine a suitable pricing measure in the real, practical world.

What does a pricing measure achieve in the case of options?



The pricing measure gives a <u>model</u> to compute the option's fair value as a function of the price of the underlying asset, the strike and the maturity

The Black-Scholes Model

Assume that the pricing measure is log-normal, i.e. **log-returns are normal**

$$S_T = S_0 e^X, \quad X \sim N\left(\mu T, \sigma^2 T\right)$$

$$E\{S_T\} = \int_{-\infty}^{+\infty} S_0 e^y e^{-\frac{(y-\mu T)^2}{2\sigma^2 T}} \frac{dy}{\sqrt{2\pi\sigma^2 T}} = S_0 e^{\mu T + \frac{\sigma^2 T}{2}} = S_0 e^{\left(\mu + \frac{\sigma^2}{2}\right)T}$$

$$\therefore \quad \mu + \frac{\sigma^2}{2} = r - q \quad \therefore \quad \mu = r - q - \frac{\sigma^2}{2}$$

$$X = Z\sigma\sqrt{T} - \frac{\sigma^2}{2}T + (r-q)T, \quad Z \sim N(0,1)$$

Call pricing with the Black-Scholes model

$$Call(S,K,T) = e^{-rT} E\left\{(S_T - K)^+\right\} = e^{-rT} \int_{-\infty}^{+\infty} \left(S e^{z\sigma\sqrt{T} - \sigma^2 T/2 + (r-q)T} - K\right)^+ e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}}$$

$$= e^{-rT} \int_{A}^{+\infty} S e^{z\sigma\sqrt{T} - \sigma^2 T/2 + (r-q)T} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} - e^{-rT} K \int_{A}^{+\infty} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} \qquad \left(A = \frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{K}{S}\right) + \frac{\sigma^2 T}{2} - (r-q)T\right)\right)$$

$$= e^{-qT} S \left(\int_{A}^{+\infty} e^{z\sigma\sqrt{T} - \sigma^2 T/2} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}}\right) - e^{-rT} K \left(\int_{A}^{+\infty} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}}\right)$$

$$= e^{-qT} S \left(\int_{A}^{+\infty} e^{-\frac{(z-\sigma\sqrt{T})^2}{2}} \frac{dz}{\sqrt{2\pi}}\right) - e^{-rT} K \left(\int_{A}^{+\infty} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}}\right)$$

$$= e^{-qT} S \left(\int_{A-\sigma\sqrt{T}}^{+\infty} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}}\right) - e^{-rT} K \left(\int_{A}^{+\infty} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}}\right)$$

$$= e^{-qT} S \left(\int_{-\infty}^{-A+\sigma\sqrt{T}} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}}\right) - e^{-rT} K \left(\int_{-\infty}^{-A} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}}\right)$$

Black-Scholes Formula

$$BSCall(S,T,K,r,q,\sigma) = Se^{-qT}N(d_1) - Ke^{-rT}N(d_2)$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{F_{0,T}}{K}\right) + \frac{\sigma^2 T}{2}\right), \quad d_2 = \frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{F_{0,T}}{K}\right) - \frac{\sigma^2 T}{2}\right), \quad F_{0,T} = Se^{(r-q)T}$$

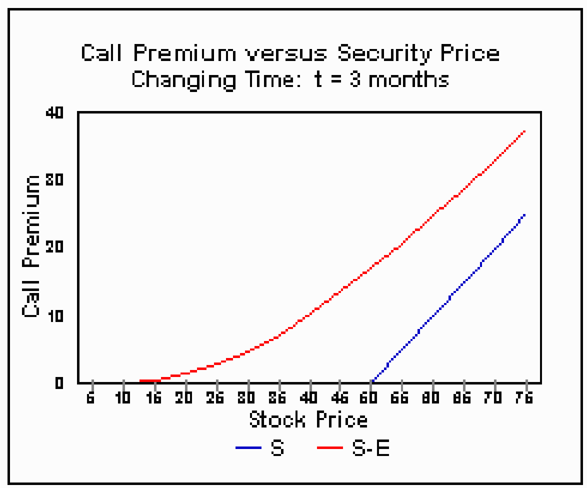
$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{z^2}{2}}\,dz$$

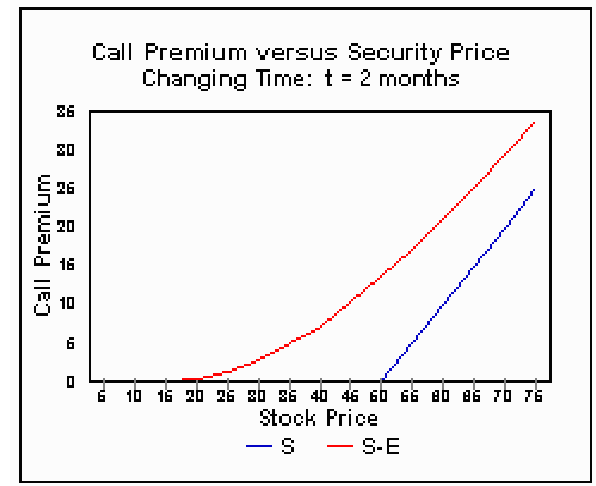
cumulative normal distribution

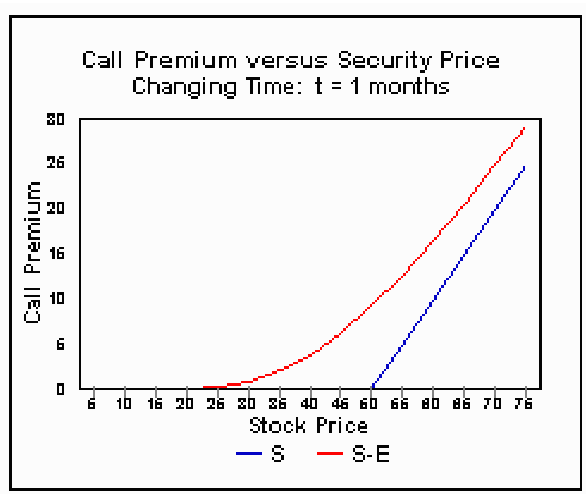
Programa para Black Scholes Call

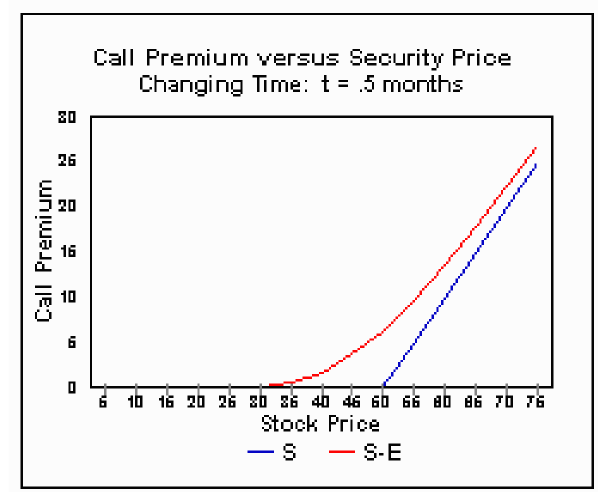
```
Public Function BSCall(ByVal S, ByVal t, ByVal k, ByVal R, ByVal D, ByVal V) As Double

S = S * Exp(-D * t)
k = k * Exp(-R * t)
V = V * Sqr(t)

Dim d1, d2 As Double
Dim x As Double

If k < 0.00001 Then
BSCall = S
End If

If V < 0.00001 Then
    If S > k Then
    BSCall = S - k
    End If
    If S <= k Then
    BSCall = 0
    End If
End If

d1 = Math.Log(S / k) / V + 0.5 * V
d2 = d1 - V

x = WorksheetFunction.NormDist(d1, 0, 1, True)
x = S * x
x = x - k * WorksheetFunction.NormDist(d2, 0, 1, True)

BSCall = x
End Function
```

Black-Scholes Formula at work









S=$48, K=$50, r=6%, sigma=40%, q=0

Time-series model for underlying price

- Derivative securities may depend on multiple expiration/cash-flow dates. Furthermore, the 1-period model described above is rigid in the sense that it cannot price American-style options.

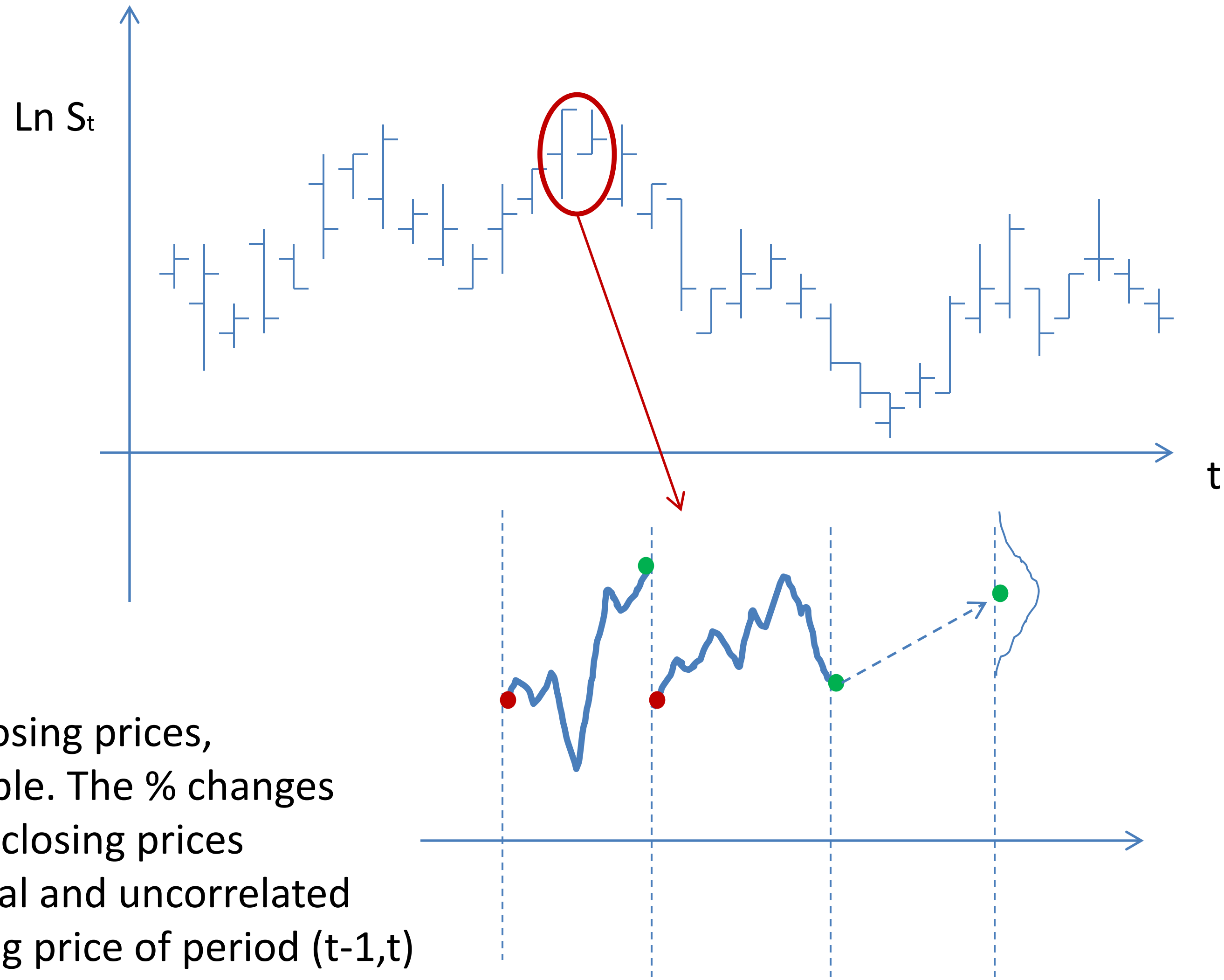
- We consider instead a more realistic approach to pricing based on the statistics of stock returns over short periods of time (e.g. 1 day).

- We assume that the underlying price has returns satisfying

$$\frac{S_{t+\Delta t} - S_t}{S_t} = \frac{\Delta S_t}{S_t} \sim N\left(\mu \Delta t, \sigma^2 \Delta t\right)$$

- We also assume that successive returns are uncorrelated.

Modeling the return of a price time-series (OHLC)



Model closing prices,
for example. The % changes
between closing prices
are normal and uncorrelated
St= closing price of period (t-1,t)

Parameterization

$$\mu\Delta t = E\left\{\frac{\Delta S_t}{S_t}\right\}, \qquad \sigma^2\Delta t = E\left\{\left(\frac{\Delta S_t}{S_t}\right)^2\right\} - \left(E\left\{\frac{\Delta S_t}{S_t}\right\}\right)^2$$

$\mu =$ annualized expected return

$\sigma =$ annualized standard deviation

1% daily standard deviation => 15.9% annualized standard deviation

$$\Delta t = \frac{1}{252}, \quad \sqrt{252} = 15.9$$

Pricing Derivatives

- Let us model the value of a derivative security as **a function of the underlying asset price and the time to expiration**

$$V_t = C(S_t, t) \qquad 0 < t < T$$

Change in market value over one period :

$$\Delta V_t = \Delta C(S_t, t)$$

$$= \frac{\partial C(S_t,t)}{\partial t}\Delta t + \frac{\partial C(S_t,t)}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 C(S_t,t)}{\partial t^2}\Delta t^2 + \frac{1}{2}\frac{\partial^2 C(S_t,t)}{\partial S^2}(\Delta S_t)^2 + ...$$

$$= \frac{\partial C(S_t,t)}{\partial t}\Delta t + \frac{\partial C(S_t,t)}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 C(S_t,t)}{\partial S^2}S_t^{\,2}\left(\frac{\Delta S_t}{S_t}\right)^2 + o(\Delta t)$$

$$= \left(\frac{\partial C(S_t,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 C(S_t,t)}{\partial S^2}S_t^{\,2}\sigma^2\right)\Delta t + \frac{\partial C(S_t,t)}{\partial S}\Delta S_t + \frac{1}{2}\frac{\partial^2 C(S_t,t)}{\partial S^2}S_t^{\,2}\left[\left(\frac{\Delta S_t}{S_t}\right)^2 - \sigma^2\Delta t\right] + o(\Delta t)$$

$$= \alpha\Delta t + \beta\Delta S_t + \varepsilon_t$$

The hedging argument

- Consider a portfolio which is long 1 derivative and short β stocks.
- Assume derivative does not pay dividends

$$\boxed{\beta = \beta_t = \frac{\partial C(S_t, t)}{\partial S_t}}$$

Profit and loss, including financing and dividends:

$$PNL = -V_t \cdot r\Delta t + \Delta V_t - \beta\left(\Delta S_t - S_t r\Delta t + S_t q\Delta t\right)$$

$$= -V_t \cdot r\Delta t + \alpha\Delta t + \beta\Delta S_t + \varepsilon_t - \beta\left(\Delta S_t - S_t r\Delta t + S_t q\Delta t\right)$$

$$= -V_t \cdot r\Delta t + \alpha\Delta t + \beta S_t (r-q)\Delta t + \varepsilon_t$$

$$= \left(-C(S_t, t)r + \frac{\partial C(S_t, t)}{\partial t} + \frac{\partial C(S_t, t)}{\partial S} S(r-q) + \frac{1}{2}\frac{\partial^2 C(S_t, t)}{\partial S^2} S_t^2 \sigma^2\right)\Delta t + \varepsilon_t$$

Analyzing the residual term ε_t

$$\varepsilon_t = S_t^2 \frac{\partial^2 C(S_t, t)}{\partial S^2}\left[\left(\frac{\Delta S_t}{S_t}\right)^2 - \sigma^2 \Delta t\right] + o(\Delta t)$$

Conditional expectation of epsilon

$$E\{\varepsilon_t \mid S_t\} = S_t^2 \frac{\partial^2 C(S_t, t)}{\partial S^2} E\left\{\left(\frac{\Delta S_t}{S_t}\right)^2 - \sigma^2 \Delta t \,\middle|\, S_t\right\} + o(\Delta t)$$

$$= S_t^2 \frac{\partial^2 C(S_t, t)}{\partial S^2} \mu^2 \Delta t^2 + o(\Delta t)$$

$$= o(\Delta t)$$

- The residual term has essentially zero expected return (vanishing exp. return in the limit Dt_>0.)

The fair value of our derivative security is…

- The PNL for the long short portfolio of **1 derivative and – beta shares** has expected value

$$
\begin{aligned}
E\{PNL\} &= \alpha\Delta t + o(\Delta t) \\
&= \left(-C(S_t,t)r + \frac{\partial C(S_t,t)}{\partial t} + \frac{\partial C(S_t,t)}{\partial S} S(r-q) + \frac{1}{2}\frac{\partial^2 C(S_t,t)}{\partial S^2} S_t^2 \sigma^2 \right)\Delta t + o(\Delta t)
\end{aligned}
$$

- This portfolio has no exposure to the stock price changes. Therefore, if *C(S$_t$,t)* represents the ``fair value'' of the derivative, the portfolio should have zero rate of return (we already took into acct its financing). Thus:

$$
\boxed{\frac{\partial C(S\ ,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S_t,t)}{\partial S^2} + (r-q)S\frac{\partial C(S\ ,t)}{\partial S} - C(S,t)r = 0}
$$

- This is the Black-Scholes partial differential equation (PDE).

American-style calls & puts

- Consider a call option on an underlying asset paying dividends continuously. Since the option can be exercised anytime, we have

$$C(S,t) \geq \max(S-K,0), \qquad t<T. \qquad (1)$$

- The terminal condition at *t=T* corresponds to the final payoff

$$C(S,T) = \max(S-K,0).$$

- Thus, the function *C(S,t)* should satisfy the **Black-Scholes PDE** in the region of the (S,t)-plane for which strict inequality holds in (1), and it should be equal to *max(S-K,0)* otherwise.

- The solution of this problem is done numerically and will be addressed in the next lecture.

The Black Scholes PDE

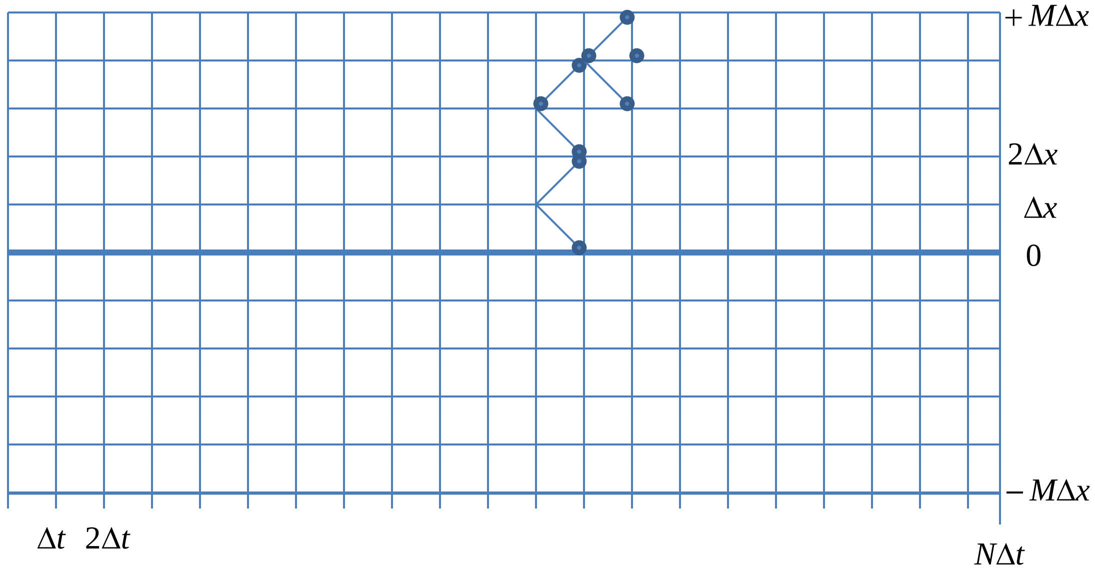
- The hedging argument for assets with normal returns presented at the end of Lecture 4 gave rise to the Black Scholes PDE

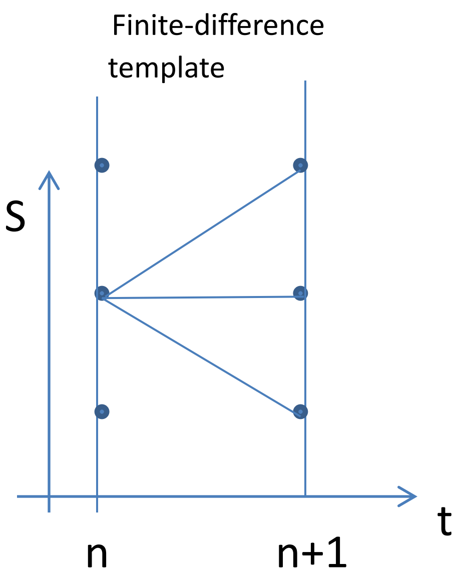
$$\frac{\partial C(S,t)}{\partial t}+\frac{\sigma^2 S^2}{2}\frac{\partial^2 C(S,t)}{\partial S^2}+(r-q)S\,\frac{\partial\ C(S,t)}{\partial S}-rC(S,t)=0$$

r=interest rate, q=dividend yield, $\sigma =$ volatility. The volatility is the annualized standard deviation of returns (it is not a market price or, rate, but rather a model input).

- We introduce a method for solving this PDE numerically on a grid.

Finite-difference scheme, or ``trinomial tree''





$$S_n^j = S_0 e^{j\Delta x}, \quad -M \le j \le +M$$

$$C_n^j \leftrightarrow C\left(S_n^j, n\Delta t\right), \; 0 \le n \le N$$

Change of variables

$$S = S_0 e^x$$

$$S\frac{\partial C}{\partial S} = S\frac{\partial C}{\partial x}\frac{\partial x}{\partial S} = S\frac{\partial C}{\partial x}\frac{1}{S} = \frac{\partial C}{\partial x}$$

$$\begin{aligned}
S^2\frac{\partial^2 C}{\partial S^2} &= S^2\frac{\partial}{\partial S}\left(\frac{1}{S}S\frac{\partial C}{\partial S}\right) = S^2\frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial C}{\partial x}\right) \\
&= S\frac{\partial}{\partial x}\left(\frac{1}{S}\frac{\partial C}{\partial x}\right) \\
&= \frac{\partial^2 C}{\partial x^2} - \frac{\partial C}{\partial x}
\end{aligned}$$

BS equation in log-price

$$\boxed{\frac{\partial C}{\partial t} + \left(r - q - \frac{1}{2}\sigma^2\right)\frac{\partial C}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 C}{\partial x^2} - rC = 0}$$

Taylor expansion & symmetric finite-difference approximations for derivatives

$$f(x) = f(0) + f'(0)x + \frac{1}{2} f''(0)x^2 + \ldots$$

$$f(-x) = f(0) - f'(0)x + \frac{1}{2} f''(0)x^2 + \ldots$$

$$\therefore$$

$$f(x) - f(-x) = 2f'(0)x + o\left(x^2\right)$$
$$f(x) + f(-x) = 2f(0) + f''(0)x^2 + o\left(x^3\right)$$

$$\therefore$$

$$f'(0) = \frac{f(x) - f(-x)}{2x} + o(x)$$
$$f''(0) = \frac{f(x) + f(-x) - 2f(0)}{x^2} + o(x)$$

Symmetric finite difference approximations for first and second derivatives

Discretization of the PDE

$$\frac{\partial C(S,t)}{\partial t} \leftrightarrow \frac{C_{n+1}^{j} - C_{n}^{j}}{\Delta t}$$

Here we do not use symmetric differences

$$\frac{\partial C(S,t)}{\partial x} \leftrightarrow \frac{C_{n+1}^{j+1} - C_{n+1}^{j-1}}{2\Delta x}$$

$$\frac{\partial^2 C(S,t)}{\partial x^2} \leftrightarrow \frac{C_{n+1}^{j+1} + C_{n+1}^{j-1} - 2C_{n+1}^{j}}{(\Delta x)^2}$$

Here use symmetric differences

$$\frac{C_{n+1}^{j} - C_{n}^{j}}{\Delta t} + (r - q - \frac{\sigma^2}{2}) \frac{C_{n+1}^{j+1} - C_{n+1}^{j-1}}{2\Delta x} + \frac{\sigma^2}{2} \frac{C_{n+1}^{j+1} + C_{n+1}^{j-1} - 2C_{n+1}^{j}}{(\Delta x)^2} - rC_{n}^{j} = 0$$

From PDE to recursive scheme

$$\frac{C_{n+1}^{j}-C_{n}^{j}}{\Delta t}+(r-q-\frac{\sigma^{2}}{2})\frac{C_{n+1}^{j+1}-C_{n+1}^{j-1}}{2\Delta x}+\frac{\sigma^{2}}{2}\frac{C_{n+1}^{j+1}+C_{n+1}^{j-1}-2C_{n+1}^{j}}{(\Delta x)^{2}}-rC_{n}^{j}=0$$

$$C_{n}^{j}=C_{n+1}^{j}+\left(\frac{\sigma^{2}\Delta t}{2(\Delta x)^{2}}+\frac{(r-q-\sigma^{2}/2)\Delta t}{2\Delta x}\right)C_{n+1}^{j+1}+\left(1-\frac{\sigma^{2}\Delta t}{(\Delta x)^{2}}\right)C_{n+1}^{j}+\left(\frac{\sigma^{2}\Delta t}{2(\Delta x)^{2}}-\frac{(r-q-\sigma^{2}/2)\Delta t}{2\Delta x}\right)C_{n+1}^{j-1}-r\Delta tC_{n}^{j}$$

$$\boxed{\begin{aligned}
&C_{n}^{j}=\frac{1}{1+r\Delta t}\left(p_{\mathrm{U}}C_{n+1}^{j+1}+p_{\mathrm{M}}C_{n+1}^{j}+p_{\mathrm{D}}C_{n+1}^{j-1}\right)\\
&\left\{\begin{aligned}
p_{\mathrm{U}}&=\frac{\sigma^{2}\Delta t}{2(\Delta x)^{2}}+\frac{(r-q-\sigma^{2}/2)\Delta t}{2\Delta x}\\
p_{\mathrm{M}}&=1-\frac{\sigma^{2}\Delta t}{(\Delta x)^{2}}\\
p_{\mathrm{D}}&=\frac{\sigma^{2}\Delta t}{2(\Delta x)^{2}}-\frac{(r-q-\sigma^{2}/2)\Delta t}{2\Delta x}
\end{aligned}\right.
\end{aligned}}$$

Interpreting the weights

- Notice that

$$p_{\mathrm{U}} + p_{\mathrm{M}} + p_{\mathrm{D}} = 1$$

- Set

$$\Delta x = \sigma_{\max}\sqrt{\Delta t}$$

$$\mu = r - q - \frac{\sigma^2}{2}$$

$$p = \frac{\sigma^2 \Delta t}{2(\Delta x)^2} = \frac{\sigma^2}{2\sigma_{\max}^2}$$

- The weights become

$$p_{\mathrm{U}} = p + \frac{\mu\sqrt{\Delta t}}{2\sigma_{\max}}$$

$$p_{\mathrm{M}} = 1 - 2p$$

$$p_{\mathrm{D}} = p - \frac{\mu\sqrt{\Delta t}}{2\sigma_{\max}}$$

Stability conditions & probabilities

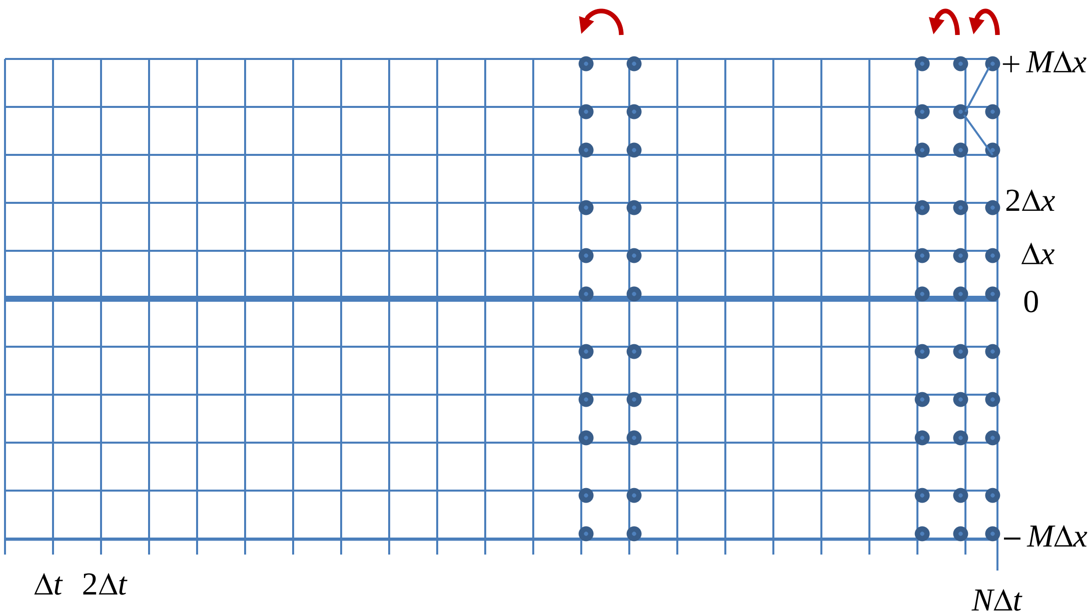
$$\left.\begin{array}{l} p < 1/2 \\ \text{and} \\ \dfrac{|\mu|\sqrt{\Delta t}}{\sigma_{\max}} < 1 \end{array}\right. \quad \Rightarrow \quad p_U > 0,\, p_M > 0,\, p_D > 0$$

- In this case, the discretization of the PDE corresponds to discounting over probabilities

$$C_n^j = \frac{1}{1+r\Delta t}\left(p_{\mathrm{U}} C_{n+1}^{j+1} + p_{\mathrm{M}} C_{n+1}^{j} + p_{\mathrm{D}} C_{n+1}^{j-1}\right)$$

- This gives a simple and intuitive interpretation of the B-S PDE

European Options



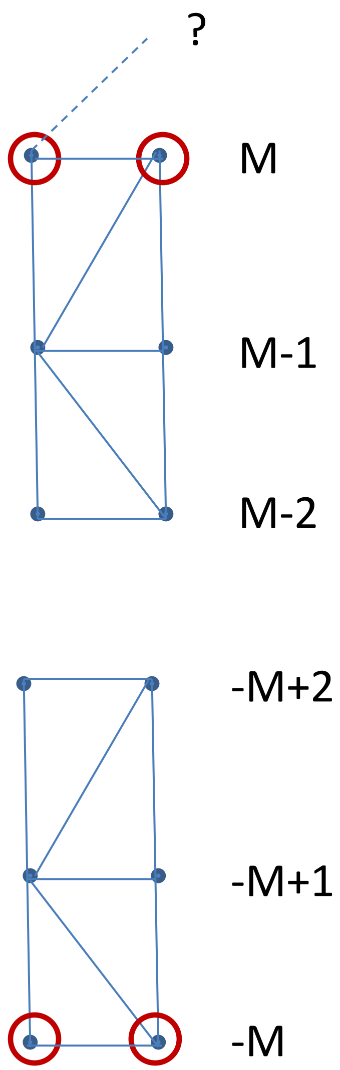
- Value at expiration date

$$C_N^j = \max\left(S_0 e^{j\sigma_{\max}\sqrt{\Delta t}} - K, 0\right), \quad -M \le j \le +M \qquad \text{(call)}$$
$$C_N^j = \max\left(K - S_0 e^{j\sigma_{\max}\sqrt{\Delta t}}, 0\right), \quad -M \le j \le +M \qquad \text{(put)}$$

- Solve recursively

$$C_n^j = \frac{1}{1 + r\Delta t}\left[p_{\mathrm{U}} C_{n+1}^{j+1} + p_{\mathrm{M}} C_{n+1}^{j} + p_{\mathrm{D}} C_{n+1}^{j-1}\right], \quad -M < j < +M,\ \ n = N-1, N-2, \ldots, 0$$

Boundary Nodes



$$C_n^{M} = 2C_n^{M-1} - C_n^{M-2} \qquad \text{(upper boundary)}$$

$$C_n^{-M} = 2C_n^{-M+1} - C_n^{-M+2} \qquad \text{(lower boundary)}$$

These boundary conditions are called ``radiation boundary conditions'' or ``zero-gamma'' boundary conditions. They assume that there is <u>no convexity at the boundary</u>, so the values at the boundary will not affect the computation significantly.

(More on this later…)

VB pseudo code (1)

```
Function BSCall(ByVal S As Double, ByVal T As Double, ByVal K As Double, ByVal r As Double, _
ByVal q As Double, ByVal sigma As Double) As Double

'set mesh = 1day
Dim dt As Double
dt = 1# / 252
'set number of time steps
Dim N As Integer
N = CInt(T / dt)
'set carry
Dim mu As Double
mu = r – q-0.5 * sigma*sigma

'set sigma max for stability requirements
Dim smax As Double
smax = 2 * Abs(mu) * sqrt(dt)
If smax < sigma * sqrt(2) Then
smax = sigma * sqrt(2)
End If
If smax = 0 Then
BSCall = -9999
End If
```

This ensures that smax is large enough

VB Code (II)

```
'allocate arrays
Dim M As Integer
M = CInt(5 * sqrt(N))

Dim S() As Double
Dim C() As Double
Dim pC() As Double

ReDim C(1 To 2 * M + 1)
ReDim pC(1 To 2 * M + 1)
ReDim S(1 To 2 * M + 1)

'probabilities
Dim PU, PM, PD As Double
Dim p As Double
p = 0.5 * sigma * sigma / (smax * smax)

PU = p + 0.5 * mu * sqrt(dt) / smax
PM = 1 - 2 * p
PD = p - 0.5 * mu * sqrt(dt) / smax
```

This sets the vertical dimension

From the discretization of
The PDE

VB Code (III)

```
'initialize call payoff
Dim D, E As Double
D = 1# / (1 + r * dt)
E = Exp(smax * sqrt(dt))


S(1) = S * Exp(-M * smax * sqrt(dt))
For j = 2 To 2 * M + 1
S(j) = S(j - 1) * E
Next j

For j = 1 To 2 * M + 1
C(j) = Max(S(j) - K, 0)
Next j

' time loop
For K = 1 To N
   'interior nodes
   For j = 2 To 2 * M
   pC(j) = PU * C(j + 1) + PM * C(j) + PD * C(j - 1)
   pC(j) = pC(j) * D
   Next j

   'boundary nodes
   pC(1) = 2 * pC(2) - pC(3)
   pC(2 * M + 1) = 2 * pC(2 * M) - pC(2 * M - 1)

   'copy array
   For j = 1 To 2 * M + 1
   C(j) = pC(j)
   Next j

Next K

BSCall = C(M + 1)
End Function
```

Discount factor and vertical mesh size

Main loop

Answer= central vertical node

Discussion

- This is called an explicit scheme, which means that we ``roll back'', solving time n in terms of time n+1
- For this to work, we need smax large enough so that the ``probabilities'' are positive (stability)
- The requirement that M=5*sqrt(N) has to do with the fact that the grid must be large enough to avoid ``feeling the boundary''
- The result at the end is the full vertical array at n=0, so we get more information than just the central node, if we wish.

American Options

- We must enforce the requirement that, at each node, the value of the option is greater than the payoff (intrinsic value

$$C_n^j \geq \max\left(S_n^j - K, 0\right) \quad \text{(call)}$$
$$C_n^j \geq \max\left(K - S_n^j, 0\right) \quad \text{(put)}$$

Let $F(S)$ be the intrinsic value. Then,

$$\boxed{C_n^j = \max\left[F\left(S_n^j\right), \frac{1}{1 + r\Delta t}\left(p_{\mathrm{U}} C_{n+1}^{j+1} + p_{\mathrm{M}} C_{n+1}^{j} + p_{\mathrm{D}} C_{n+1}^{j-1}\right)\right]}$$

Why is the numerical scheme correct?

- An American-style option is always greater than the IV

- Suppose that you know the value of the American option at time $t_{n+1} = (n+1)\Delta t$.

- A European option with payoff $F(S, t_{n+1}) = C_{n+1}^{j},\ \ S = S_0 e^{j\Delta x}$ expiring at time t_{n+1} has a value at time $t_n = n\Delta t$ equal to

$$V_n^j = \frac{1}{1+r\Delta t}\left[p_{\mathrm{U}} C_{n+1}^{j+1} + p_{\mathrm{M}} C_{n+1}^{j} + p_{\mathrm{D}} C_{n+1}^{j-1}\right]$$

- An American option gives the right to exercise at time t_n or to continue. If you continue, this is like holding the European-style derivative for one more time period. Therefore,

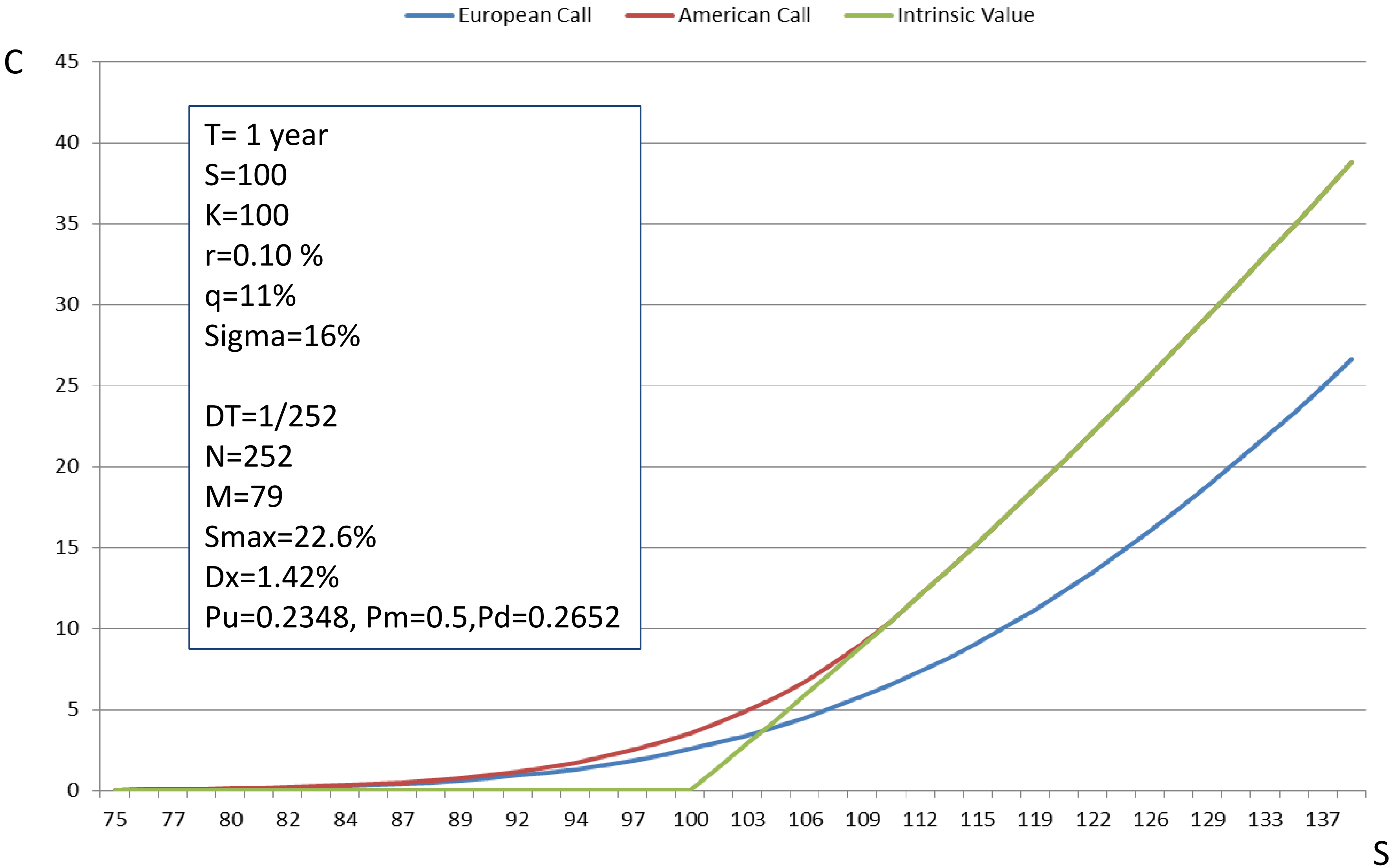
$$C_n^j = \max\left[IV\left(S_n^j\right), V_n^j\right] = \max\left[IV\left(S_n^j\right), \frac{1}{1+r\Delta t}\left[p_{\mathrm{U}} C_{n+1}^{j+1} + p_{\mathrm{M}} C_{n+1}^{j} + p_{\mathrm{D}} C_{n+1}^{j-1}\right]\right]$$

VB code for American Call

```
' time loop
For K = 1 To N
   'interior nodes
   For j = 2 To 2 * M
   pC(j) = PU * C(j + 1) + PM * C(j) + PD * C(j - 1)
   pC(j) = pC(j) * D
   Next j

   'boundary nodes
   pC(1) = 2 * pC(2) - pC(3)
   pC(2 * M + 1) = 2 * pC(2 * M) - pC(2 * M - 1)

   'copy array & compare with intrinsic value
   For j = 1 To 2 * M + 1
   C(j) = pC(j)

            If C(j)<Max(S(j)-K,0)  then
            C(j)=Max(S(j)-K,0)
            End if
   Next j

Next K

BSCall = C(M + 1)
End Function
```

This guarantees that C is at least equal to the intrinsic value. Everything else is the same.

Pricing a 1-year call numerically

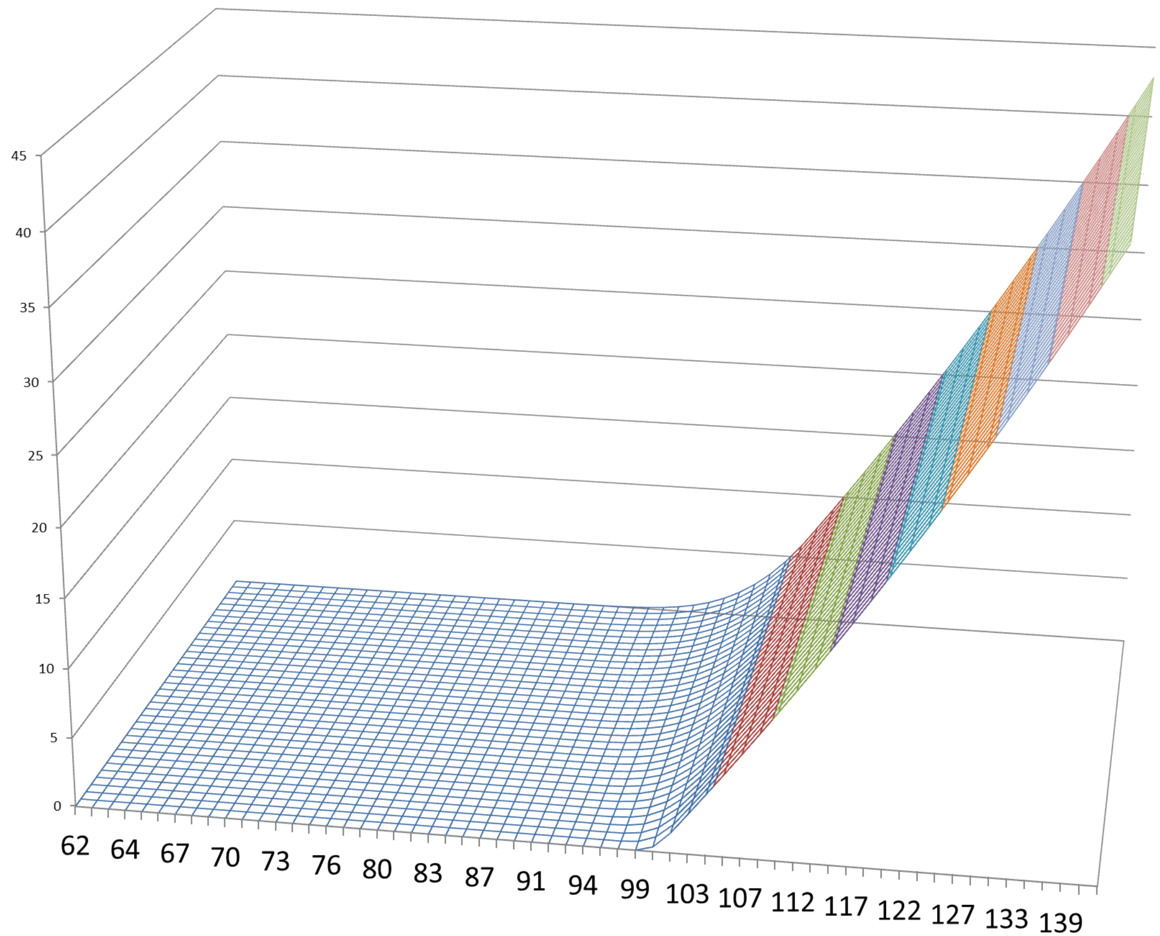


Numerical solution vs. Black-Scholes for European options

S	Numerical	Black-Scholes	Diff
74.13	0.02184	0.022141	3E-04
75.2	0.02903	0.029406	4E-04
76.28	0.03833	0.038791	5E-04
77.37	0.05027	0.050829	6E-04
78.48	0.06549	0.06616	7E-04
79.61	0.08474	0.085546	8E-04
80.75	0.10892	0.109889	1E-03
81.91	0.13909	0.14024	0.001
83.09	0.17647	0.177822	0.001
84.28	0.22244	0.224035	0.002
85.49	0.27862	0.280472	0.002
86.72	0.34677	0.348926	0.002
87.96	0.42891	0.431395	0.002
89.22	0.52724	0.530085	0.003
90.5	0.64416	0.647404	0.003
91.8	0.78228	0.785954	0.004
93.12	0.94437	0.948515	0.004
94.46	1.13339	1.138026	0.005
95.81	1.35239	1.357555	0.005
97.19	1.60454	1.610268	0.006
98.58	1.89309	1.899388	0.006
100	2.22126	2.228156	0.007
101.4	2.59228	2.59978	0.008
102.9	3.00928	3.017386	0.008
104.4	3.47525	3.483969	0.009
105.9	3.99303	4.002341	0.009
107.4	4.56521	4.575086	0.01
108.9	5.1941	5.204516	0.01
110.5	5.88171	5.892626	0.011
112.1	6.62971	6.641072	0.011
113.7	7.43938	7.45114	0.012
115.3	8.31164	8.323734	0.012
117	9.24702	9.259366	0.012
118.7	10.2456	10.25817	0.013
120.4	11.3073	11.31989	0.013
122.1	12.4313	12.44395	0.013
123.8	13.6168	13.62941	0.013
125.6	14.8626	14.87508	0.012

- Same parameters as previous example
- Compared BS with numerical scheme
- Adjust the time-step to produce acceptable error
- Use numerical code to price American options

Numerical solution as a surface



Early-exercise boundary

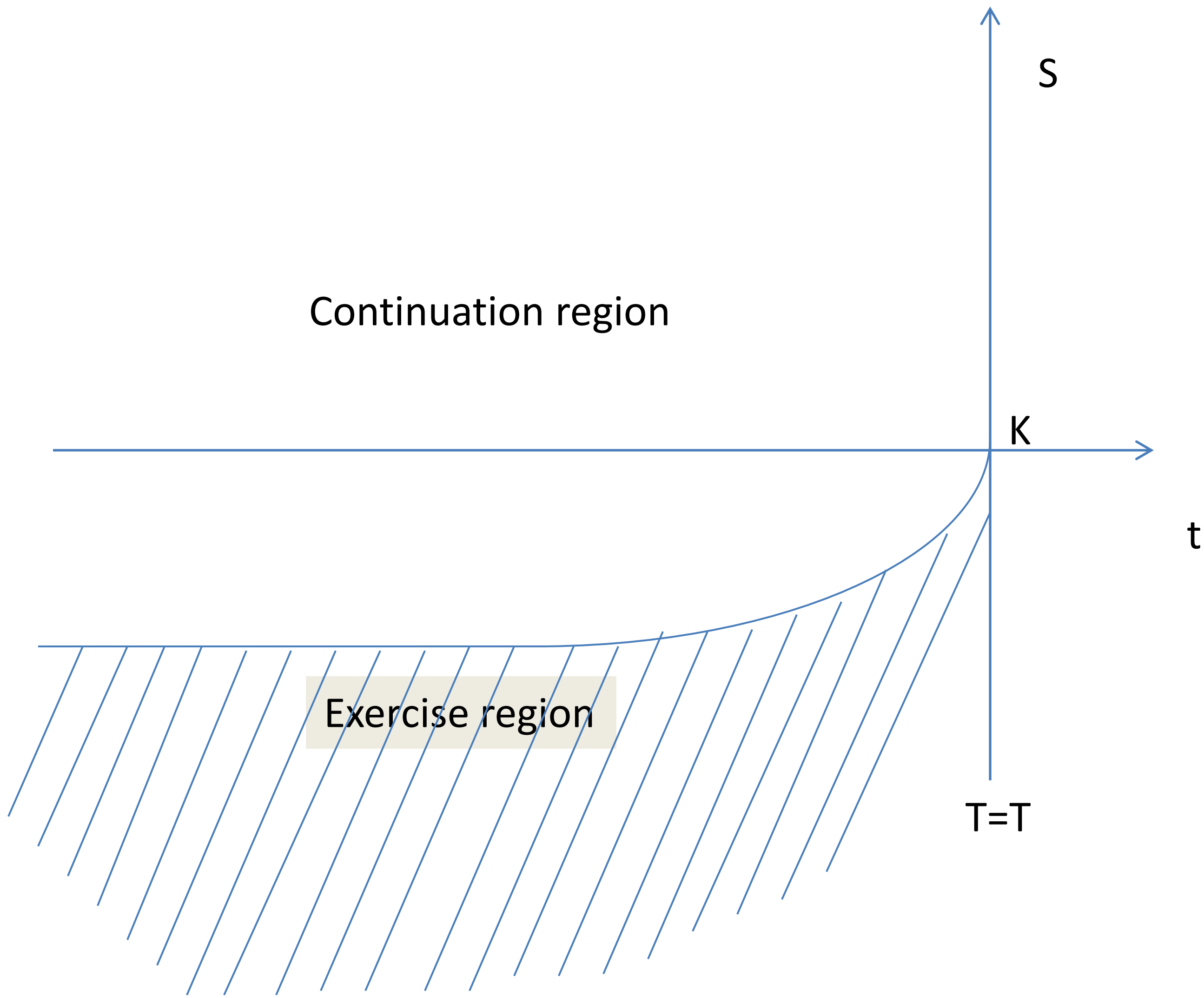
- Exercising a call option makes sense only if the stock is sufficiently high (and the dividend income is greater than the potential increase in the option price).

- Exercising a put option makes sense only if the stock is sufficiently low, so that the interest income from the cash received exceeds the expected gains from increase in option price.

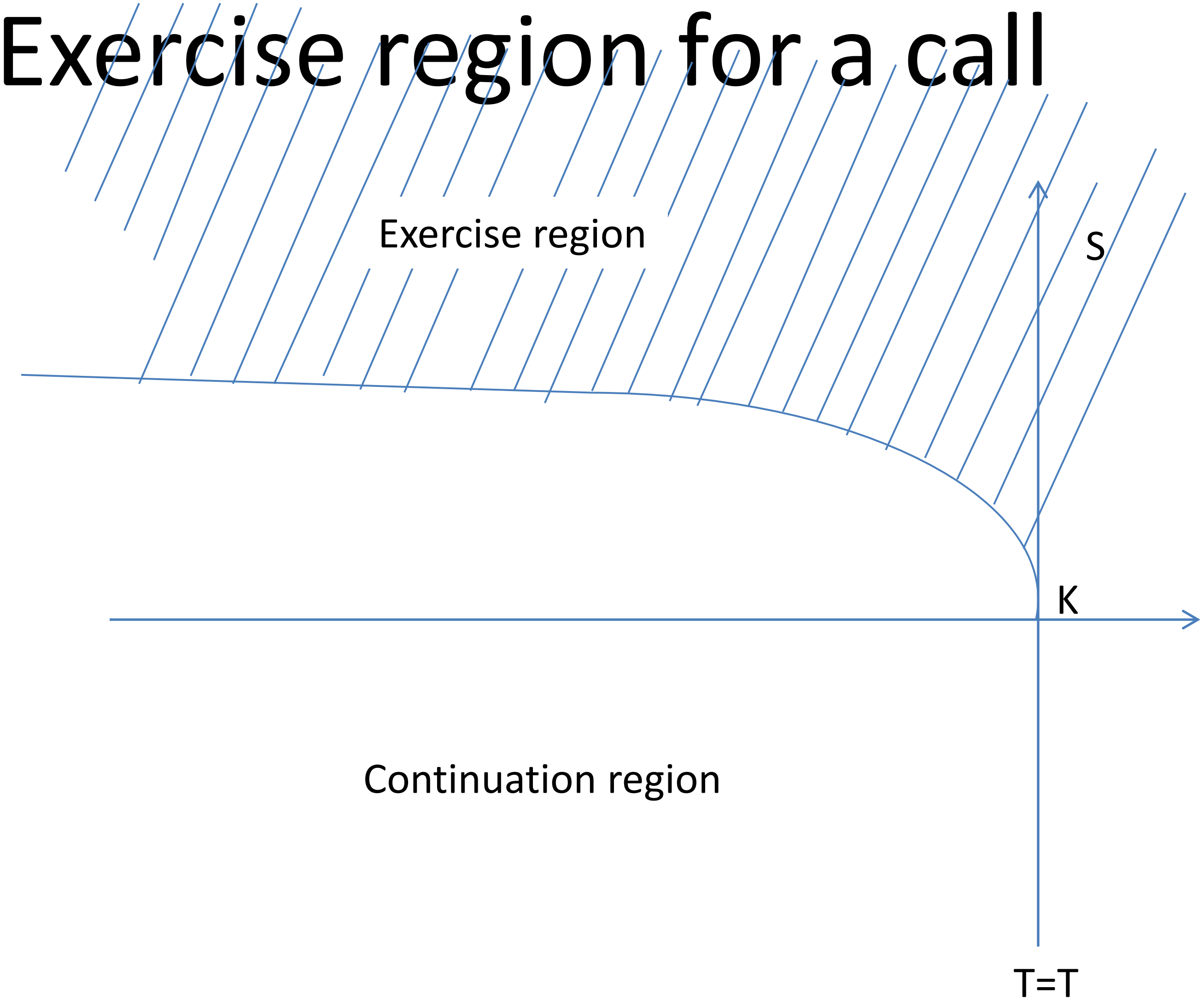
- Therefore, the **exercise regions** for calls/puts will be of the form

$$\boxed{ER_{call}=\{ (S,t):\ S>S^*(t)\} \qquad ER_{put}=\{ (S,t):\ S<S^{**}(t)\}}$$

Exercise region for a put



Exercise region for a call



Applications of the Black-Scholes pricing model

- In the case of European-style options, we obtain a compact formula for the value of options:

$$C_{eur} = BSCall(S, K, T, r, d, \sigma) \quad P_{eur} = BSPut(S, K, T, r, d, \sigma)$$

- In the case of American-style options, we have numerical scheme which depends on the same 6 parameters and gives the value with arbitrary precision.

- Of the 6 parameters, 5 of them are observable or derivable from the market (e.g. implied dividend)

- The volatility parameter is NOT observable or derivable from the market in an unequivocal way. It is an essential component of the model.

Implied Volatility

- The implied volatility of an option is the volatility that makes the Black-Scholes pricing formula true

$$C = BSCall\left(S, T, K, r, d, \sigma_{imp}\right), \quad P = BSPut\left(S, T, K, r, d, \sigma_{imp}\right)$$

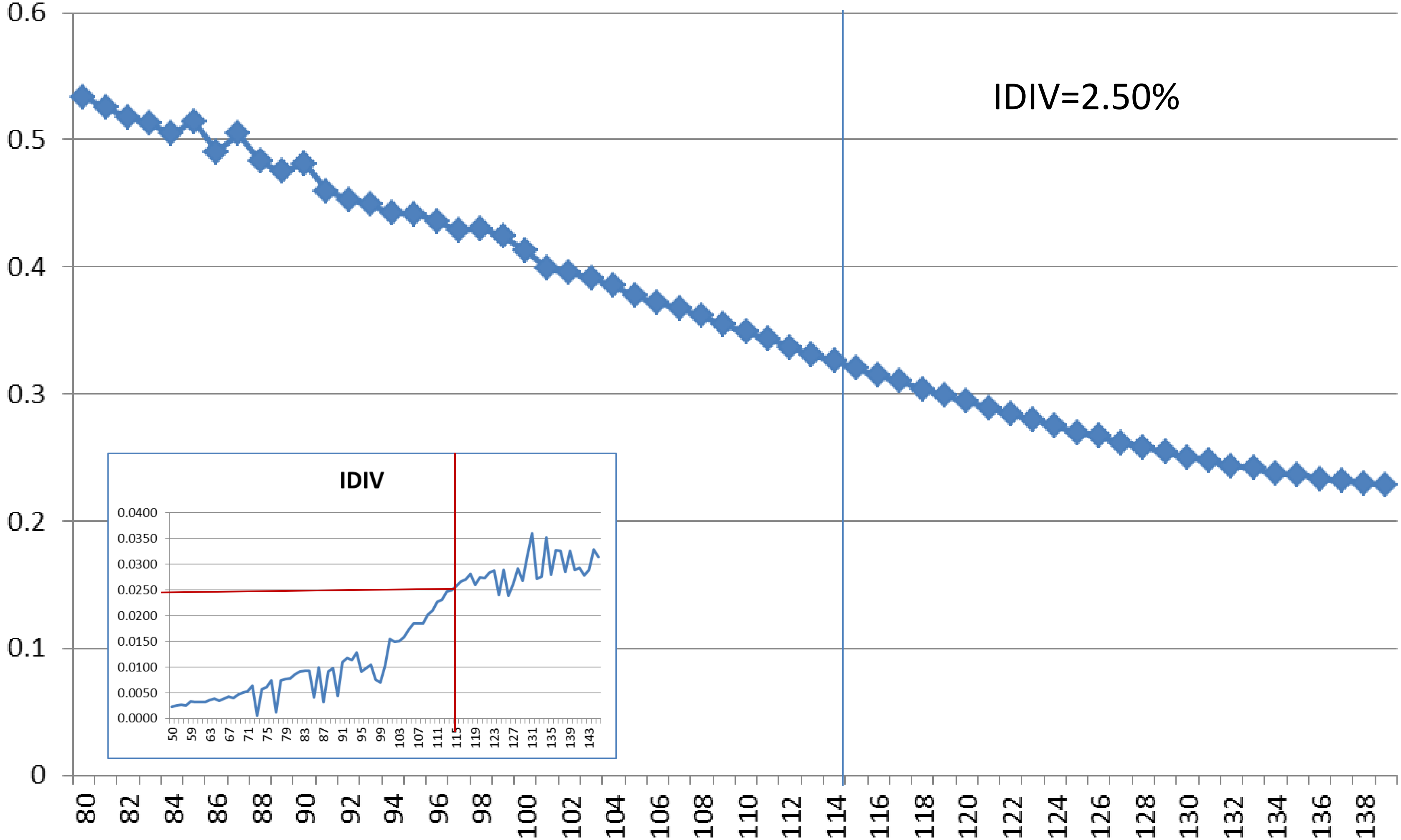
- Given (S,K,T,r,q) and the price of an option, there is a unique implied vol associated with a given price. The reason is that

$$\frac{\partial BSCall}{\partial \sigma} > 0, \quad \frac{\partial BSPut}{\partial \sigma} > 0$$

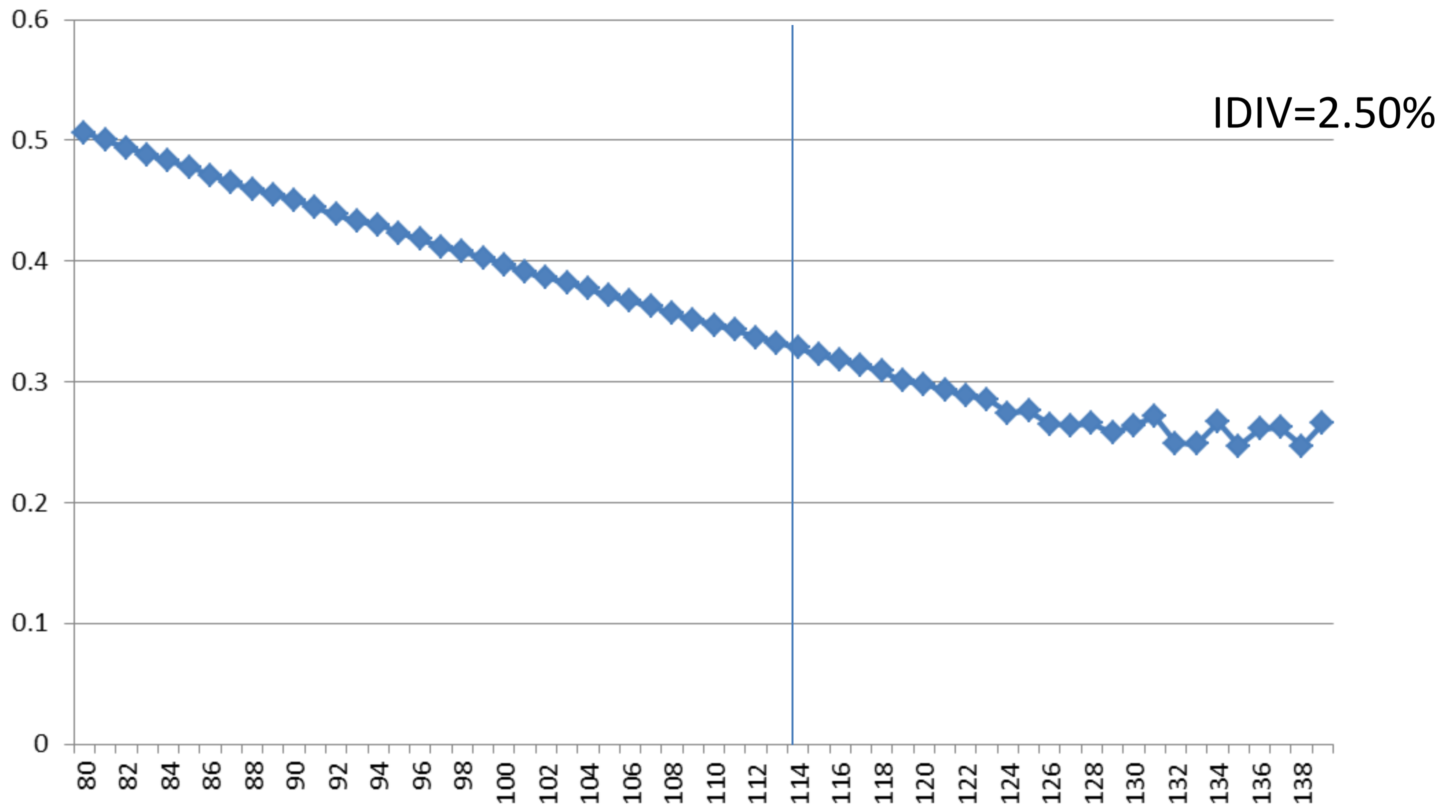
- Usually computed from mid-prices (bid+offer)/2. We can also talk about a **bid implied vol** and an **offer implied vol**, associated with bid prices and offer prices.

```
Public Function IVOL(ByVal CP As String, ByVal S, ByVal t, ByVal k, ByVal R, ByVal D, ByVal P) As Double
Dim x, sigmaH, sigmaL, tmp As Double
If CP = "C" Then
        tmp = 0
        sigmaH = 3
        sigma = 0.15
        sigmaL = 0.01
        er = 100
        icount = 0
        Do Until Abs(er) < 0.0001 Or icount = 20
        icount = icount + 1
        er = P - BSCall(S, t, k, R, D, sigma)
        If er > 0 Then
        sigmaL = sigma
        End If
        If er < 0 Then
        sigmaH = sigma
        End If
        sigma = (sigmaL + sigmaH) * 0.5
        Loop
        tmp = sigma
End If
        If CP = "P" Then
        sigmaH = 3
        sigma = 0.15
        sigmaL = 0.01
        er = 100
        icount = 0
        Do Until Abs(er) < 0.0001 Or icount = 20
        icount = icount + 1
        er = P - BSPut(S, t, k, R, D, sigma)
        If er > 0 Then
        sigmaL = sigma
        End If
        If er < 0 Then
        sigmaH = sigma
        End If
        sigma = (sigmaL + sigmaH) * 0.5
        Loop
        tmp = sigma
        End If
IVOL = tmp
End Function
```

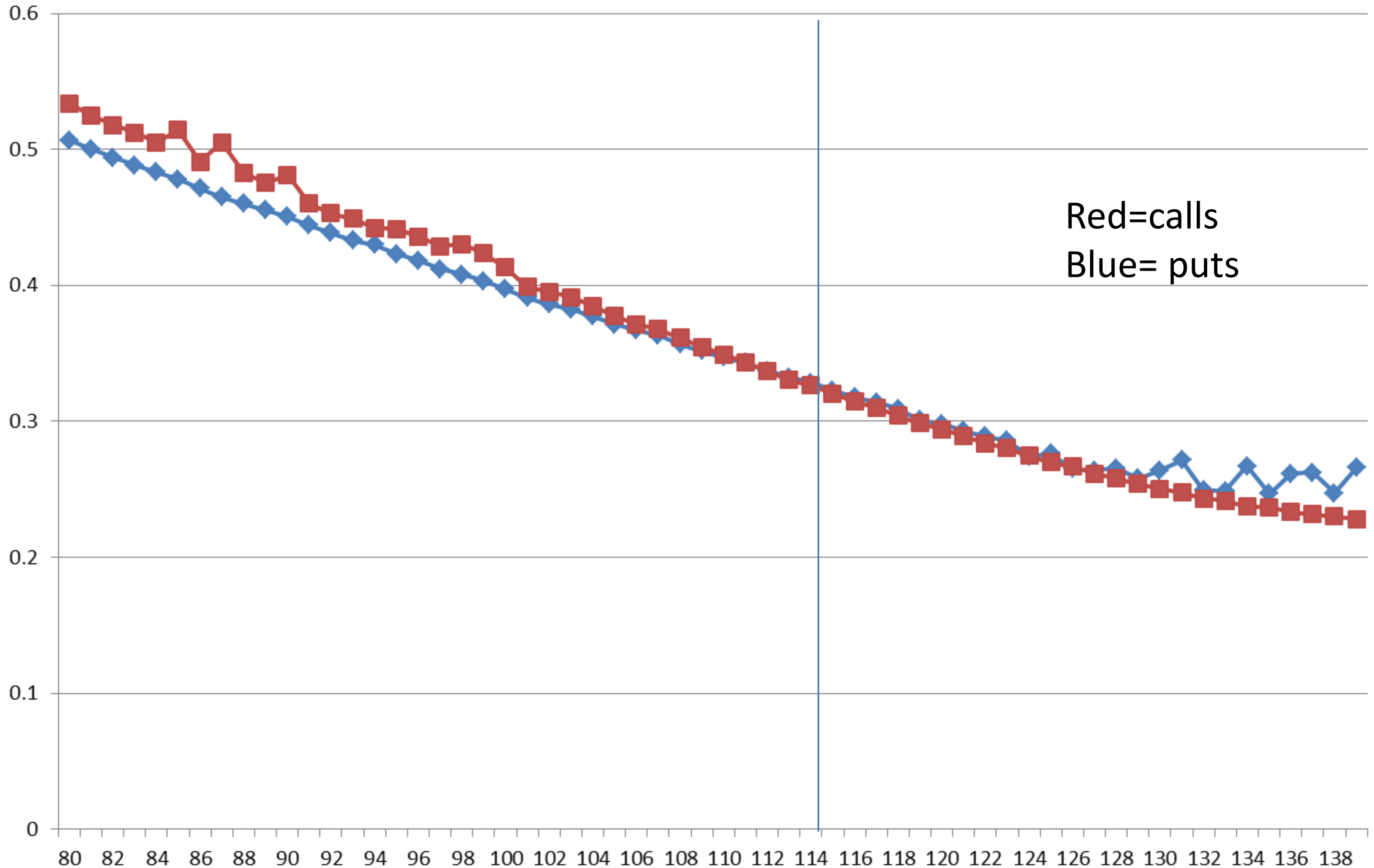
Call Implied Volatility (SPY Dec 17 calls)



Put Implied Vols (SPY Dec 17 Calls)



Calls and Puts together



Implied Volatility

- Implied volatility of OTM options are more stable than ITM

- IVOLS of calls and puts should be approximately equal due to the fact that we determined the dividend yields implicitly

- For SPY, the implied volatility is a decreasing function of the strike price. This is known as the **volatility skew** in the business.

- Volatilities are not constant across strikes, but they vary relatively smoothly.

- Option markets can be viewed as volatility markets, as we will soon see.

The Greeks and Basic Hedging

Hedging option exposure against the underlying asset

- Assume that you are long 1 call option on XYZ, with strike K, maturity T.
- Assume that the dividend yield and interest rate are known.

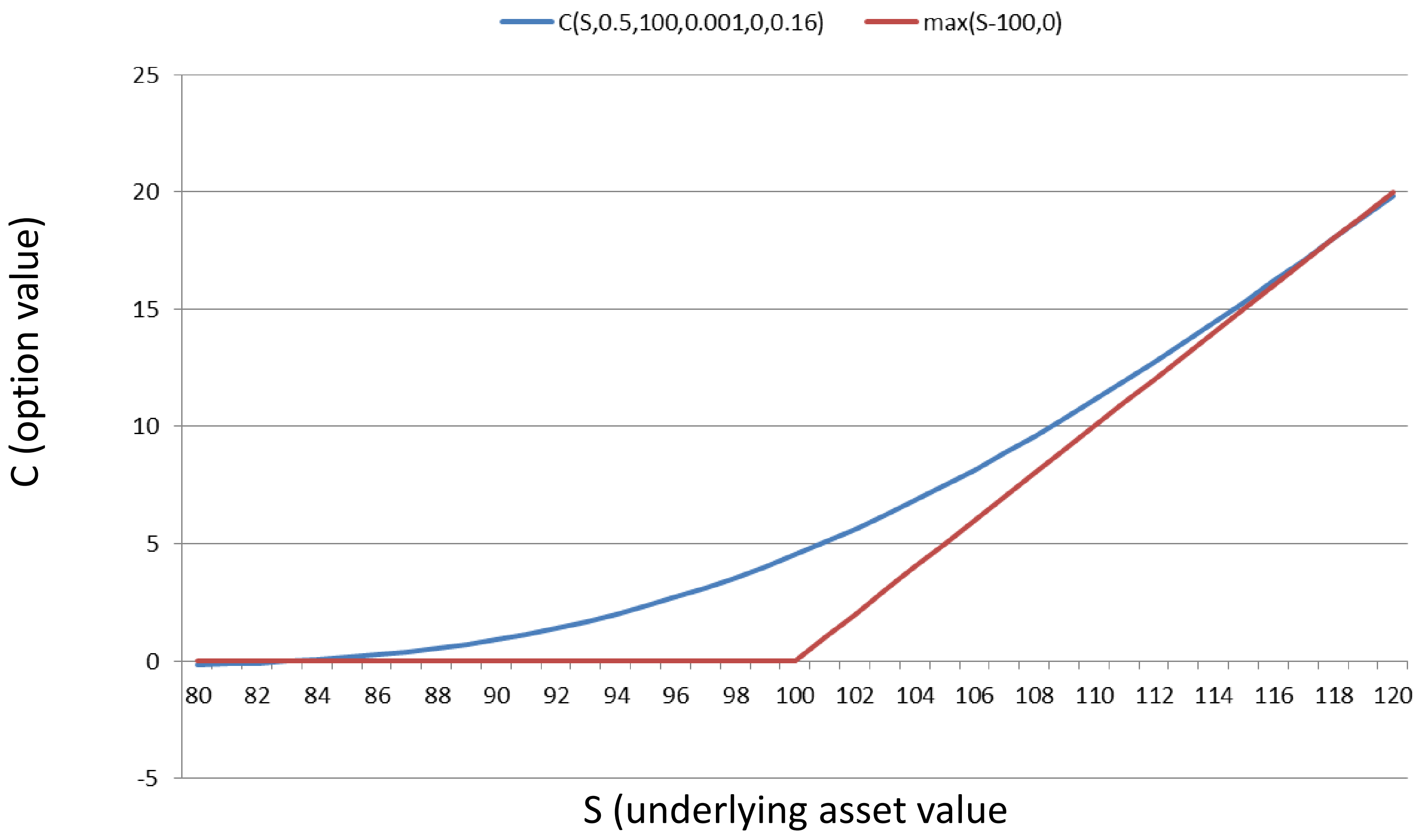
-- compute the **implied volatility** σ

$$C = BSCall(S,T,K,r,q,\sigma) = C(S,T,K,r,q,\sigma)$$

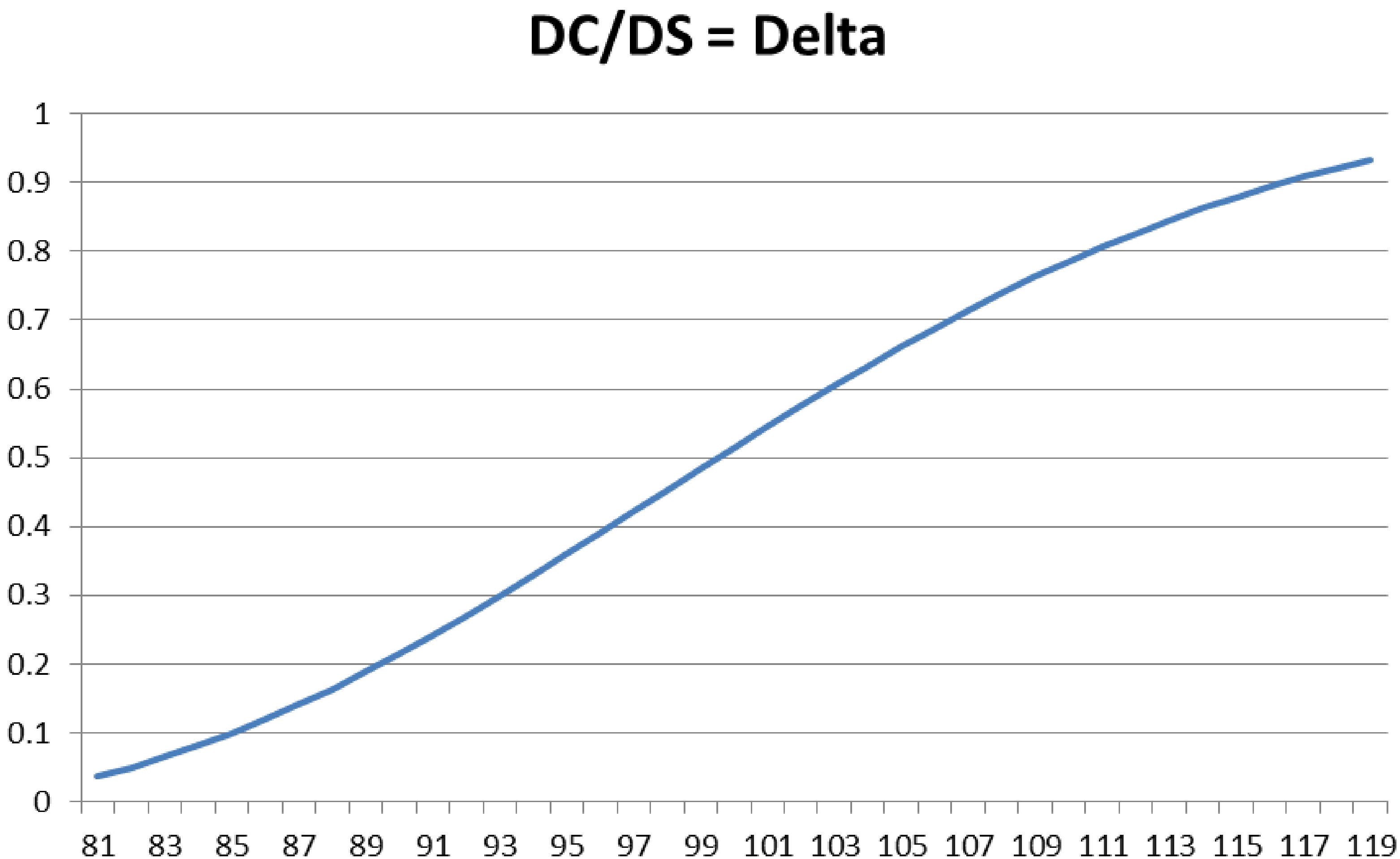
$$\Delta C \cong \frac{\partial C}{\partial S}\Delta S + \frac{\partial C}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\Delta S)^2 + o(\Delta t)$$

- The exposure to the underlying asset is represented by the first derivative with respect to price

Options are non-linear functions of the underlying asset



$$\text{Delta} = \frac{\partial C(S,T)}{\partial S}$$



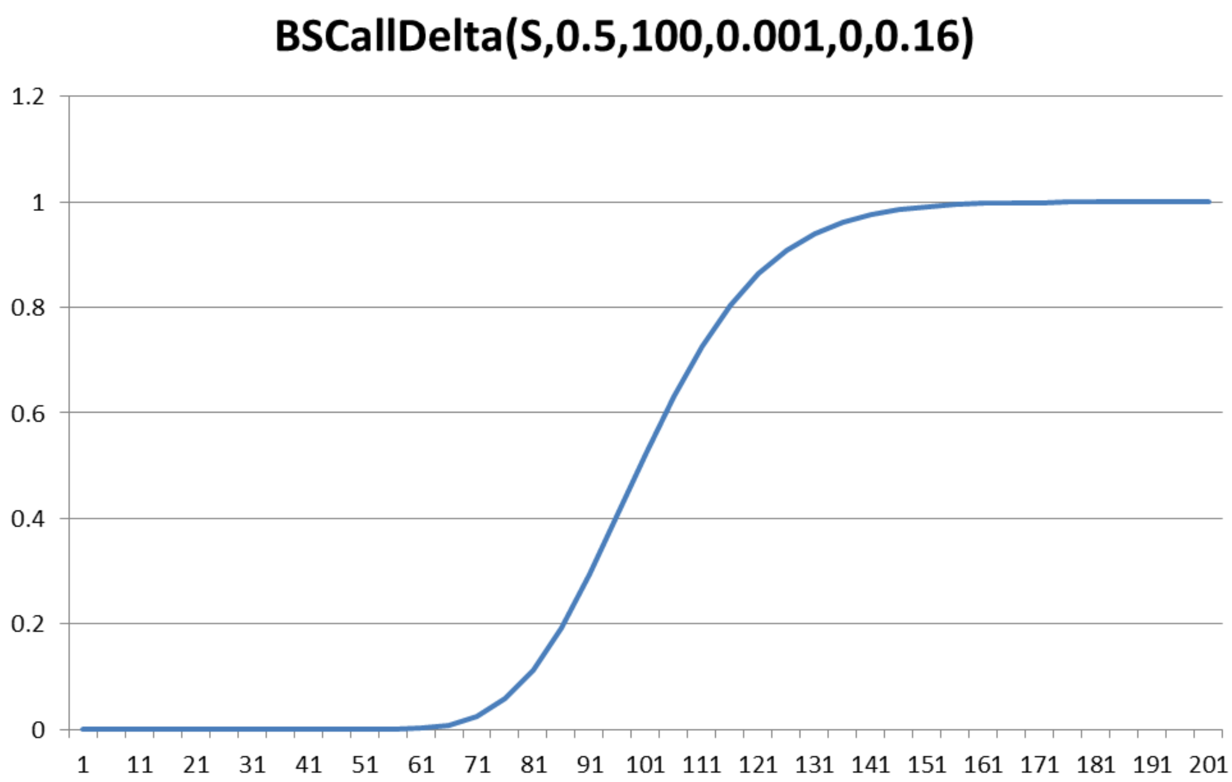
Delta for European Call (Black & Scholes)

$$C = e^{-qT} S N(d_1) - e^{-rT} K N(d_2)$$

$$\begin{aligned}
\frac{\partial C}{\partial S} &= e^{-qT} N(d_1) + e^{-qT} S N'(d_1) \frac{\partial d_1}{\partial S} - e^{-rT} K N'(d_2) \frac{\partial d_2}{\partial S} \\
&= e^{-qT} N(d_1) + \frac{e^{-qT}}{\sigma\sqrt{T}} N'(d_1) - e^{-rT} K N'(d_2) \frac{1}{S\sigma\sqrt{T}} \\
&= e^{-qT} N(d_1) + \frac{e^{-qT}}{\sigma\sqrt{T}} \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} - \frac{e^{-rT} K}{S\sigma\sqrt{T}} \frac{e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}} \\
&= e^{-qT} N(d_1) + \frac{e^{-qT}}{\sigma\sqrt{T}} \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} - \frac{e^{-rT} K}{S\sigma\sqrt{T}} \frac{e^{-\frac{\left(d_1 - \sigma\sqrt{T}\right)^2}{2}}}{\sqrt{2\pi}} \\
&= e^{-qT} N(d_1) + \frac{e^{-qT}}{\sigma\sqrt{T}} \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} - \frac{e^{-rT} K}{S\sigma\sqrt{T}} \frac{e^{-\frac{d_1^2}{2} - \ln\left(\frac{Se^{(r-q)T}}{K}\right)}}{\sqrt{2\pi}} \\
&= e^{-qT} N(d_1)
\end{aligned}$$

Delta: European Call- Black-Scholes model

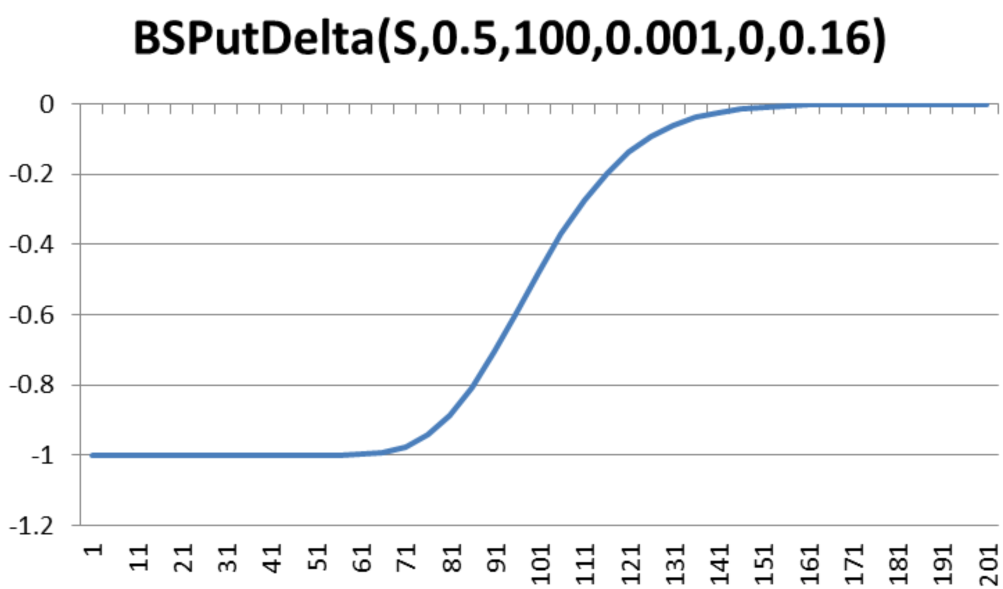
$$BSCallDelta(S,T,K,r,q,\sigma) = e^{-qT}N(d_1) = e^{-qT}N\left(\frac{1}{\sigma\sqrt{T}}\ln\left(\frac{F}{K}\right)+\frac{\sigma\sqrt{T}}{2}\right)$$



Delta For European Put (Black-Scholes)

$$P = C - Se^{-qT} + Ke^{-rT} \qquad \delta_{put} = \delta_{call} - e^{-qT}$$ (Put-call parity)

$$\delta_{put} = e^{-qT} N(d_1) - e^{-qT} = -e^{-qT} N\left(\frac{1}{\sigma\sqrt{T}} \ln\left(\frac{K}{F}\right) - \frac{\sigma\sqrt{T}}{2} \right)$$



Gamma – the change in Delta as the stock price moves

- Options are non-linear financial instruments, in the sense that they do not have a constant Delta with respect to the underlying instrument

- The second derivative of the option value with respect to the underlying price is called Gamma. It represents the rate of change of Delta as the price moves. In the European B-S model, we have

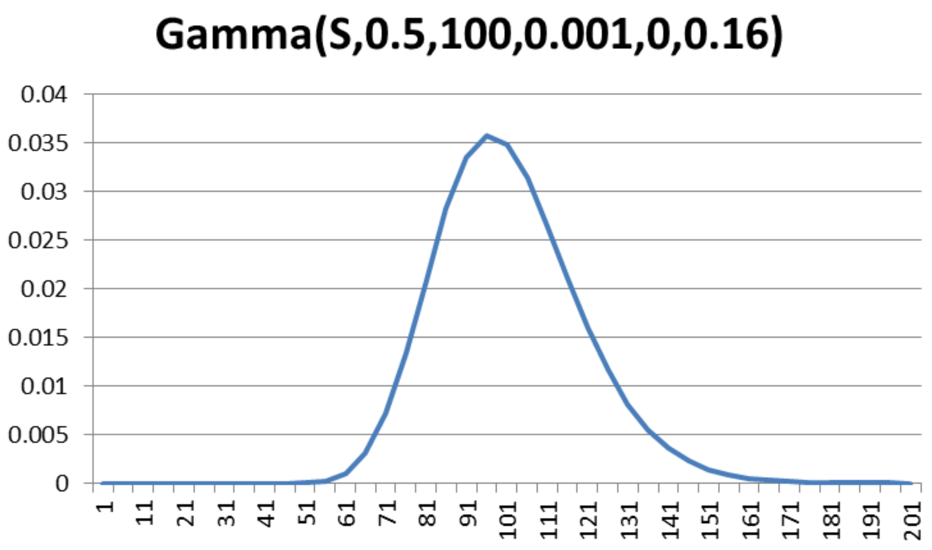
$$\Gamma(S,T,K,r,q,\sigma) = e^{-qT}\,\frac{e^{-\frac{d_1^2}{2}}}{S\sigma\sqrt{2\pi T}}$$

- The Gammas of a call and put with the same parameters are identical.

Properties of Gamma

- The option price is convex in S, so Gamma is positive for a long options position

- Gamma is mostly concentrated near the strike price, i.e. Gamma is the largest for at-the-money options. OTM and ITM options have less convexity



Delta and Gamma for American Options

- The derivatives can be computed by finite-differences (trinomial scheme)

- If we assume that the arrays are labeled C(-M to +M), S(-M to +M) for the option price and the stock price respectively, then we have

$$
\left[
\begin{aligned}
&\text{Option price} = C_0^0 \\
&\text{Delta} \qquad = \frac{C_0^1 - C_0^{-1}}{2\Delta x \cdot S_0^0} \\
&\text{Gamma} \qquad = \frac{1}{(S_0^0)^2}\left[\frac{C_0^1 + C_0^{-1} - 2C_0^0}{\Delta x^2} - \frac{C_0^1 - C_0^{-1}}{2\Delta x}\right]
\end{aligned}
\right.
$$

- These values are very close to the analytic expressions for European-style Greeks for ATM options

Example

- A trader has a position in SPY stock and options on SPY expiring in June 2012. He is long 10,000 SPY December 105 puts. He is also delta-neutral through SPY stock.

SPY=$114.25
Bid price=$3.91, Ask price=$4.00
Implied Volatility=37.7%
Delta=-0.29371
Gamma= 0.02653

Option market value = 10,000*100*3.955=$3,955,000
SPY hedge= long 293,371 shares (MV= $ 33,517,337)

- If SPY increases by 1 dollar, New Delta ~ -0.29371+0.02653= -0.26718
  New theoretical hedge= long 267,180 shares
  Difference = 26,530 shares
  To be market-neutral, the trader would need to **sell 26,530 shares at $115.25**

- If SPY decreases by 1 dollar, in order to become delta-neutral, the trader would need to **buy 26,530 shares at $113.25**

Gamma and hedged portfolios

$$\Delta C = \frac{\partial C}{\partial t}\Delta t + \frac{\partial C}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\left(\Delta S\right)^2 + \ldots.$$

$$\begin{aligned}
\Delta\Pi &= -rC\Delta t + \Delta C - \delta\left(\Delta S - rS\Delta t + qS\Delta t\right) + \ldots \\
&= \left(-rC + \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\left(\Delta S\right)^2 + S\frac{\partial C}{\partial S}(r-q) \right)\Delta t + \ldots \\
&= \left(-rC + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} + S\frac{\partial C}{\partial S}(r-q) \right)\Delta t + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\left(\Delta S\right)^2 - \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\Delta t + \ldots \\
&= \frac{S^2}{2}\frac{\partial^2 C}{\partial S^2}\left[\left(\frac{\Delta S}{S}\right)^2 - \sigma^2 \Delta t \right] + \ldots
\end{aligned}$$

Vega

- Vega is the sensitivity of an option price to changes in implied volatility

$$Vega = \frac{\partial}{\partial \sigma}\left(e^{-qT} SN(d_1) - e^{-rT} KN(d_2)\right)$$

$$= e^{-qT} SN'(d_1)\left(-\frac{\ln(F/K)}{\sigma^2\sqrt{T}} + \frac{\sqrt{T}}{2}\right) - e^{-rT} KN'(d_2)\left(-\frac{\ln(F/K)}{\sigma^2\sqrt{T}} + \frac{\sqrt{T}}{2}\right)$$

$$= e^{-qT} SN'(d_1)\sqrt{T}$$

$$\boxed{\mathrm{Vega} = e^{-qT} S \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \sqrt{T}}$$

Time-dependence

- Option **premia above par** value decrease with time-to-maturity
- Gamma increases with time to maturity (for ATM options)
- Vega decreases with time-to-maturity (for ATM options)

Short-term options are mostly sensitive to Gamma
(frequent delta hedging needed to maintain market-neutrality)

Long-term options are mostly sensitive to Vega
(value is very sensitive to the implied volatility)

Theta (time decay rate)

- Theta is the derivative of the option value with respect to time to maturity

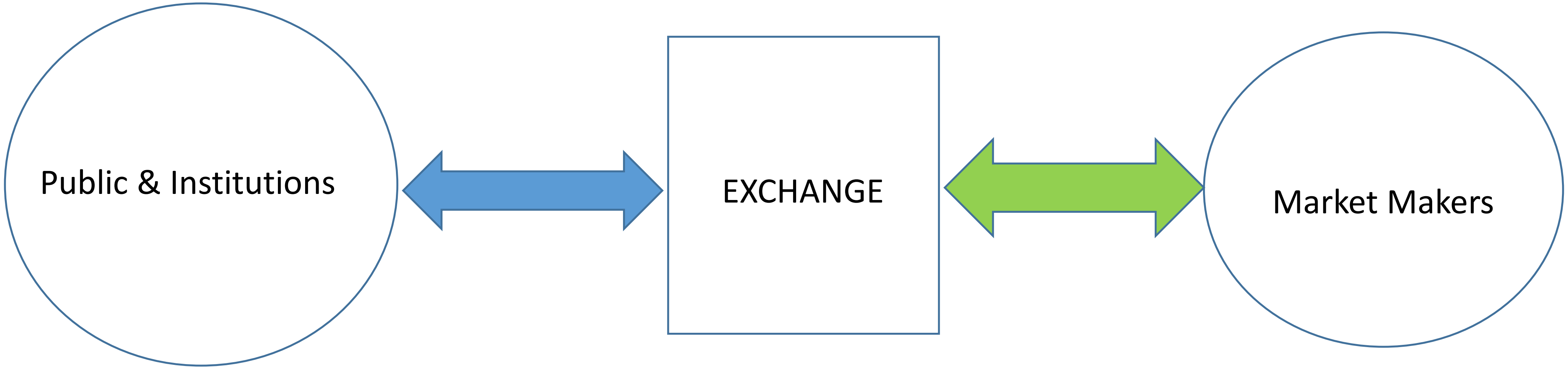
- To get intuition for Theta, assume that r and q are zero. Then, by the Black Scholes equation

$$\frac{\partial C}{\partial t}+\frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}=0$$

$$\vartheta=\frac{\partial C}{\partial T}=\frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}=\frac{\sigma^2 S^2}{2}\Gamma$$

$$\boxed{\theta=\frac{\sigma S e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi T}}}$$

Option Markets Structure (schematic)



Buy/sell options according
to their objectives, strategies

``TAKE'' PRICES

Facilitate the order flow
by taking the other side
of trades by public and
adding liquidity to the market

``MAKE'' PRICES

Volatility Lab - Implied Volatility
201.57 -1.52 (-0.75%) U1133678
SPY ARCA Implied Volatility
Expiries: Nov Dec Jan'16
Time period: 6 months
SPY ARCA Multi-Expiry Skew
Expiries: Nov Jan'16
Strikes
Date: Oct 21st
NOV20
SPY ARCA Time Lapse Skew
Days: Today Yesterday
Expiry: Nov20 '15
Today
SPY ARCA Volatility Profile
HV IV
SPY ARCA Open Interest
Puts + Calls
Implied Volatility
Historical Volatility
Industry Comparison
Time (Descending) + Action (Ascending)


IB Volatility Lab - Implied Volatility
15.20 -0.74 (-4.64%) U1133678
FCAU NYSE Implied Volatility
Expiries: Nov20 Jan15 '16
Time period: 6 months
16.50
16.00
15.50
15.20
15.00
14.50
14.00
13.50
13.00
12.50
12.00
52.5%
50%
47.5%
45.4%
42.5%
41.6%
40%
37.5%
35%
5/1
6/1
7/1
8/3
9/1
10/1
FCAU NYSE Multi-Expiry Skew
Expiries: Nov20
Strikes
Date: Oct 21st
150%
125%
100%
75%
50%
NOV20
Strike: 10.0 12.5 15.20 17.5 20.0 22.5 25.0
FCAU NYSE Time Lapse Skew
Days: Today Yesterday
Strikes
Expiry: Nov20 '15
100%
90%
80%
70%
60%
50%
Today
Strike: 12 13 14 15.20 16 17 18 19 20 21 22
FCAU NYSE Volatility Profile
HV
IV
10 30 50 75 100 150 200
52.5%
50.0%
47.5%
45.0%
42.5%
40.0%
Nov20'15
Jan15'16
Mar18'16
Jun
FCAU NYSE Open Interest
Expiries: Nov20 Jan15 '16
Strikes
Date: Oct 21st
Puts + Calls
30K
20K
10K
0
Strike: 9 11 12.5 15.20 17.5 19 21 22.5 24
Implied Volatility
Historical Volatility
Industry Comparison
Time (Descending) + Action (Ascending)


Volatility Lab - Implied Volatility
118.90 -27.84 (-18.97%) U1133678
VRX NYSE Implied Volatility
Expiries: Nov Dec Jan'16
Time period: 6 months
250.00 225.00 200.00 175.00 150.00 125.00 118.90
104.2% 100% 92.5% 90% 80% 70% 60% 50% 40% 30%
5/1 6/1 7/1 8/3 9/1 10/1
VRX NYSE Multi-Expiry Skew
Expiries: Nov Jan'16
Strikes
Date: Oct 21st
NOV20
150% 125% 100% 75%
Strike: 118.90 150 175 200 225 250 275 300
VRX NYSE Time Lapse Skew
Days: Today Yesterday
Strikes
Expiry: Nov20 '15
Today
125% 100% 75%
Strike: 118.90 150 175 200 225
VRX NYSE Volatility Profile
HV IV
10 30 50 75 100 150 200
175.0% 150.0% 125.0% 100.0% 75.0%
Nov06'15 Dec18'15 Mar18'16 Jun1
VRX NYSE Open Interest
Expiries: Nov Jan'16
Strikes
Date: Oct 21st
Puts + Calls
30K 20K 10K 0
Strike: 118.90 129 139 149 197.5 247.5
Implied Volatility
Historical Volatility
Industry Comparison
Time (Descending) + Action (Ascending)

NEW YORK (TheStreet) -- **Valeant Pharmaceuticals** (VRX - Get Report) stock is plummeting by 31.52% to $100.49 in midday trading on Wednesday, after **Citron Research** released a report accusing the company of creating a network of "phantom" pharmacies to falsify sales and avoid auditor scrutiny.

Pharmaceutical companies have determined a way to vastly overcharge patients for combinations of generic drugs that would cost much less when sold separately, the *New York Times* reported on Monday.

The manufacturers avoid insurers' and pharmacists' recommendations of generic drugs by imploring doctors to submit prescriptions directly to a mail-order pharmacy connected to the drug maker, the *Times* alleged.

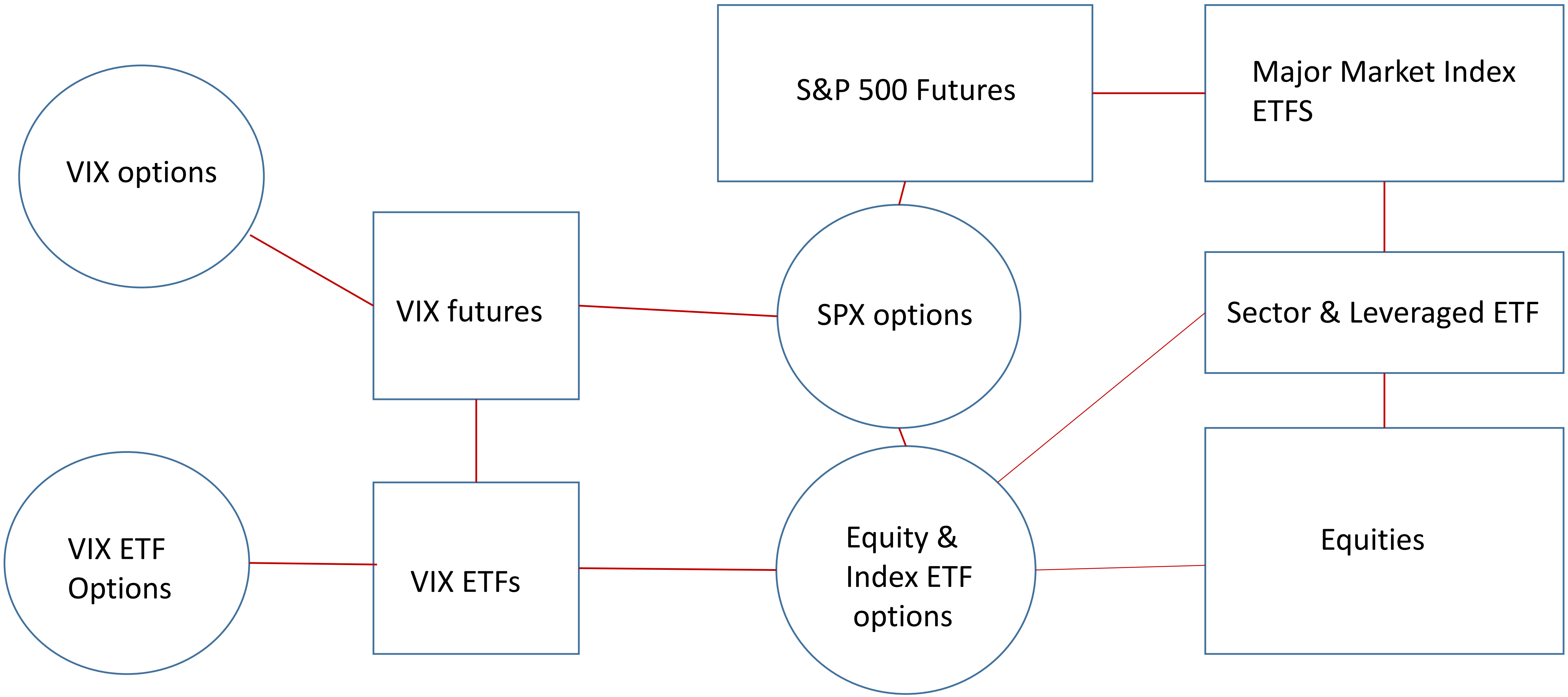
Valeant and a specialty pharmacy called **Philidor Rx Services** have such a relationship, the *Times* reported, and Citron Research reiterated this in a report this morning.

The relationship came under scrutiny following Valeant's earnings report on Monday, in which the company announced it had purchased an option to acquire Philidor in late 2014.


IB Volatility Lab - Implied Volatility
150.07 -0.38 (-0.25%) U1133678
AMGN NASDAQ.NMS Implied Volatility
Expiries: Nov Dec Jan'16
Time period: 6 months
175.00 170.00 165.00 160.00 155.00 150.07 145.00 140.00 135.00
42.5% 40% 37.5% 35% 33.7% 32.5% 30.7% 30% 27.5% 25% 22.5%
5/1 6/1 7/1 8/3 9/1 10/1
AMGN NASDAQ.NMS Multi-Expiry Skew
Expiries: Nov Jan'16
Strikes Date: Oct 21st
70% 60% 50% 40% 30%
NOV20
Strike: 100 125 150.07 175 200 225
AMGN NASDAQ.NMS Time Lapse Skew
Days: Today Yesterday
Strikes Expiry: Nov20 '15
55% 50% 45% 40% 35% 30% 25%
Today
Strike: 120 130 140 150.07 160 170 180 190
AMGN NASDAQ.NMS Volatility Profile
HV IV
10 30 50 75 100 150 200
42.5% 40.0% 37.5% 35.0% 32.5% 30.0%
Nov06'15 Dec18'15 Jan15'16 Apr15'16
AMGN NASDAQ.NMS Open Interest
Expiries: Nov Jan'16
Strikes Date: Oct 21st
Puts + Calls
15K 10K 5K 0
Strike: 113 120 127 134 141 150.07 180
Implied Volatility
Historical Volatility
Industry Comparison
Time (Descending) + Action (Ascending)


Volatility Lab - Implied Volatility
68.00 -0.91 (-1.32%) U1133678
XLV ARCA Implied Volatility
Expiries: Nov Dec Jan'16
Time period: 6 months
XLV ARCA Multi-Expiry Skew
Expiries: Nov Jan'16
Strikes
Date: Oct 21st
NOV20
XLV ARCA Time Lapse Skew
Days: Today Yesterday
Strikes
Expiry: Nov20 '15
Today
XLV ARCA Volatility Profile
HV IV
XLV ARCA Open Interest
Expiries: Nov Jan'16
Strikes
Date: Oct 21st
Puts + Calls
Implied Volatility
Historical Volatility
Industry Comparison

The Equity Volatility World



Volatility Indexes and Statistics: important elements of modern options markets

- Track ranges and dynamics of equities implied volatilities as time-series
- Statistics for implied volatilities of Indexes and ETFs
- Construct Indexes of Implied Volatilities
- In some cases, trade directly the Implied Volatility Indexes using exchange-traded products.
- US CBOE Volatility Index (VIX)
- Eurozone DTB Eurostoxx Volatility Index (VStoxx)
- Hong Kong Hang Seng Volatility Index (VHSI)
- In the US there are even volatility indexes for some stocks such as AAPL.

Volatility Indexes on U.S. Stock Indexes

		Delayed Quotes		
Ticker	**Index**	**Sym**	**Last**	**Pt. Change**
VIX®	Cboe Volatility Index®	VIX	15.85	0.00
VXN℠	Cboe NASDAQ Volatility Index	VXN	20.63	0.00
VXO℠	Cboe S&P 100 Volatility Index	VXO	18.01	0.00
VXD℠	Cboe DJIA Volatility Index	VXD	16.05	0.00
RVX℠	Cboe Russell 2000 Volatility Index	RVX	19.41	0.00
VIX9D℠	CBOE S&P 500 9-Day Volatility Index	VIX9D	15.36	0.00
VIX3M℠	Cboe 3-Month Volatility Index	VIX3M	17.09	0.00
VIX6M℠	CBOE S&P 500 6-Month Volatility Index	VIX6M	17.72	0.00
VIX1Y℠	Cboe 1-Year Volatility Index℠	VIX1Y	18.27	0.00

Volatility Indexes on Non- U.S. Stock ETFs

		Delayed Quotes		
Ticker	**Index**	**Sym**	**Last**	**Pt. Change**
VXEFA	Cboe EFA ETF Volatility Index	VXEFA	14.90	0.00
VXEEM	Cboe Emerging Markets ETF Volatility Index	VXEEM	20.55	0.00
VXFXI	Cboe China ETF Volatility Index	VXFXI	22.73	0.00
VXEWZ	Cboe Brazil ETF Volatility Index	VXEWZ	32.30	0.00

Volatility Indexes on Interest Rates

Delayed Quotes

Ticker	Index	Sym	Last	Pt. Change
TYVIX	Cboe/CBOT 10-year U.S. Treasury Note Volatility Index	TYVIX	4.22	0.00
SRVIX	Cboe Interest Rate Swap Volatility Index	SRVIX	66.69	0.00

Volatility Indexes on Commodity-related ETFs

Delayed Quotes

Ticker	Index	Sym	Last	Pt. Change
OVX	Cboe Crude Oil ETF Volatility Index	OVX	33.19	0.00
GVZ	Cboe Gold ETF Volatility Index	GVZ	9.00	0.00
VXSLV	Cboe Silver ETF Volatility Index	VXSLV	15.33	0.00
VXGDX	Cboe Gold Miners ETF Volatility Index	VXGDX	22.87	0.00
VXXLE	Cboe Energy Sector ETF Volatility Index	VXXLE	21.87	0.00

Volatility Indexes on Currency-related Futures/ETFs

Delayed Quotes

Ticker	Index	Sym	Last	Pt. Change
EUVIX	Cboe/CME FX Euro Volatility Index℠	EUVIX	5.75	0.00
JYVIX	Cboe/CME FX Yen Volatility Index℠	JYVIX	7.44	0.00
BPVIX	Cboe/CME FX British Pound Volatility Index℠	BPVIX	7.75	0.00
EVZ	Cboe EuroCurrency ETF Volatility Index	EVZ	5.68	0.00

Volatility Indexes on Single Stocks

Ticker	Index	Sym	Last	Pt. Change
		Delayed Quotes		
VXAZN	Cboe Equity VIX® on Amazon	VXAZN	27.21	0.00
VXAPL	Cboe Equity VIX® on Apple	VXAPL	31.74	0.00
VXGS	Cboe Equity VIX® on Goldman Sachs	VXGS	26.12	0.00
VXGOG	Cboe Equity VIX® on Google	VXGOG	22.86	0.00
VXIBM	Cboe Equity VIX® on IBM	VXIBM	23.20	0.00

Volatility of VIX

Ticker	Index	Sym	Last	Pt. Change
		Delayed Quotes		
VVIX	Cboe VIX of VIX Index	VVIX	90.19	0.00

Futures and Options on Cboe's Volatility Indexes

Listed options on volatility indexes are offered for trading on Cboe, while futures on volatility indexes are traded at the Cboe Futures Exchange (CFE).

Futures and options on Cboe's volatility indexes have several features that distinguish them from most equity and index options. Here are unique features of VIX options:

- Pricing Based on Forward VIX Value
- Pricing Can Be Different for a Number of Reasons
- Wednesday Settlement
- Special Opening Quotation Price
- High Volatility of Volatility

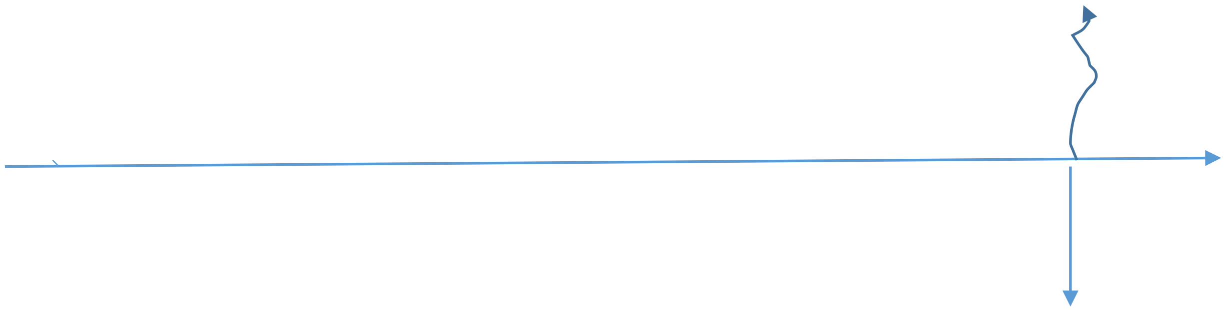
Focus on VIX, VIX futures and ETNs

What is VIX?

- VIX is an index which tracks the implied volatilities of first- and second- month expiration of SPX options.
- Old VIX (since 1992): a weighted average of ATM implied volatilities
- New VIX (since 2000): based on **Variance Swap formula**
- VIX Futures started trading in 2004
- VIX Options started trading in 2006
- VIX ETNs started trading in 2009

Due to the fact that VIX has a traded future and that there is a link between the VIX and the implied volatility of the S&P 500 options which is also connected to the implied volatility of the constituents, VIX plays a crucial role in today's equity markets. It also is the Fear Gauge of the market: buying VIX is a bearish trade.

Variance Swap



Floating leg = Variance

Fixed leg = Fixed amount (VSV)

$$Floating\ Leg = \sum_{i=1}^{N} R_i^2$$

$$R_i = \frac{S_{i\Delta t}}{S_{(i-1)\Delta t}} - 1, i = 1, \dots, N, \qquad N\Delta t = T$$

$$Fixed\ Leg = E\left\{\sum_{i=1}^{N} R_i^2\right\}$$

Calculation of Fixed Leg (I)

F= forward price of SPX

$$d\ln F = \frac{dF}{F} - \frac{1}{2}\left(\frac{dF}{F}\right)^2 + \mathrm{o}(\Delta t)$$

$$\ln\left(\frac{F_T}{F_0}\right) = \sum_{i=1}^{N}\frac{dF}{F} - \frac{1}{2}\sum_{i=1}^{N}\left(\frac{dF}{F}\right)^2 + \mathrm{o}(1)$$

$$\mathrm{E}\left(\ln\left(\frac{F_T}{F_0}\right)\right) = E\left(\sum_{i=1}^{N}\frac{dF}{F}\right) - \frac{1}{2}E\left(\sum_{i=1}^{N}\left(\frac{dF}{F}\right)^2\right)$$
$$+ \mathrm{o}(1)$$

$$\mathrm{E}\left(\ln\left(\frac{F_T}{F_0}\right)\right) = -\frac{1}{2}E\left(\sum_{i=1}^{N}\left(\frac{dF}{F}\right)^2\right) + \mathrm{o}(1)$$

Calculation of Fixed Leg of VS (II)

$$\frac{dF}{F} = \frac{dS}{S} + (r - d)\Delta t$$

$$\left(\frac{dF}{F}\right)^2 = \left(\frac{dS}{S}\right)^2 + o(\Delta t)$$

$$\sum_{i=1}^{N} \left(\frac{dS}{S}\right)^2 = \sum_{i=1}^{N} \left(\frac{dF}{F}\right)^2 + o(1)$$

$$E\left(\sum_{i=1}^{N} \left(\frac{dS}{S}\right)^2\right) = E\left(\sum_{i=1}^{N} \left(\frac{dF}{F}\right)^2\right) + o(1) = 2E\left(ln\left(\frac{F_0}{F_T}\right)\right) + o(1)$$

Fixed Leg of VS (III)

As $\Delta t \to 0$, $$Fixed\ Leg = 2\ E\left[ln\left(\frac{F_0}{F_T}\right)\right] = 2\ E\left[ln\left(\frac{F_0}{S_T}\right)\right]$$

Taylor Expansion:

$$ln\frac{S}{F} = S - F\ - \int_F^S (S - V)\frac{dV}{V^2}$$

$$ln\frac{S}{F} = S - F\ - \int_F^\infty (S - V)^+\frac{dV}{V^2} - \int_0^F (V - S)^+\frac{dV}{V^2}$$

$$Fixed\ Leg = 2\ E\left[ln\left(\frac{F_0}{S_T}\right)\right] = 2\int_{F_0}^\infty E(S_T - V)^+\frac{dV}{V^2} + 2\int_0^{F_0} E(V - S_T)^+\frac{dV}{V^2}$$

$$\boxed{E(S - F) = 0}$$

The VIX Formula

- Variance Swap Volatility

$$\boxed{\sigma_T^2 = \frac{2e^{rT}}{T}\int_0^\infty OTM(K,T,S)\frac{dK}{K^2}}$$

- Here $OTM(K,T,S)$ represents the value of the OTM (forward) option with strike K, or ATM if S=F.

- CBOE created a discrete version of the VSV in which the sum replaces the integral and the maturity is 30 days. Since there are no 30 day options, VIX uses first two maturities*

$$VIX = \sqrt{w_1 \sum_{i=1}^{n} OTM(K_i, T_1, S)\frac{\Delta K}{K_i^2} + w_2 \sum_{i} OTM(K_i, T_2, S)\frac{\Delta K}{K_i^2}}$$

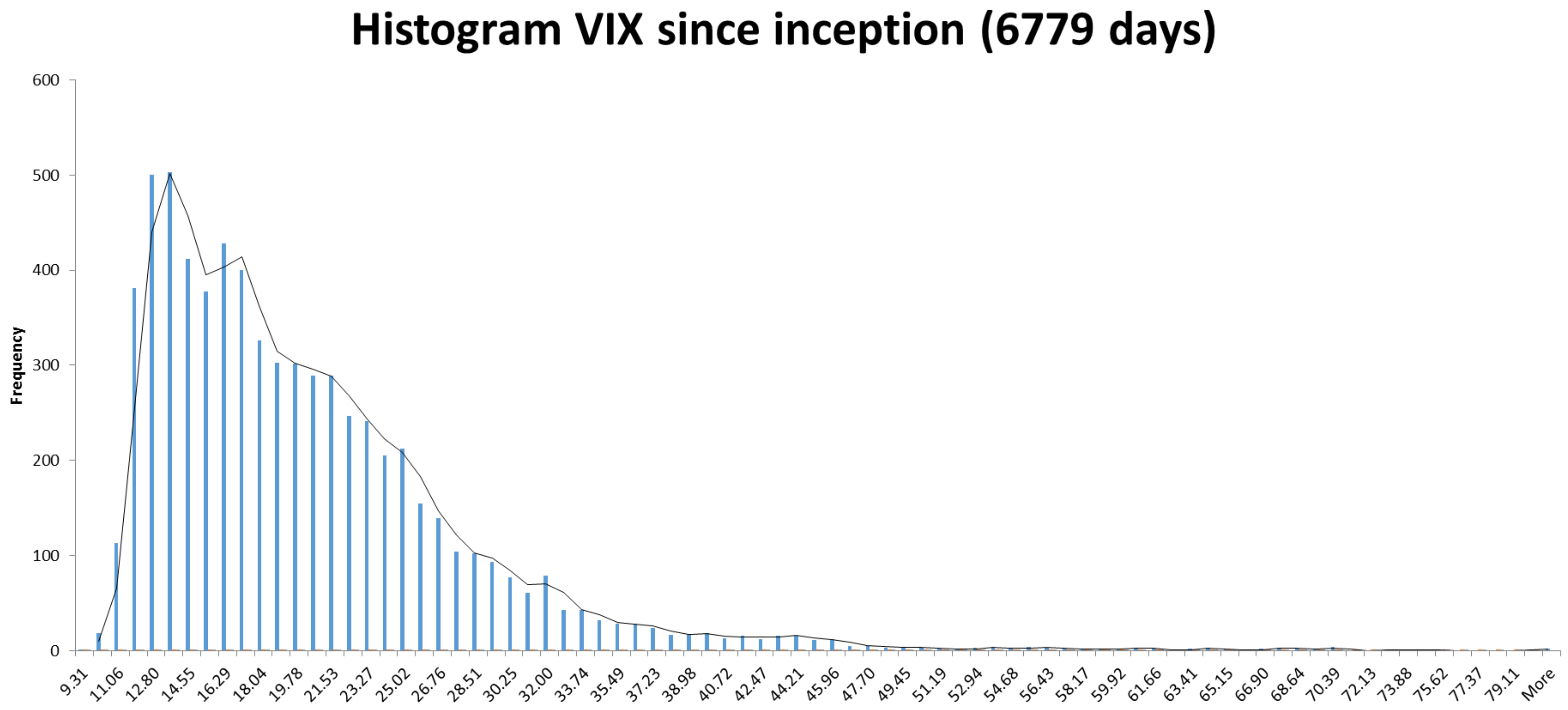
* My understanding is that recently they could have added more maturities using weekly options as well.

```
Function VS(ByVal N As Double, ByVal R As Double, ByVal t As Double, ByRef strike() As Double, _
ByRef callbid() As Double, ByRef callask() As Double, ByRef putbid() As Double, _
ByRef putask() As Double, ByRef coi() As Long, ByRef poi() As Long) As Double

'1. forward

Dim cmid() As Double
ReDim cmid(1 To N)
Dim pmid() As Double
ReDim pmid(1 To N) As Double

For i = 1 To N
cmid(i) = 0.5 * (callbid(i) + callask(i))
pmid(i) = 0.5 * (putbid(i) + putask(i))
Next i

Dim Min, index As Integer

Min = Abs(cmid(1) - pmid(1))
index = 1
For i = 2 To N
If (Abs(cmid(i) - pmid(i)) < Min) Then
Min = Abs(cmid(i) - pmid(i))
index = i
End If
Next i

Dim f As Double

f = strike(index) + Exp(R * t) * (cmid(index) - pmid(index))

Dim index0 As Integer
'' Support of vix
i = 1
Do While (strike(i) < f)
i = i + 1
Loop
index0 = i - 1
```

Code for for the CBOE discretized
Variance Swap formula
(1 of 3)

```
Dim Lmarker

Lmarker = 1

For i = 2 To index0 - 1
If putbid(index0 - i + 1) = 0 And putbid(index0 - i) = 0 Then
Lmarker = index0 - i + 2
i = index0 + 1
End If
Next i

' upper cutoff
Dim Umarker
Umarker = N
For i = index0 To N - 2
If callbid(i + 1) = 0 And callbid(i + 2) = 0 Then
Umarker = i
i = N + 1
End If
Next i

' lower sum
Dim distrike, tmp As Double
'dstrike = strike(1) - strike(0)
tmp = 0
For i = Lmarker To index0 - 1
If i > 1 Then
dstrike = (strike(i + 1) - strike(i - 1)) * 0.5
End If
If i = 1 Then
dstrike = strike(2) - strike(i)
End If
tmp = tmp + pmid(i) * dstrike / (strike(i) * strike(i))
Next i
'atmf strike
dstrike = (strike(index0 + 1) - strike(index0 - 1)) * 0.5
tmp = tmp + 0.5 * (pmid(index0) + cmid(index0)) * dstrike / (strike(index0) * strike(index0))
```

```
' lower sum
Dim distrike, tmp As Double
'dstrike = strike(1) - strike(0)
tmp = 0
For i = Lmarker To index0 - 1
If i > 1 Then
dstrike = (strike(i + 1) - strike(i - 1)) * 0.5
End If
If i = 1 Then
dstrike = strike(2) - strike(i)
End If
tmp = tmp + pmid(i) * dstrike / (strike(i) * strike(i))
Next i
'atmf strike
dstrike = (strike(index0 + 1) - strike(index0 - 1)) * 0.5
tmp = tmp + 0.5 * (pmid(index0) + cmid(index0)) * dstrike / (strike(index0) * strike(index0))
'upper sum
For i = index0 + 1 To Umarker
If i < N Then
dstrike = (strike(i + 1) - strike(i - 1)) * 0.5
End If
If i = N Then
dstrike = strike(N) - strike(N - 1)
End If
tmp = tmp + cmid(i) * dstrike / (strike(i) * strike(i))
Next i

VS = tmp - 0.5 * Exp(-R * t) * (f / strike(index0) - 1) ^ 2
'in PV terms

End Function
```


Histogram VIX since inception (6779 days)
Frequency
0
100
200
300
400
500
600
9.31
11.06
12.80
14.55
16.29
18.04
19.78
21.53
23.27
25.02
26.76
28.51
30.25
32.00
33.74
35.49
37.23
38.98
40.72
42.47
44.21
45.96
47.70
49.45
51.19
52.94
54.68
56.43
58.17
59.92
61.66
63.41
65.15
66.90
68.64
70.39
72.13
73.88
75.62
77.37
79.11
More

Statistics of VIX Levels

VIX Descriptive Stats

Long term mean	Mean	19.70256232
	Standard Error	0.095343095
	Median	17.84
Most likely level	Mode	12.42
	Standard Deviation	7.850043462
	Sample Variance	61.62318236
Fat tails	Kurtosis	7.753084648
Upside risk	Skewness	2.111767067
	Range	71.550001
	Minimum	9.31
	Maximum	80.860001
	Count	6779
	Largest(10)	69.25
	Smallest(10)	10.02
	Confidence Level(95.0%)	0.186902407

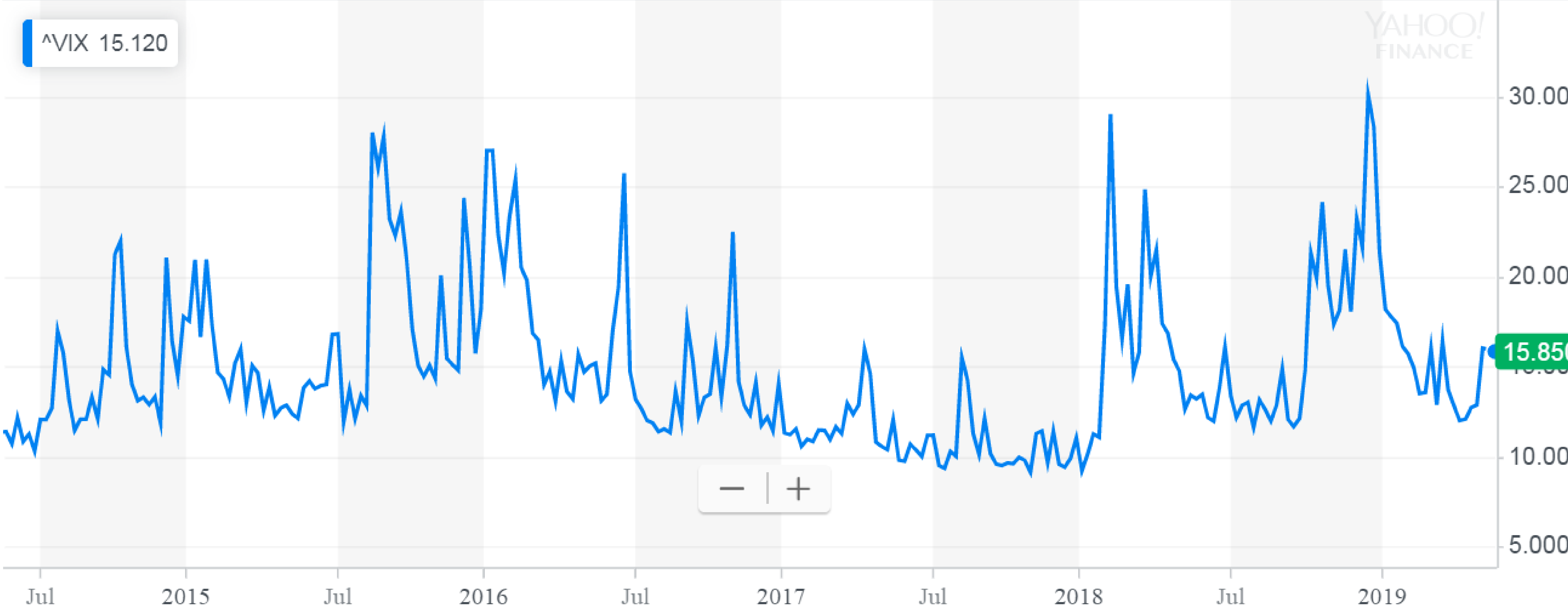
CBOE Volatility Index (^VIX)

Chicago Options - Chicago Options Delayed Price. Currency in USD

15.85 -1.07 (-6.32%)

At close: May 24 4:14PM EDT

Indicators Comparison Date Range 1D 5D 1M 3M 6M YTD 1Y 2Y **5Y** Max 1W



VIX as a weighted average of volatilites

Emphasizing the dependence on the implied vol:

$$\sigma_T^2 = \frac{2e^{rT}}{T}\int_0^{\infty} OTM(K,T,\sigma(K,T))\frac{dK}{K^2}$$

and that, trivially,

$$\sigma_T^2 = \frac{2e^{rT}}{T}\int_0^{\infty} OTM(K,T,\sigma_T)\frac{dK}{K^2}$$

conclude that

$$0 = \int_0^{\infty} [OTM\big(K,T,\sigma(K,T)\big) - \ OTM(K,T,\sigma_T)]\frac{dK}{K^2}$$

VIX as weighted average of IVOLS

$$0 = \int_0^\infty \left[OTM\big(K,T,\sigma(K,T)\big) - \; OTM(K,T,\sigma_T)\right] \frac{dK}{K^2}$$

From last slide

$$0 = \int_0^\infty Vega\big(K,T,\sigma^*(K,T)\big)\,(\sigma(K,T) - \sigma_T)\,\frac{dK}{K^2}$$

Mean value thm.

$$\boxed{\sigma_T = \frac{\int_0^\infty \sigma(K,T)\, Vega\big(K,T,\sigma^*(K,T)\big)\frac{dK}{K^2}}{\int_0^\infty Vega\big(K,T,\sigma^*(K,T)\big)\frac{dK}{K^2}}}$$

This formula can be simplified considerably by passing to log-moneyness variables and using the explicit form of Vega. This is left as an exercise to the reader. Due to the concentration of Vega around the 50-delta strikes, this weighted average is very concentrated on near the money options.
T=30 days (VIX) is a relatively small maturity in terms of the asymptotics.

Bloomberg quant seminar: January 22,2018

Statistics of VIX Futures and Applications to trading VIX ETNs

Marco Avellaneda
NYU-Courant

Joint work with Andrew Papanicolaou, NYU-Tandon School of Engineering,

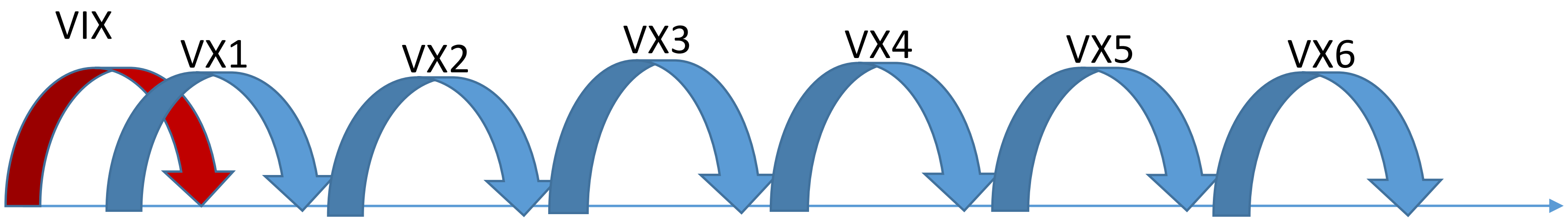
Xinyuan Zhang, NYU-Courant, Xinyu Fan, NYU-Courant

Outline

- VIX Futures (CBOE)
- VIX ETNS (VXX, XIV, UVXY, SVXY, TVIX, EWZ)
- Modeling the VIX curve and applications to ETN trading/investing

VIX Futures (CBOE, symbol:VX)

- Contract notional value = VX $\times$ 1,000
- Tick size= 0.05 (USD 50 dollars)
- Final settlement price = Spot VIX $\times$ 1,000
- Monthly settlements, on Wednesday at 8AM, prior to the 3rd Friday (classical option expiration date)
- Exchange: Chicago Futures Exchange (CBOE)
- Cash-settled (obviously!)



- Each VIX futures covers 30 days of volatility after the settlement date.
- Settlement dates are 1 month apart.
- Recently, weekly settlements have been added in the first two months.

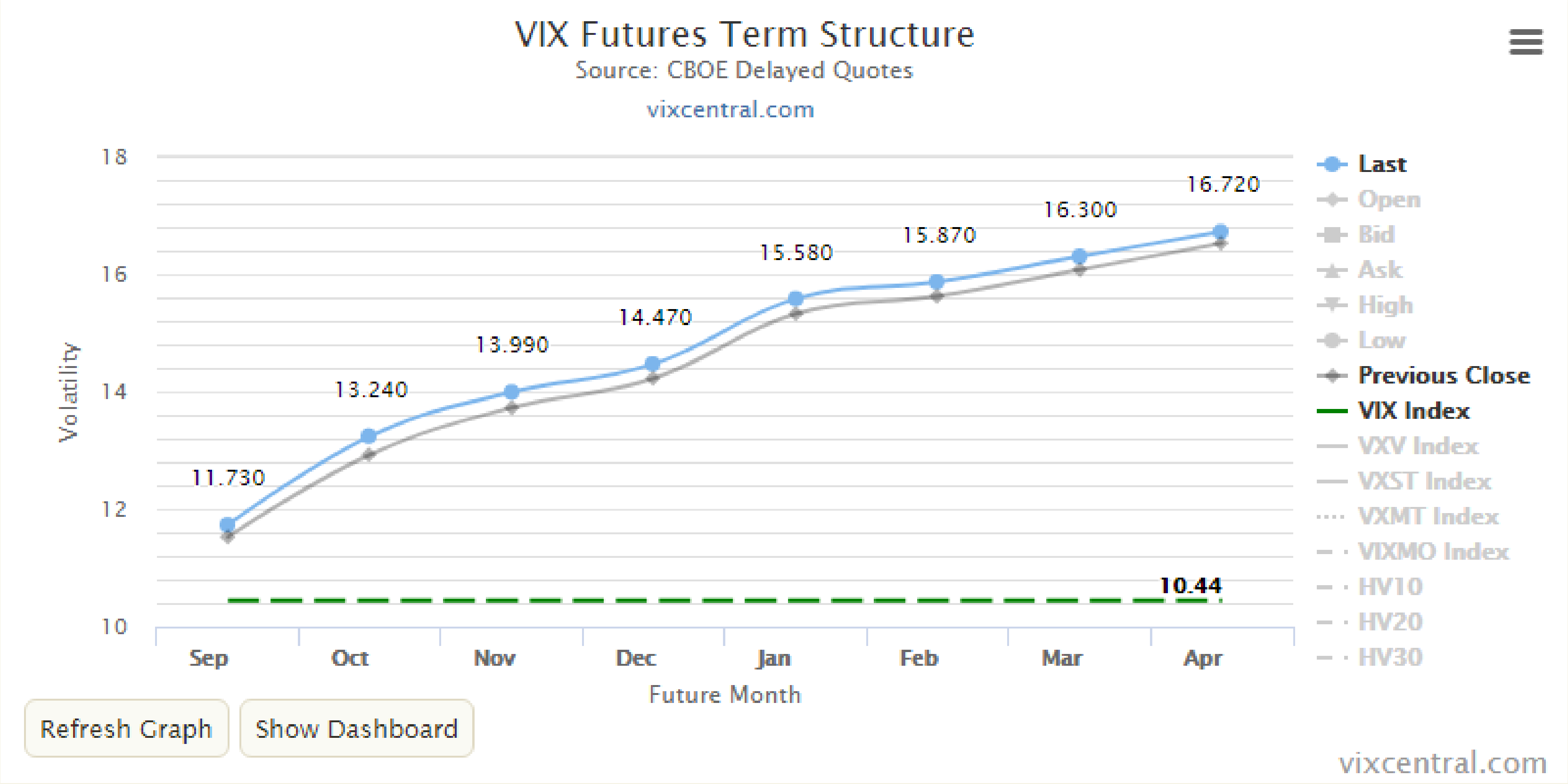
VIX Central

Questions and Feedback

Follow @VixCentral

VIX futures 6:30 PM
Thursday Sep 14, 2017

VIX Term Months | VIX Term All | Historical Prices | Contango | Help | Blogroll



% Contango	1	12.87%	2	5.66%	3	3.43%	4	7.67%	5	1.86%	6	2.71%	7	2.58%	8
Difference	1	1.51	2	0.75	3	0.48	4	1.11	5	0.29	6	0.43	7	0.42	8

Month 7 to 4 contango	12.65%	4.22%

Settlement dates:

Sep 20, 2017

Oct 18, 2017

Nov 17, 2017

Dec 19, 2017

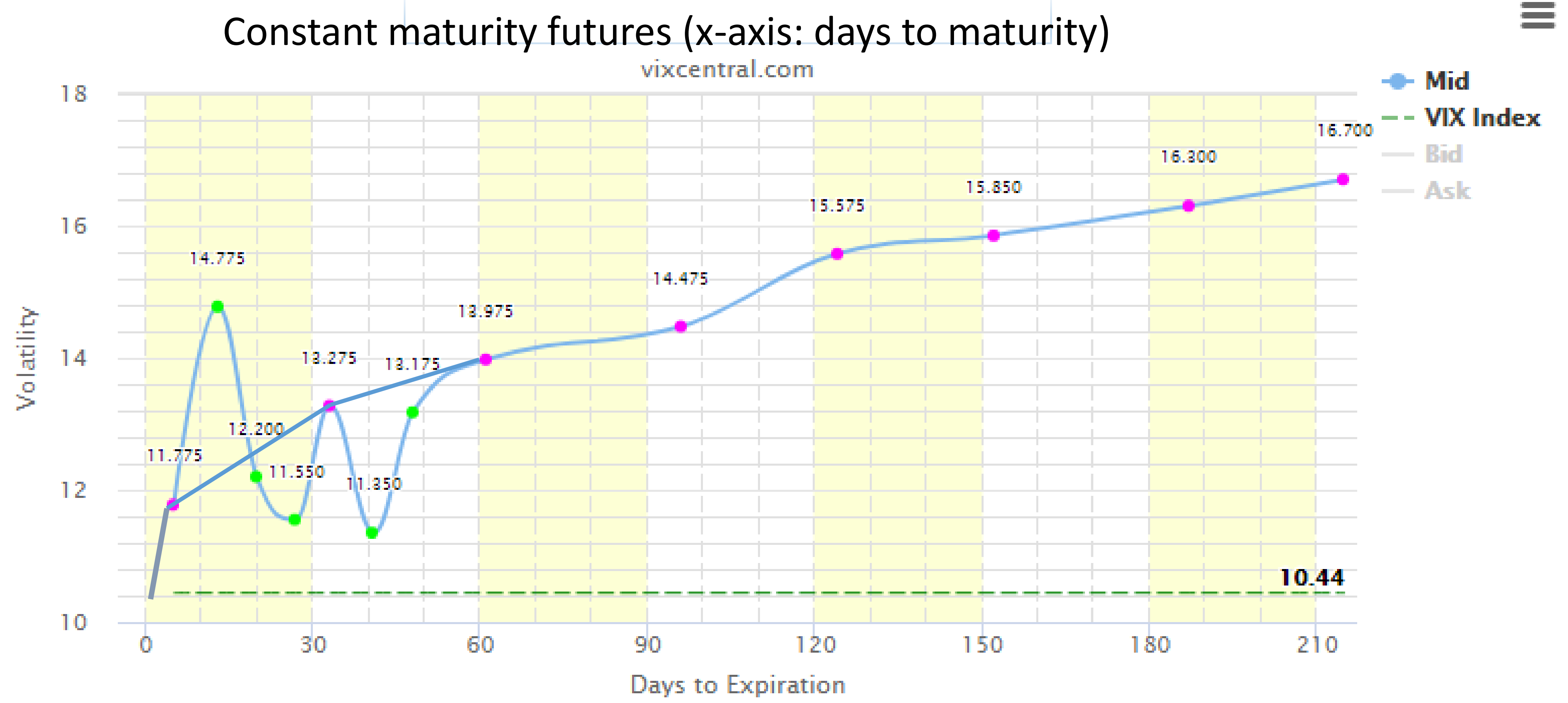
Jan 16, 2018

Feb 13, 2018

Mar 20, 2018

April 17, 2018

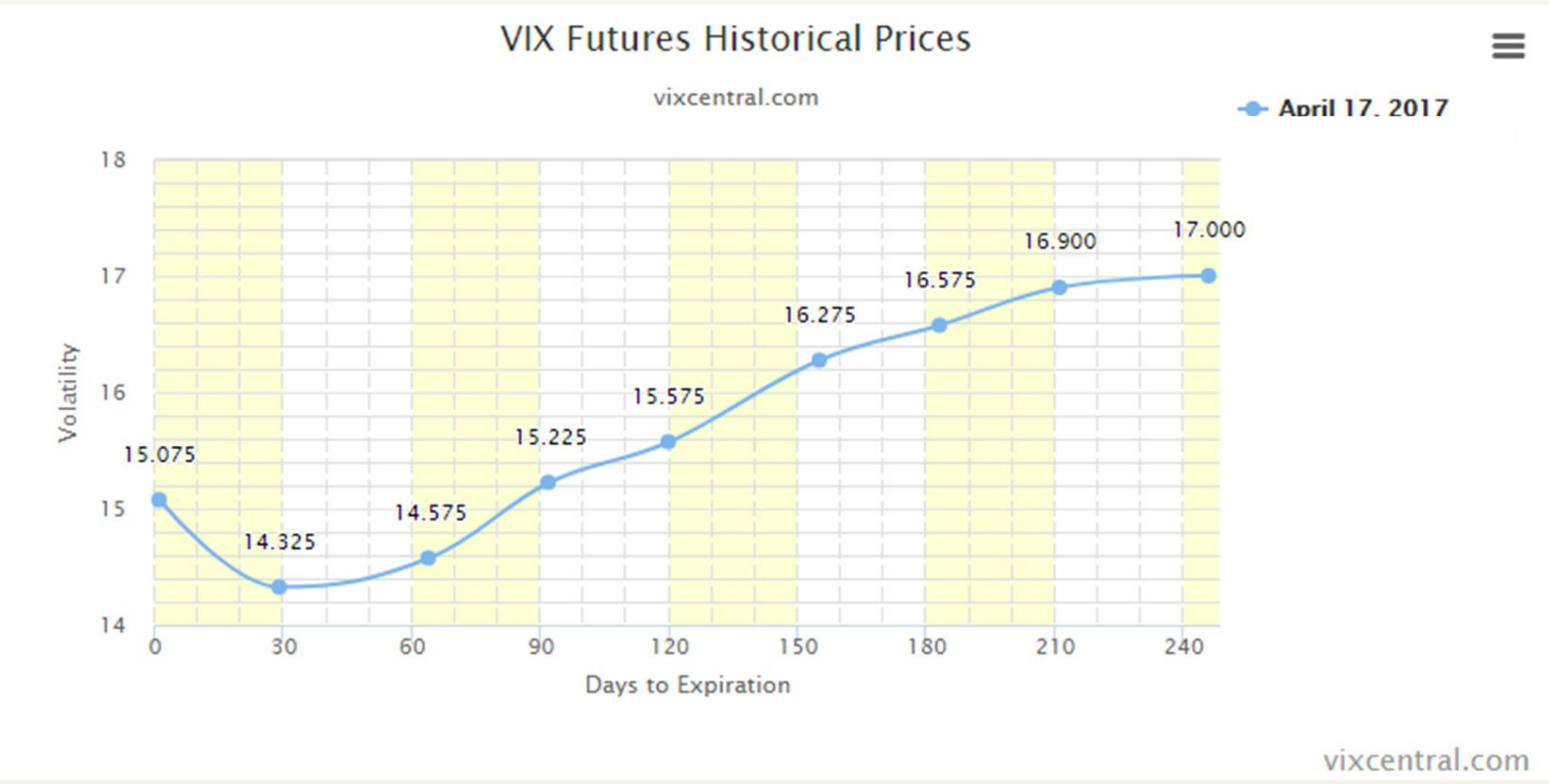
VIX Term Months | VIX Term All | Historical Prices | Contango | Help | Blogroll



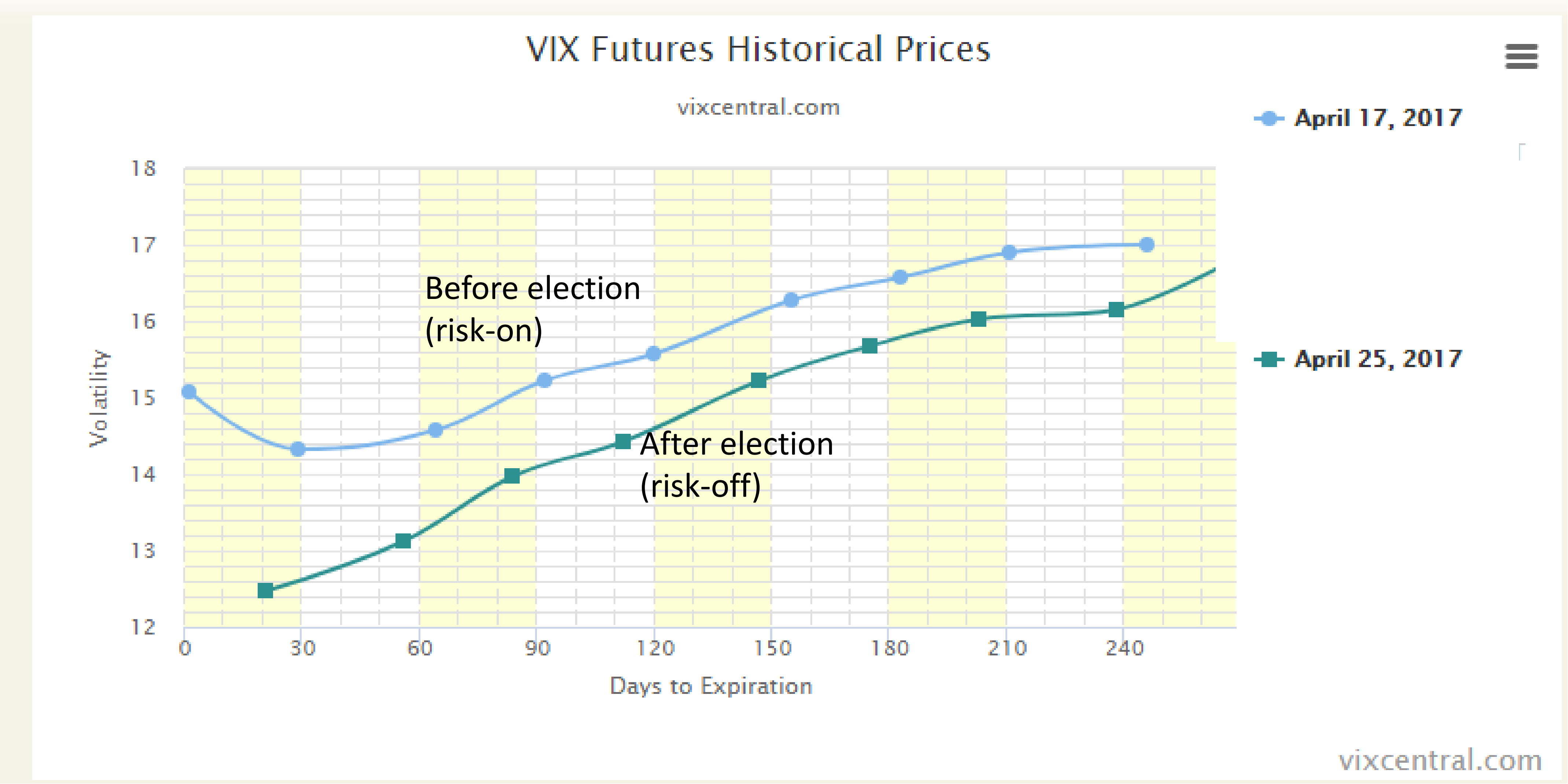
Note: Recently introduced weeklies are illiquid and should not be used to build CMF curve

vixcentral.com

Partial Backwardation: French election, 1st round



Term-structures before & after French election



VIX futures: Lehman week, and 2 months later

The VIX futures cycle (a.k.a. Risk-on/Risk-off)

1. Start here

- Market is ``trending/quiet” (risk-on), volatility is low , VIX term structure is in contango (i.e. upward sloping)
- The possibility of market becoming more risky arises; 30-day S&P implied vols rise
- Risk-off: VIX spikes, CMF flattens in the front , then `curls up’, eventually going into backwardation
- Backwardation is usually partial (CMF decreases only for short maturities), but can be total in extreme cases (2008)
- Uncertainty eventually resolves itself, CMF curve drops and steepens, risk-on comes back
- Most likely state (contango) is restored

2. End here

3. Repeat…..

Statistics of VIX Futures are studied with CMF curves

- Constant-maturity futures, V^{τ}, linearly interpolating quoted futures prices

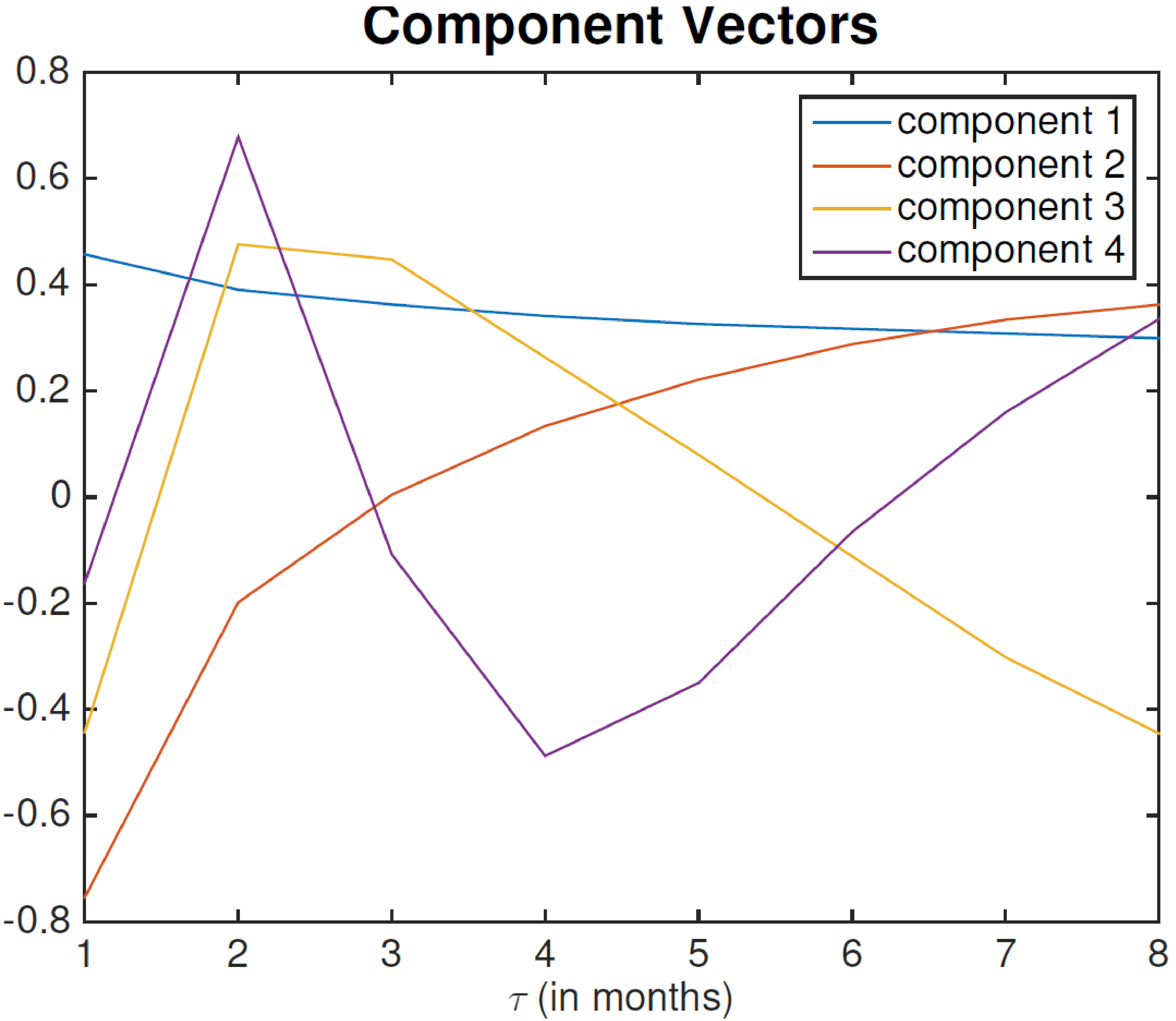
$$\boxed{V_t^{\tau} = \frac{\tau_{k+1} - \tau}{\tau_{k+1} - \tau_k} VX_k(t) + \frac{\tau - \tau_k}{\tau_{k+1} - \tau_k} VX_{k+1}(t)}$$

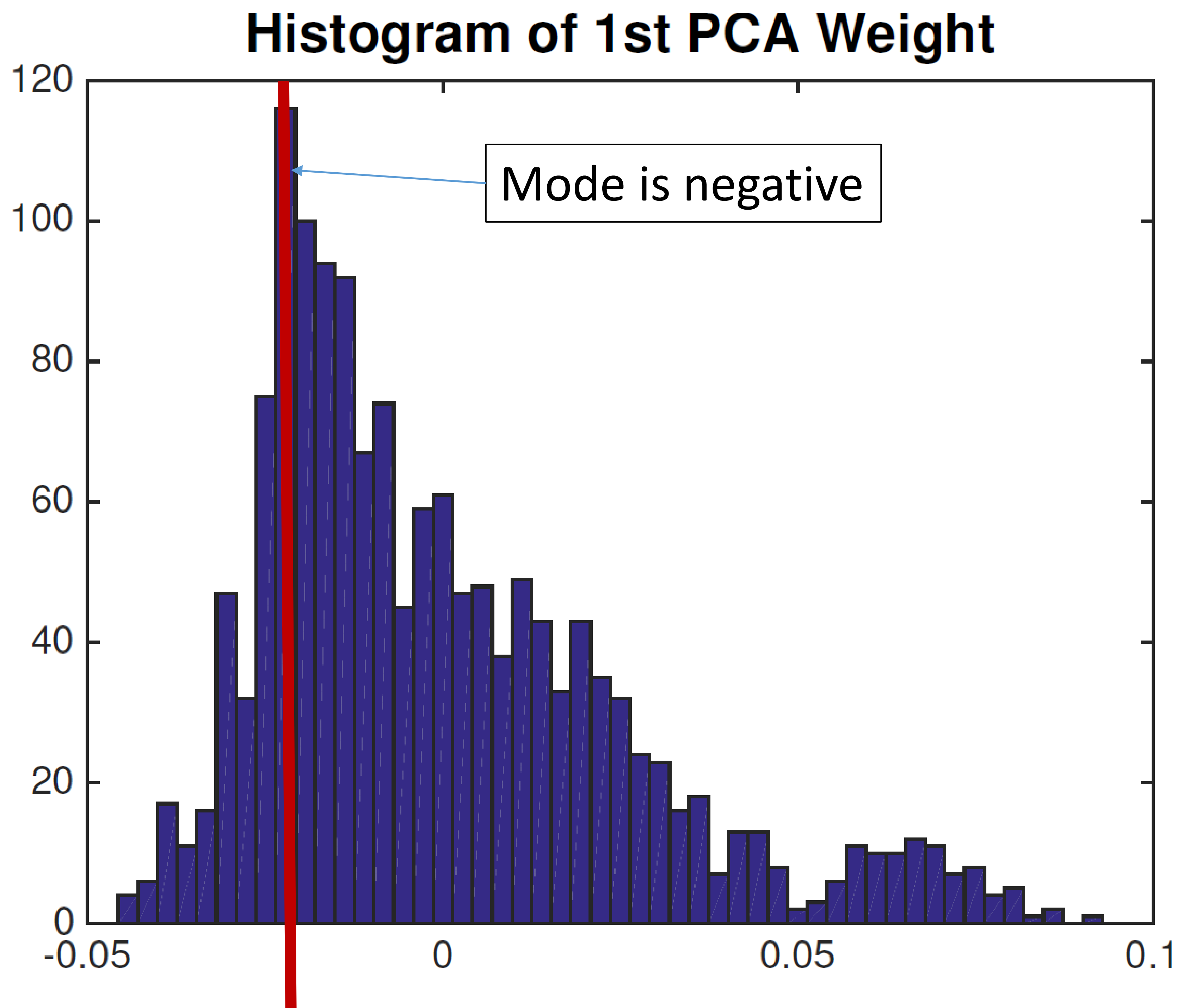
VX_k(t)= kth futures price on date t, $VX_0 = \text{VIX}, \quad \tau_0 = 0, \quad \tau_k = \text{tenor of kth futures}$

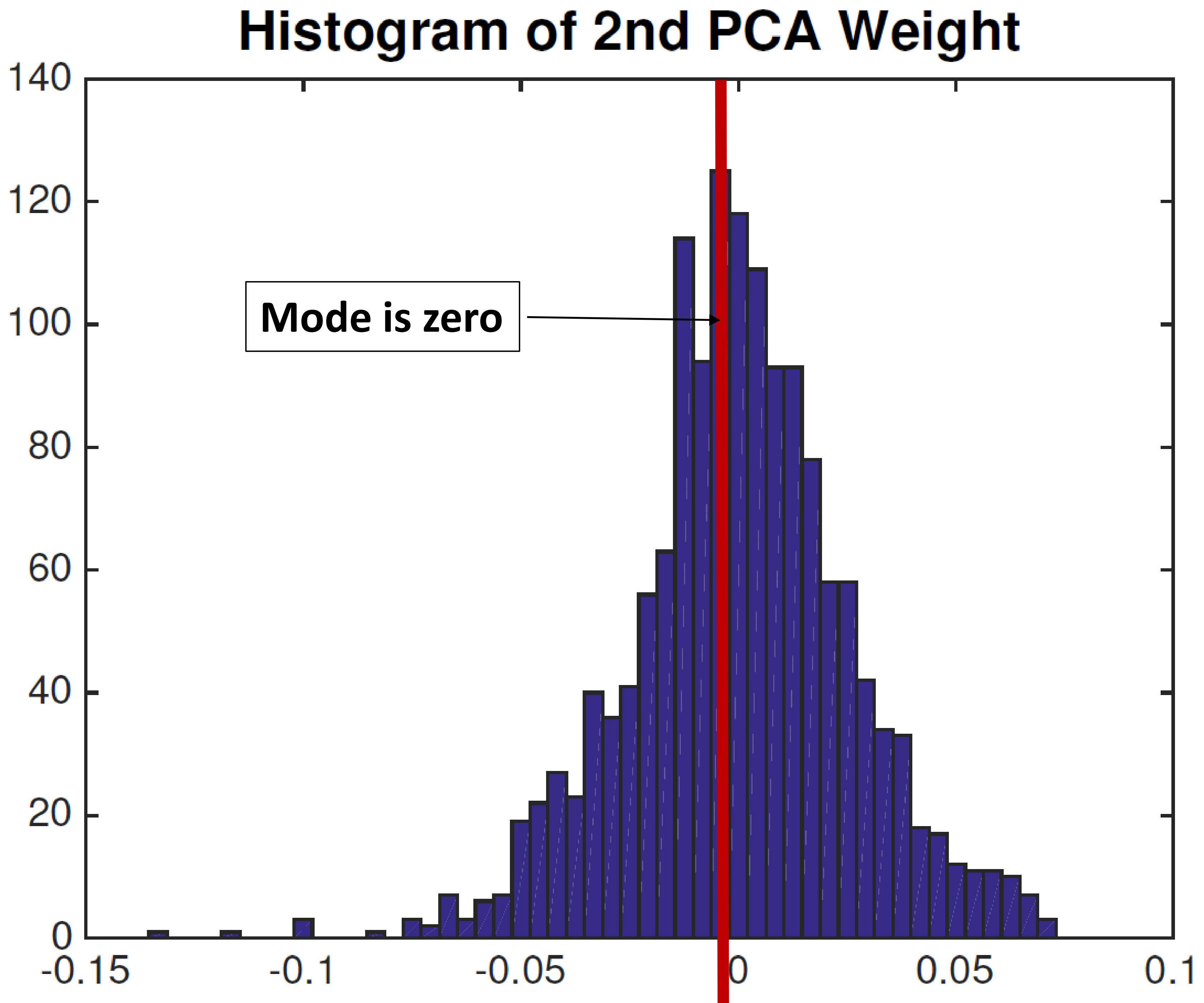
PCA: quantify CMF fluctuations

- Select standard tenors τ_k, $k = 0, 30, 60, 90, 120,150, 180, 210$
- Data: Feb 8 2011 to Dec 15 2016
- Estimate: $$lnV_{t_i}^{\tau_k} = \overline{lnV^{\tau_k}} + \sum_{l=1}^{8} a_{il}\Psi_l^k$$
- Slightly different from Alexander and Korovilas (2010) who did the PCA of 1-day log-returns.

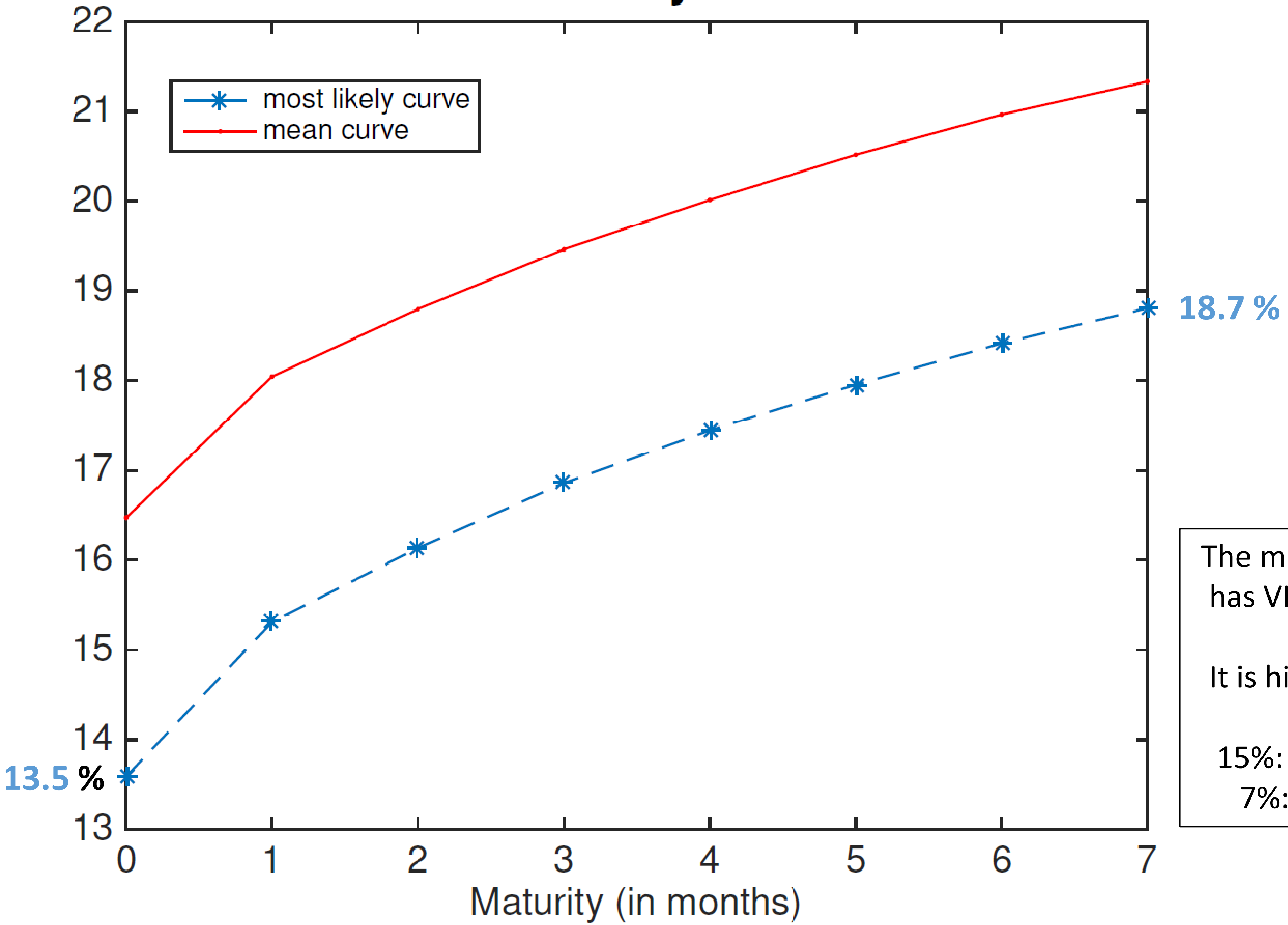
Eigenvalue	% variance expl
1	72
2	18
3	6
4	1
5 to 8	<1


Component Vectors
component 1
component 2
component 3
component 4
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
1
2
3
4
5
6
7
8
τ (in months)


Histogram of 1st PCA Weight
Mode is negative
120
100
80
60
40
20
0
-0.05
0
0.05
0.1


Histogram of 2nd PCA Weight
Mode is zero
140
120
100
80
60
40
20
0
-0.15
-0.1
-0.05
0
0.05
0.1

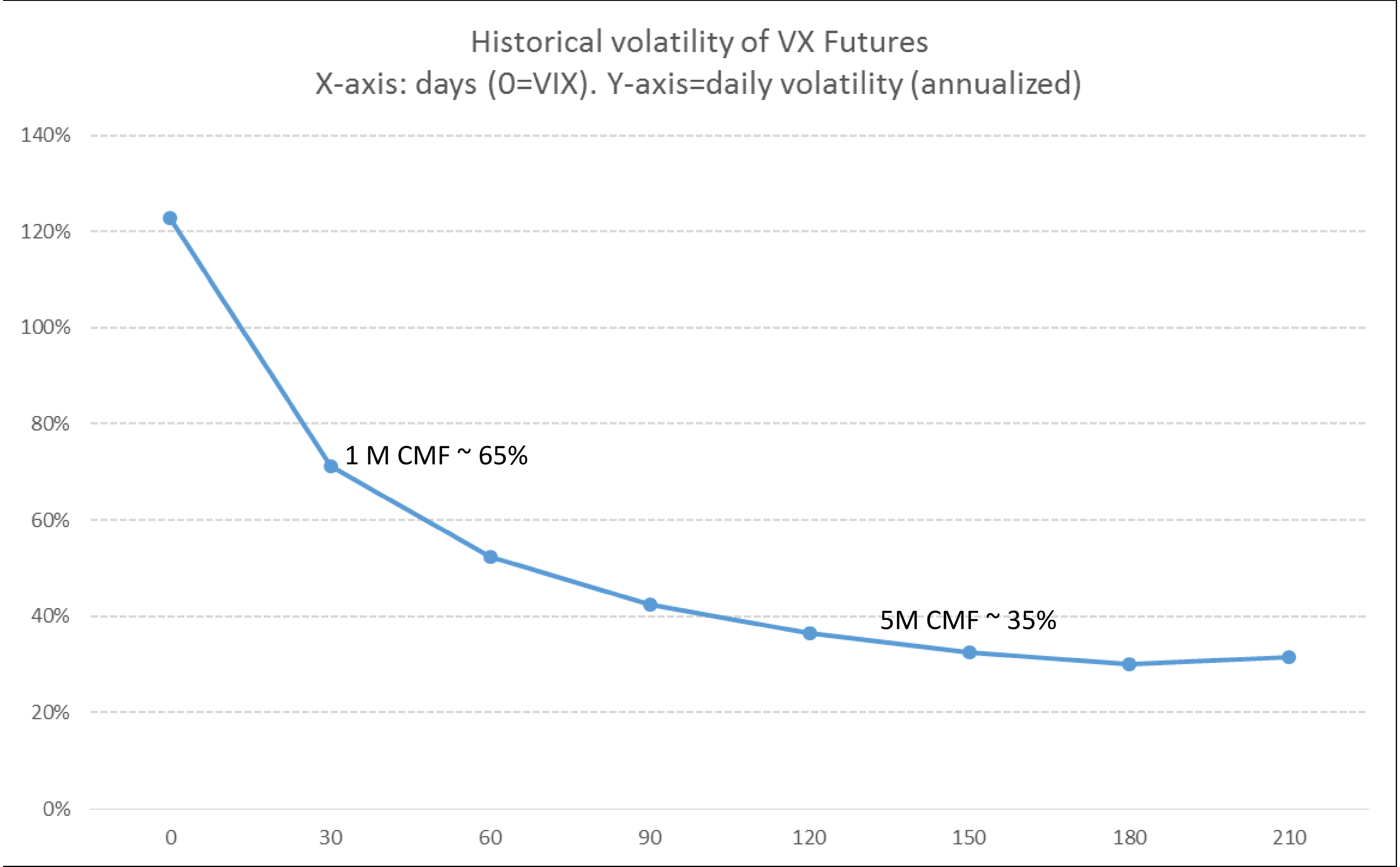
Most Likely Curve (Data: Feb 8 2011 to Dec 15 2016)



The most likely curve has VIX at 13.5

It is highly concave,

15%: 0 to 1 month
7%: 1 to 2 months


Historical volatility of VX Futures
X-axis: days (0=VIX). Y-axis=daily volatility (annualized)
140%
120%
100%
80%
60%
40%
20%
0%
1 M CMF ~ 65%
5M CMF ~ 35%
0
30
60
90
120
150
180
210

ETFs/ETNs based on futures: the `equitization' of VIX

- Funds track an ``investable index'', corresponding to a rolling futures strategy

- Invest in a basket of futures contracts

$$\frac{dI}{I} = r\,dt + \sum_{i=1}^{N} a_i \frac{dF_i}{F_i}$$

a_i = fraction (%) of assets in ith future

- Normalization of weights:

$$\sum_{i=1}^{N} a_i = \beta,$$

β = leverage coefficient

ETFs/ETNs have a target average maturity

- Assume $\beta = 1$, let b_i= fraction of **total number of contracts** invested in i[th] futures:

$$b_i = \frac{n_i}{\sum n_j} = \frac{I\, a_i}{F_i}.$$

- The average maturity θ is typically fixed, resulting in a rolling strategy.

$$\theta = \sum_{i=1}^{N} b_i\,(T_i - t) = \sum_{i=1}^{N} b_i \tau_i$$

Example 1: VXX (maturity = 1M, long VIX futures, daily rolling)

$$\frac{dI}{I} = rdt + \frac{b(t)dF_1 + (1-b(t))dF_2}{b(t)F_1 + (1-b(t))F_2}$$

Weights are based on 1-M CMF , no leverage

$$b(t) = \frac{T_2 - t - \theta}{T_2 - T_1}$$

$$\theta = 1 \text{ month} = 30/360$$

Weights correspond to CMF (linear int.)

$$V_t^\theta = b(t)F_1 + (1-b(t))F_2$$

We have

$$dV_t^\theta = b(t)dF_1 + (1-b(t))dF_2 + b'(t)F_1 - b'(t)F_2$$

Hence

$$\frac{dV_t^\theta}{V_t^\theta} = \frac{b(t)dF_1 + (1-b(t))dF_2}{b(t)F_1 + (1-b(t))F_2} + \frac{F_2 - F_1}{b(t)F_1 + (1-b(t))F_2}\frac{dt}{T_2 - T_1}$$

Dynamic link between Index and CMF equations (long 1M CMF, daily rolling)

$$\frac{dI}{I} = r dt + \frac{b(t)dF_1 + (1 - b(t))dF_2}{b(t)F_1 + (1 - b(t))F_2}$$

$$= r\, dt + \frac{dV_t^\theta}{V_t^\theta} - \frac{F_2 - F_1}{b(t)F_1 + (1 - b(t))F_2} \frac{dt}{T_2 - T_1}$$

$$\boxed{\frac{dI}{I} = r\, dt + \frac{dV_t^\theta}{V_t^\theta} - \left.\frac{\partial \ln V_t^\tau}{\partial \tau}\right]_{\tau=\theta} dt}$$

Slope of the CMF is the relative drift between index and CMF

Example 2 : XIV, Short 1-M rolling futures

This is a fund that follows a DAILY rolling strategy, sells futures, targets 1-month maturity

$$\frac{dJ}{J} = r\,dt - \frac{dV_t^\theta}{V_t^\theta} + \left.\frac{\partial \ln V_t^\tau}{\partial\,\tau}\right]_{\tau=\theta} dt$$

$\theta = 1 \text{ month} = 30/360$

In order to maintain average maturities/leverage, funds must ``reload'' on futures contracts which tend to spot VIX and then expire. **Hence, under contango, long ETNs decay, short ETNs increase.**

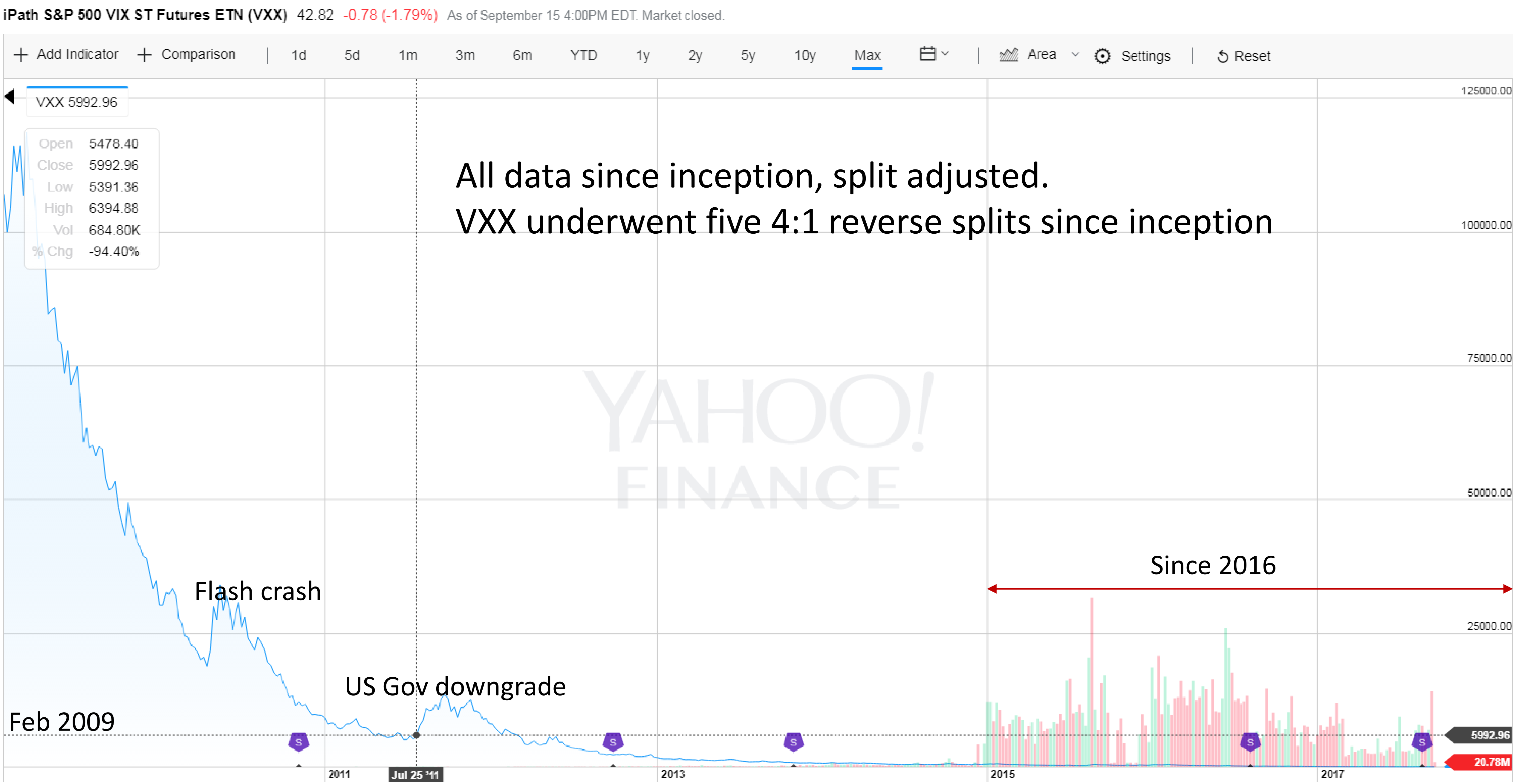
Stationarity/ergodicity of CMF and consequences

Integrating the *I*-equation for VXX and the corresponding *J*-equation for XIV (inverse):

$$VXX_0 - e^{-r\,t}\,VXX_t = VXX_0\left[1 - \frac{V_t^\tau}{V_0^\tau}\,exp\left(-\int_0^t \frac{\partial \ln V_s^\theta\,ds}{\partial\tau}\right)\right]$$

$$e^{-r\,t}\,XIV_t - XIV_0 = XIV_0\left[\frac{V_0^\tau}{V_t^\tau}\,exp\left(\int_0^t \frac{\partial \ln V_s^\theta\,ds}{\partial\tau}\right) - 1\right]$$

If VIX is stationary and ergodic, and $E\left(\frac{\partial \ln V_s^\theta}{\partial\tau}\right) > 0,$ static buy-and-hold XIV or short-and-hold VXX produce sure profits in the long run, with probability 1.


iPath S&P 500 VIX ST Futures ETN (VXX) 42.82 -0.78 (-1.79%) As of September 15 4:00PM EDT. Market closed.
Add Indicator
Comparison
1d
5d
1m
3m
6m
YTD
1y
2y
5y
10y
Max
Area
Settings
Reset
VXX 5992.96
Open 5478.40
Close 5992.96
Low 5391.36
High 6394.88
Vol 684.80K
% Chg -94.40%
All data since inception, split adjusted.
VXX underwent five 4:1 reverse splits since inception
Since 2016
Flash crash
US Gov downgrade
Feb 2009
125000.00
100000.00
75000.00
50000.00
25000.00
5992.96
20.78M
2011
Jul 25 '11
2013
2015
2017

A closer look shows self-similar pattern (last 2 1/2 years)


VelocityShares Daily Inverse VIX ST ETN (XIV) 89.61 +1.69 (1.92%) As of September 15 4:00PM EDT. Market closed.
Add Indicator
Comparison
1d
5d
1m
3m
6m
YTD
1y
2y
5y
10y
Max
Area
Settings
Reset
XIV 13.89
Open 14.31
Close 13.89
Low 13.33
High 14.43
Vol 41.11M
% Chg 25.80%
korea
le pen
china
trump
brexit
100.00
90.00
89.61
80.00
70.00
60.00
50.00
40.00
30.00
20.00
13.89
10.00
5.00M
2011
Aug 27 '12
2013
2015
2017
YAHOO! FINANCE

Modeling the CMF curve dynamics

- VIX ETNs are exposed to (i) volatility of VIX (ii) slope of the CMF curve

- 1-factor model is not sufficient to capture observed ``partial backwardation'' and ``bursts'' of volatility

- Parsimony suggests a 2-factor model

- Assume mean-reversion to investigate the stationarity assumptions

- Sacrifice other ``stylized facts'' (fancy vol-of-vol) to obtain analytically tractable formulas.

`Classic' log-normal 2-factor model for VIX

$$VIX_t = exp(X_{1\,t} + X_{2\,t})$$

$$dX_1 = \sigma_1 dW_1 + k_1(\mu_1 - X_1)dt$$

$$dX_2 = \sigma_2 dW_2 + k_2(\mu_2 - X_2)dt$$

$$dW_1\, dW_2 = \rho\, dt$$

X_1 = factor driving mostly VIX or short-term futures fluctuations (slower)

X_2 = factor driving mostly CMF slope fluctuations (faster)

These factors should be positively correlated.

Use Q-measure to model CMFs

$$V^{\tau} = E^{Q}\{VIX_{\tau}\} = E^{Q}\{exp(X_{1\,\tau} + X_{2\,\tau})\}$$

Ensuring no-arbitrage between Futures, Q = ``pricing measure'' with MPR

$$V^{\tau} = V^{\infty}\, exp\left[e^{-\bar{k}_1\tau}(X_1 - \bar{\mu}_1) + e^{-\bar{k}_2\tau}(X_2 - \bar{\mu}_2) - \frac{1}{2}\sum_{ji=1}^{2}\frac{e^{-\bar{k}_i\tau}e^{\overline{-k}_j\tau}}{\bar{k}_i+\bar{k}_j}\sigma_i\sigma_j\rho_{ij}\right]$$

`Overline parameters' correspond to Q-measure. Assuming a linear market price of risk, the risk factors X are distributed like OU processes with ``renormalized'' parameters under Q.

Estimating the model means: find $k_1, \mu_1, k_2, \mu_2, \bar{k}_1, \bar{\mu}_1, \bar{k}_2, \bar{\mu}_2, \sigma_1, \sigma_2, \rho, V^{\infty}$ using historical data (P measure)

Estimating the model, 2011-2016 (post 2008)

- Kalman filtering approach

Estimated Θ
Input data: 2/2011 to 12/2016, with VIX and CMFs 1m to 7m.

$\bar{\mu}_1$	3.8103	μ_1	3.2957
$\bar{\mu}_2$	-0.7212	μ_2	-0.7588
$\bar{\kappa}_1$	1.1933	κ_1	0.4065
$\bar{\kappa}_2$	10.8757	κ_2	13.1019
σ_1	0.6776		
σ_2	0.8577		
ρ	0.4462		

Estimated Θ
Input data: 2/2011 to 12/2016, with VIX, 3m and 6m CMFs.

$\bar{\mu}_1$	3.8094	μ_1	3.2524
$\bar{\mu}_2$	-0.7100	μ_2	-0.7155
$\bar{\kappa}_1$	1.0215	κ_1	0.3219
$\bar{\kappa}_2$	10.8739	κ_2	13.0799
σ_1	0.5931		
σ_2	0.9066		
ρ	0.4266		

Estimating the model, 2007 to 2016 (contains 2008)

Estimated Θ
Input data: 7/2007 to 7/2016, with VIX and CMFs 1m to 6m.

$\bar{\mu}_1$	-6.6216	μ_1	-7.1367
$\bar{\mu}_2$	9.7372	μ_2	9.7608
$\bar{\kappa}_1$	0.6543	κ_1	0.2010
$\bar{\kappa}_2$	5.9052	κ_2	5.9389
σ_1	0.5525		
σ_2	0.9802		
ρ	0.6015		

Estimated Θ
Input data: 7/2007 to 7/2016, with VIX, 1m and 6m CMFs.

$\bar{\mu}_1$	2.4581	μ_1	1.8685
$\bar{\mu}_2$	0.8002	μ_2	0.7555
$\bar{\kappa}_1$	0.5505	κ_1	0.1081
$\bar{\kappa}_2$	10.0013	κ_2	12.3600
σ_1	0.4294		
σ_2	0.7998		
ρ	0.5073		

Stochastic differential equations for ETNs (e.g. VXX)

$$\frac{dI}{I} = r\,dt + \frac{dV_t^\theta}{V_t^\theta} - \left.\frac{\partial \ln V_t^\tau}{\partial\,\tau}\right]_{\tau=\theta} dt$$

Substituting closed-form solution in the ETN index equation we get:

$$\frac{dI}{I} = r\,dt + \sum_{i=1}^{2} e^{-\bar{k}_i\theta}\sigma_i dW_i + \sum_{i=1}^{2} e^{-\bar{k}_i\theta}\left[\,\left(\bar{k}_i - k_i\right)X_i + \left(k_i\mu_i - \bar{k}_i\bar{\mu}_i\right)\right] dt$$

$$\text{Equilibrium local drift} = \sum_{i=1}^{2} e^{-\bar{k}_i\theta}\left[\bar{k}_i(\mu_i - \bar{\mu}_i)\right] + r$$

$$\sigma_I^2 = \sum_{ji=1}^{2} e^{-\bar{k}_i\tau} e^{\,\overline{-k}_j\tau}\,\sigma_i\sigma_j\rho_{ij}$$

Results of the Numerical Estimation for VIX ETNs

Forecast profitability for short VXX/long XIV, in equilibrium

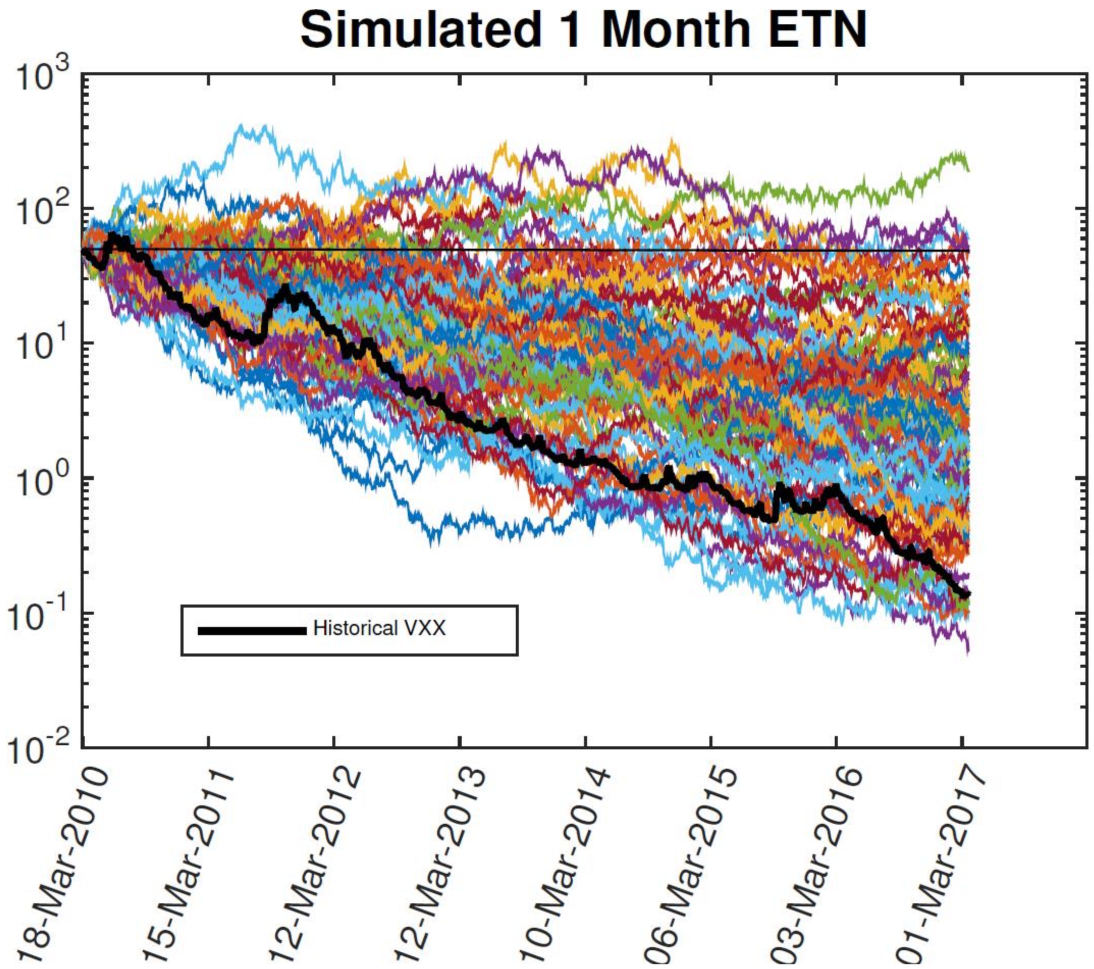
	Jul 07 to Jul 16 VIX, CMF 1M to 6M	Jul 07 to Jul 16 VIX, 1M, 6M	Feb 11 to Dec 16 VIX, CMF 1M to 7M	Feb 11 to Jul 16 VIX, 3M, 6M
Excess Return	0.30	0.32	0.56	0.53
Volatility	1.00	0.65	0.82	0.77
Sharpe ratio (short trade)	0.29	0.50	0.68	0.68

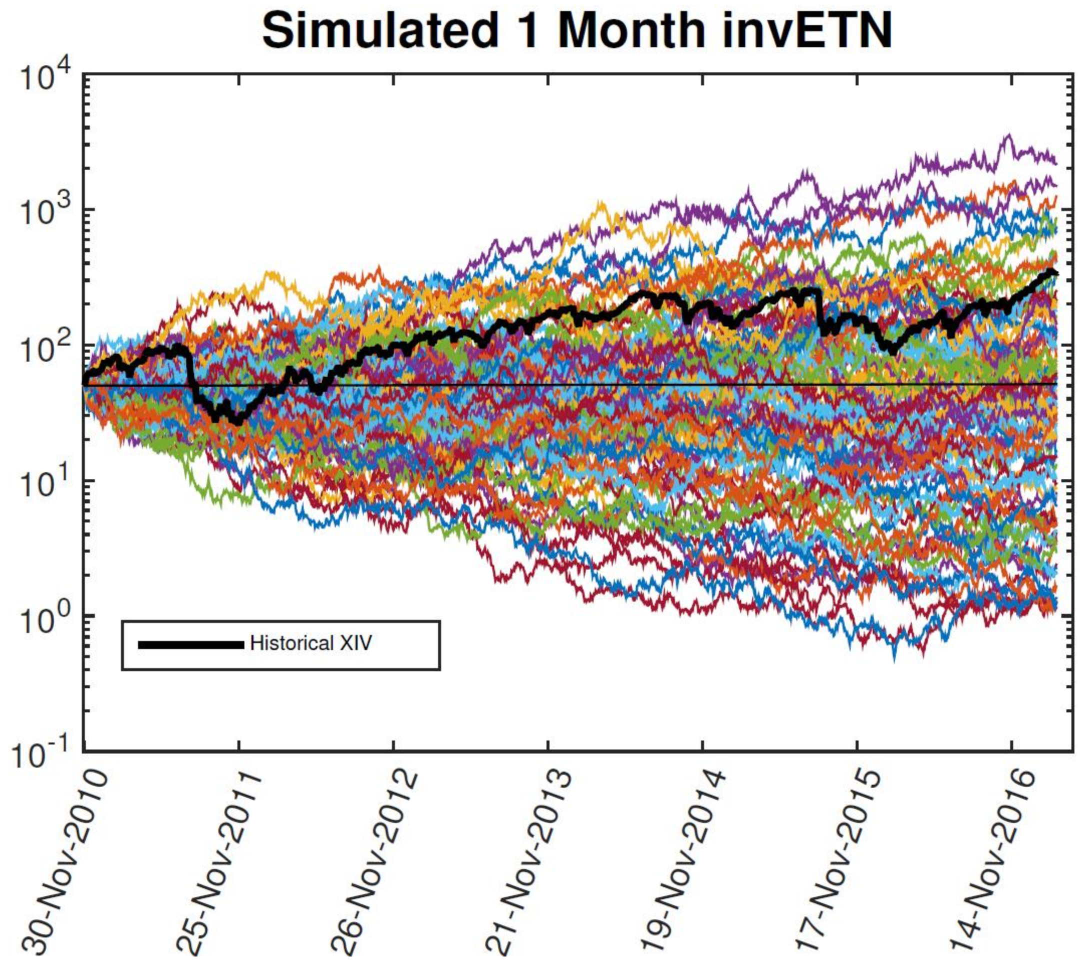
Notes :

(1) For shorting VXX one should reduce the ``excess return'' by the average borrowing cost which is 3%.
It is therefore better to be long XIV (note however that XIV is less liquid, but trading volumes in XIV are increasing.

(2) Realized Sharpe ratios are higher. For instance the Sharpe ratio for Short VXX (with 3% borrow) from Feb 11 To May 2017 is 0.90. This can be explained by low realized volatility in VIX and the fact that the model predicts significant fluctuations in P/L over finite time-windows.

Variability of rolling futures strategies predicted by model (static ETN strategies).





Black=actual historical, colored= simulated paths

Passing from OU factors to CMF curve shapes

$$\begin{bmatrix} \ln V_t^\tau \\ \frac{\partial \ln V_t^\tau}{\partial \tau} \end{bmatrix} = \begin{bmatrix} e^{-\bar{\kappa}_1 \tau} & e^{-\bar{\kappa}_2 \tau} \\ -\bar{\kappa}_1 e^{-\bar{\kappa}_1 \tau} & -\bar{\kappa}_2 e^{-\bar{\kappa}_2 \tau} \end{bmatrix} \begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix} +$$

$$\begin{bmatrix} \ln V^\infty - \bar{\mu}_1 e^{-\bar{\kappa}_1 \tau} - \bar{\mu}_2 e^{-\bar{\kappa}_2 \tau} - \frac{1}{2} \frac{\sigma_1^2}{2\bar{\kappa}_1} e^{-2\bar{\kappa}_1 \tau} - \frac{1}{2} \frac{\sigma_2^2}{2\bar{\kappa}_2} e^{-2\bar{\kappa}_2 \tau} - \frac{\rho_{12}\sigma_1\sigma_2}{\bar{\kappa}_1 + \bar{\kappa}_2} e^{-(\bar{\kappa}_1 + \bar{\kappa}_2)\tau} \\ \bar{\kappa}_1 \bar{\mu}_1 e^{-\bar{\kappa}_1 \tau} + \bar{\kappa}_2 \bar{\mu}_2 e^{-\bar{\kappa}_2 \tau} + \frac{1}{2}\sigma_1^2 e^{-2\bar{\kappa}_1 \tau} + \frac{1}{2}\sigma_2^2 e^{-2\bar{\kappa}_2 \tau} + \rho_{12}\sigma_1\sigma_2 e^{-(\bar{\kappa}_1 + \bar{\kappa}_2)\tau} \end{bmatrix} + \epsilon$$

$$\frac{dI}{I} = r dt + \sum_{i=1}^{2} e^{-\bar{\kappa}_i \theta} \sigma_i dW_i + \sum_{i=1}^{2} e^{-\bar{\kappa}_i \theta} [(\bar{\kappa}_i - \kappa_i) X_i + \kappa_i \mu_i - \bar{\kappa}_i \bar{\mu}_i] dt$$

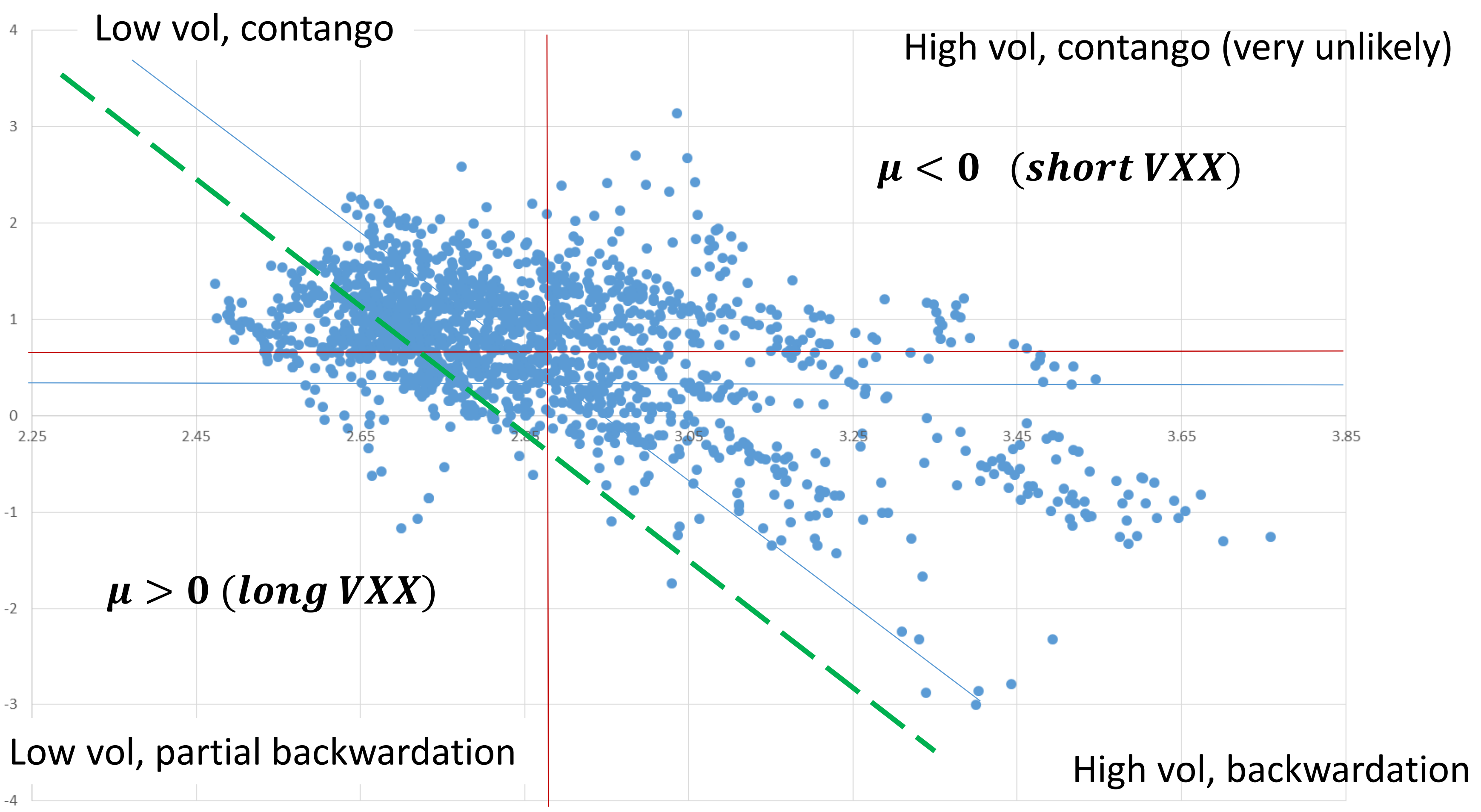
Solve for X in the first equation, substitute in second equation.

Expressing VXX dynamics in terms of the shape of the CMF curve as an SDE

$$\frac{dI}{I}\Big|_{\tau=\frac{1}{12}} = \{-2.53[\ln V_t^0 - E[\ln V_t^0]] - 0.61[\frac{\partial \ln V_t^\tau}{\partial \tau}\Big|_{\tau=\frac{1}{12}} - E[\frac{\partial \ln V_t^\tau}{\partial \tau}\Big|_{\tau=\frac{1}{12}}]] - 0.57\}dt + 0.60 dW_t$$

- This is an actionable result for quantitative trading.
- Negative drift for high volatility and/or high contango

“Phase Diagram”: X=ln V(30), Y=d lnV(30)/dτ



Dynamic trading of VIX ETNs can produce Sharpe ratios as high as 1.7 (not shown)

A few references…

[1] The CBOE Volatility Index- VIX, White Paper, Chicago Board of Options Exchange, http://www.cboe.com/micro/vix/vixwhite.pdf

[2] Alexander, C. and Korovilas, Understanding ETNs on VIX Futures (SSRN, 2012)

[3] Avellaneda, M., and Papanicolaou, A., Statistics of VIX Futures and Applications to Trading Volatility ETNs (SSRN, Sep 2017)

[4] http://www.eurexchange.com/exchange-en/products/vol/vstoxx/vstoxx--futures-and-options
.
[5] https://www.investing.com/indices/stoxx-50-volatility-3stoxx-eur-historical-data

3. VIX Options

VXX Options

- Options on VIX futures settle and are priced based on VIX futures
- (i.e. the ``forward price'' is the co-terminal VIX price). Black 76 works.

- What about options on VXX? **Contango plays a role** in the evolution of VXX prices, and thus the fair value of an option (regardless of volatility).

- Our point: The term-structure of VIX is volatile, so the **forward pricing of VXX is not be that straight-forward.**

- **Strategy A:** derive a better ``physical measure'' (or forecast) for VXX based on econometric analysis of VIX futures curves and roll formula.
  Trading strategy: buy cheap options and don't delta hedge

- **Strategy B**: use the physical measure as prior and build a risk-neutral measure for trading VXX options relative to VIX options.
  Trading strategy: trade VXX options vs. VIX options (delta-neutral)

Joint work with Shinshun Jiang, NYU

Build a good physical measure for VXX

- VXX is based on an index, which represents a theoretical roll between the two front VIX contracts.

- Fit the term-structure of futures to a multivariate GARCH (1,1) model.

  Step 1: Perform PCA on the futures curve, parameterized by constant maturity

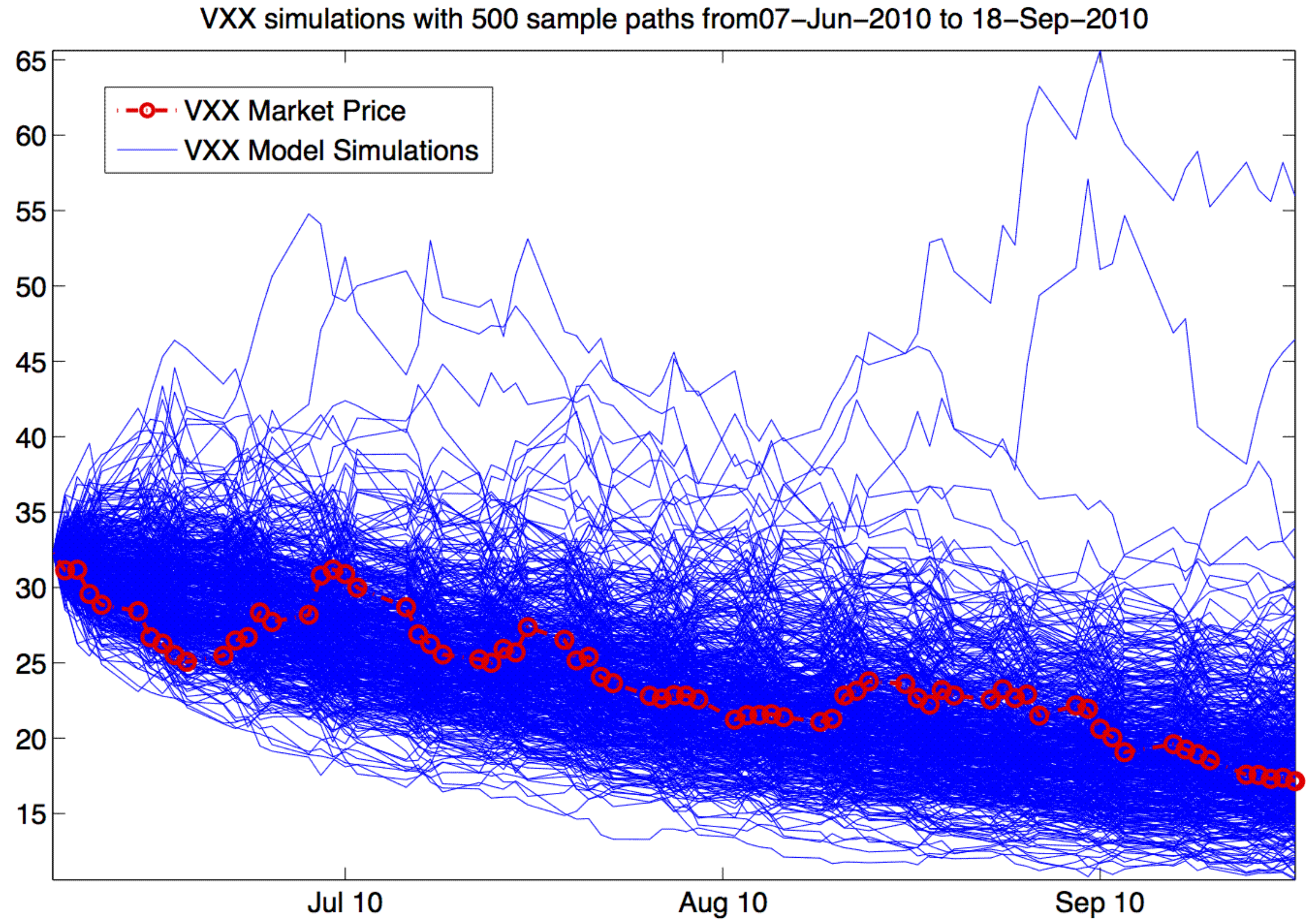
  Step 2: Derive significant Principal Components for VIX term-structure

  Step 3: Estimate the evolution of the Principal Component loadings as uncorrelated GARCH (1,1) processes

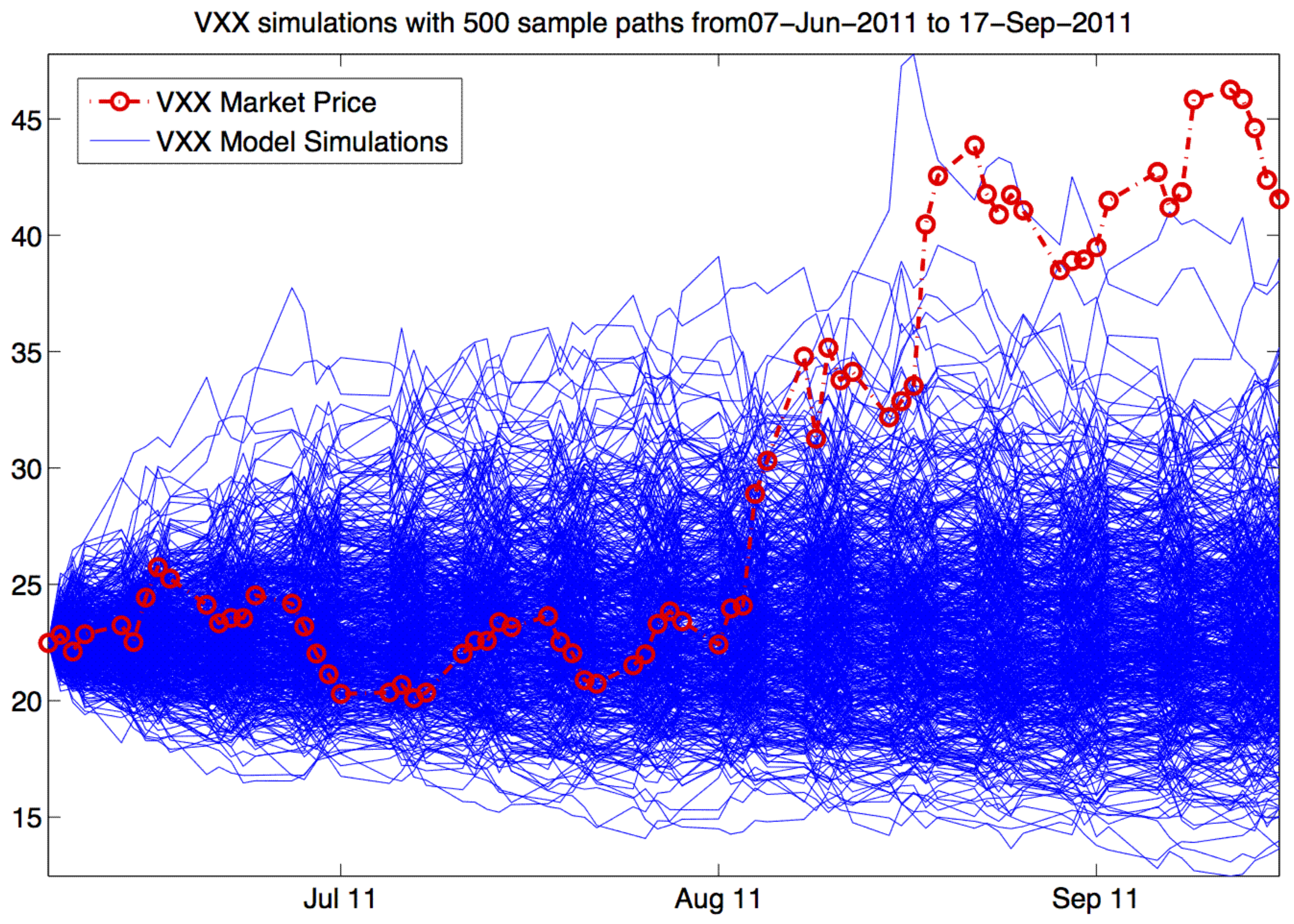
This gives a tool for **forecasting VXX.**

For PCA: see Carol Alexander and Dmitris Korovilas (2002)
For GARCH on term structures: see Avellaneda and Zhou (1996)

Actual VXX is within the forecast prices (out of sample simulation)



Effect of the Downgrade of the US Treasury by S&P in August 2011



VXX Option Valuation

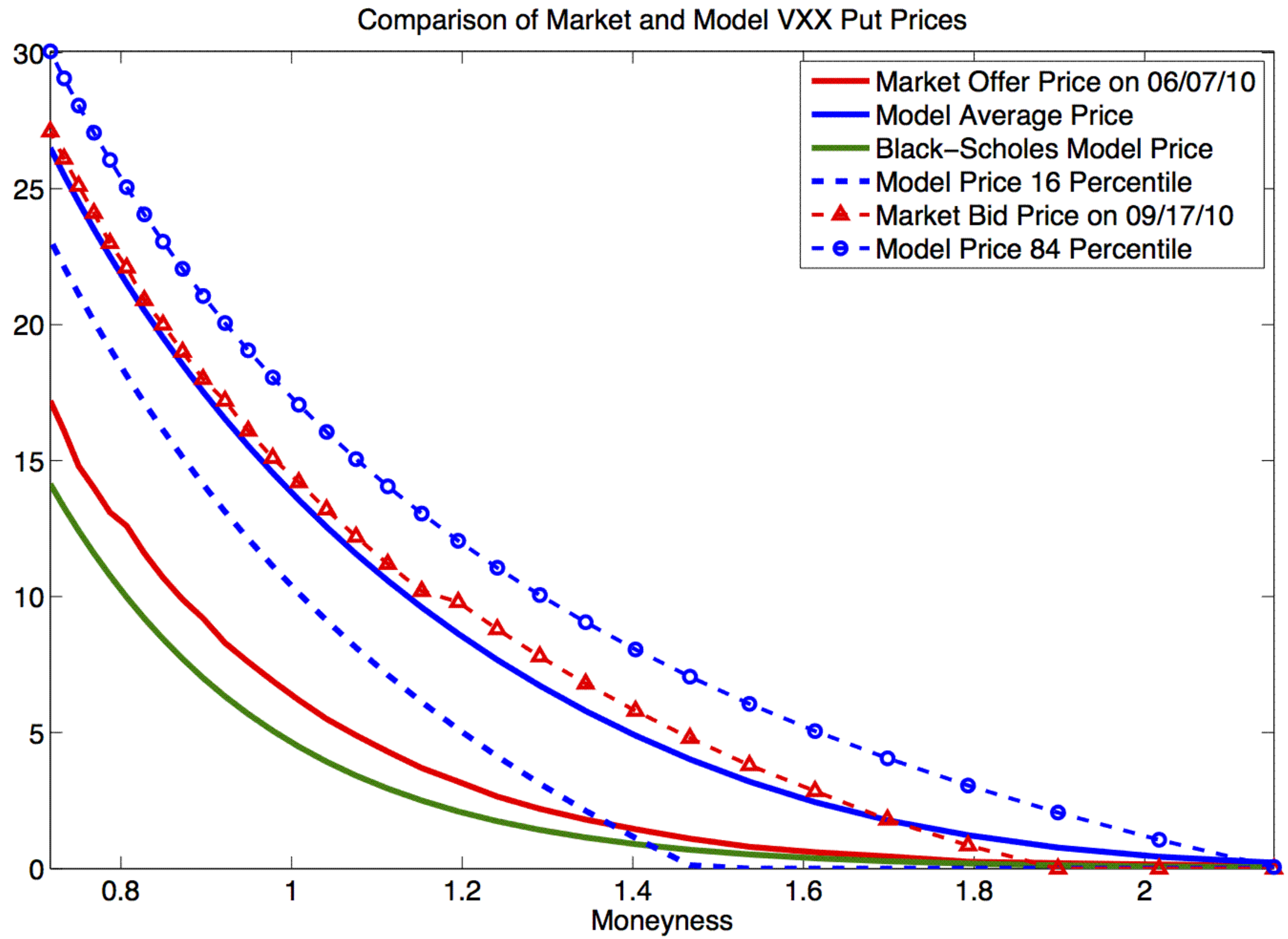
- **Market value:** this is the value of the VXX option from the market, on the pricing date.

- **Black Scholes with historical volatility:** based on an estimation window in the past, use the BS formula with historical volatility to determine a subjective price for VXX options on the pricing date.

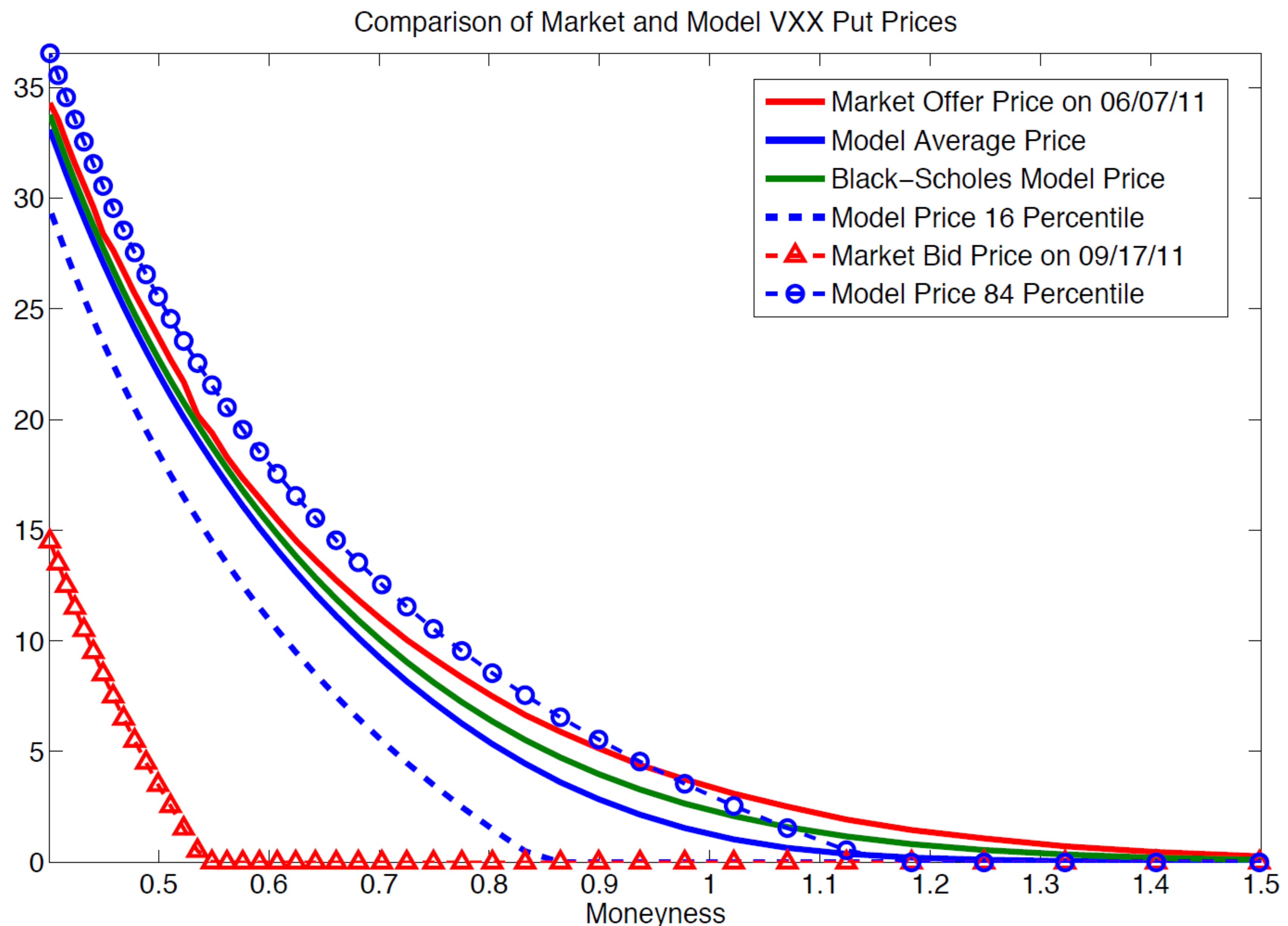
- **New Model:** Use GARCH (1,1) estimation of the movements of the VIX futures curve to simulate scenarios of evolution of the VXX beyond the pricing date. Price options by averaging final payoff over GARCH statistics.

Negative drift of VXX suggests opportunities in buying puts/selling calls.

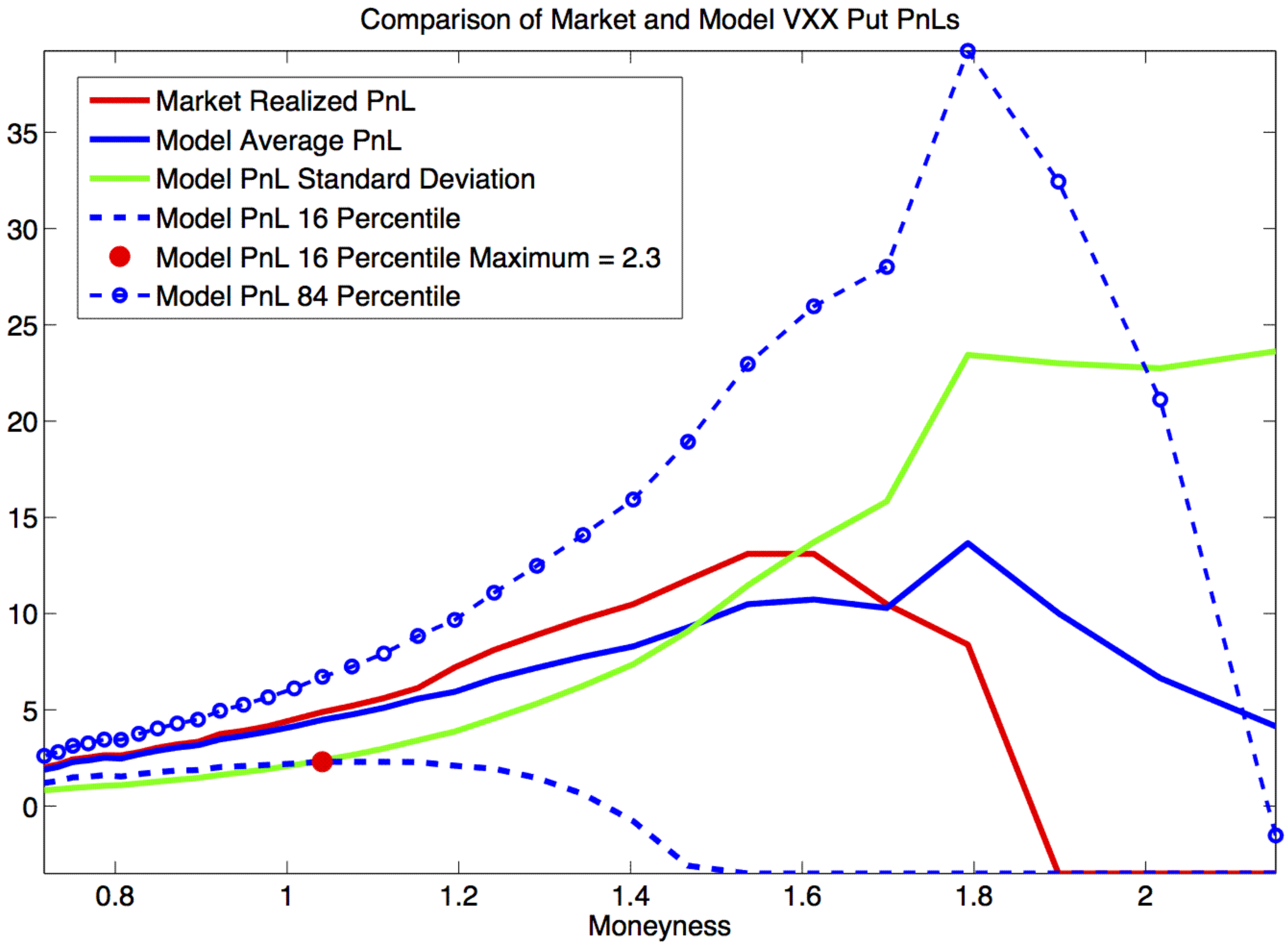
Price Date: June 7, 2010



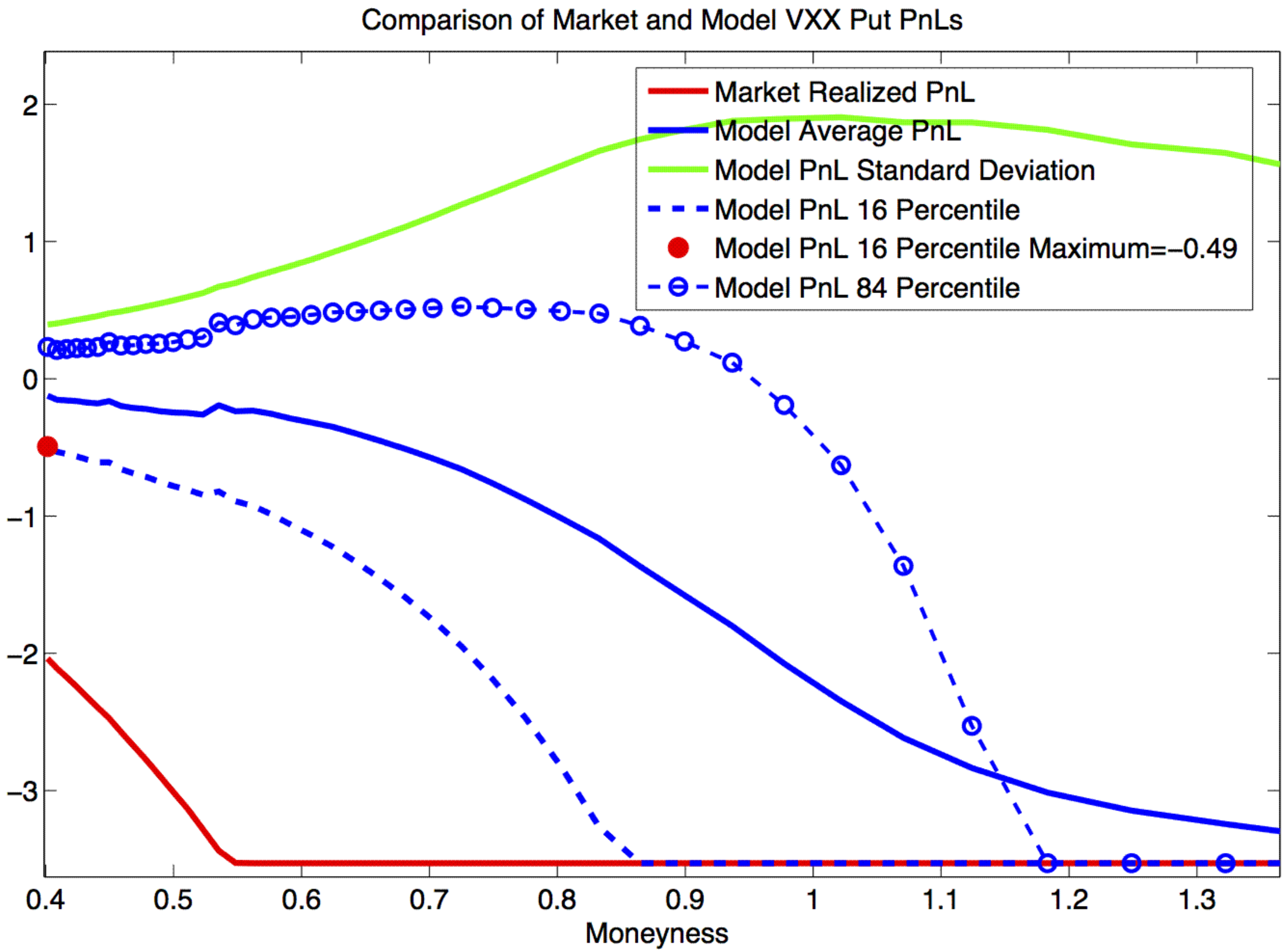
Price Date: June 7, 2011



PNL analysis of Put-buying strategy (6/7/2010)



PNL analysis of Put-buying strategy 6/7/2011



Backtesting Strategy A: May 2008, Oct 2012

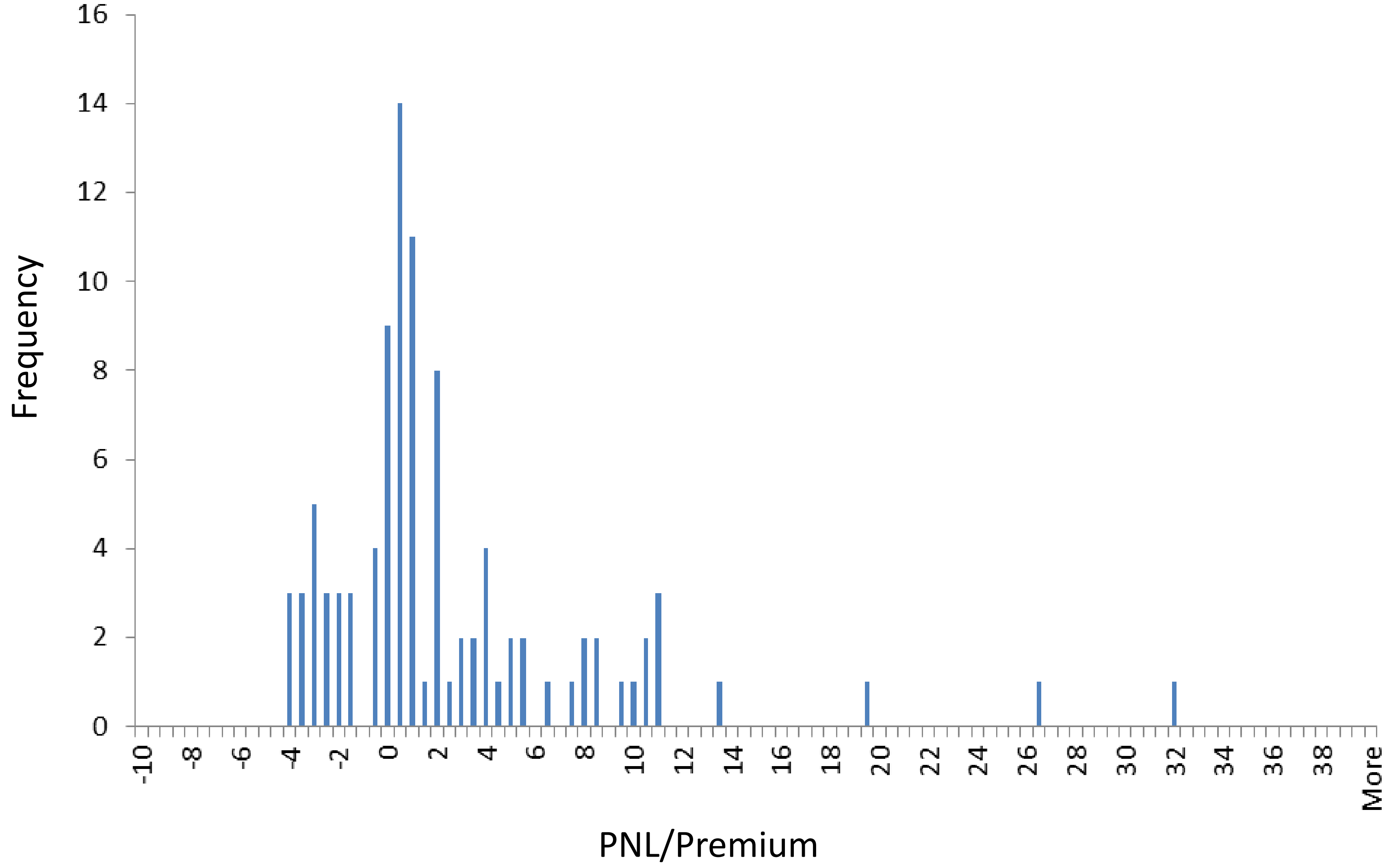
99 Weeks, one Pricing Date per week.

White= profit

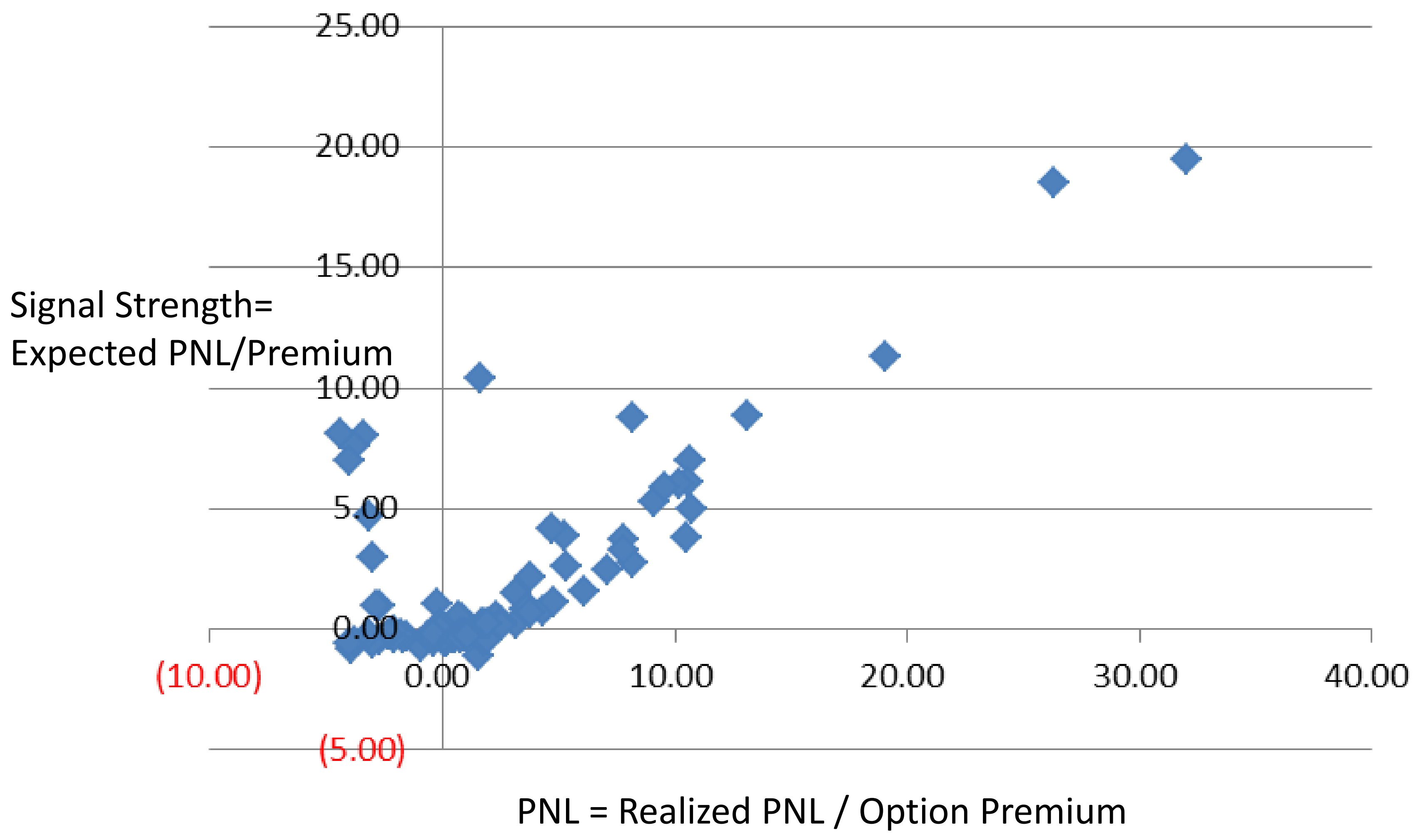
Red = loss

		PNL			Moneyness										
cDate	exDate	PnL	prcMax	Moneyness	0.50	0.60	0.70	0.80	0.90	1.00	1.10	1.20	1.30	1.40	1.50
5/28/2010	9/18/2010	8.16	8.78	1.43	1.50	1.16	1.58	1.94	2.59	3.46	4.83	6.30	7.01	7.90	7.32
6/7/2010	9/18/2010	13.11	8.88	1.54	32.15	7.43	1.96	2.64	3.38	4.41	5.46	7.31	9.04	10.44	12.56
6/14/2010	9/18/2010	10.65	6.98	1.29	4.10	1.52	1.95	2.61	3.62	4.72	6.32	8.70	10.75	12.41	11.17
6/21/2010	9/18/2010	10.59	6.10	1.21	1.89	1.41	1.83	2.58	3.55	5.74	7.77	10.27	11.54	7.92	(4.04)
6/28/2010	9/18/2010	19.02	11.33	1.28	7.16	2.00	2.70	4.05	5.70	8.13	11.60	15.42	19.19	20.59	17.17
7/6/2010	9/18/2010	26.27	18.53	1.30	11.06	2.54	3.42	4.81	6.92	9.37	14.84	20.82	26.06	29.39	30.15
7/13/2010	9/18/2010	10.21	6.00	1.09	2.04	1.93	2.62	3.66	4.99	7.33	10.54	12.75	14.36	8.37	(5.37)
7/20/2010	9/18/2010	32.00	19.45	1.26	2.66	2.36	3.24	4.69	6.79	11.03	16.10	24.56	31.11	19.16	(6.00)
7/27/2010	9/18/2010	9.11	5.25	1.03	1.32	1.83	2.59	3.47	5.72	8.07	11.44	11.88	(0.75)	(6.79)	(6.79)
8/3/2010	9/18/2010	8.16	2.76	0.98	1.22	1.71	2.55	4.22	5.88	8.81	13.25	10.76	(7.83)	(7.83)	(7.83)
11/10/2010	1/22/2011	1.51	(1.08)	0.64	(4.30)	0.55	2.30	3.07	3.99	5.03	5.78	5.89	2.83	(4.88)	(4.93)
11/17/2010	1/22/2011	3.13	0.21	0.69	1.03	2.04	3.30	5.02	6.78	9.69	12.80	15.90	17.85	11.71	(5.26)
11/24/2010	3/19/2011	0.45	0.02	0.62	10.80	0.68	0.62	0.58	0.65	0.22	(0.70)	(2.57)	(3.13)	(3.13)	(3.13)
12/2/2010	3/19/2011	0.52	(0.06)	0.61	(2.01)	0.51	0.69	0.65	0.78	0.40	(0.57)	(2.64)	(3.36)	(3.36)	(3.36)
12/9/2010	3/19/2011	0.15	(0.55)	0.57	(2.88)	0.17	0.13	(0.04)	(0.50)	(1.64)	(3.59)	(3.60)	(3.60)	(3.60)	(3.60)
12/16/2010	3/19/2011	0.18	(0.12)	0.55	(2.38)	0.29	0.32	0.25	(0.22)	(1.33)	(3.66)	(3.87)	(3.87)	(3.87)	(3.87)
12/23/2010	3/19/2011	(0.12)	(0.06)	0.56	(2.27)	0.10	0.11	(0.03)	(0.69)	(2.18)	(4.18)	(4.19)	(4.19)	(4.19)	(4.19)
12/31/2010	3/19/2011	0.05	(0.02)	0.53	(0.60)	0.14	0.13	(0.02)	(0.71)	(2.44)	(4.61)	(4.62)	(4.62)	(4.62)	(4.62)
1/7/2011	3/19/2011	(0.06)	(0.14)	0.55	(0.50)	(0.12)	(0.21)	(0.56)	(1.67)	(4.29)	(5.07)	(5.07)	(5.07)	(5.07)	(5.07)
1/14/2011	3/19/2011	(0.80)	(0.51)	0.44	(0.89)	(1.30)	(1.90)	(3.24)	(5.61)	(5.63)	(5.63)	(5.63)	(5.63)	(5.63)	(5.63)
1/24/2011	3/19/2011	(0.92)	(0.57)	0.44	(1.00)	(1.50)	(2.21)	(3.81)	(6.64)	(6.67)	(6.67)	(6.67)	(6.67)	(6.67)	(6.67)
1/31/2011	3/19/2011	(0.97)	(0.76)	0.45	(1.02)	(1.53)	(2.24)	(3.89)	(7.28)	(7.66)	(7.66)	(7.66)	(7.66)	(7.66)	(7.66)
2/7/2011	6/18/2011	0.12	(0.51)	0.41	(0.02)	0.17	0.09	0.00	(0.16)	(0.77)	(2.00)	(2.75)	(2.75)	(2.75)	(2.75)
2/14/2011	6/18/2011	0.09	(0.53)	0.40	0.06	0.06	0.06	(0.10)	(0.48)	(1.28)	(2.85)	(2.90)	(2.90)	(2.90)	(2.90)
2/22/2011	5/21/2011	1.80	(0.50)	0.65	(104.74)	(4.53)	2.03	2.12	3.57	4.63	4.89	2.86	4.02	(2.23)	(4.09)
3/1/2011	5/21/2011	1.96	(0.44)	0.65	15.71	2.74	2.35	3.15	4.14	5.33	6.32	7.00	5.54	(2.15)	(4.44)
3/8/2011	6/18/2011	0.47	0.03	0.45	0.35	0.68	0.89	1.13	1.21	1.32	0.57	(1.25)	(3.53)	(3.53)	(3.53)
3/15/2011	6/18/2011	0.79	0.11	0.49	0.53	0.79	1.43	1.85	2.30	3.13	3.59	3.49	1.44	(3.79)	(3.79)
3/22/2011	6/18/2011	0.64	(0.07)	0.46	0.26	0.98	1.30	1.65	2.02	2.62	2.38	0.76	(4.09)	(4.09)	(4.09)
3/29/2011	6/18/2011	0.43	(0.36)	0.42	0.51	0.67	0.84	1.06	1.19	0.83	(0.53)	(4.44)	(4.44)	(4.44)	(4.44)
4/5/2011	6/18/2011	0.25	(0.40)	0.40	0.14	0.38	0.63	0.66	0.58	(0.28)	(3.48)	(4.86)	(4.86)	(4.86)	(4.86)
4/12/2011	6/18/2011	0.34	(0.10)	0.41	0.09	0.50	0.71	0.86	1.11	0.52	(2.08)	(5.37)	(5.37)	(5.37)	(5.37)
4/19/2011	6/18/2011	0.10	(0.33)	0.38	0.12	0.16	0.07	0.05	(0.65)	(2.72)	(6.00)	(6.00)	(6.00)	(6.00)	(6.00)
4/27/2011	6/18/2011	(0.37)	(0.54)	0.33	(0.76)	(1.11)	(1.56)	(2.81)	(5.58)	(6.92)	(6.92)	(6.92)	(6.92)	(6.92)	(6.92)
5/4/2011	6/18/2011	(0.29)	(0.40)	0.35	(0.49)	(0.67)	(0.96)	(1.81)	(4.04)	(8.00)	(8.00)	(8.00)	(8.00)	(8.00)	(8.00)
5/11/2011	9/17/2011	(1.57)	(0.33)	0.42	(2.15)	(2.79)	(2.79)	(2.79)	(2.79)	(2.79)	(2.79)	(2.79)	(2.79)	(2.79)	(2.79)
5/18/2011	9/17/2011	(1.67)	(0.36)	0.41	(2.39)	(2.95)	(2.95)	(2.95)	(2.95)	(2.95)	(2.95)	(2.95)	(2.95)	(2.95)	(2.95)
5/25/2011	9/17/2011	(1.79)	(0.23)	0.41	(2.59)	(3.13)	(3.13)	(3.13)	(3.13)	(3.13)	(3.13)	(3.13)	(3.13)	(3.13)	(3.13)
6/2/2011	8/20/2011	(3.96)	(0.80)	0.49	(4.18)	(4.56)	(4.56)	(4.56)	(4.56)	(4.56)	(4.56)	(4.56)	(4.56)	(4.56)	(4.56)
6/9/2011	9/17/2011	(2.08)	(0.38)	0.39	(3.16)	(3.60)	(3.60)	(3.60)	(3.60)	(3.60)	(3.60)	(3.60)	(3.60)	(3.60)	(3.60)
6/16/2011	9/17/2011	(2.06)	(0.10)	0.46	(2.44)	(3.58)	(3.87)	(3.87)	(3.87)	(3.87)	(3.87)	(3.87)	(3.87)	(3.87)	(3.87)
6/23/2011	9/17/2011	(2.34)	(0.25)	0.42	(3.22)	(4.19)	(4.19)	(4.19)	(4.19)	(4.19)	(4.19)	(4.19)	(4.19)	(4.19)	(4.19)
6/30/2011	9/17/2011	(2.69)	(0.45)	0.38	(4.40)	(4.56)	(4.56)	(4.56)	(4.56)	(4.56)	(4.56)	(4.56)	(4.56)	(4.56)	(4.56)
7/8/2011	9/17/2011	(3.03)	(0.57)	0.36	(5.07)	(5.07)	(5.07)	(5.07)	(5.07)	(5.07)	(5.07)	(5.07)	(5.07)	(5.07)	(5.07)
7/15/2011	9/17/2011	(3.16)	(0.24)	0.41	(4.47)	(5.63)	(5.63)	(5.63)	(5.63)	(5.63)	(5.63)	(5.63)	(5.63)	(5.63)	(5.63)
7/22/2011	9/17/2011	(3.76)	(0.55)	0.37	(6.27)	(6.32)	(6.32)	(6.32)	(6.32)	(6.32)	(6.32)	(6.32)	(6.32)	(6.32)	(6.32)
7/29/2011	9/17/2011	(4.03)	(0.60)	0.42	(5.60)	(7.20)	(7.20)	(7.20)	(7.20)	(7.20)	(7.20)	(7.20)	(7.20)	(7.20)	(7.20)
8/5/2011	12/17/2011	(2.69)	1.01	0.92	(0.75)	(1.34)	(2.00)	(2.69)	(2.69)	(2.69)	(2.69)	(2.69)	(2.69)	(2.69)	(2.69)
8/12/2011	12/17/2011	(2.83)	1.00	0.90	(0.83)	(0.86)	(1.36)	(2.08)	(2.83)	(2.83)	(2.83)	(2.83)	(2.83)	(2.83)	(2.83)
8/19/2011	12/17/2011	(3.00)	2.96	1.18	52.35	1.51	(0.23)	(0.55)	(1.07)	(1.88)	(2.97)	(3.00)	(3.00)	(3.00)	(3.00)
8/26/2011	12/17/2011	(3.19)	4.71	1.32	(0.13)	(0.14)	(0.35)	(0.73)	(1.35)	(2.39)	(3.19)	(3.19)	(3.19)	(3.19)	(3.19)
9/2/2011	12/17/2011	(3.40)	8.05	1.38	(0.11)	(0.10)	(0.27)	(0.52)	(1.14)	(2.26)	(3.40)	(3.40)	(3.40)	(3.40)	(3.40)
9/12/2011	12/17/2011	(3.75)	7.59	1.36	0.41	0.45	0.46	0.37	0.08	(0.56)	(1.82)	(3.71)	(3.75)	(3.75)	(3.75)
9/19/2011	12/17/2011	(4.04)	6.96	1.26	0.06	0.11	0.16	(0.09)	(0.60)	(1.77)	(3.92)	(4.04)	(4.04)	(4.04)	(4.04)
9/26/2011	12/17/2011	(4.39)	8.13	1.28	9.47	0.90	1.07	1.21	1.22	0.85	(0.14)	(2.53)	(4.39)	(4.39)	(4.39)
10/3/2011	12/17/2011	1.59	10.44	1.35	(86.74)	(0.87)	2.35	2.68	3.71	4.48	4.94	4.79	3.25	(0.99)	(4.80)
10/10/2011	1/21/2012	5.23	3.84	1.14	0.99	1.41	1.94	2.66	3.42	4.16	4.97	5.57	5.92	5.30	3.45
10/17/2011	1/21/2012	4.71	4.18	1.15	0.92	1.08	1.91	2.67	3.32	3.97	4.52	4.74	4.28	2.84	(0.12)
10/24/2011	1/21/2012	3.78	2.20	1.03	0.82	1.25	1.73	2.44	3.02	3.60	3.93	3.66	2.21	(1.19)	(4.04)
10/31/2011	1/21/2012	3.18	1.52	0.91	0.81	1.18	1.84	2.31	3.01	3.36	3.65	3.35	0.98	(3.24)	(4.39)
11/7/2011	1/21/2012	5.28	2.62	1.00	1.20	1.82	2.39	3.44	4.35	5.32	6.15	6.63	6.04	3.10	(3.34)
11/14/2011	1/21/2012	7.80	3.71	1.06	1.31	2.02	2.92	4.19	5.37	6.74	8.32	9.90	10.64	9.70	4.14
11/21/2011	3/17/2012	9.56	5.89	1.36	1.34	1.80	2.30	2.96	3.82	4.84	6.19	7.36	8.77	9.82	10.84
11/29/2011	3/17/2012	10.69	5.01	1.31	1.42	1.94	2.55	3.27	4.13	5.44	7.23	8.72	10.51	12.15	13.71
12/6/2011	3/17/2012	7.76	3.31	1.10	1.30	1.83	2.44	3.28	4.78	6.14	7.83	9.64	11.66	13.57	14.62
12/13/2011	3/17/2012	10.46	3.82	1.16	1.35	2.06	2.75	3.66	5.37	7.08	9.05	11.44	14.18	16.99	19.45
12/20/2011	3/17/2012	7.09	2.43	1.01	1.27	1.80	2.51	3.87	5.28	6.97	9.05	11.41	13.73	15.11	14.68
12/28/2011	3/17/2012	6.09	1.15	0.90	1.46	1.87	2.78	4.23	6.14	8.24	10.04	14.23	18.24	21.02	21.14
1/5/2012	3/17/2012	2.33	0.50	0.69	1.14	1.77	2.62	4.19	5.71	7.74	10.27	12.86	14.02	10.73	(2.82)
1/12/2012	3/17/2012	2.55	0.33	0.64	1.21	1.69	3.12	4.26	5.83	7.79	10.50	12.76	12.64	0.65	(5.54)
1/20/2012	3/17/2012	1.69	0.20	0.57	0.80	1.53	2.72	3.71	4.82	6.08	7.15	6.20	(2.70)	(6.32)	(6.32)
1/27/2012	4/21/2012	0.44	(0.01)	0.31	0.92	1.34	1.97	3.04	4.25	5.81	7.61	9.49	10.18	5.43	(3.59)
2/3/2012	4/21/2012	0.34	(0.09)	0.32	0.47	1.41	2.09	2.74	3.57	4.47	5.26	4.83	(0.18)	(4.57)	(4.62)
2/10/2012	4/21/2012	1.99	0.26	0.59	1.13	2.01	3.13	4.51	6.11	8.18	10.98	14.13	18.16	19.17	7.99
2/17/2012	4/21/2012	3.55	0.79	0.74	1.22	1.81	2.95	4.08	5.66	7.48	9.84	12.08	13.30	8.82	(3.14)
2/27/2012	5/19/2012	(0.21)	1.04	0.84	0.12	0.29	0.22	(0.04)	(0.20)	(1.20)	(3.36)	(4.39)	(4.39)	(4.39)	(4.39)
3/5/2012	6/16/2012	0.84	0.47	0.67	0.41	0.81	0.85	1.22	1.32	1.25	0.81	(0.36)	(2.72)	(3.49)	(3.50)
3/12/2012	6/16/2012	0.69	0.55	0.70	0.19	0.26	0.69	0.70	0.54	0.10	(0.97)	(3.16)	(3.75)	(3.75)	(3.75)
3/19/2012	6/16/2012	(0.10)	0.26	0.55	0.08	(0.02)	0.12	(0.43)	(0.68)	(1.41)	(4.04)	(4.04)	(4.04)	(4.04)	(4.04)
3/26/2012	6/16/2012	(0.61)	(0.28)	0.30	(0.93)	(1.43)	(2.21)	(3.49)	(4.39)	(4.39)	(4.39)	(4.39)	(4.39)	(4.39)	(4.39)
4/2/2012	6/16/2012	(0.49)	(0.35)	0.26	(0.85)	(0.97)	(1.57)	(2.64)	(4.40)	(4.80)	(4.80)	(4.80)	(4.80)	(4.80)	(4.80)
4/10/2012	6/16/2012	(0.09)	(0.01)	0.41	(0.05)	0.12	0.37	0.10	(0.61)	(2.01)	(4.63)	(5.37)	(5.37)	(5.37)	(5.37)
4/17/2012	6/16/2012	(0.36)	(0.24)	0.28	(0.52)	(0.54)	(1.02)	(1.94)	(3.66)	(5.98)	(6.00)	(6.00)	(6.00)	(6.00)	(6.00)
4/24/2012	7/21/2012	0.73	(0.14)	0.44	0.93	1.23	1.59	1.94	2.32	2.54	2.42	1.54	(1.13)	(4.00)	(4.09)
5/1/2012	7/21/2012	0.50	(0.38)	0.43	0.60	0.85	1.08	1.26	1.24	0.97	(0.27)	(3.62)	(4.43)	(4.44)	(4.44)
5/8/2012	7/21/2012	0.84	(0.31)	0.46	0.97	1.32	1.70	2.11	2.56	2.77	2.46	0.58	(4.35)	(4.85)	(4.86)
5/15/2012	7/21/2012	2.51	0.16	0.65	0.89	2.06	2.91	3.67	5.02	6.49	8.02	9.35	9.51	4.86	(4.90)
5/22/2012	8/18/2012	3.72	0.92	0.76	2.38	2.31	3.11	4.09	5.39	7.00	9.02	11.27	14.27	17.28	19.26
5/30/2012	8/18/2012	4.35	0.74	0.79	1.72	2.37	3.22	4.47	5.86	7.65	10.04	12.73	15.98	19.02	20.64
6/6/2012	8/18/2012	3.76	0.71	0.70	1.95	2.74	3.74	5.02	6.73	9.31	12.44	16.52	21.69	27.01	31.95
6/13/2012	9/22/2012	4.77	1.13	0.82	1.84	2.53	3.38	4.46	5.81	7.53	9.83	12.66	16.17	20.27	25.19
6/20/2012	9/22/2012	1.74	0.20	0.53	1.56	2.15	2.93	3.90	5.26	7.06	9.38	12.58	16.21	20.66	21.18
6/27/2012	9/22/2012	1.95	0.20	0.53	1.71	2.44	3.37	4.52	5.99	8.04	10.48	13.79	17.85	22.10	25.10
7/5/2012	9/22/2012	0.91	(0.14)	0.36	1.52	2.26	3.07	4.13	5.61	7.65	10.50	14.01	17.77	20.42	16.35
7/12/2012	9/22/2012	0.98	(0.27)	0.35	1.68	2.40	3.32	4.52	6.13	8.34	11.57	15.05	18.51	19.59	14.19
7/19/2012	9/22/2012	0.80	(0.36)	0.33	1.54	2.14	2.95	3.99	5.59	7.92	10.45	12.22	11.02	3.36	(3.32)
7/26/2012	9/22/2012	0.97	(0.23)	0.32	1.99	2.82	3.94	5.43	7.82	11.23	16.31	23.45	33.97	32.81	22.74
8/2/2012	9/22/2012	1.09	(0.30)	0.32	2.23	3.13	4.45	6.27	9.09	13.33	19.21	27.39	38.10	36.76	25.83
		2.27			(0.60)	0.34	0.70	1.06	1.44	1.96	2.58	3.24	3.43	2.60	0.81

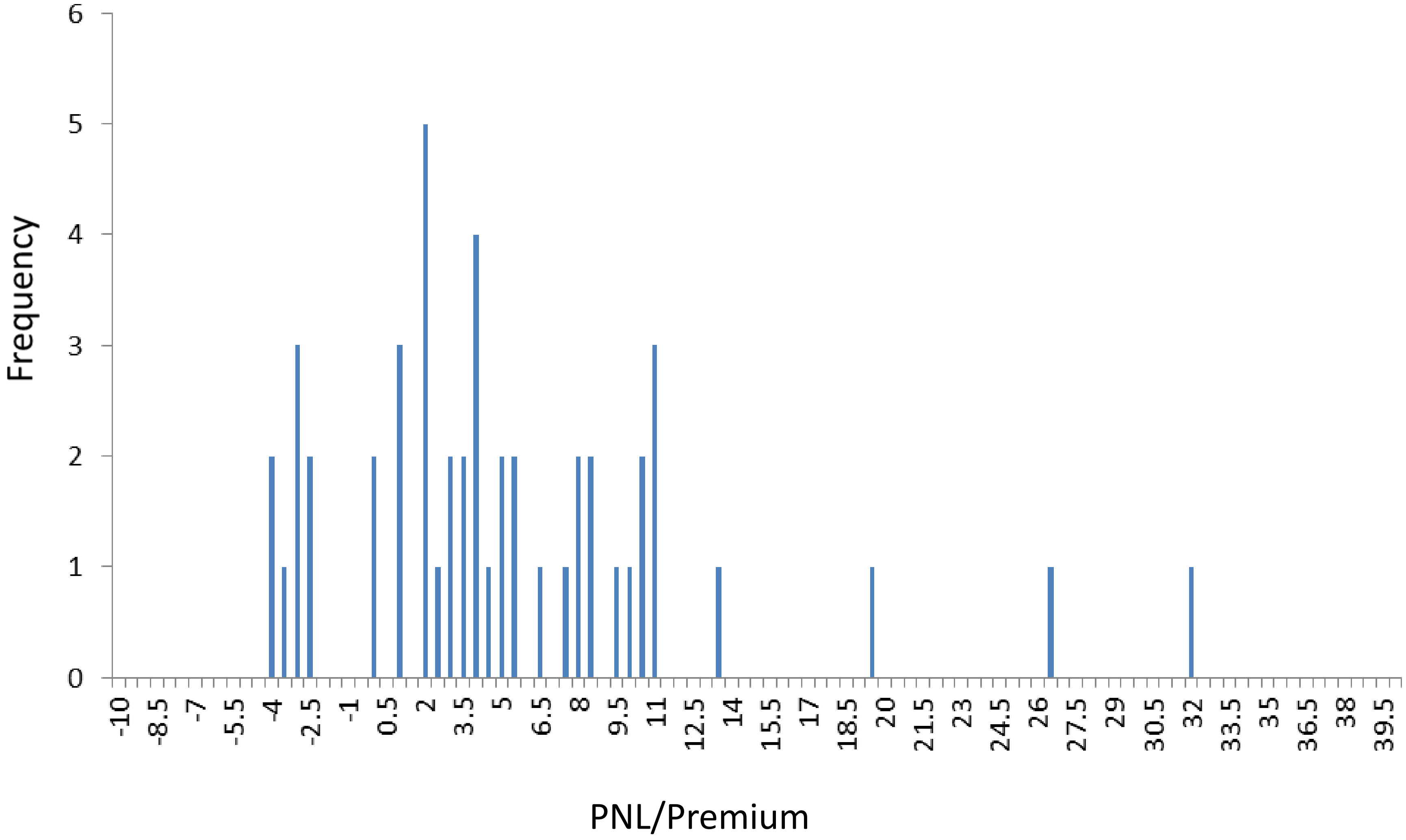
Histogram of 99 trades
(Best trade selected on each date based on model)



X-Y Plot (signal strength/PNL)



Histogram of best trades with signal threshold (model 16% quantile > 1.10; 49 trades)



Options on VIX Futures
Convexity Adjustment between Variance and Futures

Variance Swap Volatility and VIX Futures

Forward Variance Swap from T to $T + \Delta T$

$$E\left(\sum_{i=N}^{N+M}\left(\frac{dS}{S}\right)^2\right) = E\left\{E_T\left(\sum_{i=N}^{N+M}\left(\frac{dS}{S}\right)^2\right)\right\}$$

$$= E\{VIX_T^2\}\,\Delta T$$

Forward Variance vs. Futures Squared

$$VX_T = E\{VIX_T\}$$

VIX futures with settlement date T is the expected value of VIX_T

$$E\{VIX_T^2\} = (E\{VIX_T\})^2 + E\{VIX_T^2\} - (E\{VIX_T\})^2$$

$$= VX_T^2 + \mathrm{Variance}(VIX_T)$$

Forward variance is futures squared + the VIX variance

The Variance of VIX as the value of a portfolio of VX futures

$$\text{Variance}(VIX_T) = E(VIX_T - VX_T\ )^2 = \int_0^\infty (v - \bar{v})^2\, f(v) dv \qquad \bar{v} = VX_T$$

But $$(v - \bar{v})^2 = 2\int_0^{\bar{v}} (k - v)^+\, dk + 2\int_{\bar{v}}^{\infty} (v - k)^+\, dk$$

So

$$\boxed{\text{Variance}(VIX_T) = 2e^{rT} \int_0^\infty OTM(VX_T, k, T)\, dk}$$

Relation between Variance, VIX futures and VIX vol

$$E\{VIX_T^2\} = VX_T^2 + \text{Variance}(VIX_T)$$

$$= VX_T^2 + 2e^{rT}\int_0^\infty OTM_{vix}(VX_T, k, T)\,dk$$

Adding all forward variances

$$\boxed{T\sigma_{T,vs}^2 = \Delta T\sum_{n=0}^{N-1} VX_{n\Delta T}^2 + 2\Delta T\sum_{n=0}^{N-1} e^{rn\Delta T}\int_0^\infty OTM_{vix}(VX_{n\Delta T}, k, n\Delta T)\,dk}$$

Final link between SPX options, VIX futures and VIX options

$$2e^{rT}\int_0^{\infty} OTM_{spx}(SPX_T, Q, T)\,\frac{dQ}{Q^2} = \frac{1}{N}\sum_{n=0}^{N-1} VX_{n\Delta T}^2 + \frac{2}{N}\sum_{n=0}^{N-1} e^{rn\Delta T}\int_0^{\infty} OTM_{vix}(VX_{n\Delta T}, k, n\Delta T)\,dk$$

SPX (S&P 500) options

VIX futures

VIX options

Open question: how can this be converted into an arbitrage strategy?? (Too many moving parts!)

Compact formula using instantaneous futures (`weeklies')

$$\int_0^\infty OTM_{spx}(SPX_T, Q, T)\frac{dQ}{Q^2} = \frac{1}{2T}\int_0^T VX_t^2\, dt + \int_0^T e^{-r(T-t)} \int_0^\infty OTM_{vix}(VX_t, k, t)\, dk\, dt$$

The fair value of the variance of S&P 500 options is equal to the sum of

-- the variance implied by VIX futures contracts (strip of futures squared)

-- a term depending on the volatility of volatility (involving all strikes/maturities before T.

Conclusions on VIX trading

- VIX is an index representing an average of short-term volatilities of S&P500 index options. It has become a widely accepted ``fear gauge'' in markets.
- VIX is traded via variance swaps (OTC) and futures (CBOE). Futures extend to 1.5 years, with liquid first 7 settlement month
- There are also VIX options
- There are also many VIX-linked ETFs, the most important being VXX. Most of the ETFs have options.
- The arbitrage between SPX vol and VIX is complicated due to liquidity and due to the great number of options involved in getting an ``exact relation''.
- Term-structure arb via futures and futures/ETF trades are feasible and show interesting signals.
- VIX trading continues to evolve. For instance Brexit (UK referendum) has affected VIX in various ways.

Dispersion Trading (index options vs equity options

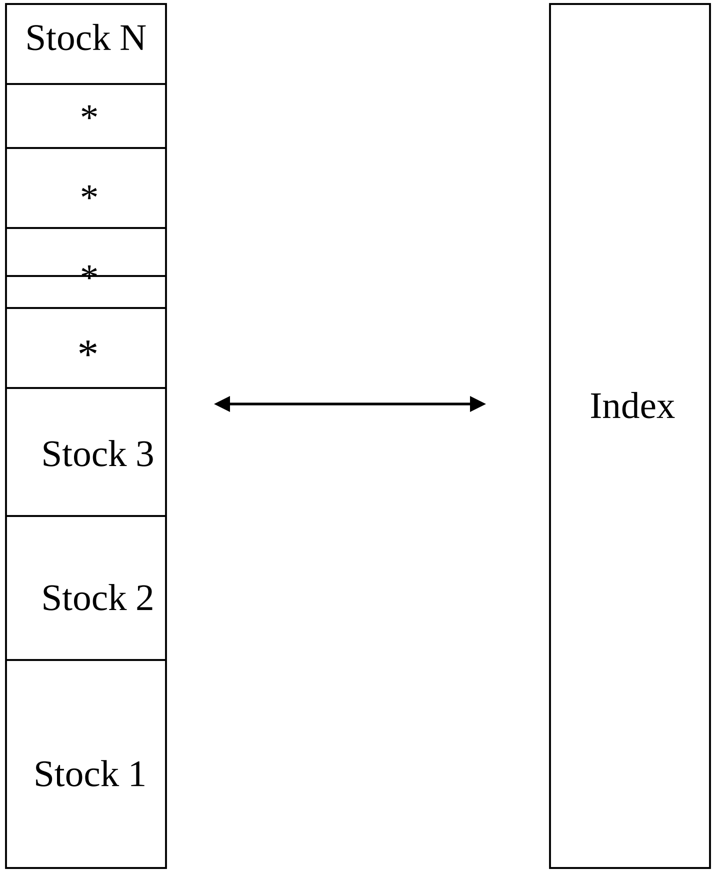
Motivation: to profit from price differences in volatility markets
using index options and options on individual stocks

Opportunities: Market segmentation, temporary shifts in correlations
between assets, idiosyncratic news on individual stocks

Basic Trades:

- Sell index option, buy options on index components (“sell correlation”)
- Buy index option, sell options on index components (“buy correlation”)

Index Arbitrage versus Dispersion Trading



Index Arbitrage:
Reconstruct
an index product (ETF)
using the
component stocks

Dispersion Trading:
Reconstruct an index option
using options on the
component stocks

COMS	CMGI	LGTO	PSFT
ADPT	CNET	LVLT	PMCS
ADCT	CMCSK	LLTC	QLGC
ADLAC	CPWR	ERICY	QCOM
ADBE	CMVT	LCOS	QTRN
ALTR	CEFT	MXIM	RNWK
AMZN	CNXT	MCLD	RFMD
APCC	COST	MEDI	SANM
AMGN	DELL	MFNX	SDLI
APOL	DLTR	MCHP	SEBL
AAPL	EBAY	MSFT	SIAL
AMAT	DISH	MOLX	SSCC
AMCC	ERTS	NTAP	SPLS
ATHM	FISV	NETA	SBUX
ATML	GMST	NXTL	SUNW
BBBY	GENZ	NXLK	SNPS
BGEN	GBLX	NWAC	TLAB
BMET	MLHR	NOVL	USAI
BMCS	ITWO	NTLI	VRSN
BVSN	IMNX	ORCL	VRTS
CHIR	INTC	PCAR	VTSS
CIEN	INTU	PHSY	VSTR
CTAS	JDSU	SPOT	WCOM
CSCO	JNPR	PMTC	XLNX
CTXS	KLAC	PAYX	YHOO

NASDAQ-100 Index (NDX) and ETF (QQQ)

- QQQ ~ 1/40 * NDX
- Capitalization-weighted
- QQQ trades as a stock
- QQQ options: largest daily traded volume in U.S.

Sector Exchange Traded Funds

~ 20 - 40 stocks in same sector

Weightings by:

- capitalization
- equal-dollar
- equal-stock

SOX
ALTR
AMAT
AMD
INTC
KLAC
LLTC
LSCC
LSI
MOT
MU
NSM
NVLS
RMBS
TER
TXN
XLNX

XNG
APA
APC
BR
BRR
EEX
ENE
EOG
EPG
KMI
NBL
NFG
OEI
PPP
STR
WMB

XOI
AHC
BP
CHV
COC.B
XOM
KMG
OXY
P
REP
RD
SUN
TX
TOT
UCL
MRO

Index Option Arbitrage (a.k.a. Dispersion Trading)

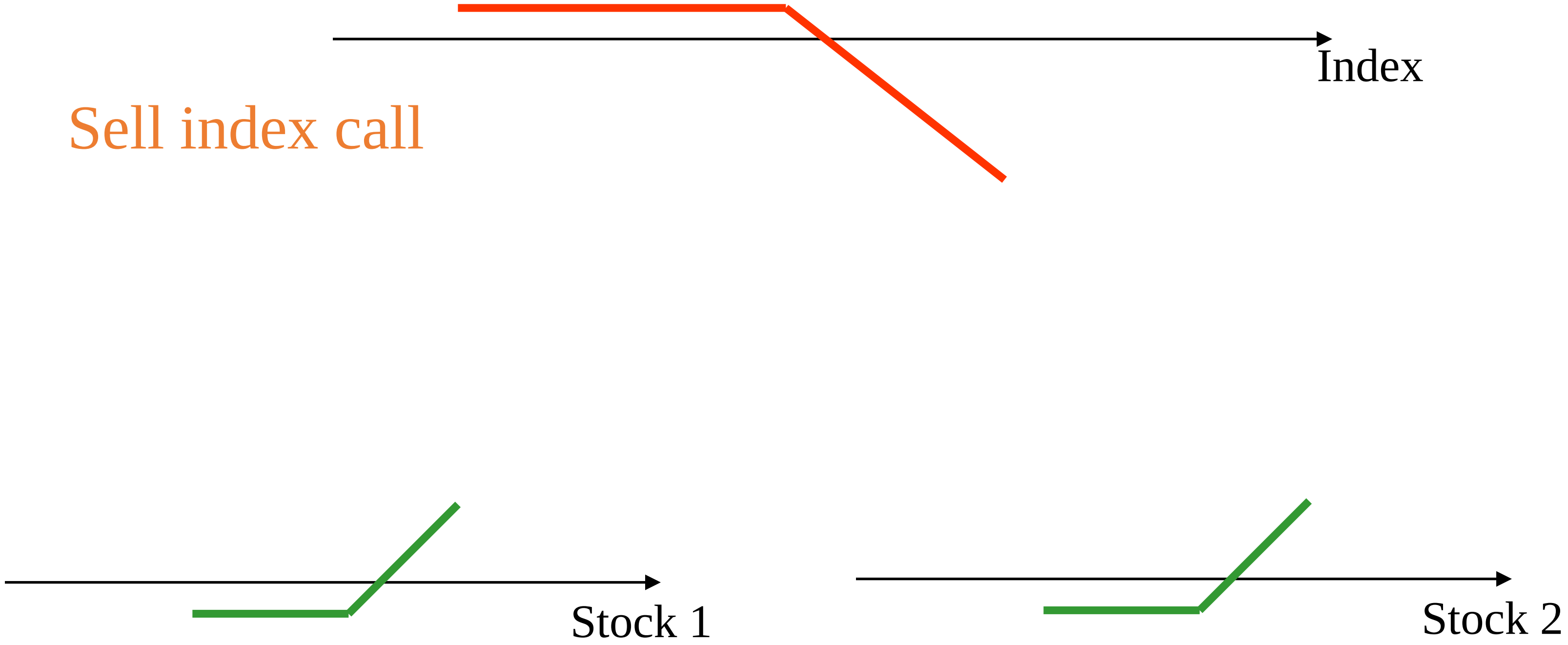
- Takes advantage of differences in implied volatilities of index options and implied volatilities of individual stock options

- Main source of arbitrage: correlations between asset prices vary with time due to corporate events, earnings, and ``macro'' shocks

- Full or partial index reconstruction

The trade in pictures

Sell index call



Buy calls on different stocks.

Also, buy index/sell stocks

First approximation to hedging: ``Intrinsic Value Hedge''

$$I = \sum_{i=1}^{M} w_i S_i \qquad w_i = \text{number of shares, scaled by ``divisor''}$$

$$K = \sum_{j=1}^{M} w_i K_i \quad \Rightarrow$$

IVH: use index weights for option hedge

$$\max\left(I - K, 0\right) \leq \sum_{j=1}^{M} w_i \max\left(S_i - K_i, 0\right)$$

$$\boxed{C_I\left(I, K, T\right) \leq \sum_{j=1}^{M} w_i C_i\left(S_i, K_i, T\right)}$$

IVH:
premium from index is less than premium from components
"Super-replication"

Makes sense for deep- -in-the-money options

Intrinsic-Value Hedging is `exact' only if stocks are perfectly correlated

$$I(T)=\sum_{i=1}^{M} w_i S_i(T)=\sum_{i=1}^{M} w_i F_i e^{\sigma_i N_i-\frac{1}{2}\sigma_i^2 T}$$

$$\rho_{ij}\equiv 1 \Rightarrow \quad N_i \equiv N = \text{standardized normal}$$

$$\text{Solve for } X \text{ in}: \quad K=\sum_{i=1}^{M} w_i F_i e^{\sigma_i X-\frac{1}{2}\sigma_i^2 T}$$

$$\text{Set}: \quad K_i = F_i e^{\sigma_i X-\frac{1}{2}\sigma_i^2 T}$$

$$\therefore$$

$$\max\left(I(T)-K,0\right)=\sum_{i=1}^{M} w_i \max\left(S_i(T)-K_i,0\right) \quad \forall T$$

Similar to Jamshidian (1989) for pricing bond options in 1-factor model

IVH : Hedge with ``equal-delta'' options

$$K_i = F_i e^{\sigma_i X \sqrt{T} - \frac{1}{2}\sigma_i^2 T} \qquad \therefore \qquad X = \frac{1}{\sigma_i \sqrt{T}} \ln\left(\frac{K_i}{F_i}\right) + \frac{1}{2}\sigma_i \sqrt{T}$$

$$-X = \frac{1}{\sigma_i \sqrt{T}} \ln\left(\frac{F_i}{K_i}\right) - \frac{1}{2}\sigma_i \sqrt{T} = d_2$$

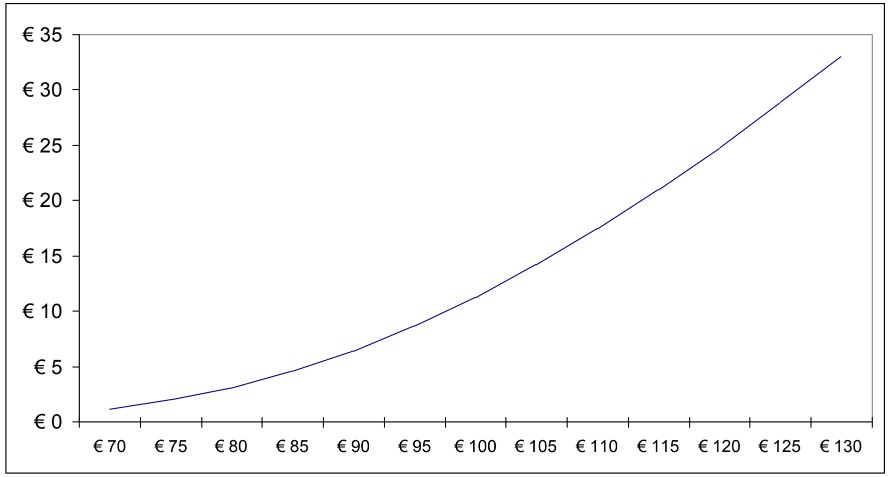
$$\mathrm{N}(d_2) = \text{constant}$$

$$\text{log - moneyness} \approx \text{constant}$$

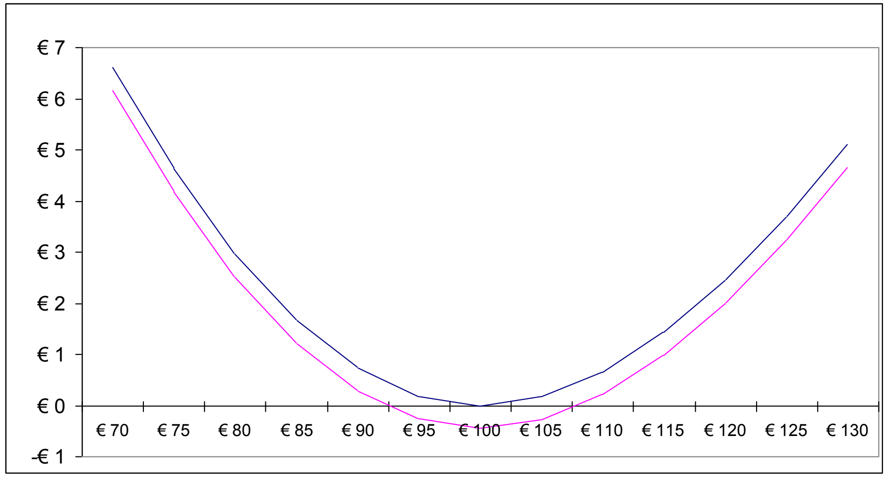
$$\text{Deltas} \approx \text{constant}$$

What happens after you enter a trade: Risk/return in hedged option trading

Unhedged call option



Hedged option



Profit-loss for a hedged single option position (Black –Scholes)

$$P/L \approx \theta \cdot \left(n^2 - 1\right) + NV \cdot \frac{d\sigma}{\sigma}$$

$$\theta = \text{time-decay (dollars)}, \qquad n = \frac{\Delta S}{S\sigma\sqrt{\Delta t}}, \qquad NV = \text{normalized Vega} = \sigma \frac{\partial C}{\partial \sigma}$$

n ~ standardized move

Gamma P/L for an Index Option

Assume $d\sigma = 0$

$$\text{Index Gamma P/L} = \theta_I\left(n_I^2 - 1\right)$$

$$n_I = \sum_{i=1}^{M} \frac{p_i \sigma_i}{\sigma_I} n_i \qquad p_i = \frac{w_i S_i}{\sum_{j=1}^{M} w_j S_j}$$

$$\sigma_I^2 = \sum_{ij=1}^{M} p_i p_j \sigma_i \sigma_j \rho_{ij}$$

$$\text{Index P/L} = \theta_I \sum_{i=1}^{M} \frac{p_i^2 \sigma_i^2}{\sigma_I^2}\left(n_i^2 - 1\right) + \theta_I \sum_{i \neq j} \frac{p_i p_j \sigma_i \sigma_j}{\sigma_I^2}\left(n_i n_j - \rho_{ij}\right)$$

Gamma P/L for Dispersion Trade

$$\text{i}^{\text{th}} \text{ stock P/L} \approx \theta_i \cdot \left(n_i^2 - 1\right)$$

$$\text{Dispersion Trade P/L } \approx \sum_{i=1}^{M} \left(\theta_i + \frac{p_i^2 \sigma_i^2}{\sigma_I^2} \theta_I \right) \left(n_i^2 - 1\right) + \theta_I \sum_{i \neq j} \frac{p_i\, p_j \sigma_i\, \sigma_j}{\sigma_I^2} \left(n_i n_j - \rho_{ij}\right)$$

diagonal term:
realized single-stock
movements vs.
implied volatilities

off-diagonal term:
realized cross-market
movements vs.
implied correlation

Introducing the Dispersion Statistic

$$\boxed{D^2 = \sum_{i=1}^{N} p_i \left(X_i - Y\right)^2}$$

$$X_i = \frac{\Delta S_i}{S_i}, \quad Y = \frac{\Delta I}{I}$$

$$D^2 = \sum_{i=1}^{N} p_i \sigma_i^2 n_i^2 - \sigma_I^2 n_I^2$$

$$\begin{aligned}
\text{P/L} &= \sum_{i=1}^{N} \theta_i \left(n_i^2 - 1\right) + \theta_I \left(n_I^2 - 1\right) \\
&= \sum_{i=1}^{N} \theta_i n_i^2 + \theta_I n_I^2 - \Theta \qquad\qquad \Theta \equiv \sum_{i=1}^{N} \theta_i + \theta_I \\
&= \sum_{i=1}^{N} \theta_i n_i^2 + \frac{\theta_I}{\sigma_I^2} \sum_{i=1}^{N} p_i \sigma_i^2 n_i^2 - \frac{\theta_I}{\sigma_I^2} \sum_{i=1}^{N} p_i \sigma_i^2 n_i^2 + \theta_I n_I^2 - \Theta \\
&= \sum_{i=1}^{N} \left(\frac{\theta_I p_i \sigma_i^2 n_i^2}{\sigma_I^2} + \theta_i \right) n_i^2 - \frac{\theta_I}{\sigma_I^2} D^2 - \Theta
\end{aligned}$$

Summary of Gamma P/L for Dispersion Trade

$$\text{Gamma P/L} = \sum_{i=1}^{N} \left(\frac{\theta_I p_i \sigma_i^2 n_i^2}{\sigma_I^2} + \theta_i \right) n_i^2 - \frac{\theta_I}{\sigma_I^2} D^2 - \Theta$$

"Idiosyncratic" Gamma

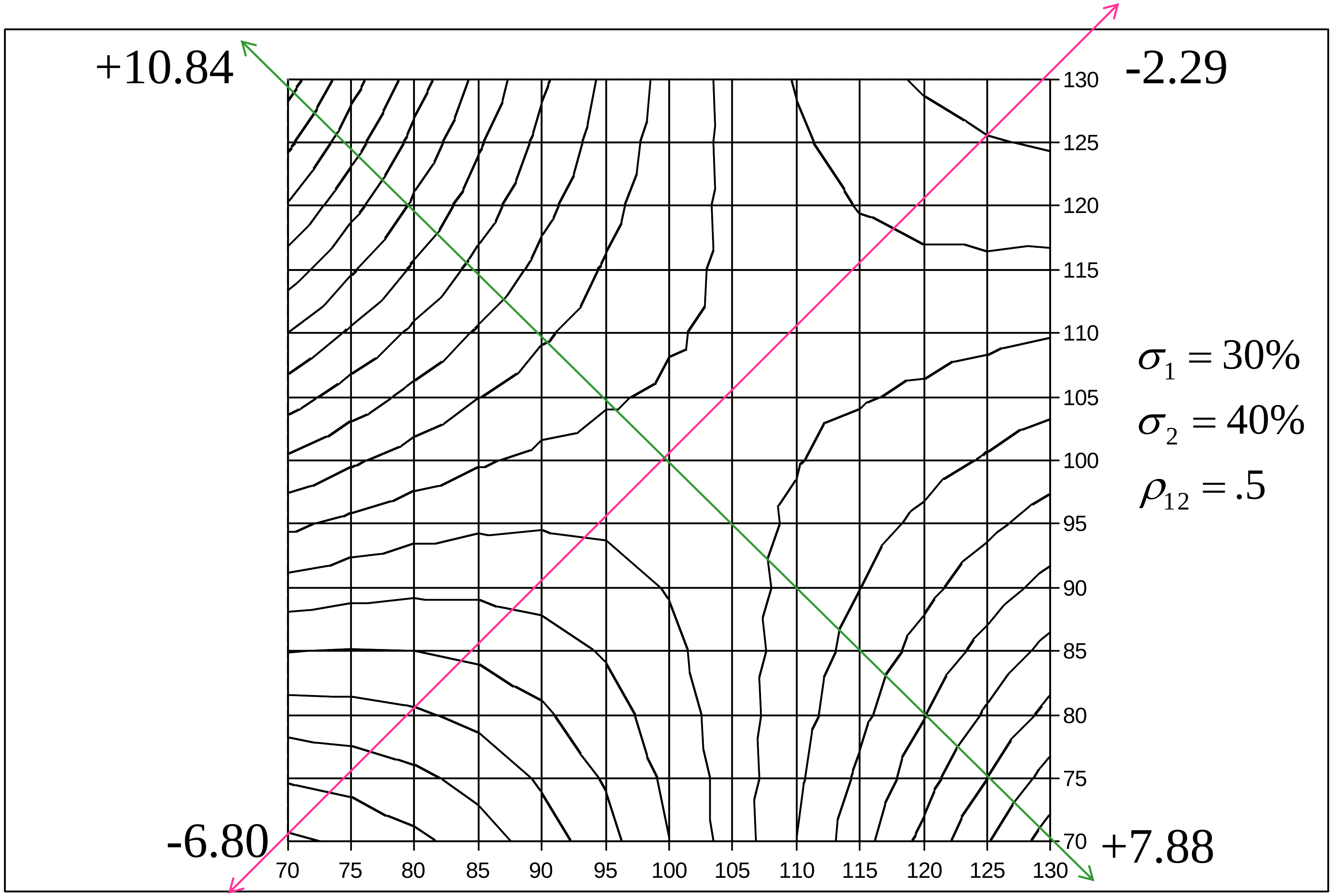
Dispersion Gamma

Time-Decay

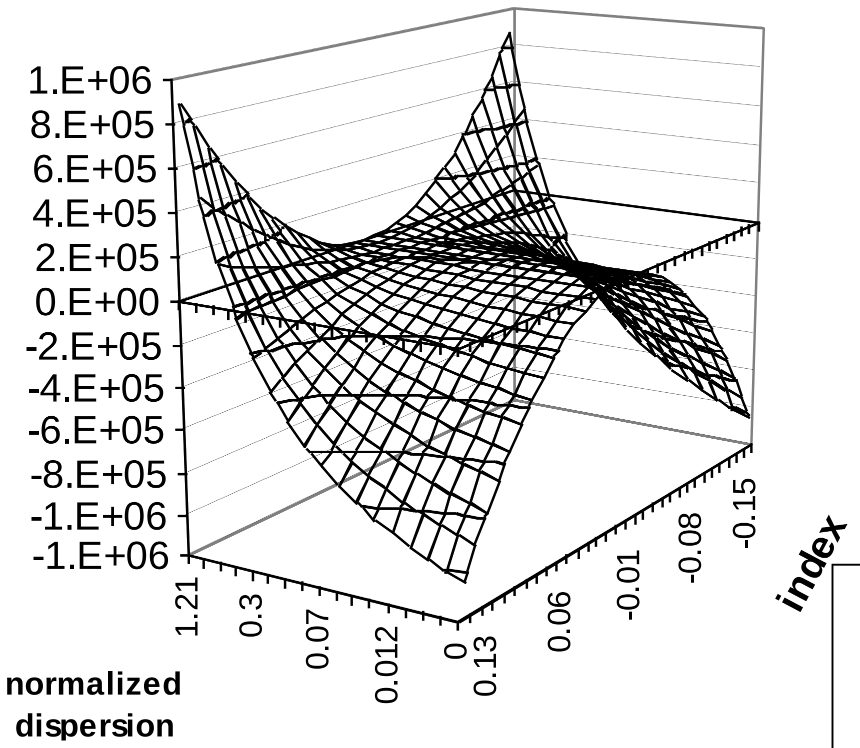
Example: ``Pure long dispersion" (zero idiosyncratic Gamma):

$$\theta_i = -\theta_I \frac{p_i \sigma_i^2}{\sigma_I^2} \qquad \Theta = |\theta_I| \left(\frac{\sum_i p_i \sigma_i^2}{\sigma_I^2} - 1 \right) \geq |\theta_I| \left(\frac{\left(\sum_i p_i \sigma_i \right)^2}{\sigma_I^2} - 1 \right) > 0$$

Gamma Risk: Negative exposure for ‘parallel’ shifts, positive ‘exposure’ to transverse shifts



Gamma-Risk for Baskets



$$X_i = \frac{\Delta S_i}{S_i} \qquad Y = \frac{\Delta I}{I}$$

$$D = \sum_{i=1}^{N} p_i \left(X_i - Y\right)^2$$

$$D/Y^2 = \sum_{i=1}^{N} p_i \left(X_i / Y - 1\right)^2$$

D= Dispersion, or cross-sectional move,
D/(Y*Y)= Normalized Dispersion

From realistic portfolio

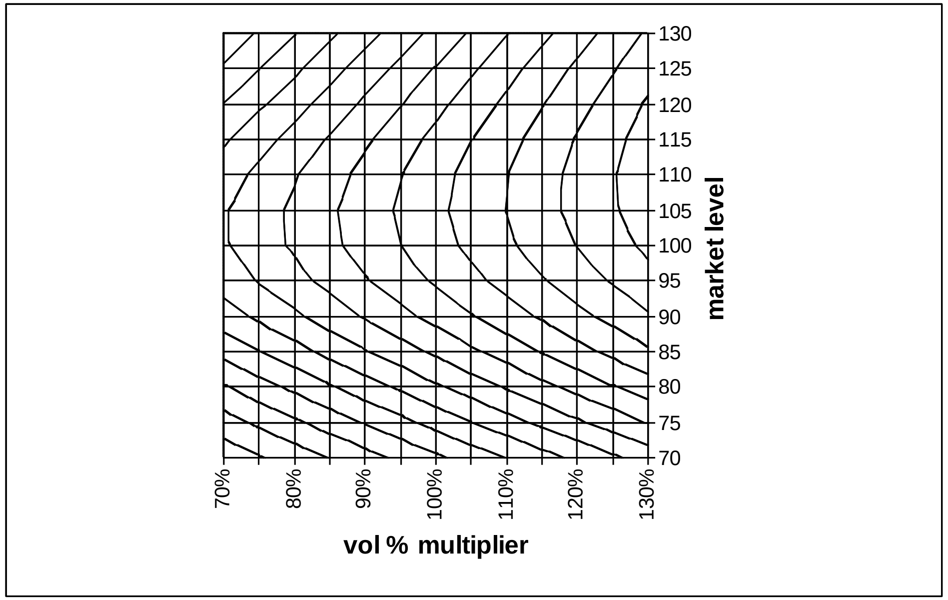
Vega Risk

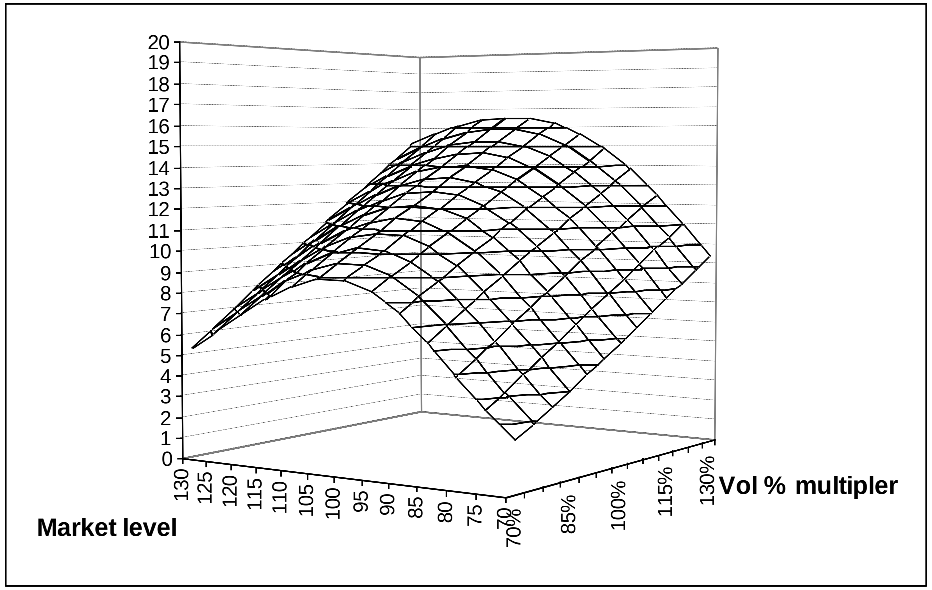
Sensitivity to volatility: move all single-stock implied volatilities by the same percentage amount

$$
\begin{aligned}
\text{Vega P/L} &= \sum_{j=1}^{M} \text{Vega}_j \Delta\sigma_j + \text{Vega}_I \Delta\sigma_I \\
&= \sum_{j=1}^{M} (NV)_j \frac{\Delta\sigma_j}{\sigma_j} + (NV)_I \frac{\Delta\sigma_I}{\sigma_I} \\
&= \left[\sum_{j=1}^{M} (NV)_j + (NV)_I \right] \frac{\Delta\sigma}{\sigma}
\end{aligned}
$$

$$
NV = \text{normalized vega} = \sigma \frac{\partial V}{\partial \sigma}
$$

Market/Volatility Risk





- Short Gamma on a perfectly correlated move
- Monotone-increasing dependence on volatility (IVH)

But there is more…

``Rega'': Sensitivity to correlation

$$\rho_{ij} \to \rho_{ij} + \Delta\rho \quad i \neq j$$

$$\sigma_I^2 \to \sum_{ij=1}^{M} p_i p_j \sigma_i \sigma_j \rho_{ij} + \left(\sum_{i \neq j} p_i p_j \sigma_i \sigma_j \right) \Delta\rho$$

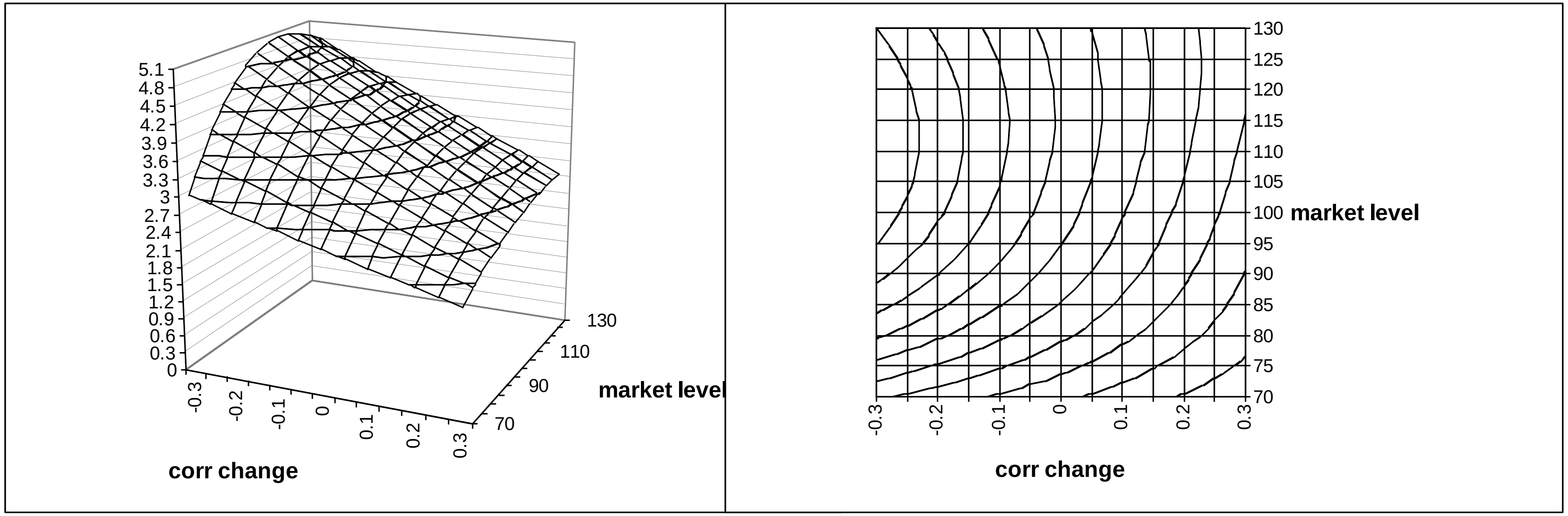
$$\Delta\sigma_I^2 = \left[\left(\sigma_I^{(1)}\right)^2 - \left(\sigma_I^{(0)}\right)^2 \right] \Delta\rho, \qquad \sigma_I^{(1)} = \sum_{j=1}^{M} p_j \sigma_j, \qquad \sigma_I^{(0)} = \sqrt{\sum_{j=1}^{M} p_j^2 \sigma_j^2}$$

$$\frac{\Delta\sigma_I}{\sigma_I} = \frac{1}{2} \frac{\left(\sigma_I^{(1)}\right)^2 - \left(\sigma_I^{(0)}\right)^2}{\sigma_I^2} \Delta\rho$$

$$\text{Correlation P/L} = \frac{1}{2} (NV)_I \frac{\left(\sigma_I^{(1)}\right)^2 - \left(\sigma_I^{(0)}\right)^2}{\sigma_I^2} \Delta\rho$$

$$\boxed{\text{Rega} = \frac{1}{2} \left(\frac{\left(\sigma_I^{(1)}\right)^2 - \left(\sigma_I^{(0)}\right)^2}{\sigma_I^2} \right) \times (NV)_I}$$

Market/Correlation Sensitivity



- Short Gamma on a perfectly correlated move
- Monotone-decreasing dependence on correlation

Steepest Descent Approximation

How can I price dispersions efficiently? Which are the ``correct'' strikes in single-name options corresponding to a given index option?

U.S. Equities: Main Sectors & Their Indices

- Major Indices: SPX, DJX, NDX
  SPY, DIA, QQQ (Exchange-Traded Funds)

- Sector Indices & Index Trackers:
  Semiconductors: SMH, SOX
  Biotech: BBH, BTK
  Pharmaceuticals: PPH, DRG
  Financials: BKX, XBD, XLF, RKH
  Oil & Gas: XNG, XOI, OSX
  High Tech, WWW, Boxes: MSH, HHH, XBD, XCI
  Retail: RTH

All these indices have options

What is the relation between index options and options on the components?

Standard (log-normal) Volatility Formula for Index Options

$$\sigma_I^2 = \sum_{j=1}^{N} p_j^2 \sigma_j^2 + \sum_{i \neq j} p_i \, p_j \sigma_i \, \sigma_j \rho_{ij} \qquad (*)$$

Does not apply when volatilities are strike-dependent

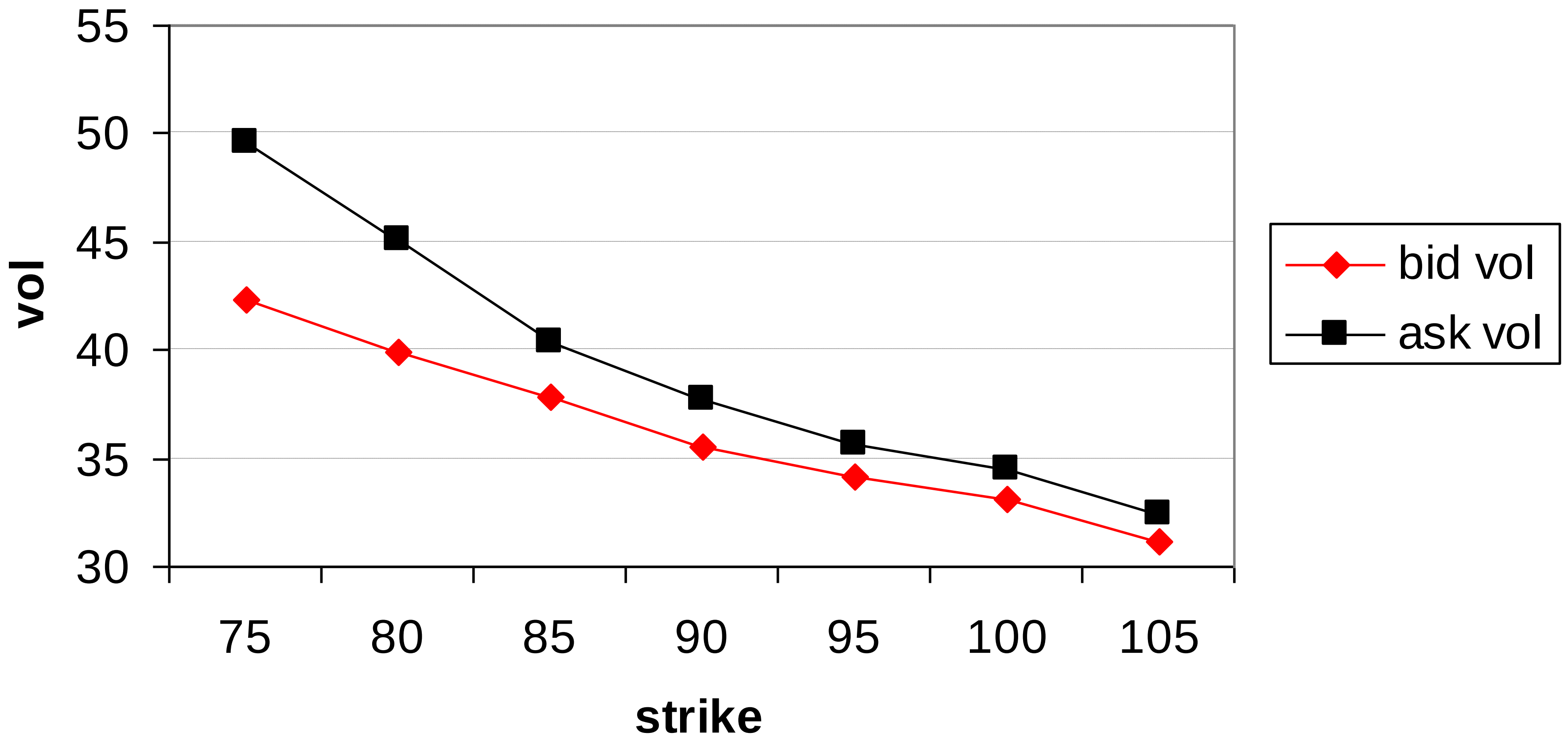
How can we incorporate volatility skew information into (*)?

BBH : Basket of 20 Biotechnology Stocks

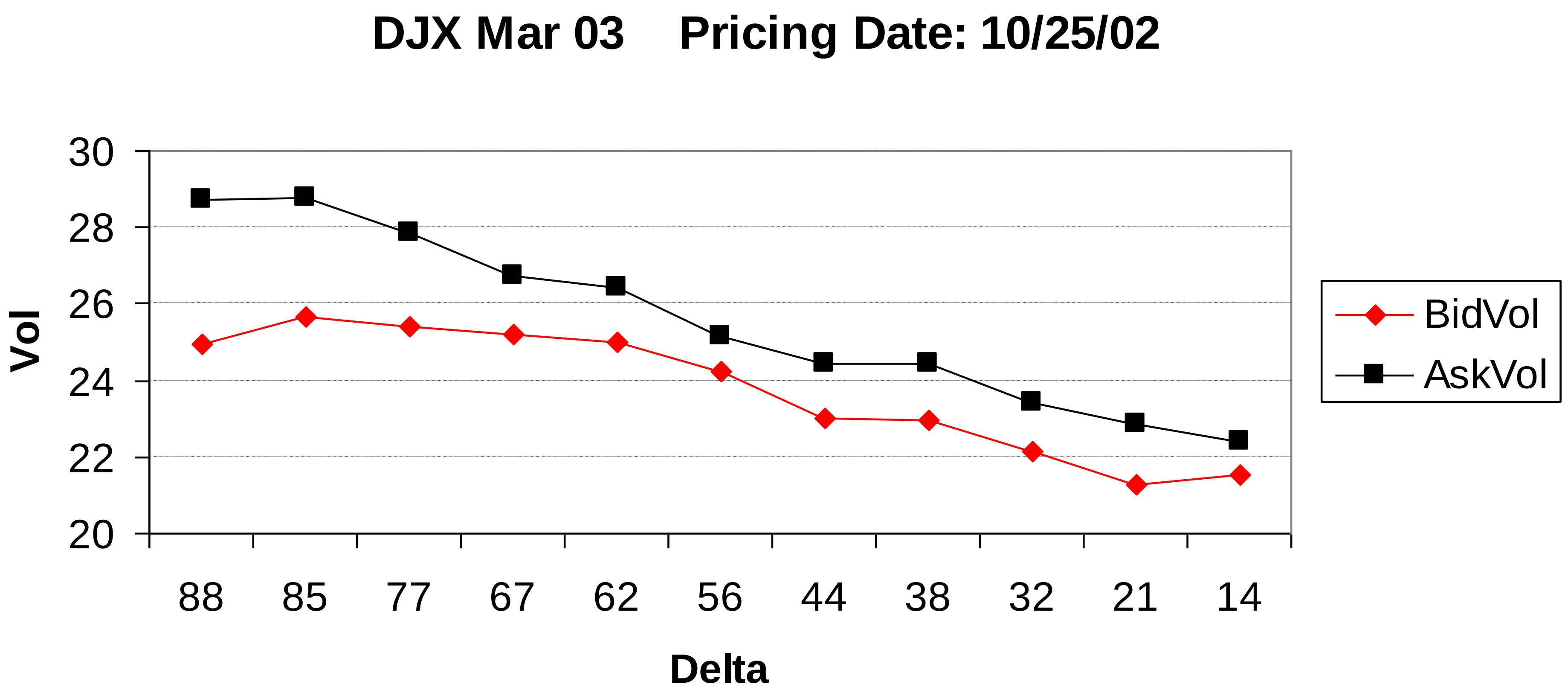
Ticker	Shares	ATM ImVol	Ticker	Shares	ATM ImVol
ABI	18	55	GILD	8	46
AFFX	4	64	HGSI	8	84
ALKS	4	106	ICOS	4	64
AMGN	**46**	40	IDPH	12	72
BGEN	13	41	MEDI	15	82
CHIR	16	37	MLNM	12	92
CRA	4	55	QLTI	5	64
DNA	44	53.5	SEPR	6	84
ENZN	3	81	SHPGY	6.8271	47
GENZ	14	56	BBH	-	32

BBH March 2003 Implied Vols

Pricing Date: Jan 22 03 10:42 AM



Implied Volatility Curve for Options on Dow Jones Average



Stylized facts about equity volatility curves

- Implied volatility curves are typically downward sloping (vol skew)

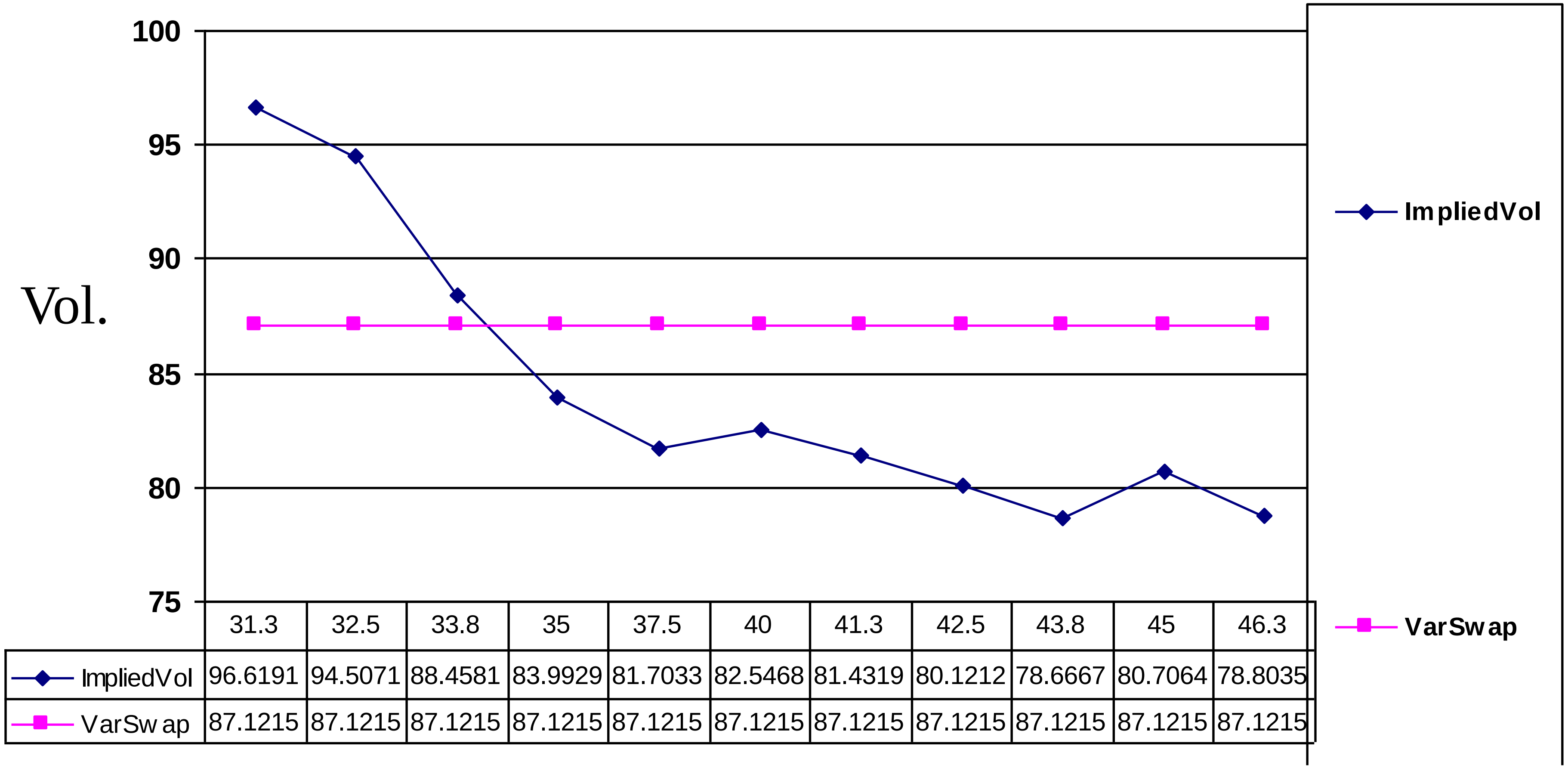
- Counterexamples: precious metal stocks are upward sloping

- There is little curvature (or smile). Skew is important.

AOL Jan 2001 Options:

Implied volatility curve on Dec 20,2000

Market close

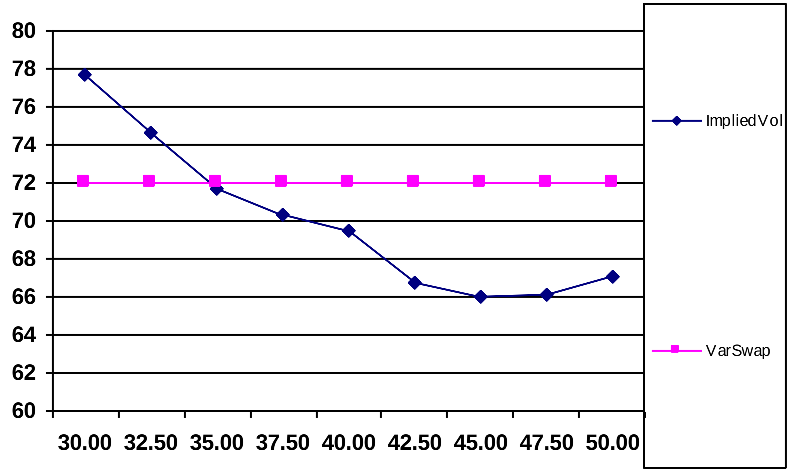


	31.3	32.5	33.8	35	37.5	40	41.3	42.5	43.8	45	46.3
ImpliedVol	96.6191	94.5071	88.4581	83.9929	81.7033	82.5468	81.4319	80.1212	78.6667	80.7064	78.8035
VarSwap	87.1215	87.1215	87.1215	87.1215	87.1215	87.1215	87.1215	87.1215	87.1215	87.1215	87.1215

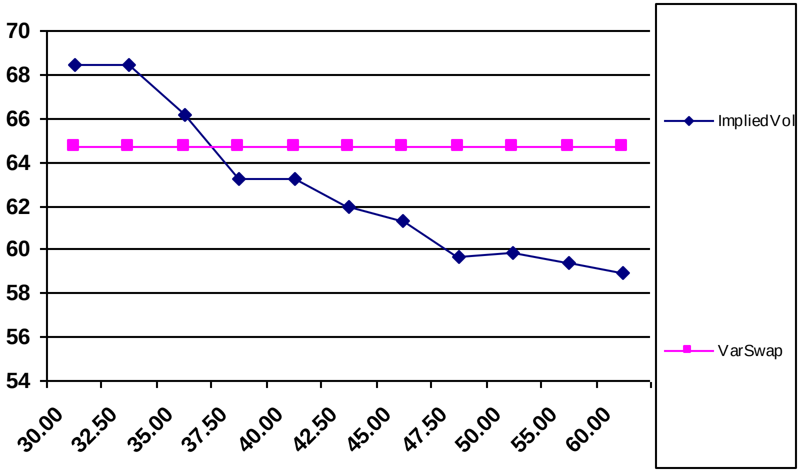
Strike

Stock probability is not lognormal

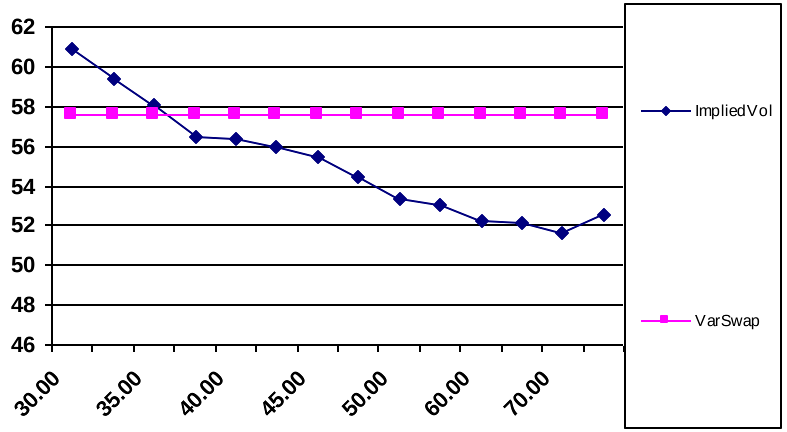
Expiration
2/17/01



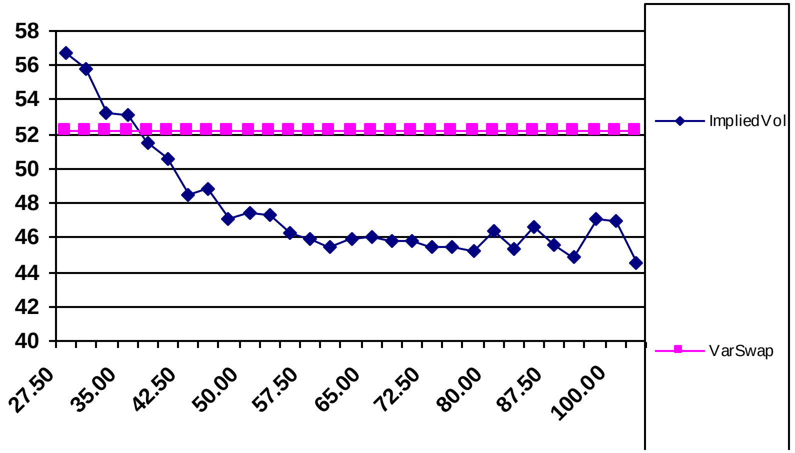
Expiration
4/21/01



Expiration
7/21/01



Expiration
1/19/02



The AOL ``volatility skew” for several expiration dates

Volatility Modeling

1.Joint stock-volatility
`dynamics'

$$\frac{dS}{S} = \sigma_t dW$$

A. $\sigma_t = \sigma(S,t)$ Dupire's Local Volatility $\sigma(S,t) = \sigma(t)\left(\frac{S}{S_0}\right)^{\gamma}$

B. $\frac{d\sigma_t}{\sigma_t} = \kappa dZ_t$ Stochastic Volatility

2. Implied vol. curve

$$\sigma_{\text{implied}}(K,T) = \sigma_{\text{implied}}(S,T)\cdot\left(1 + a\ln(K/S)\right)$$

Joint stock-volatility dynamics gives rise to an implied volatility curve

Relation between Stochastic Volatility and Local Volatility

$$\frac{dS_t}{S_t} = \sigma_t dZ_t$$

$$\boxed{\sigma^2_{\text{loc}}(S,t) = E\left\{ \sigma_t^2 \mid S_t = S \right\}}$$

Derman, Kani & Kamal1997, Britten -Jones and Neuberger, 2000, Gatheral 2000, Lim 2003

Application to Index Options

$$I = \sum_{i=1}^{n} w_i S_i$$

Index = weighted sum of stock prices (constant weights)

Diffusion eq.
for each stock
reflects vol skew
(local vol)

$$\begin{cases} \dfrac{dS_i}{S_i} = \sigma_i(S_i, t)dW_i + \mu_i dt, \\ E(dW_i dW_j) = \rho_{ij} dt \end{cases} \qquad \mu_i = r - d_i,$$

$$\frac{dI}{I} = \sigma_I(S, t)dZ + \mu_I(S, t)dt$$

$$\sigma_I^2(S, t) = \frac{\sum_{ij} \sigma_i(S_i, t)\sigma_j(S_j, t)w_i S_i w_j S_j \rho_{ij}}{I^2} \qquad \mu_I(S, t) = \frac{\sum_i \mu_i w_i S_i}{I}$$

Characterization of the equivalent local volatility for the index

$$\sigma^2_{I,\mathrm{loc}}(I,t) = E\left\{ \frac{\sum_{ij} \sigma_i(S_i(t),t)\sigma_j(S_j(t),t) S_i(t) S_j(t) w_i w_j \rho_{ij}}{I^2} \middle| \sum_i w_i S_i(t) = I \right\}$$

$$= E\left\{ \sum_{ij} p_i(S(t)) p_j(S(t)) \sigma_i(S_i(t),t)\sigma_j(S_j(t),t)\rho_{ij} \middle| \sum_i w_i S_i(t) = I \right\}$$

$$p_i(S) = \frac{w_i S_i}{\sum_j w_j S_j}, \qquad i = 1,\ldots,n.$$

- sigma_I can be seen as a ‘stochastic vol’ driving the index
- sigma_I_loc is then the ``equivalent local vol”

Varadhan's Formula and Large Deviations

$$\begin{cases} dX_i = \sum_{j=1}^{n} \sigma_i^j(X,t)\,dW_j \\ X_i(0) = x_i \end{cases} \qquad E\{dW_j dW_k\} = \rho_{jk}\,dt$$

Dupire local volatility model for each stock

$$\log \text{Prob.}\{X(t) = y \mid X(0) = x\} \approx -\frac{d^2(x,y)}{2t}, \qquad (\overline{\sigma})^2 t << 1$$

$$d^2(x,y) = \inf_{\gamma(0)=x,\gamma(1)=y} \int_0^1 \sum_{ij=1}^{n} g_{ij}(\gamma(s))\,\dot{\gamma}^i(s)\,\dot{\gamma}^j(s)\,ds$$

Riemannian metric

$$g(x) = a^{-1}(x) \qquad a_{ij}(x) = \sigma_i(x,0)\,\sigma_j(x,0)\,\rho_{ij}$$

In practice: dimensionless time ~ 0.02

Steepest-descent approximation

Change to log-scale:

$$x_i \equiv \log\left(\frac{S_i}{S_i(0)e^{\mu_i t}}\right) = \log\left(\frac{S_i}{F_i(t)}\right) \qquad i = 1,2,\ldots,n.$$

Formally,

$$\sigma^2_{I,\mathrm{loc}}(I,t) = \frac{E\left\{\sigma_I^2\,\delta\left(I(t)-I\right)\right\}}{E\left\{\delta\left(I(t)-I\right)\right\}}$$

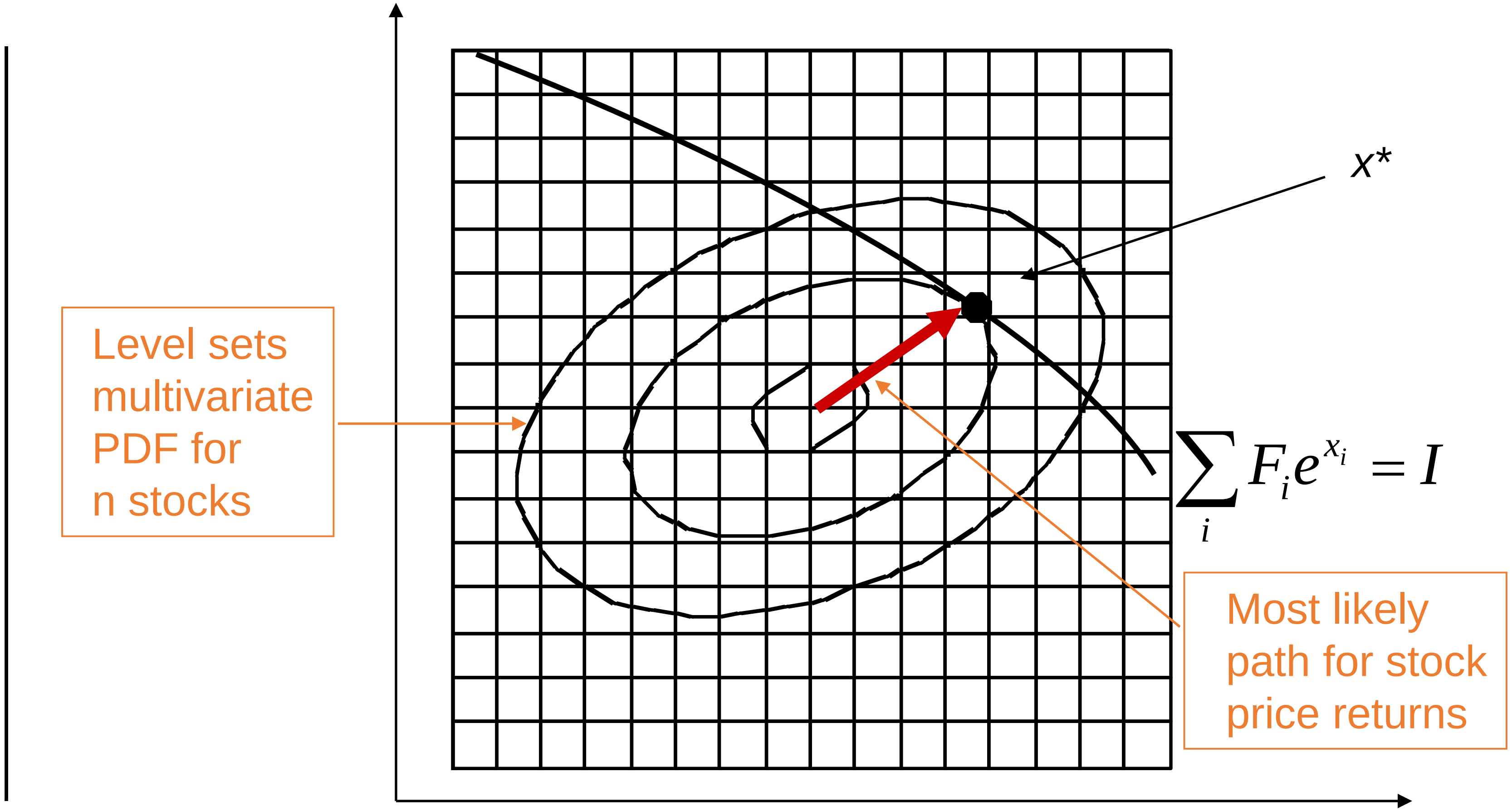
Applying Varadhan's Formula,

$$\sigma^2_{I,\mathrm{loc}}(I,t) \cong \sigma_I^2\left(S^*,t\right) \qquad S_i^* = S_i(0)e^{\mu_i t}e^{x_i^*}$$

where

$$x^* = \arg\min\left\{ d^2(0,x) \,\middle|\, \sum_i w_i S_i(0) e^{\mu_i t} e^{x_i} = I \right\}$$

Steepest Descent=Most Likely Stock Price Configuration



Replace conditional distribution by “Dirac function” at most likely configuration

Characterization of MLC

Euler-Lagrange equations: find $\left(x^*, \lambda\right)$ such that

$$\begin{cases} \displaystyle\int_0^{x_i^*} \frac{du}{\sigma_i(u)} = \lambda \sum_{j=1}^{n} p_j\left(x^*\right)\sigma_j\left(x_j^*\right)\rho_{ij} \qquad i = 1,\ldots,n \\ \displaystyle\sum_{i=1}^{n} w_i S_i(0) e^{x_i^* + \mu_i t} = I \end{cases}$$

$$\boxed{\sigma_{I,\mathrm{loc}}^2(I,t) = \sum_{ij=1}^{n} p_i\left(x^*\right) p_j\left(x^*\right) \sigma_i\left(x_i^*\right) \sigma_j\left(x_j^*\right) \rho_{ij}}$$

Linearization gives CAPM-like characterization

$$\sigma_I^2(0) \equiv \sum_{ij=1}^{n} p_i(0) p_j(0) \sigma_i(0) \sigma_j(0) \rho_{ij}$$

$$\bar{x} \equiv \ln\left(\frac{I}{I(0)e^{\bar{\mu}t}}\right)$$

$$x_i^* \cong \frac{\bar{x}}{\sigma_I^2(0)} \sum_{j=1}^{n} \rho_{ij} p_j(0) \sigma_i(0) \sigma_j(0) = \frac{\bar{x}}{\sigma_I^2(0)} Cov(x_i, \bar{x})$$

$$\boxed{x_i^* = \hat{\beta}_i \bar{x}}$$

$$\hat{\beta} = Cov\left(\frac{\Delta S}{S}, \frac{\Delta I}{I}\right) / \left[Var\left(\frac{\Delta I}{I}\right)\right]$$

Most likely config. : described by the risk-neutral regression coefficients
of stock returns with the index return (``micro” CAPM)

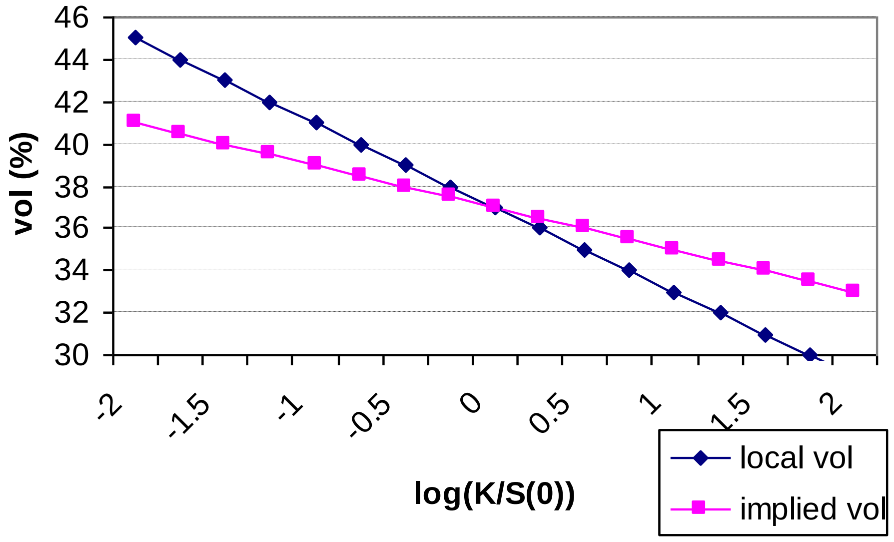
From local volatilities back to Black-Scholes implied volatilties

- Seek direct relation between implied volatilities of single-stock options and implied volatility of index options

- Tool: Berestycki-Busca-Florent large-deviations result for single-stock ("1/2 slope rule")

$$\sigma^{\text{impl.}}(x) \approx \left(\frac{1}{x} \int_0^x \frac{du}{\sigma(u)} \right)^{-1}$$

$$\sigma^{\text{impl.}}(x) \approx \frac{1}{2}\left(\sigma^{\text{impl.}}(0) + \sigma(x)\right)$$



Alternatively: integrate LV along most likely path

$$\left(\sigma^{\mathrm{impl}}(x,T)\right)^2 \approx \frac{1}{T}\int_0^T \sigma_{\mathrm{loc}}^2\left(x^*(s),s\right)ds$$

$$\approx \frac{1}{T}\int_0^T \sigma_{\mathrm{loc}}^2\left(xs,s\right)ds$$

-- For small dimensionless time, the price diffusion is localized in a neighborhood of the most likely path

-- this implies the ½ slope rule as trapezoidal approximation to the integral

Computing The Index Volatility

$$\left(\sigma_I^{\mathrm{impl}}\left(\bar{x},T\right)\right)^2 \approx \frac{1}{T}\int_0^T \sum_{ij=1}^{N} \sigma_i\left(x_i^*(s),s\right)\sigma_j\left(x_j^*(s),s\right) p_i p_j \rho_{ij}\, ds$$

$$\approx \frac{1}{T}\int_0^T \sum_{ij=1}^{N} \sigma_i\left(\beta_i \bar{x}s,s\right)\sigma_j\left(\beta_j \bar{x}s,s\right) p_i p_j \rho_{ij}\, ds$$

$$= \sum_{ij=1}^{N} \left[\frac{1}{T}\int_0^T \sigma_i\left(\beta_i \bar{x}s,s\right)\sigma_j\left(\beta_j \bar{x}s,s\right) ds\right] p_i p_j \rho_{ij}$$

$$\approx \sum_{ij=1}^{N} \sigma_i^{\mathrm{impl}}\left(\beta_i \bar{x},T\right)\sigma_j^{\mathrm{impl}}\left(\beta_j \bar{x},T\right) p_i p_j \rho_{ij} Q_{ij}\left(\bar{x},T\right)$$

$$\therefore \quad Q_{ij}\left(\bar{x},T\right) \equiv \frac{\left[\frac{1}{T}\int_0^T \sigma_i\left(\beta_i \bar{x}s,s\right)\sigma_j\left(\beta_j \bar{x}s,s\right) ds\right]}{\left[\frac{1}{T}\int_0^T \sigma_i^2\left(\beta_i \bar{x}s,s\right) ds\right]^{1/2}\left[\frac{1}{T}\int_0^T \sigma_j^2\left(\beta_j \bar{x}s,s\right) ds\right]^{1/2}}$$

Reconstruction Rule for Index Volatility

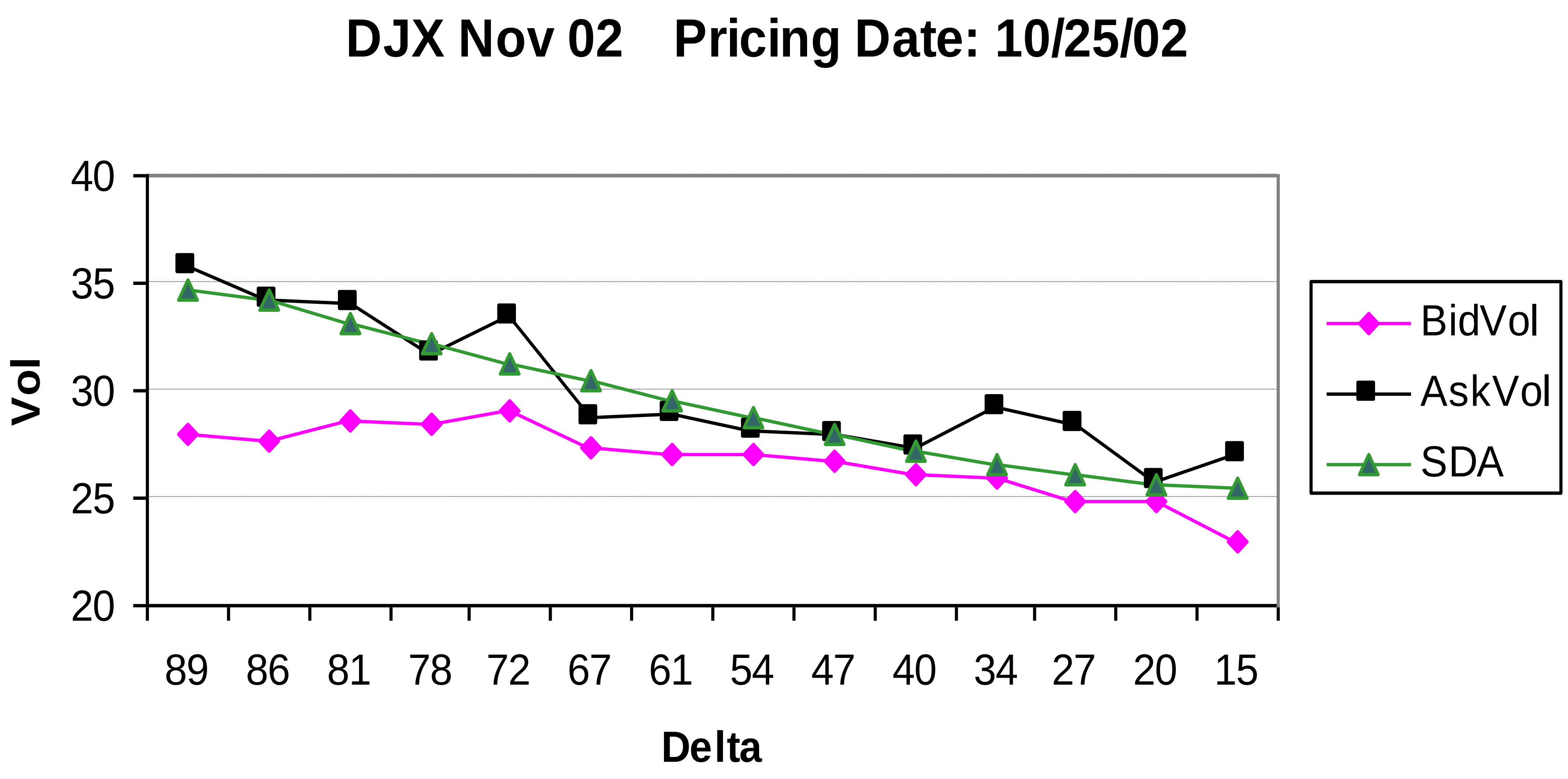
-- SD approximation is consistent with $Q_{ij}\left(\bar{x},T\right)\approx 1$

$$\boxed{\left(\sigma_I^{\mathrm{impl}}(\bar{x},T)\right)^2 \approx \sum_{ij=}^{N}\sigma_i^{\mathrm{impl}}(\beta_i\bar{x},T)\sigma_j^{\mathrm{impl}}(\beta_j\bar{x},T)p_i p_j \rho_{ij}}$$

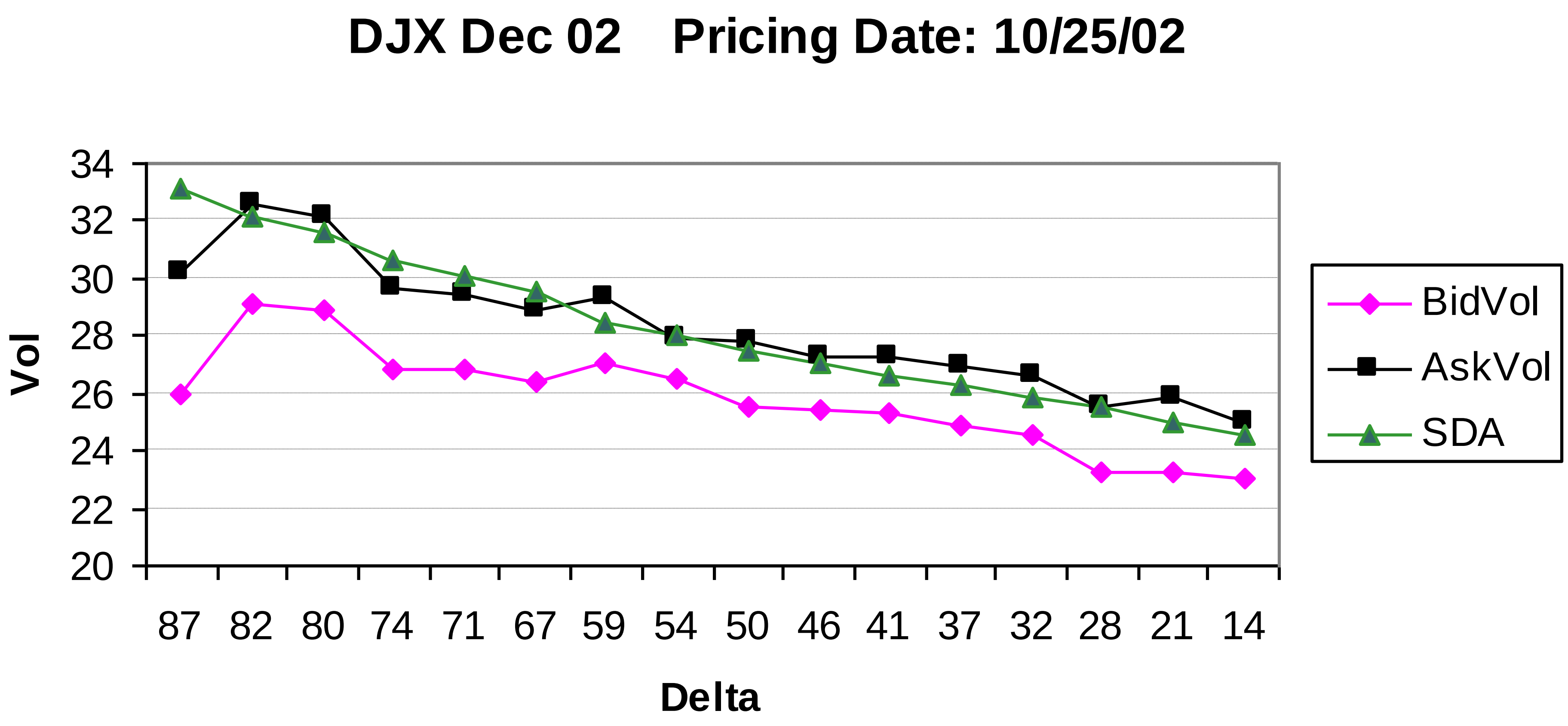
An $\bar{x}$ percent OTM strike for index corresponds to a $\beta_1\bar{x}$ percent OTM strike for stock 1, etc.

DJX: Dow Jones Industrial Average

T=1 month

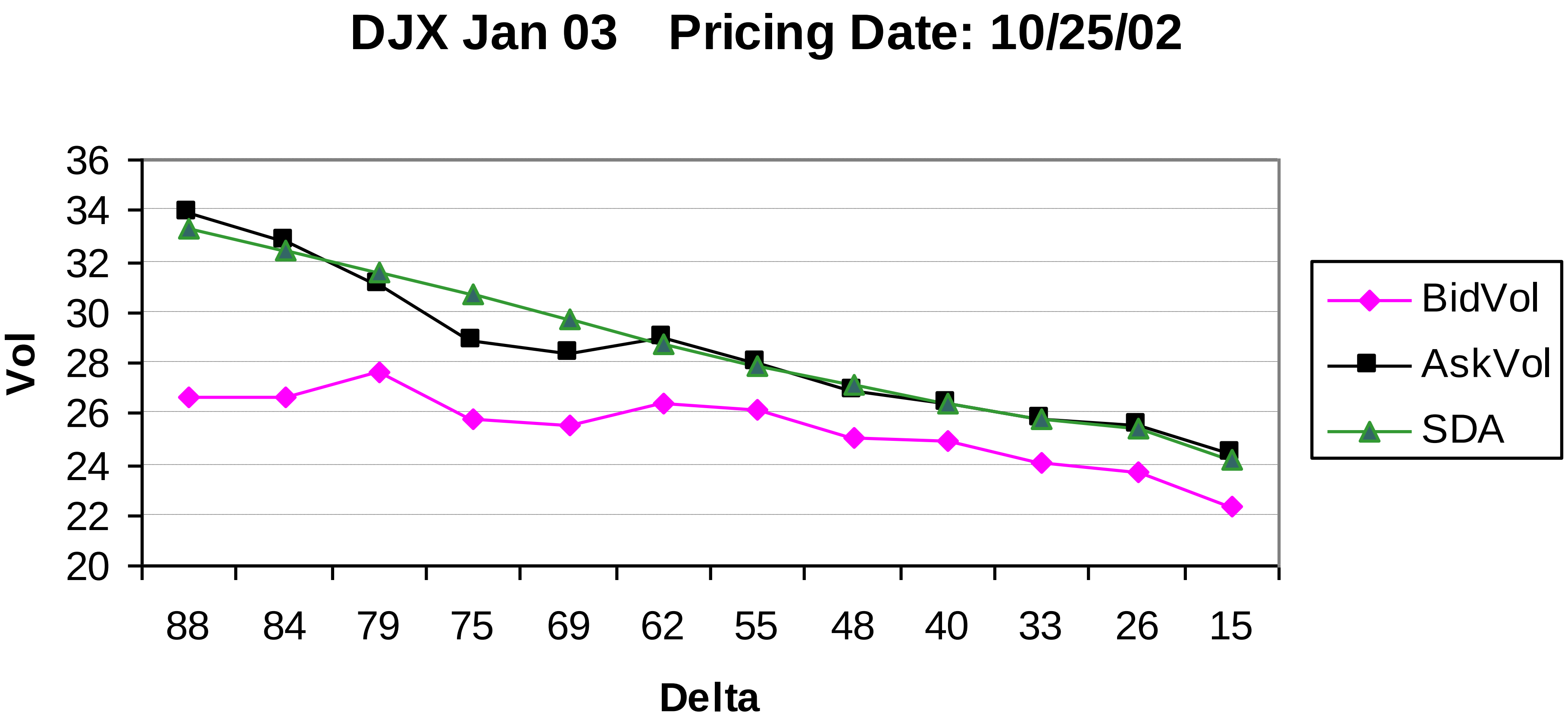


T= 2 months



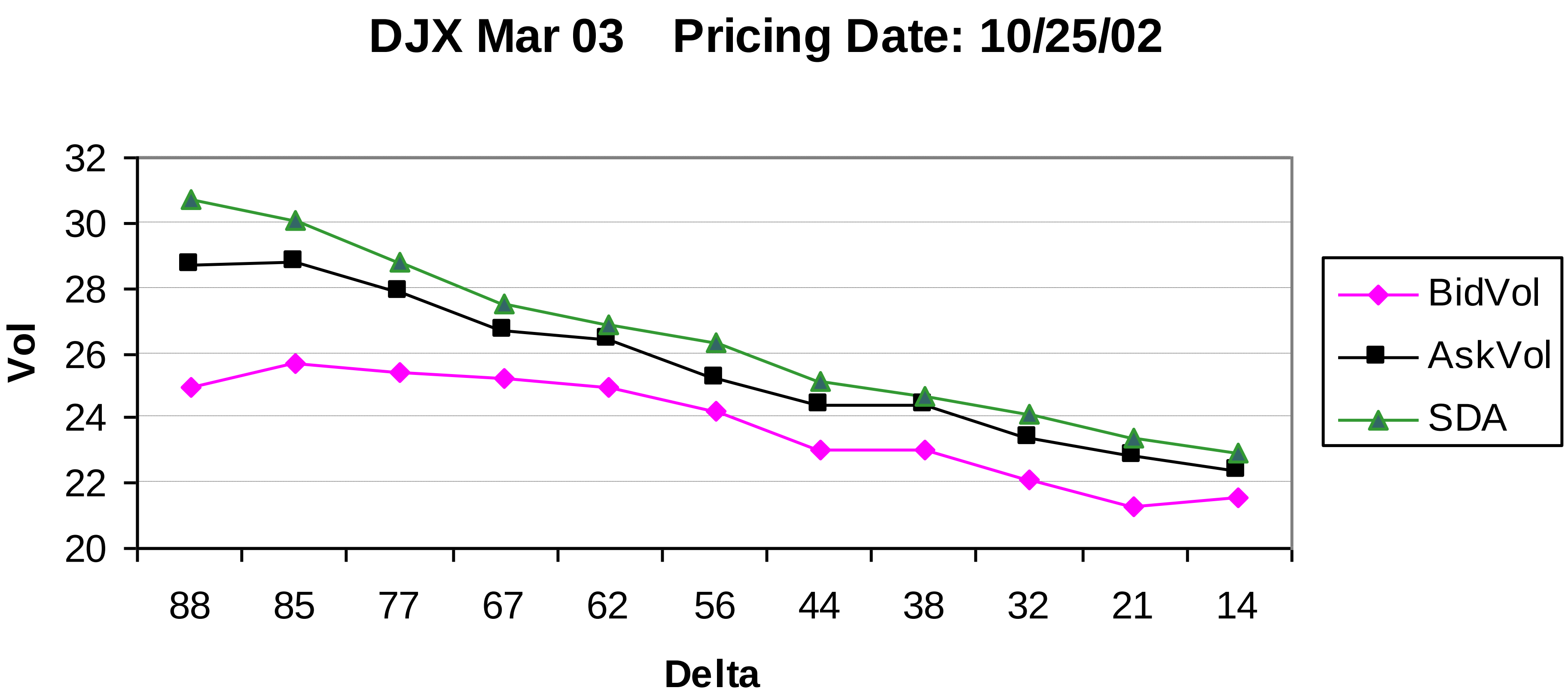
DJX Dec 02   Pricing Date: 10/25/02
34
32
30
28
26
24
22
20
Vol
87
82
80
74
71
67
59
54
50
46
41
37
32
28
21
14
Delta
BidVol
AskVol
SDA

T=3 months



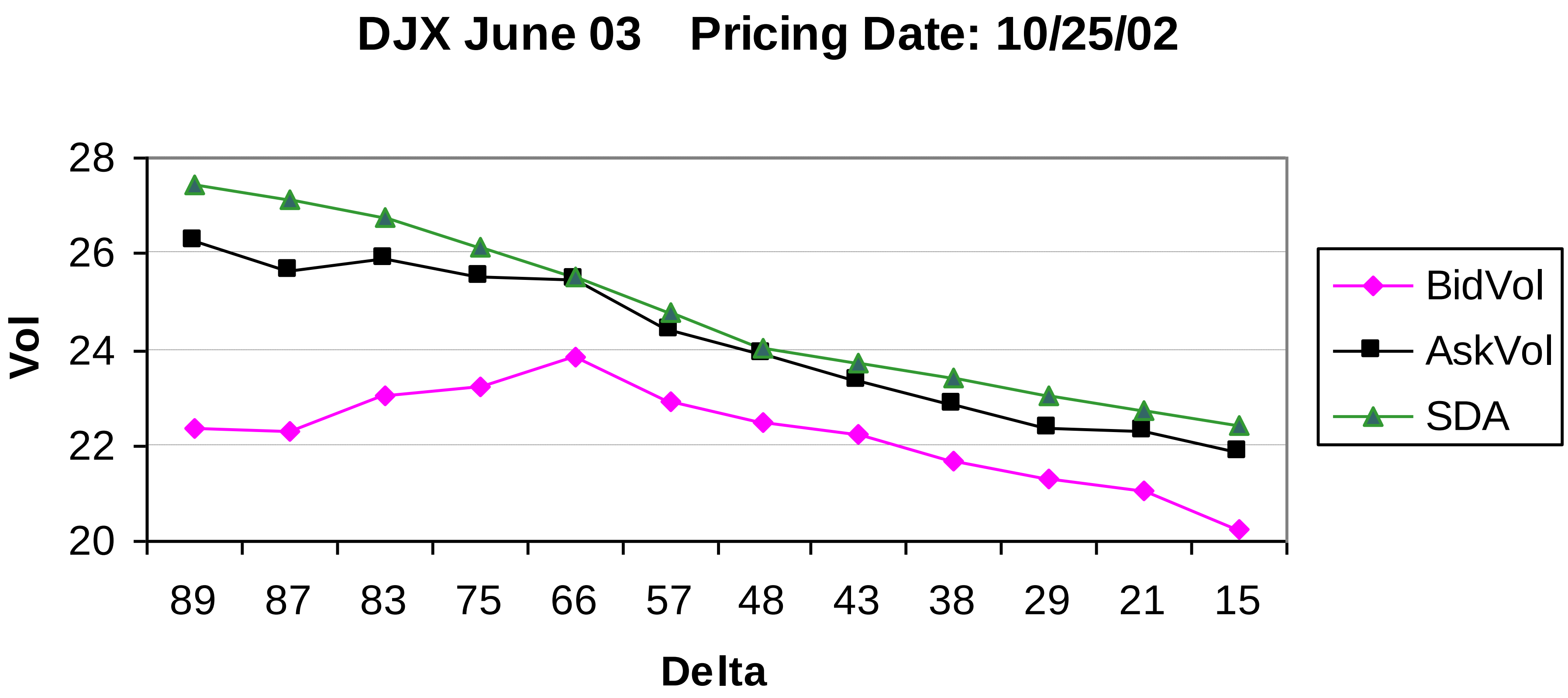
DJX Jan 03 Pricing Date: 10/25/02
36
34
32
30
28
26
24
22
20
Vol
88
84
79
75
69
62
55
48
40
33
26
15
Delta
BidVol
AskVol
SDA

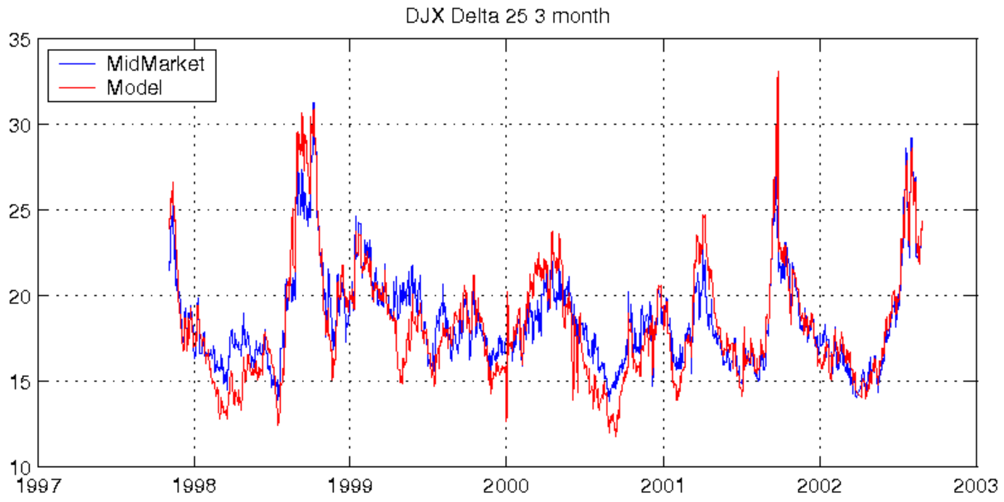
T= 5 months

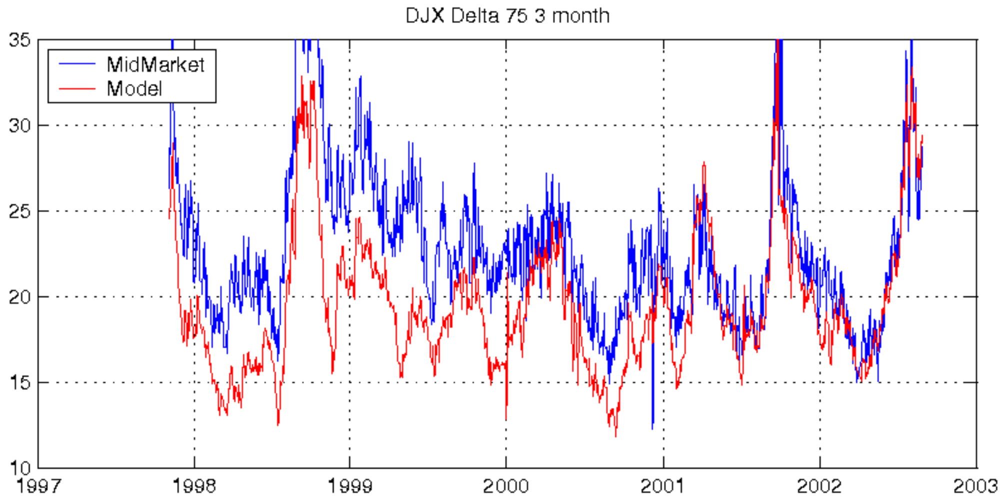


DJX Mar 03   Pricing Date: 10/25/02
Vol
32
30
28
26
24
22
20
88
85
77
67
62
56
44
38
32
21
14
Delta
BidVol
AskVol
SDA

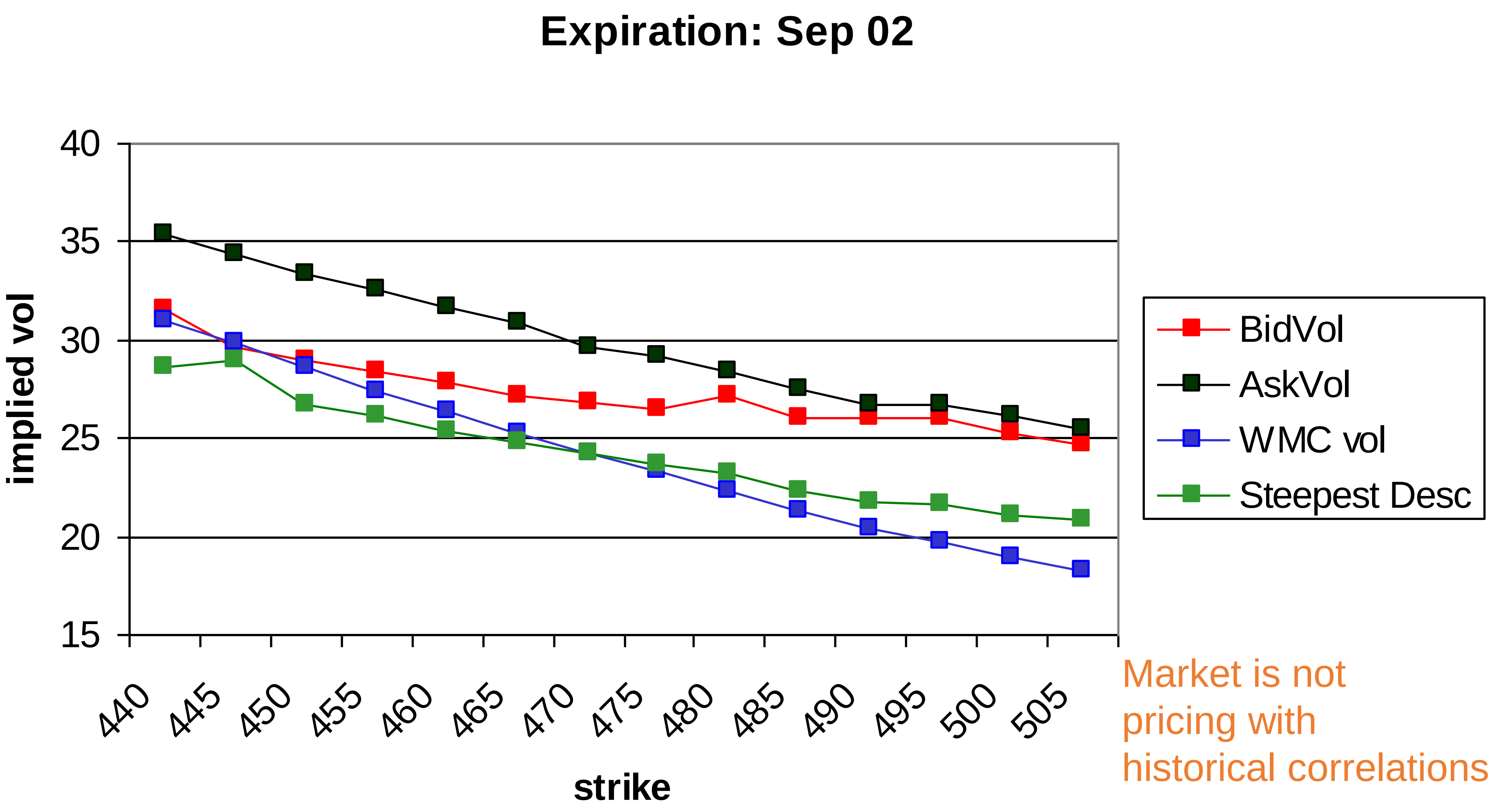
T=7 months


DJX June 03 Pricing Date: 10/25/02
Vol
28
26
24
22
20
89
87
83
75
66
57
48
43
38
29
21
15
Delta
BidVol
AskVol
SDA


DJX Delta 25 3 month
MidMarket
Model
35
30
25
20
15
10
1997
1998
1999
2000
2001
2002
2003


DJX Delta 75 3 month
MidMarket
Model
35
30
25
20
15
10
1997
1998
1999
2000
2001
2002
2003

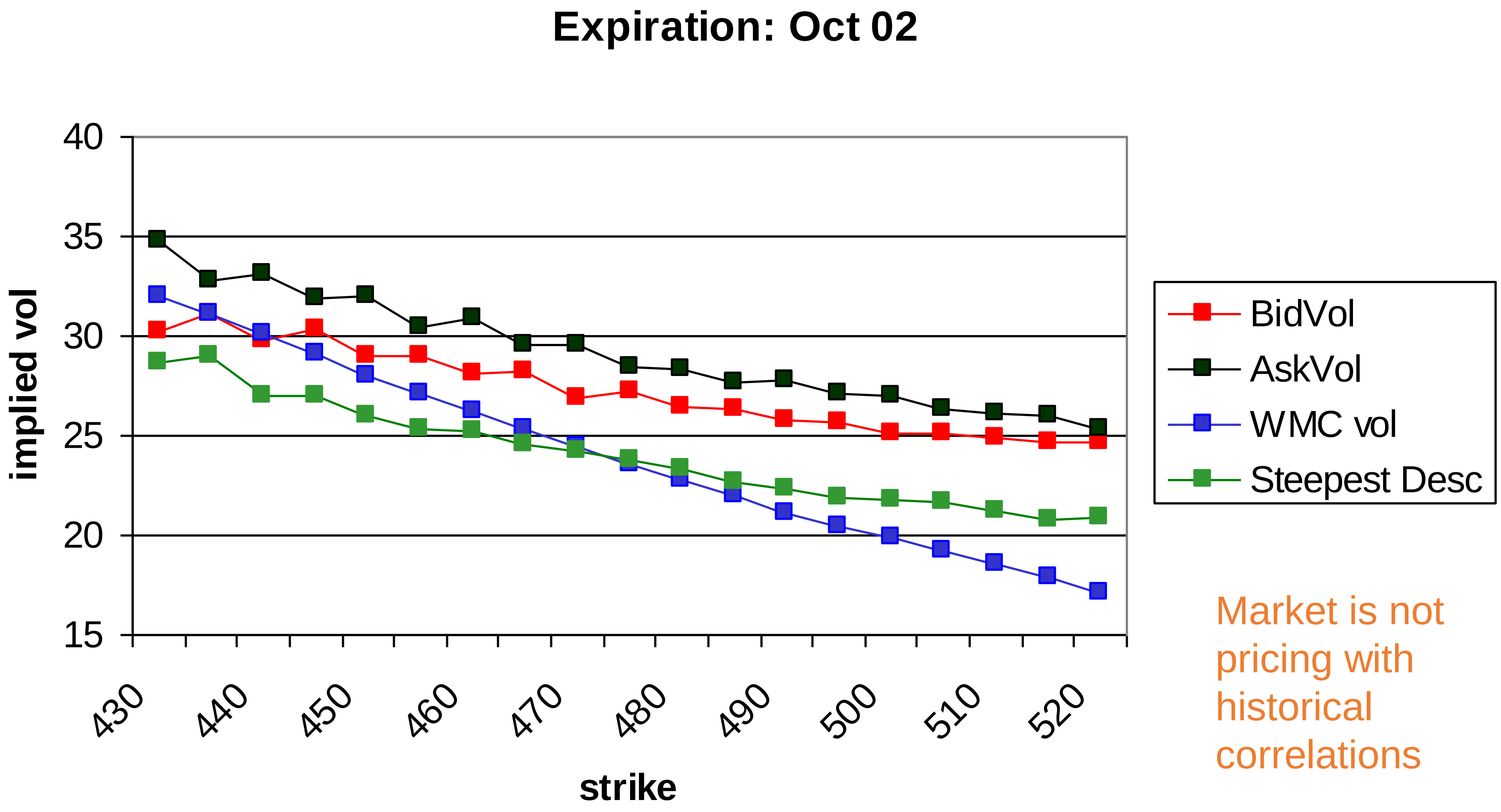
S&P 100 Index Options
(Quote date: Aug 20, 2002)



Market is not pricing with historical correlations

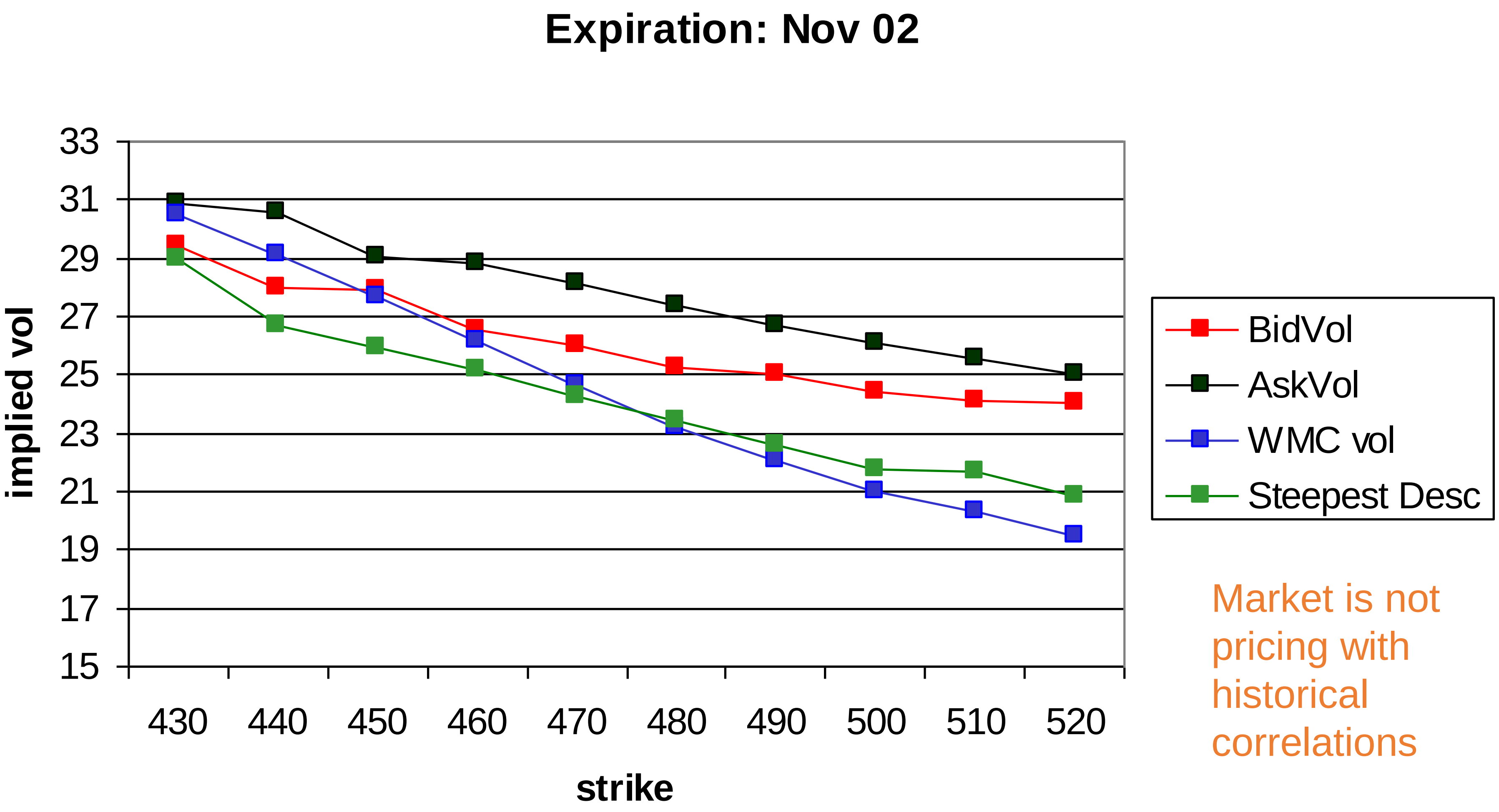
S&P 100 Index Options

(Quote date: Aug 20, 2002)



Market is not pricing with historical correlations

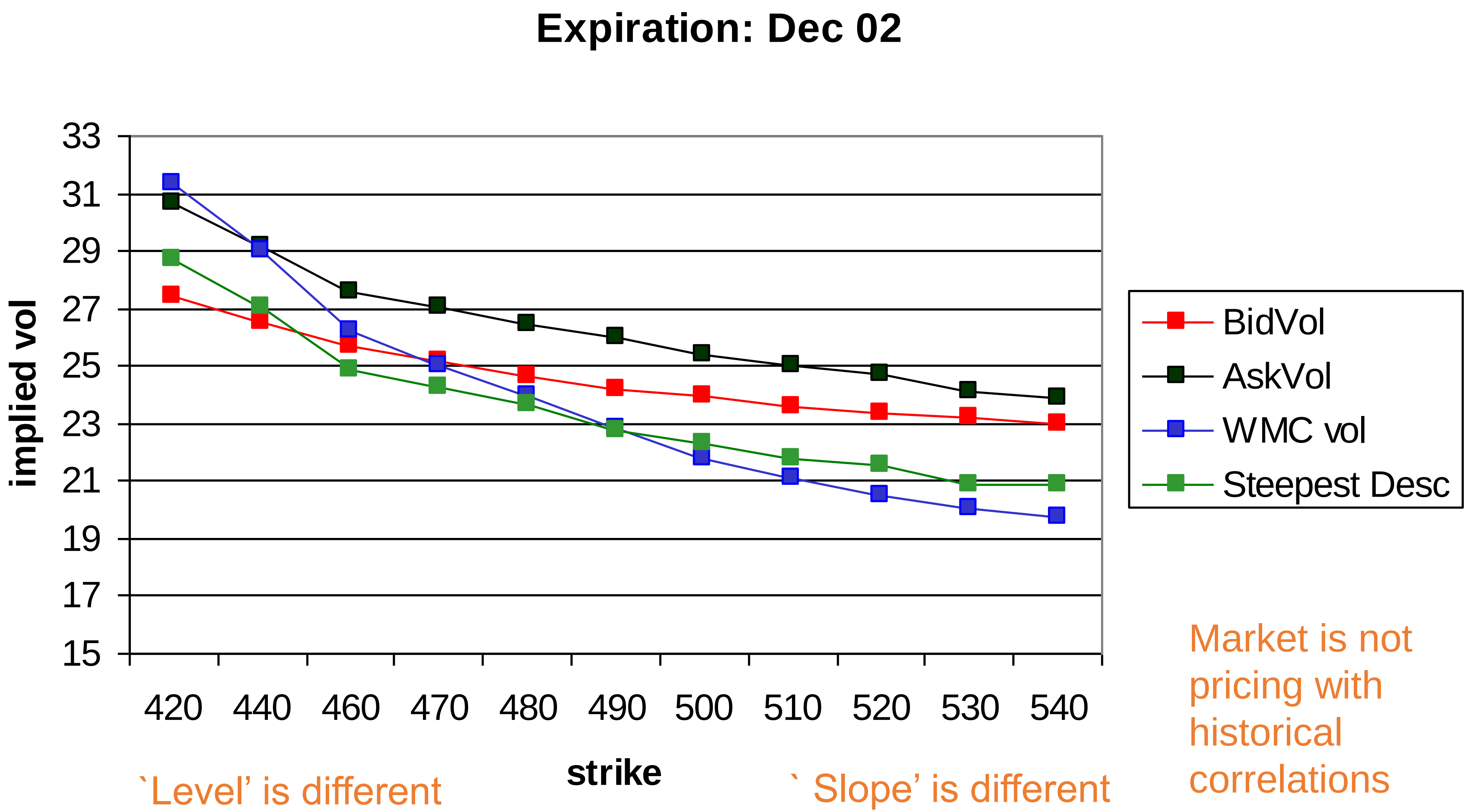
S&P 100 Index Options
(Quote date: Aug 20, 2002)



Market is not pricing with historical correlations

S&P 100 Index Options

(Quote date: Aug 20, 2002)



`Level' is different

` Slope' is different

Market is not pricing with historical correlations

Implied Correlation: a single correlation coefficient consistent with index vol

$$\left(\sigma_I^{\text{impl}}\right)^2 = \sum_{i=1}^{N} p_i^2 \left(\sigma_i^{\text{impl}}\right)^2 + \bar{\rho} \sum_{i \neq j}^{N} p_i p_j \sigma_i^{\text{impl}} \sigma_j^{\text{impl}}$$

$$\therefore \quad \bar{\rho} = \frac{\left(\sigma_I^{\text{impl}}\right)^2 - \sum_{i=1}^{N} p_i \left(\sigma_i^{\text{impl}}\right)^2}{\sum_{i \neq j}^{N} p_i p_j \sigma_i^{\text{impl}} \sigma_j^{\text{impl}}} = \frac{\left(\sigma_I^{\text{impl}}\right)^2 - \sum_{i=1}^{N} p_i^2 \left(\sigma_i^{\text{impl}}\right)^2}{\left(\sum_{i=1}^{N} p_i \sigma_i^{\text{impl}}\right)^2 - \sum_{i=1}^{N} p_i^2 \left(\sigma_i^{\text{impl}}\right)^2}$$

Approximate formula:

$$\bar{\rho} \approx \left(\frac{\sigma_I^{\text{impl}}}{\sum_{i=1}^{N} p_i \sigma_i^{\text{impl}}} \right)^2$$

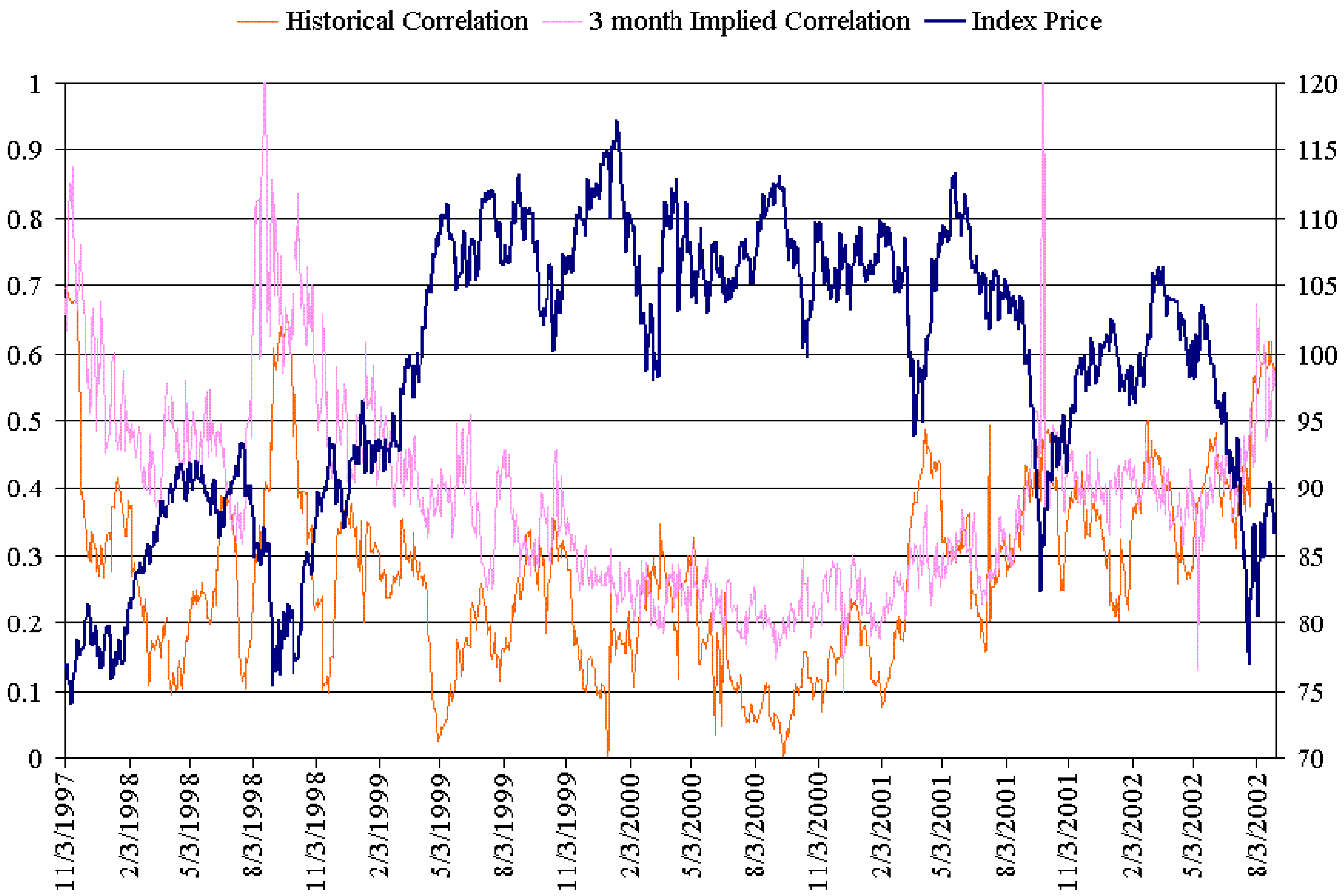
Implied correlation can be defined for different strikes, using SDA

Strike-dependent implied correlation

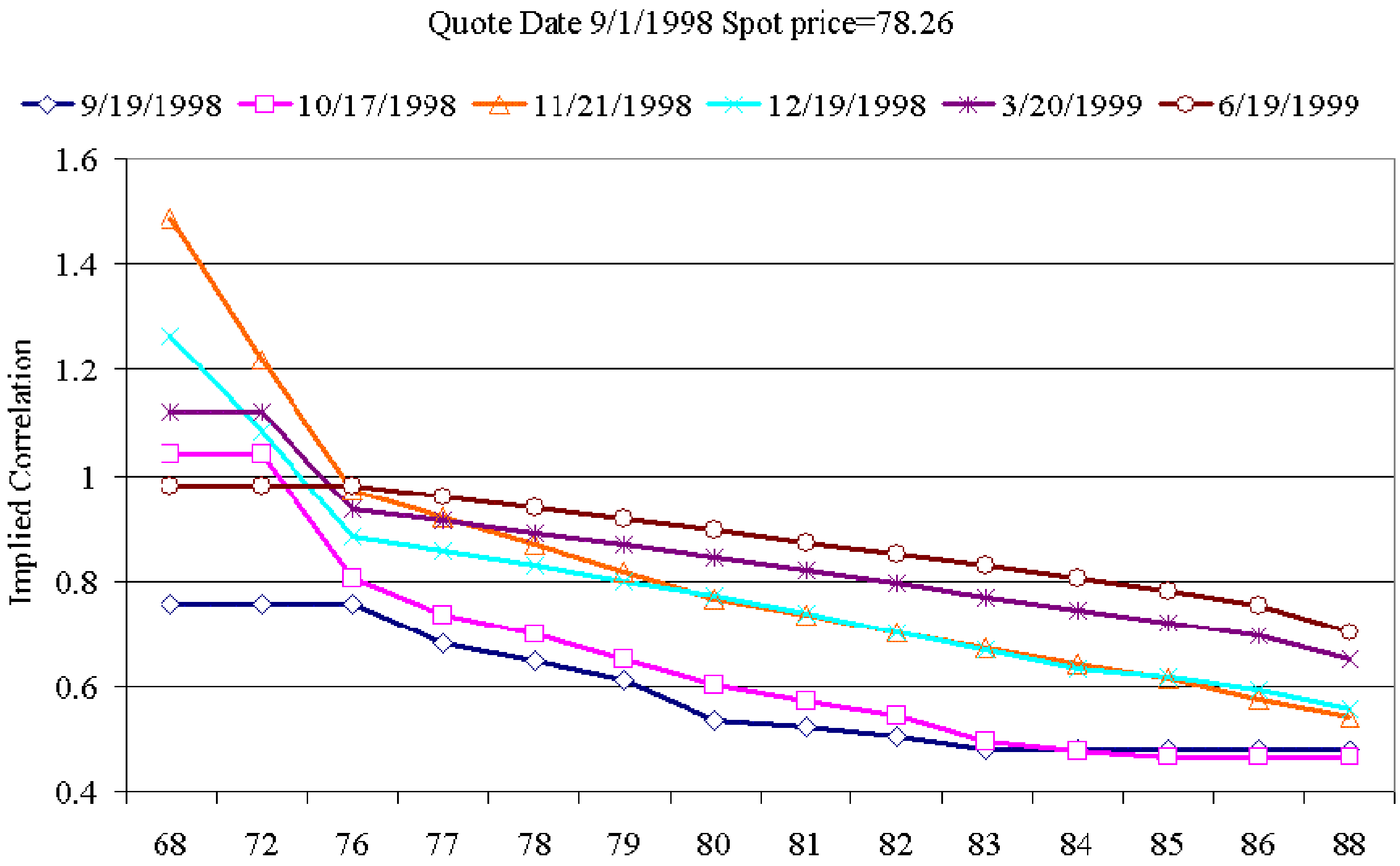
Since we observe the implied volatilities of index options and the implied volatilities of its constituent stocks as a function of strike/maturity, we can define implied Correlation as follows

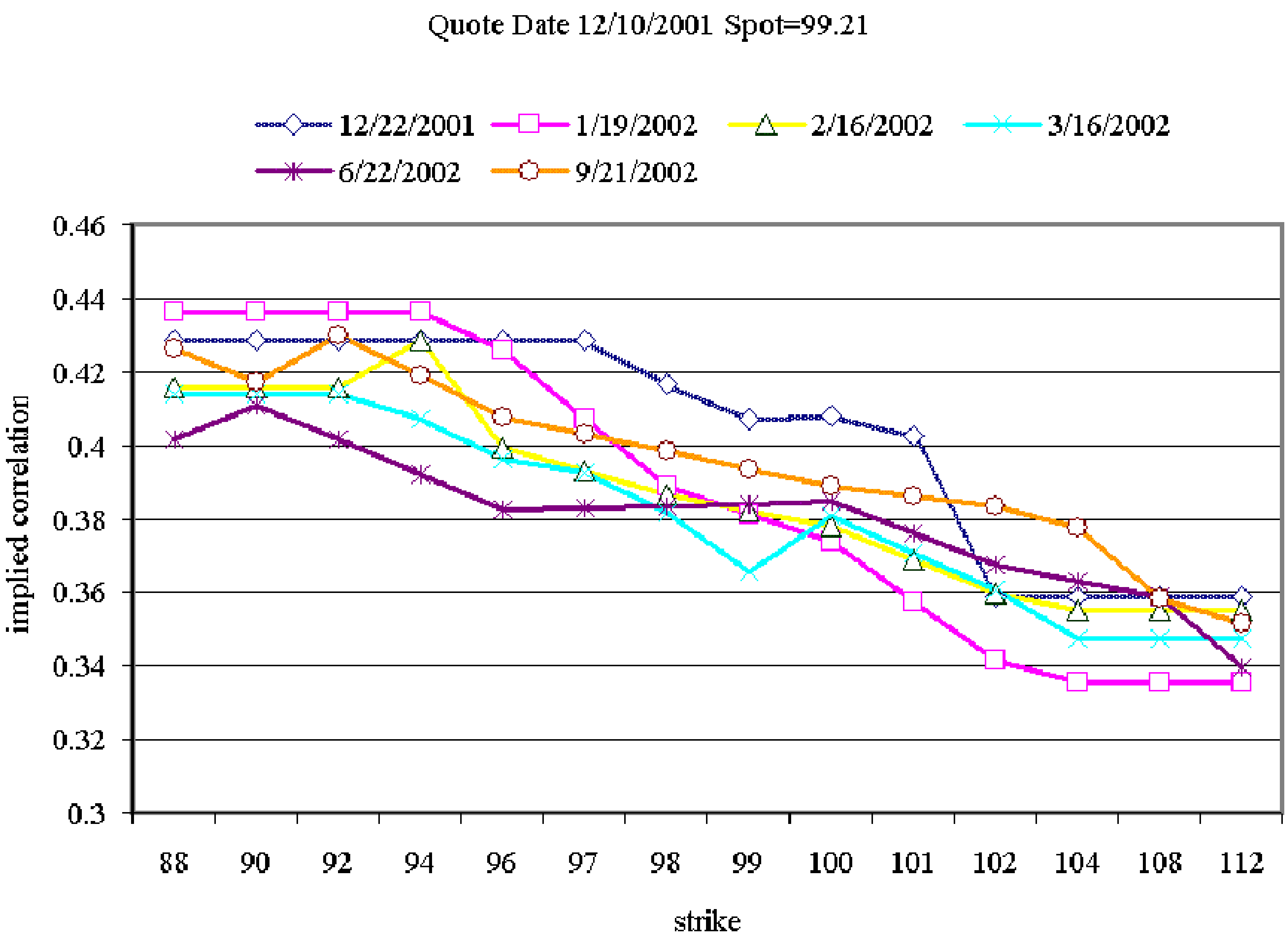
$$\bar{\rho}(\bar{x}) = \left(\frac{\sigma_I(\bar{x})}{\sum_i p_i \sigma_i(\beta_i \bar{x})} \right)^2$$

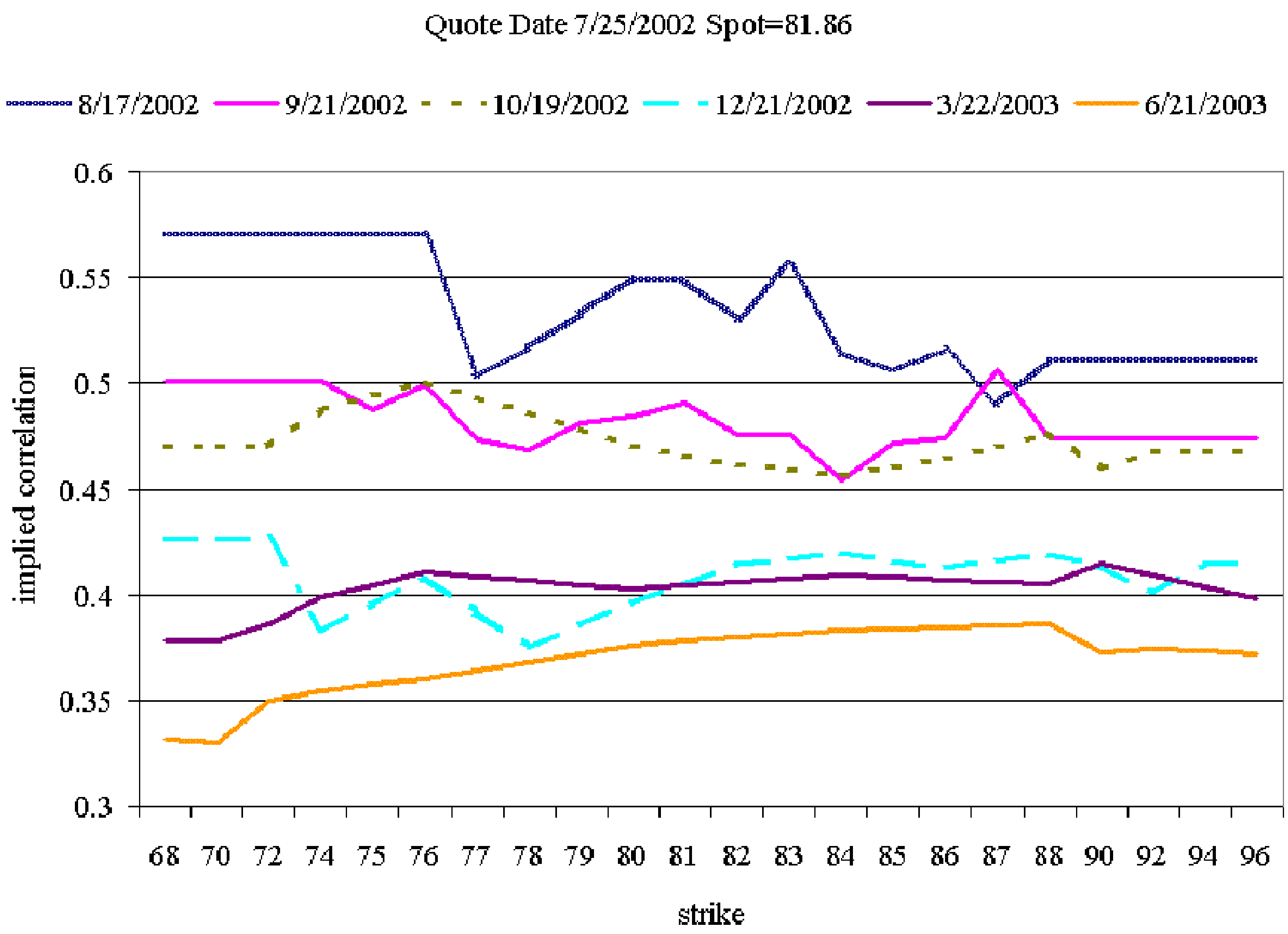
Dow Jones Index



Dow Jones Index: Correlation Skew




Quote Date 12/10/2001 Spot=99.21
12/22/2001
1/19/2002
2/16/2002
3/16/2002
6/22/2002
9/21/2002
implied correlation
0.46
0.44
0.42
0.4
0.38
0.36
0.34
0.32
0.3
88
90
92
94
96
97
98
99
100
101
102
104
108
112
strike


Quote Date 7/25/2002 Spot=81.86
8/17/2002
9/21/2002
10/19/2002
12/21/2002
3/22/2003
6/21/2003
0.6
0.55
0.5
0.45
0.4
0.35
0.3
implied correlation
68
70
72
74
75
76
77
78
79
80
81
82
83
84
85
86
87
88
90
92
94
96
strike

A model for ``Correlation skew'': Stochastic Volatility Systems

$$\frac{dS_i}{S_i} = \sigma_i dW_i \qquad\qquad \frac{d\sigma_i}{\sigma_i} = \kappa_i dZ_i$$

$$E\left(dW_i dW_j\right) = \rho_{ij} dt \qquad\qquad E\left(dW_i dZ_j\right) = r_{ij} dt$$

$$\bar{x} = \frac{dI}{I}, \qquad x_i = \frac{dS_i}{S_i} \qquad y_i = \frac{d\sigma_i}{\sigma_i}$$

Look for most likely configuration of <u>stocks and vols</u> $\left(x_1,...,x_n,y_1,...,y_n\right)$ corresponding to a given index displacement $\bar{x}$

Most likely configuration for Stochastic Volatility Systems

$$x_i^{\,*} = \beta_i \bar{x} \qquad \beta_i = \frac{\sigma_i \rho_{iI}}{\sigma_I}$$

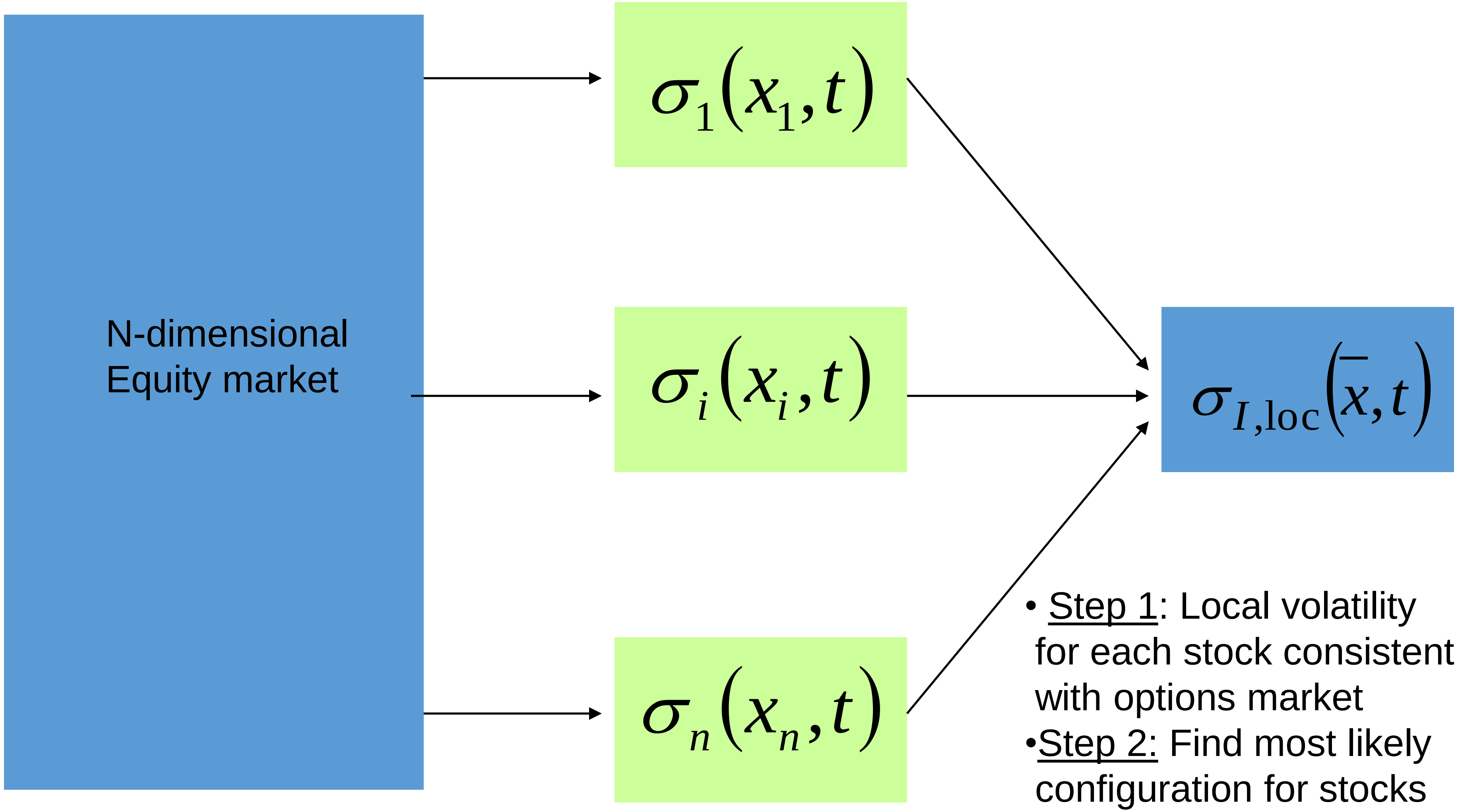
$$y_i^{\,*} = \gamma_i \bar{x} \qquad \gamma_i = \frac{\kappa_i r_{iI}}{\sigma_I}$$

Most likely configuration for stocks moves and volatility moves, given the index move

$$\sigma_{I,\mathrm{loc}}^2(\bar{x},t) \cong \sum_{ij=1}^{n} p_i p_j \sigma_i(0,t) \sigma_j(0,t) e^{\gamma_i \bar{x}} e^{\gamma_j \bar{x}} \rho_{ij}$$

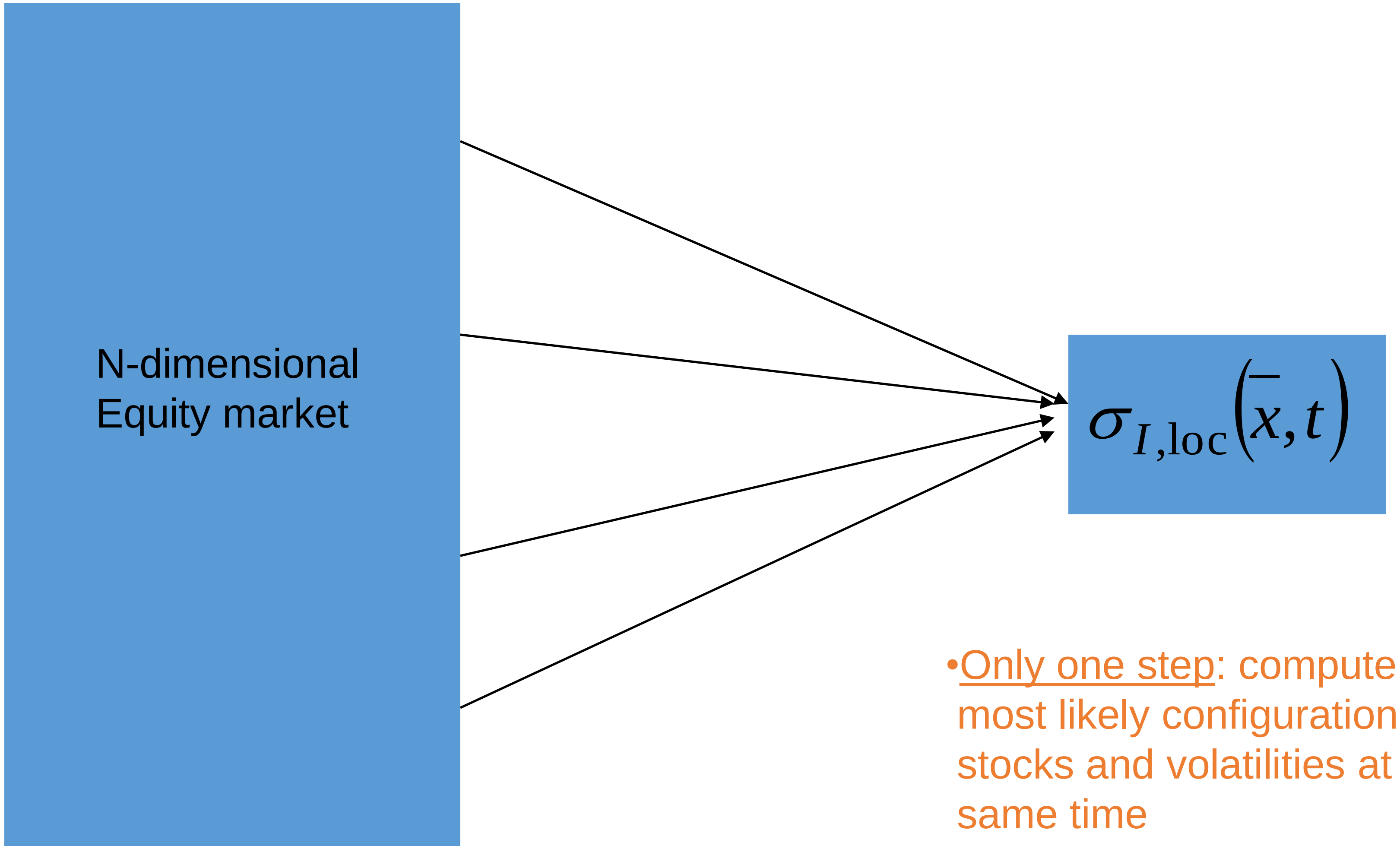
SDA

Method I: Dupire & Most Likely Configuration for Stock Moves



- Step 1: Local volatility for each stock consistent with options market
- Step 2: Find most likely configuration for stocks

Method II: Stochastic Volatility System and joint MLC for Stocks and Volatilities



- Only one step: compute the most likely configuration of stocks and volatilities at the same time

Methods I and II are not `equivalent'

Dupire local vol. for single names

$$\sigma_{i,\mathrm{loc}}(x_i,t) \approx \sigma_i(0,t)e^{\varpi_i x_i} \qquad \varpi_i = \frac{\kappa_i r_{ii}}{\sigma_i}$$

Index vol., Method I

$$\sigma^2_{I,\mathrm{loc}}(\bar{x},t) = \sum_{ij} p_i p_j \sigma_i(0,t)\sigma_j(0,t)\rho_{ij} e^{\varpi_i \beta_i \bar{x}} e^{\varpi_j \beta_j \bar{x}}$$

Index vol., Method II

$$\sigma^2_{I,\mathrm{loc}}(\bar{x},t) = \sum_{ij} p_i p_j \sigma_i(0,t)\sigma_j(0,t)\rho_{ij} e^{\gamma_i \bar{x}} e^{\gamma_j \bar{x}}$$

Stochastic Volatility Systems give rise to Index-dependent correlations

$$\sigma_{I,\mathrm{loc}}^2\left(\bar{x},t\right) \approx \sum_{ij} p_i p_j \sigma_i\left(0,t\right)\sigma_j\left(0,t\right)\rho_{ij} e^{\gamma_i \bar{x}} e^{\gamma_j \bar{x}}$$

Method II

$$\approx \sum_{ij} p_i p_j \sigma_i\left(0,t\right) e^{\beta_i \varpi_i \bar{x}} \sigma_j\left(0,t\right) e^{\beta_j \varpi_j \bar{x}} \rho_{ij} e^{\gamma_i \bar{x}} e^{\gamma_j \bar{x}} e^{-\beta_i \varpi_i \bar{x}} e^{-\beta_i \varpi_i \bar{x}}$$

$$\approx \sum_{ij} p_i p_j \sigma_{i,\mathrm{loc}}\left(\beta_i \bar{x},t\right) \sigma_{i,\mathrm{loc}}\left(\beta_i \bar{x},t\right) \rho_{ij}\left(\bar{x}\right)$$

$$\boxed{\rho_{ij}\left(\bar{x}\right) \equiv \rho_{ij} e^{\left(\gamma_i + \gamma_j - \beta_i \varpi_i - \beta_j \varpi_j\right)\bar{x}}}$$

Equivalence holds only under additional assumptions on stock-volatility correlations

$$\varpi_i \beta_i = \frac{\kappa_i r_{ii}}{\sigma_i} \frac{\sigma_i \rho_{iI}}{\sigma_I} = \frac{\kappa_i r_{ii} \rho_{iI}}{\sigma_I}$$

Method I

$$\gamma_i = \frac{\kappa_i r_{iI}}{\sigma_I}$$

Method II

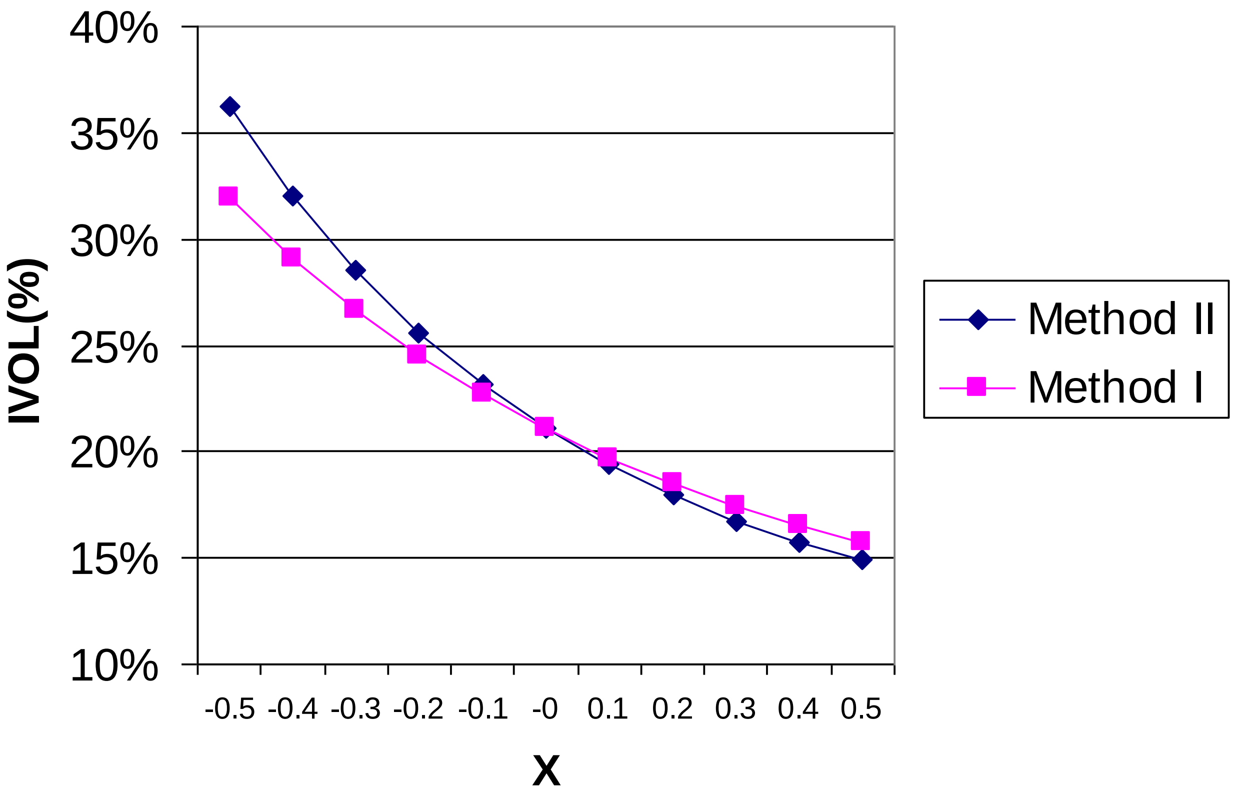
$$\boxed{\begin{gathered} r_{iI} = r_{ii}\rho_{iI} \\ r_{ij} = r_{ii}\rho_{ij} \end{gathered}}$$

Conditions under which both methods give equivalent valuations

Numerical Example

$$\sigma_1 = 20\%, \sigma_2 = 30\%, \rho = 40\%$$

$$r = \begin{bmatrix} -0.7 & -0.5 \\ -0.6 & -0.7 \end{bmatrix}, \quad \kappa_1 = \kappa_2 = 50\%$$



Lee, Wang and Karim

RISK, Dec 2003

- Propose a stochastic average correlation
- Linear econometric fit:

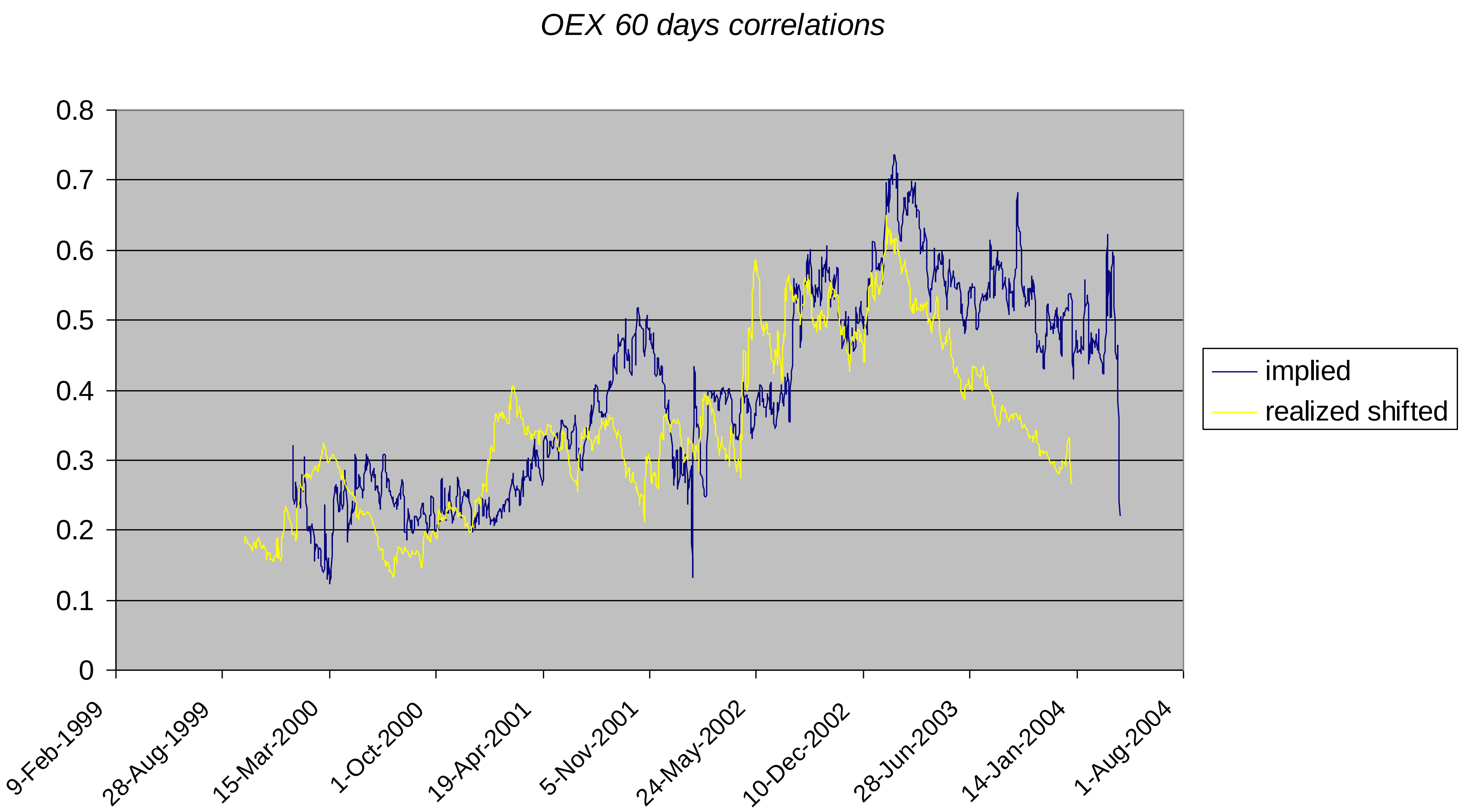
$$\overline{\rho} = \alpha + \beta \ln I + \varepsilon$$

Rho_bar is the
`average' correlation

$$OEX: \qquad \beta = -0.66$$

$$BKX: \qquad \beta = -0.34$$

This model gives rise to an index-dependent implied correlation via SDA



OEX 60 days correlations
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
9-Feb-1999
28-Aug-1999
15-Mar-2000
1-Oct-2000
19-Apr-2001
5-Nov-2001
24-May-2002
10-Dec-2002
28-Jun-2003
14-Jan-2004
1-Aug-2004
implied
realized shifted

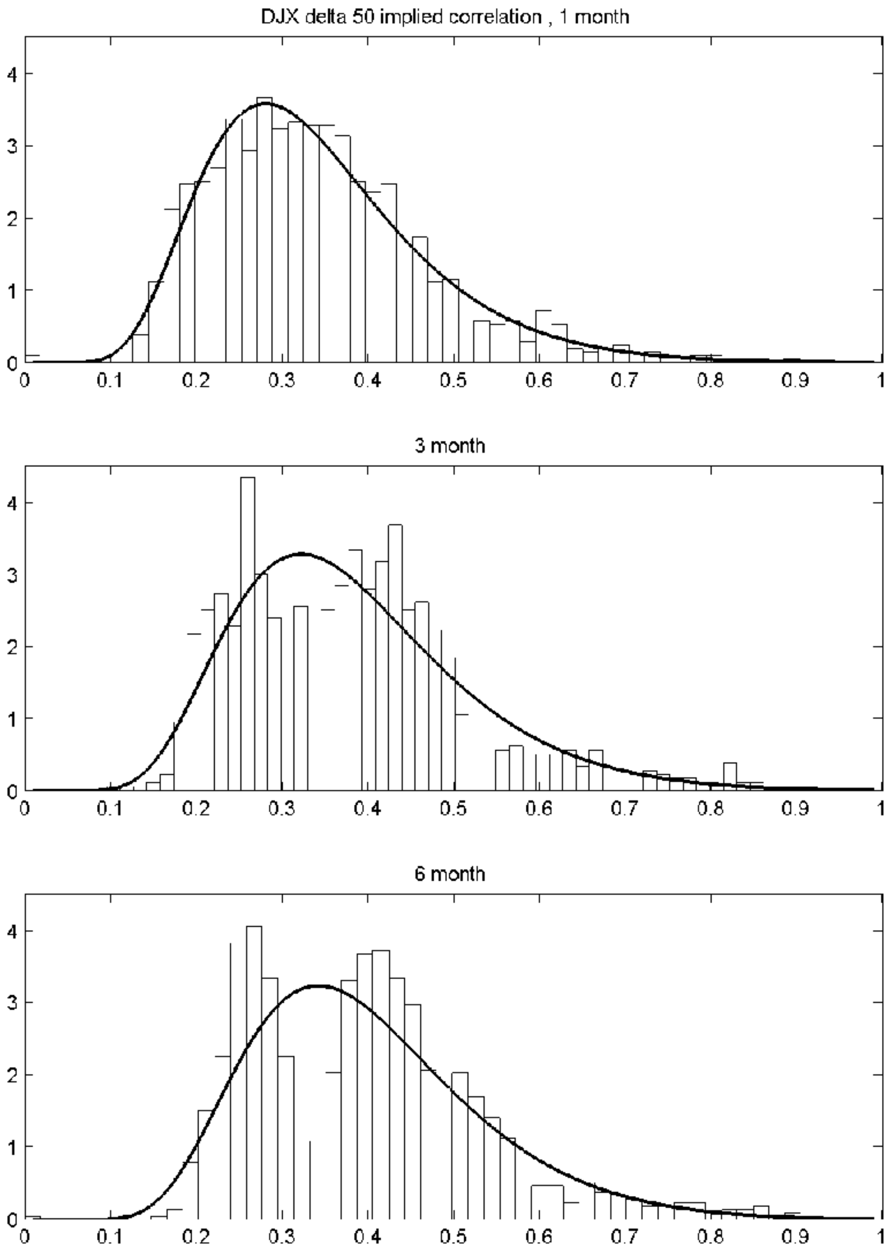
J. Lim: Statistical distribution of implied correlations

NYU Thesis, 2003

$$f\left(\overline{\rho}\right) = \text{p.d.f. for implied correlation}$$

$$\text{Parametric model}: \ \overline{\rho} \sim \frac{2}{\pi}\text{Arctan}\left(X\right)$$

$$X \sim N\left(\mu_0, \sigma_0^2\right)$$



DJX Implied Correlation (1998-2003)

1 month

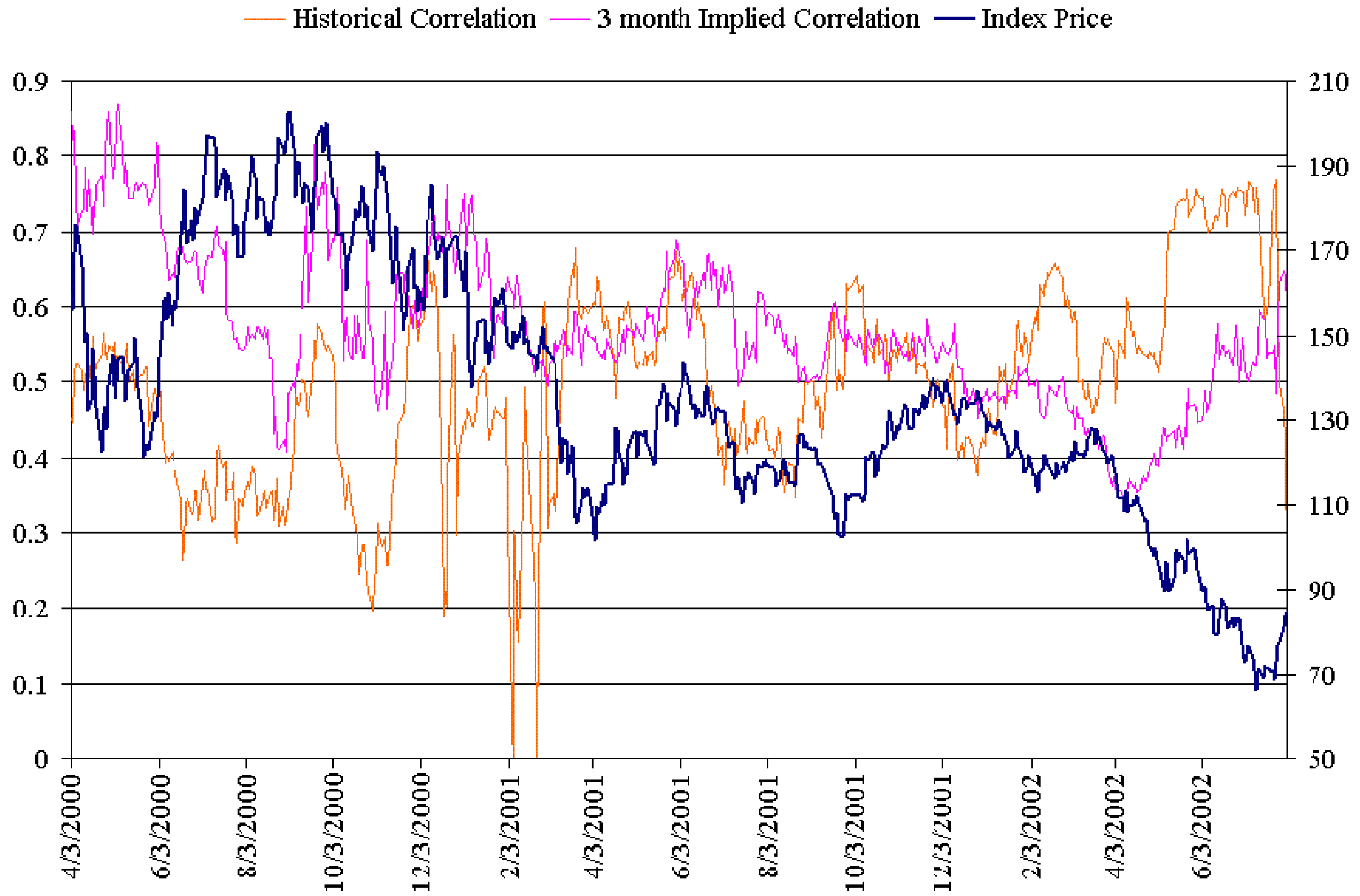
2 months

-- Heavy right-tails, low mean

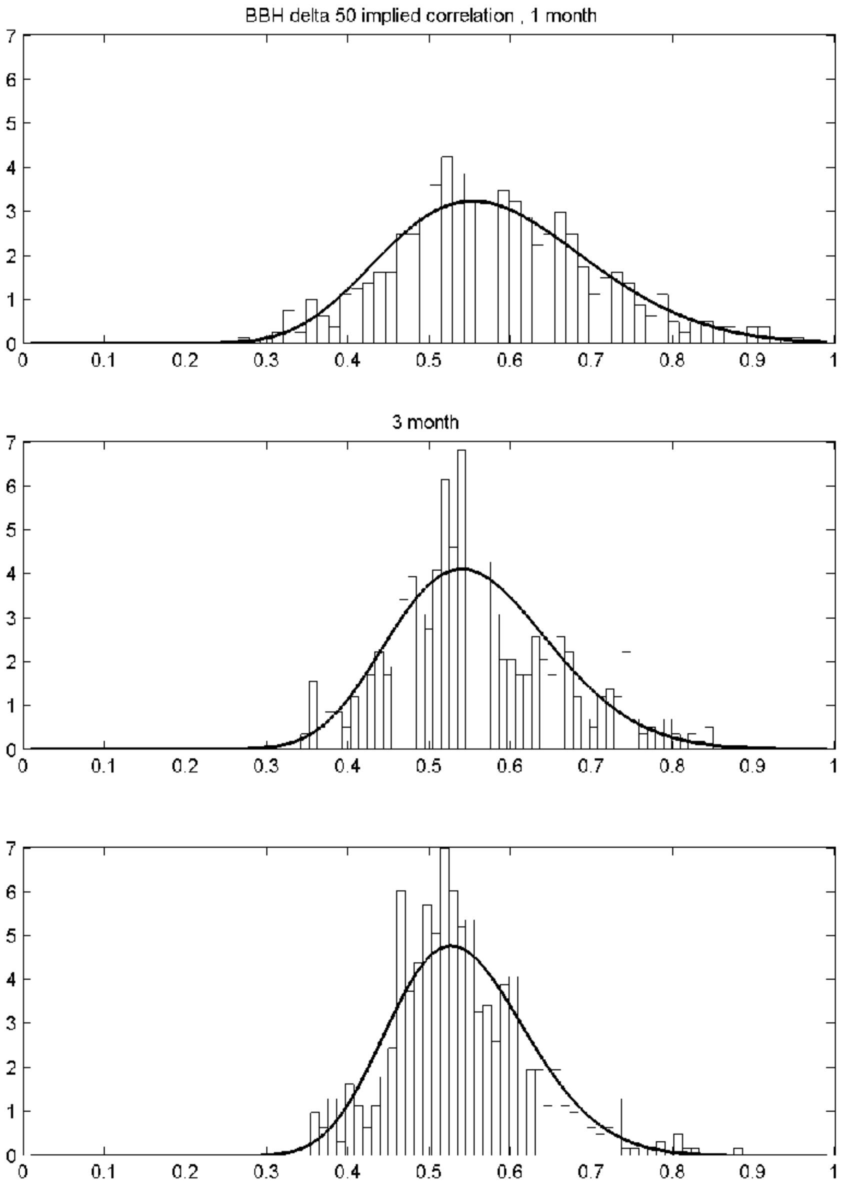
-- characteristic of major indices

3 months

BBH Biotech Index

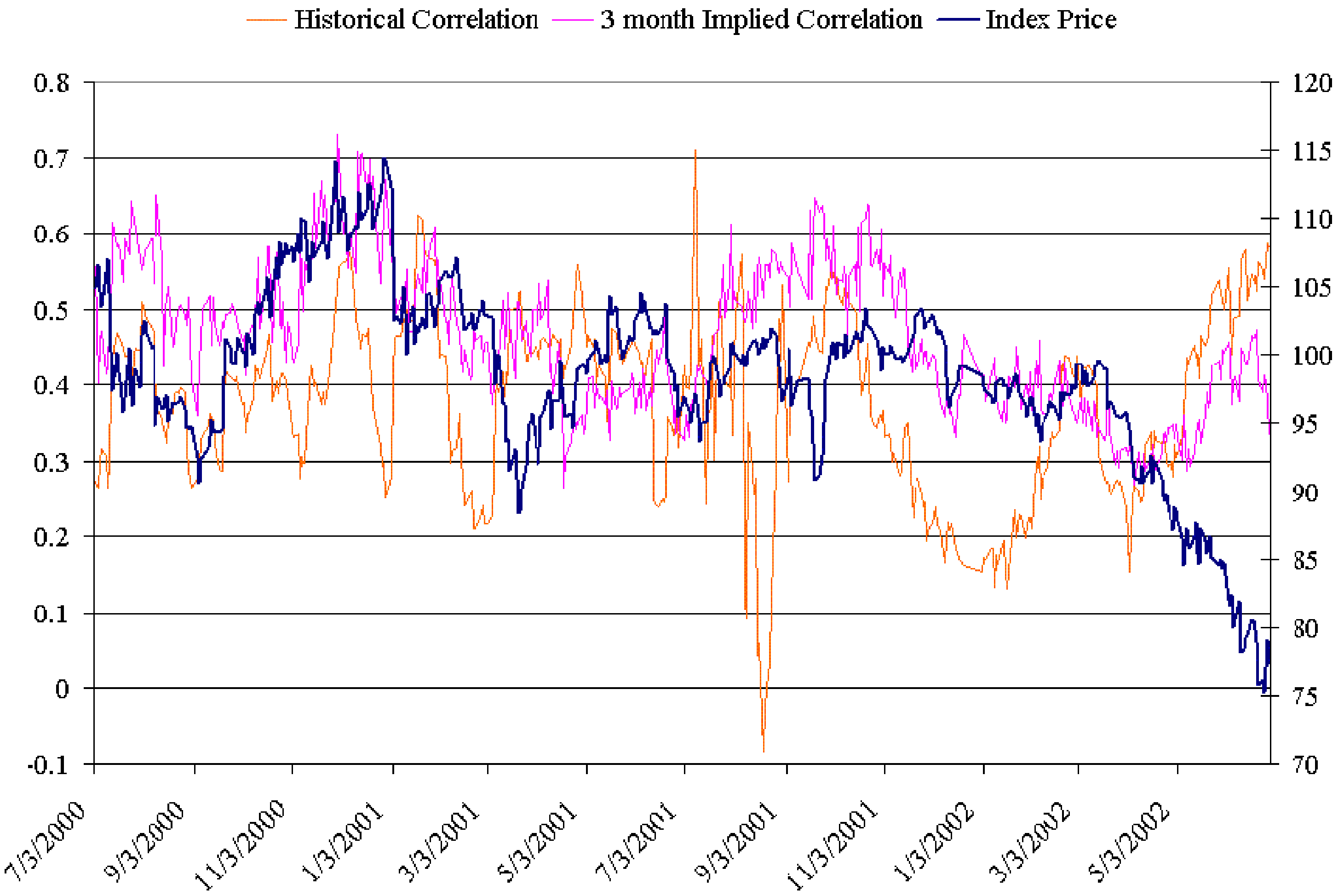


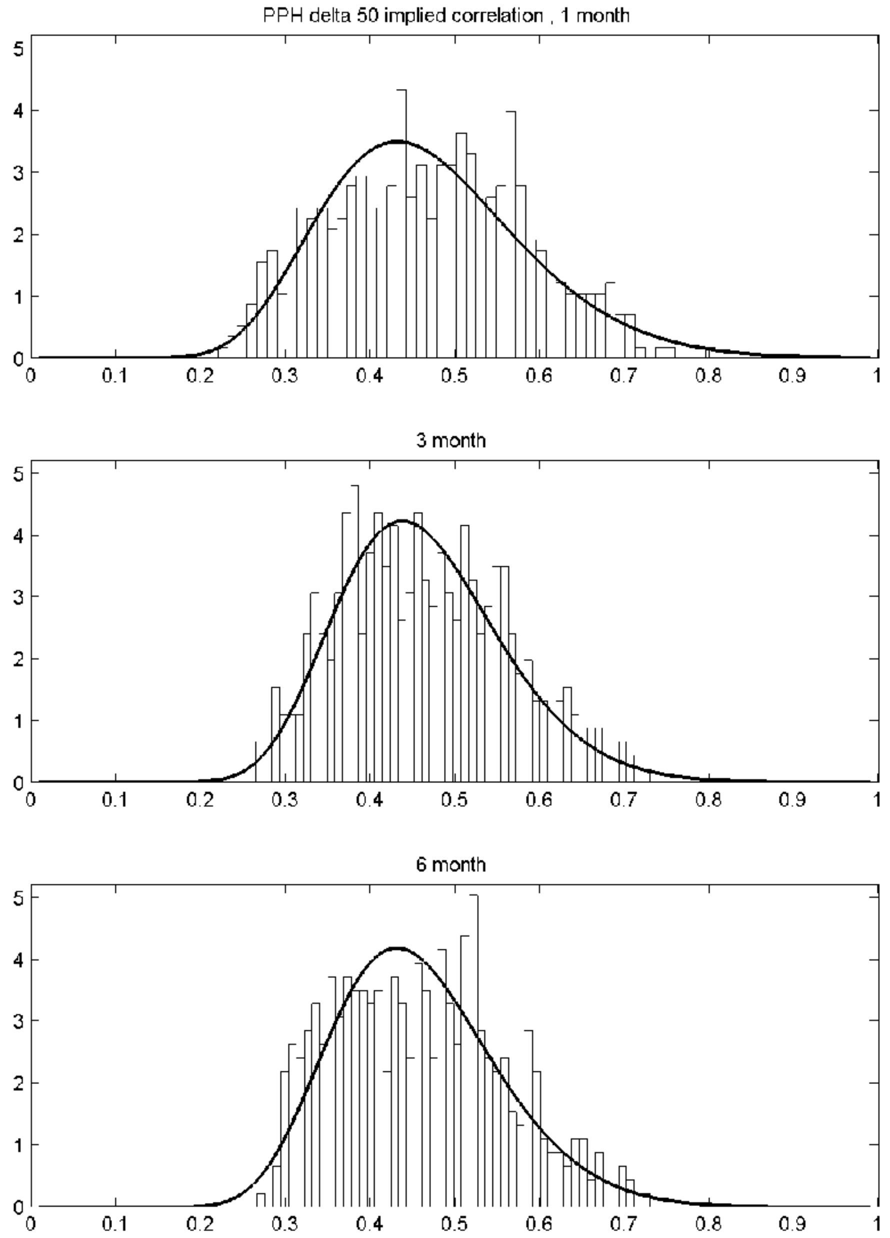
BBH Biotech Index



Mean=0.5-0.55
Heavy tails

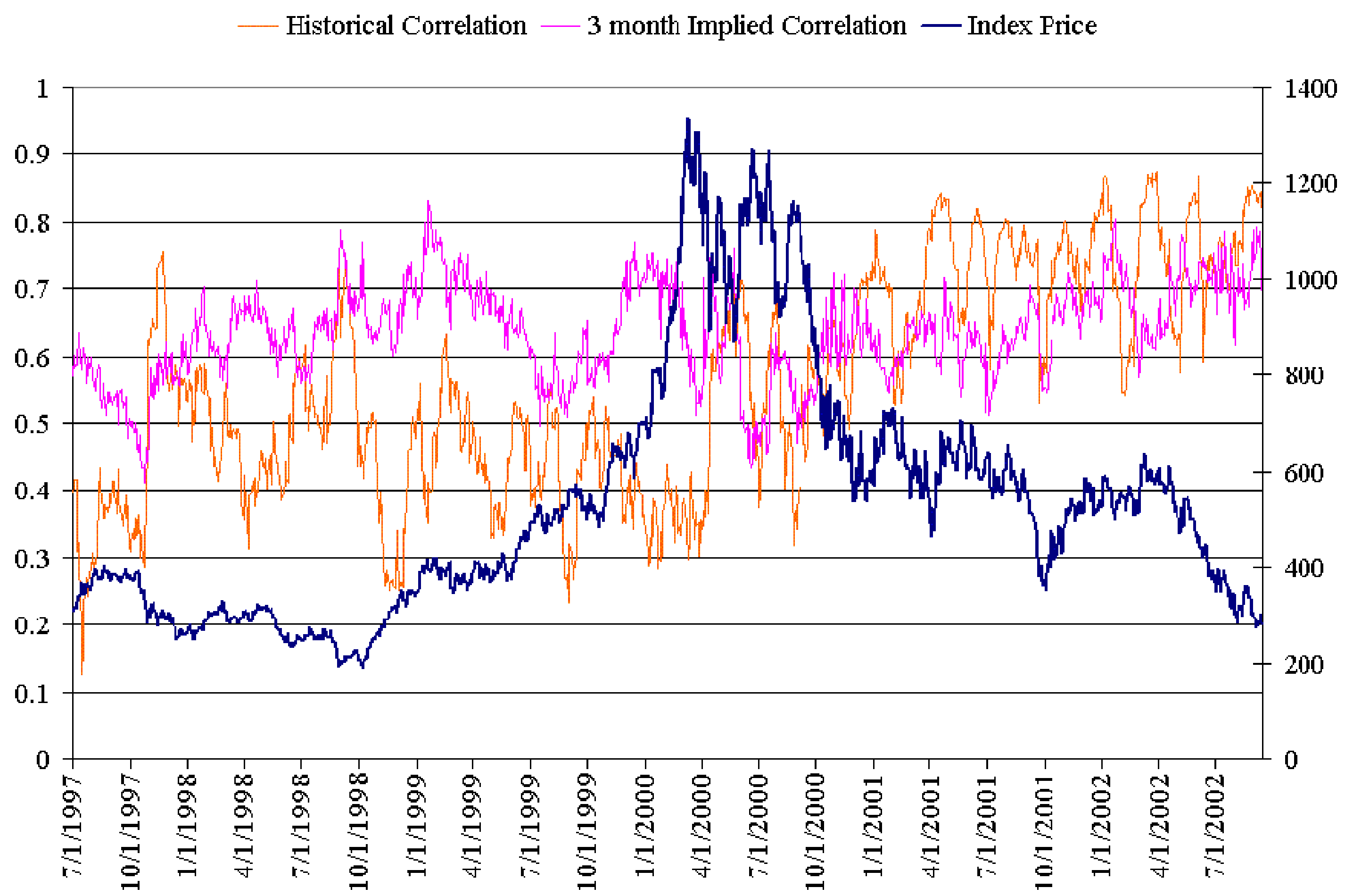
PPH Pharmaceutical Index

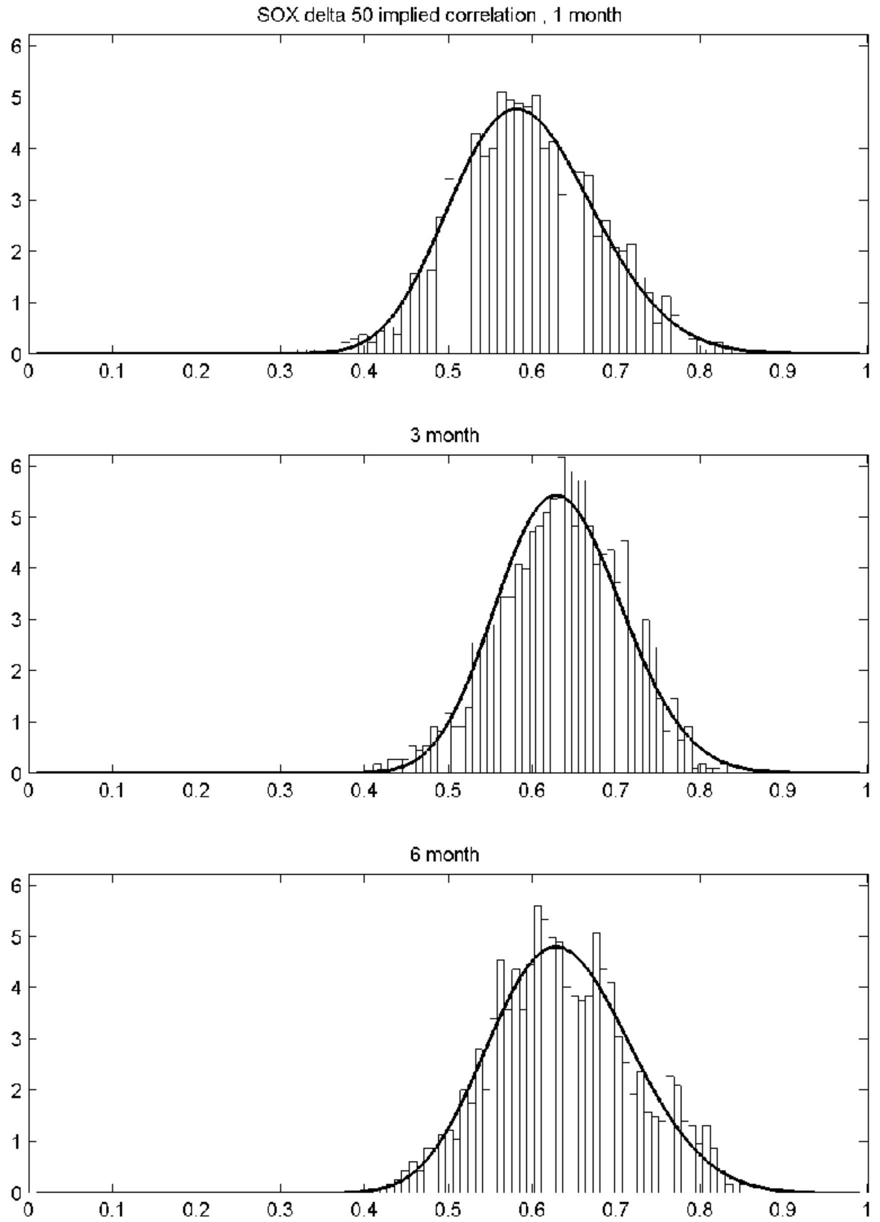




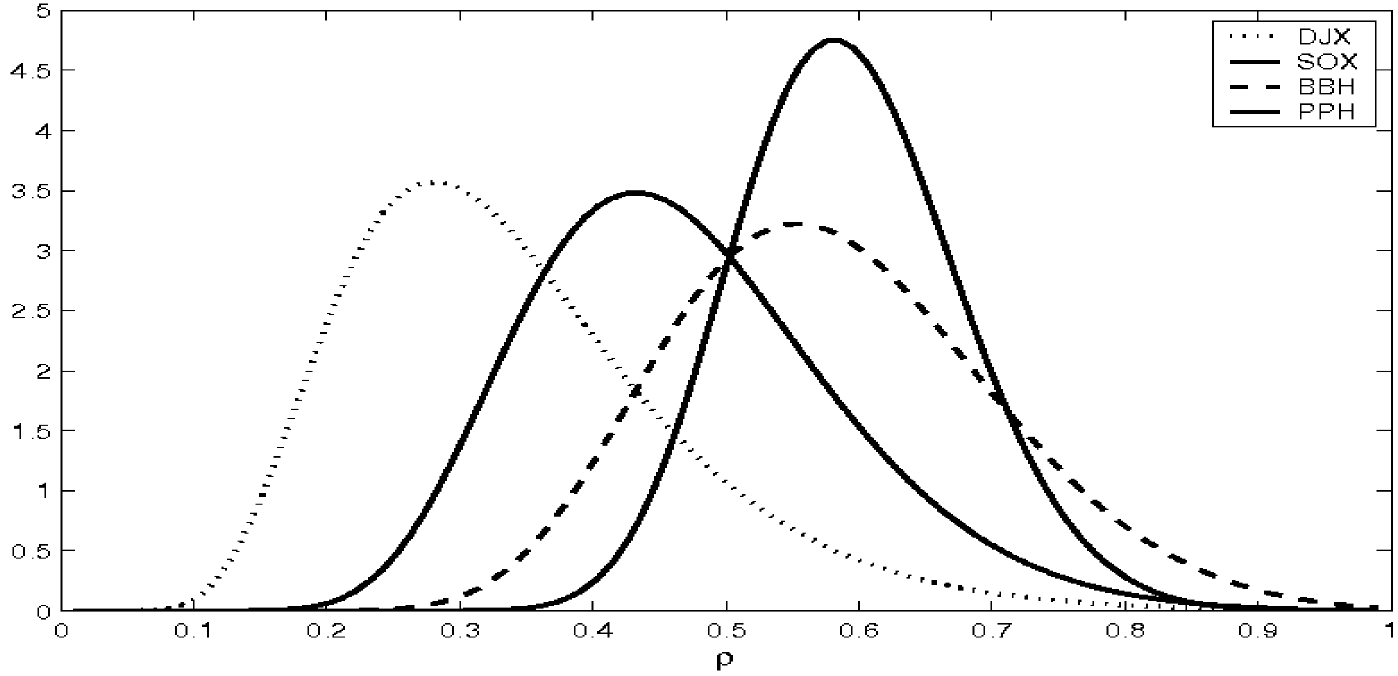
PPH Index correlation

SOX Semiconductor Index




SOX delta 50 implied correlation , 1 month
3 month
6 month
0
1
2
3
4
5
6
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

DJX, PPH, SOX, BBH



The shapes of implied correlation distribution for different sectors

The Vega-Neutral Dispersion Portfolio

How many single-name options for each name should I buy when I sell the index option?

Hedging Dispersions

- We know two methods that can be used to price dispersions (WMC, SDA)
- SDA also shows that the important strikes to hedge are such that

$$ln\frac{K_i}{F_i}=\beta_i\ ln\frac{K_I}{F_I}$$

- Question of interest: what are the ``correct'' proportions of index/single-name options that we need to hold in a dispersion trade to properly manage risk?

Vega Risk

As the market evolves:

$$\text{MTM} = -dC_I + \sum_{j=1}^{n} n_j dC_j$$

We focus on volatility & correlation risk. Recall that

$$\sigma_I^2 = \sum_{j=1}^{N} p_j^2 \sigma_j^2 + \sum_{i \neq j} p_i\, p_j \sigma_i\, \sigma_j \rho_{ij}$$

$$\sigma_I d\sigma_I = \sum_{i,j=1}^{n} p_i\, p_j \rho_{ij} \sigma_i\, d\sigma_j + \frac{1}{2} \sum_{i \neq j} p_i\, p_j \sigma_i\, \sigma_j d\rho_{ij}$$

$$MTM = -\frac{\partial C_I}{\partial \sigma_I} d\sigma_I + \sum_j n_j \frac{\partial C_j}{\partial \sigma_j} d\sigma_j$$

Assume implied correlation stays constant

$$= -\frac{\partial C_I}{\partial \sigma_I} \frac{1}{\sigma_I} \sum_{ij} p_i p_j \rho_{ij} \sigma_i d\sigma_j + \sum_j n_j \frac{\partial C_j}{\partial \sigma_j} d\sigma_j$$

$$= \sum_j \left(-\frac{\partial C_I}{\partial \sigma_I} \frac{p_j}{\sigma_I} \sum_i p_i \rho_{ij} \sigma_i + n_j \frac{\partial C_j}{\partial \sigma_j} \right) d\sigma_j$$

$$= \sum_j \left(-\frac{\partial C_I}{\partial \sigma_I} \frac{p_j}{\sigma_I} \rho_{Ij} \sigma_I + n_j \frac{\partial C_j}{\partial \sigma_j} \right) d\sigma_j$$

$$= \sum_j \left(-\frac{\partial C_I}{\partial \sigma_I} p_j \rho_{Ij} + n_j \frac{\partial C_j}{\partial \sigma_j} \right) d\sigma_j$$

$$\boxed{Vega(j) = p_j \rho_{Ij} Vega(I)}$$

Scaling relations for the Greeks in Black's Formula

$$C = F\left(\sigma^2 T\right)$$

$$\frac{\partial C}{\partial \sigma} = 2\sigma T F'\left(\sigma^2 T\right) \qquad \frac{\partial C}{\partial T} = \sigma^2 F'\left(\sigma^2 T\right)$$

$$\therefore \boxed{\sigma \frac{\partial C}{\partial \sigma} = 2T \frac{\partial C}{\partial T} = -2N_{days}\theta}$$

Links Time-Decay and Vega

Links Gamma and Vega

$$\frac{\partial C}{\partial T} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} \qquad \therefore \boxed{\Gamma = \frac{2}{\sigma^2} \frac{\partial C}{\partial T} = -\frac{1}{\sigma T} \frac{\partial C}{\partial \sigma}}$$

This solution is also Theta-neutral

$$Vega(j) = p_j \rho_{Ij} \, Vega(I)$$

Calculations
are ``up to sign''!

$$\text{Recall:} \quad \sigma \frac{\partial C}{\partial \sigma} = 2T \frac{\partial C}{\partial T} = 2N_{days}\theta$$

$$\therefore \qquad \sigma_j V_j = \sigma_j \rho_{Ij} p_j V_I = \frac{\sigma_j \rho_{Ij} p_j}{\sigma_I} \sigma_I V_I$$

$$\boxed{\begin{aligned} \sum_j \theta_j &= \left(\sum_j \frac{\sigma_j \rho_{Ij} p_j}{\sigma_I} \right) \theta_I \\ &= \left(\sum_j \frac{\sigma_j \rho_{Ij} p_j \sigma_I}{\sigma_I^2} \right) \theta_I = \frac{\sigma_I^2}{\sigma_I^2} \theta_I = \theta_I \end{aligned}}$$

What about implied correlation risk?

- Assume a parallel shift in correlation (or a shift in average correlation)
- The change in index volatility satisfies

$$\sigma_I d\sigma_I = \left(\frac{1}{2} \sum_{i \neq j} p_i p_j \sigma_i \sigma_j \right) d\bar{\rho}$$

$$\approx \frac{1}{2} \left(\sum_j p_j \sigma_j \right)^2 d\bar{\rho}$$

$$d\sigma_I \approx \frac{\sigma_I}{2} \left(\frac{\sum_j p_j \sigma_j}{\sigma_I} \right)^2 d\bar{\rho} \approx \frac{\sigma_I}{2\bar{\rho}} d\bar{\rho}$$

$$\frac{\partial C_I}{\partial \bar{\rho}} = \frac{\partial C_I}{\partial \sigma_I} \frac{d\sigma_I}{d\bar{\rho}} = \frac{\partial C_I}{\partial \sigma_I} \frac{\sigma_I}{2\bar{\rho}} \quad \therefore \boxed{\mathrm{Re}\,ga = Vega(I) \cdot \left(\frac{\sigma_I}{2\bar{\rho}} \right)}$$

Conclusion for Vega Exposure

- The neutralization of Vega across names gives rise to a portfolio such that

$$\boxed{\begin{aligned} &Vega(j) = -p_j \rho_{jI} Vega(I) \\ &\sum_{j=1}^{n} \theta_j + \theta_I = 0 \end{aligned}}$$

- The <u>correlation exposure </u>(change in index implied vol without a change in the SN implied vols) still remains. Namely: this risk is proportional to

$$\frac{\sigma_I}{2\bar{\rho}} Vega(I)$$

- The <u>Gamma exposure </u>can also be computed using the above ratios.

Gamma Exposure

- Rewrite the hedge proportions in terms of time decay:

$$\theta_j = -p_j \beta_j \theta_I \qquad \text{with} \quad \beta_j = \frac{\rho_{jI}\sigma_j}{\sigma_I}$$

$$\mathrm{i}^{\mathrm{th}} \text{ stock Gamma P/L} \approx \theta_i \cdot \left(n_i^2 - 1\right) \qquad n_i = \frac{R_i}{\sigma_i \sqrt{\Delta t}}$$

$$\text{Dispersion Gamma P/L} \approx \sum_{i=1}^{n} \left(\theta_i + \frac{p_i^2 \sigma_i^2}{\sigma_I^2} \theta_I \right) \left(n_i^2 - 1\right) + \theta_I \sum_{i \neq j} \frac{p_i\, p_j \sigma_i\, \sigma_j}{\sigma_I^2} \left(n_i n_j - \rho_{ij}\right)$$

$$= -\theta_I \sum_{i=1}^{n} \left(p_i \beta_i - \frac{p_i^2 \sigma_i^2}{\sigma_I^2} \right) \left(n_i^2 - 1\right) + \theta_I \sum_{i \neq j} \frac{p_i\, p_j \sigma_i\, \sigma_j}{\sigma_I^2} \left(n_i n_j - \rho_{ij}\right)$$

$$\approx -\theta_I \sum_{i=1}^{n} p_i \beta_i \left(n_i^2 - 1\right) + \theta_I \sum_{i \neq j} \frac{p_i\, p_j \sigma_i\, \sigma_j}{\sigma_I^2} \left(n_i n_j - \rho_{ij}\right)$$

Gamma

Cross-gamma

Total Exposure for a Vega-Neutral Dispersion

$$n_i = \frac{R_i}{\sigma_i \sqrt{\Delta t}}$$

$$\text{Convention : short index option} \leftrightarrow \quad \theta_{\mathrm{I}} < 0$$

$$PNL = -\theta_I \sum_{i=1}^{n} p_i \beta_i \left(n_i^2 - 1\right) + \theta_I \sum_{i \neq j} \frac{p_i\, p_j \sigma_i\, \sigma_j}{\sigma_I^2} \left(n_i n_j - \rho_{ij}\right) + N_{days} \cdot \theta_I \frac{d\overline{\rho}}{\overline{\rho}}$$

$$= -\theta_I \left(\sum_{i=1}^{n} p_i \beta_i \left(n_i^2 - 1\right) - \sum_{i \neq j} \frac{p_i\, p_j \sigma_i\, \sigma_j}{\sigma_I^2} \left(n_i n_j - \rho_{ij}\right) - N_{days} \frac{d\overline{\rho}}{\overline{\rho}} \right)$$

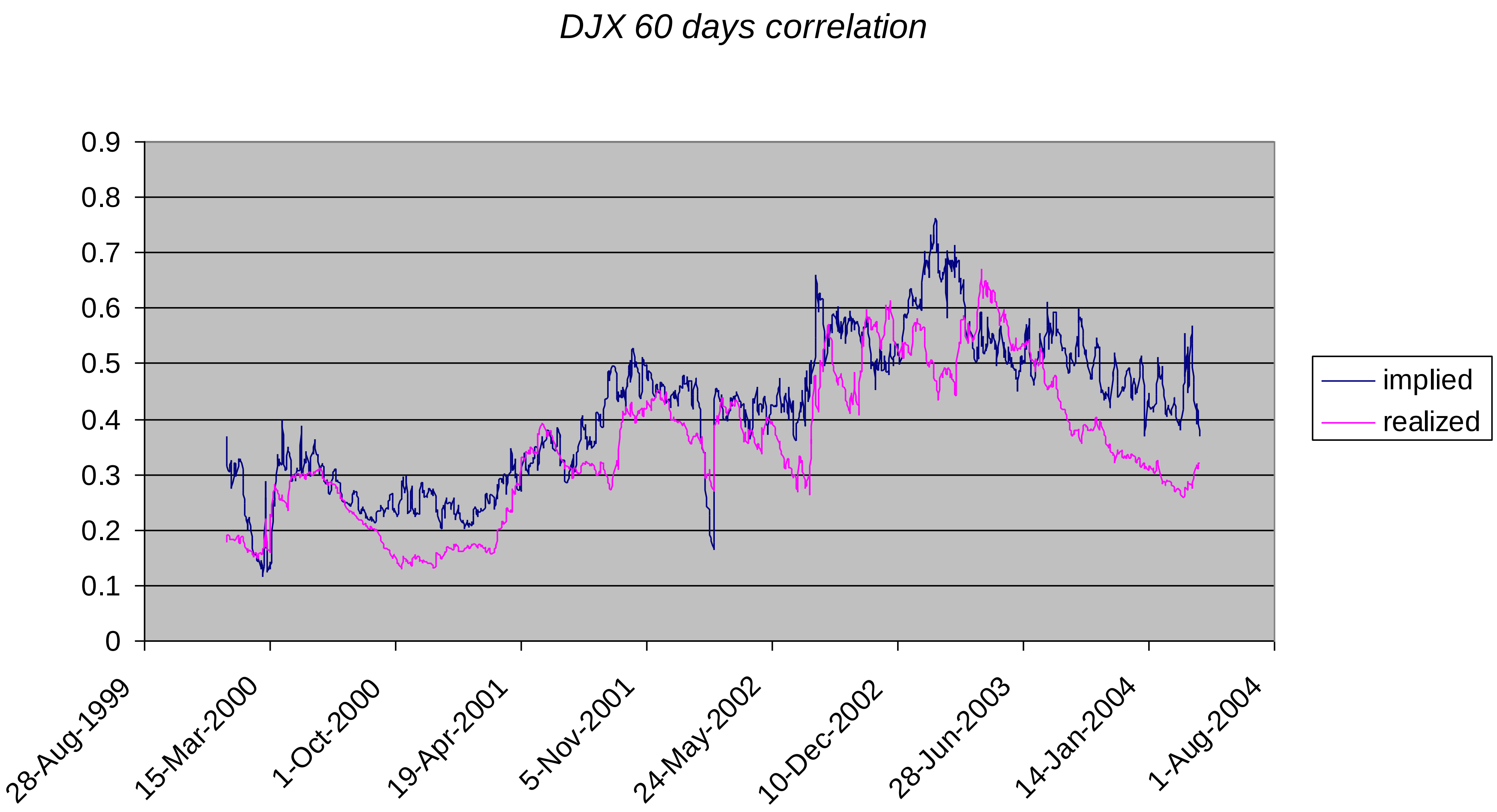
Time-decay of Index option

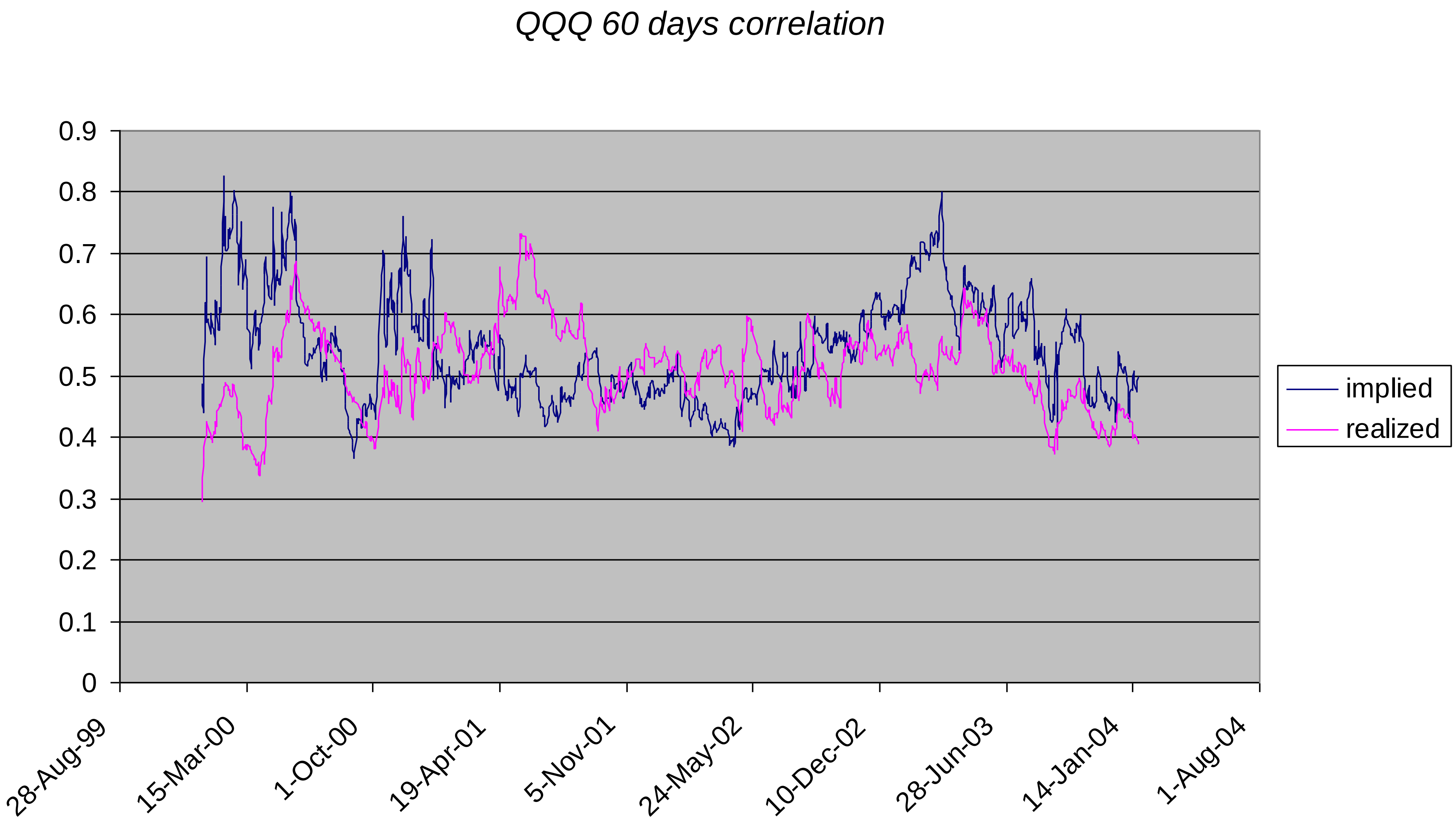
Gamma

Cross-gamma

Rega

Dispersion Trades:
Realized Correlation and Performance Attribution


DJX 60 days correlation
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
28-Aug-1999
15-Mar-2000
1-Oct-2000
19-Apr-2001
5-Nov-2001
24-May-2002
10-Dec-2002
28-Jun-2003
14-Jan-2004
1-Aug-2004
implied
realized



QQQ 60 days correlation
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
28-Aug-99
15-Mar-00
1-Oct-00
19-Apr-01
5-Nov-01
24-May-02
10-Dec-02
28-Jun-03
14-Jan-04
1-Aug-04
implied
realized

Correlation Analysis for Large Groups of Stocks

Separate the **systematic** components of stock returns from the company-specific, or **idiosyncratic** components

$$R_i = \beta_i R_{Mkt} + \varepsilon_i$$

Project returns on single Market Factor (CAPM)

$$R_i = \sum_{j=1}^{m} \beta_{ij} F_j + \varepsilon_i$$

Project returns on Multiple (sector, size) Factors (APT)

In principle, market-neutral portfolios should have no exposure to market factors ("defactoring")

The Correlation Matrix

$R_{it} =$ daily stock returns in panel form

$i = 1, \ldots, N, \; t = 1, \ldots, T$

$$\overline{R_i} = \frac{1}{T} \sum_{i=1}^{T} R_{it}, \qquad \overline{\sigma_i}^2 = \frac{1}{T-1} \sum_i \left(R_{it} - \overline{R_i}\right)^2$$

$$\overline{\rho}_{ij} = \frac{1}{T-1} \sum_i \frac{\left(R_{it} - \overline{R_i}\right)\left(R_{jt} - \overline{R_j}\right)}{\overline{\sigma_i}\,\overline{\sigma_j}}$$

Principal Component Analysis

$$\lambda_1 > \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_N$$

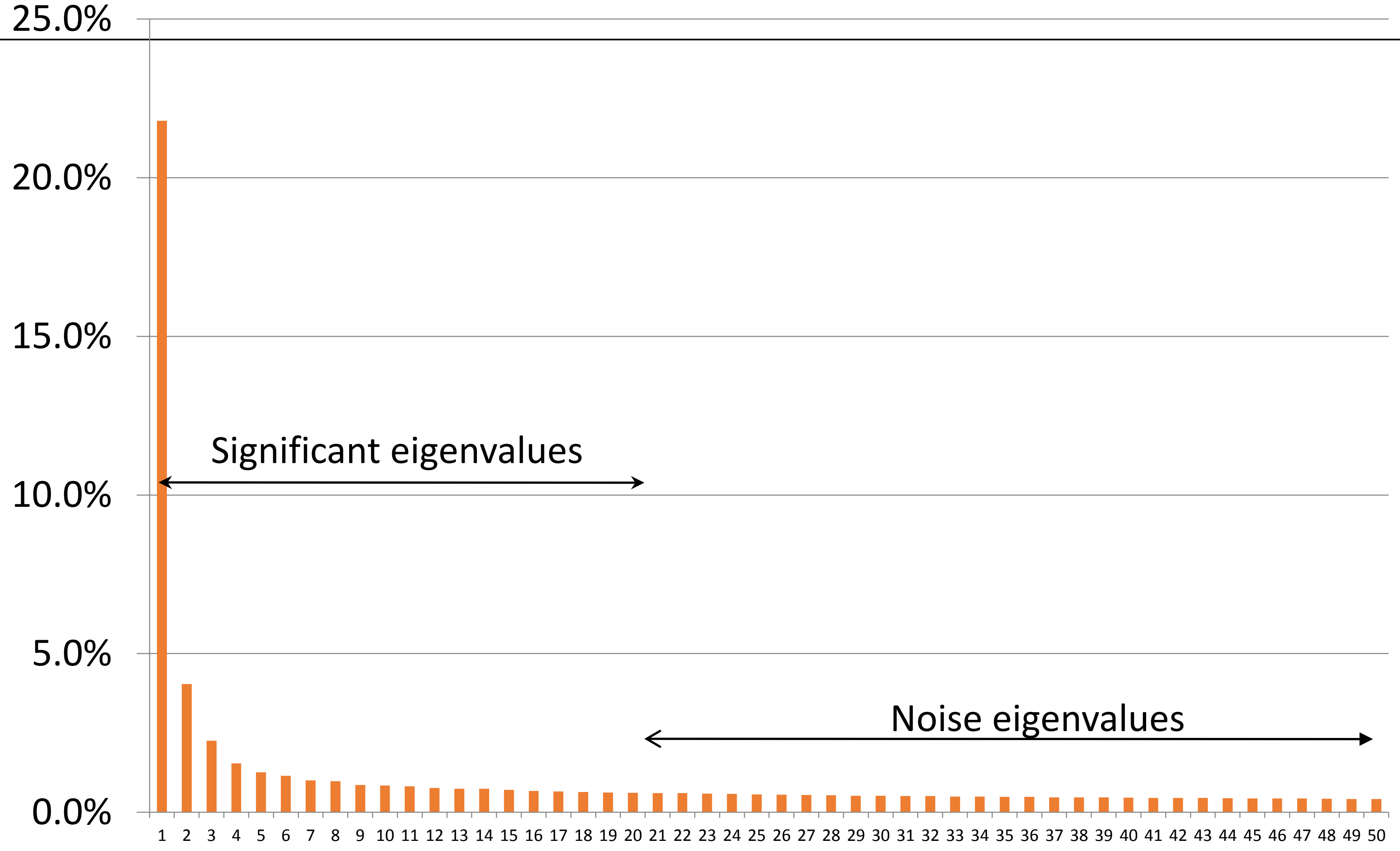
Eigenvalues are all non-negative

$$V^{(j)} = \left(v_1^{(j)}, v_2^{(j)}, \ldots, v_N^{(j)}\right)$$

Orthogonal eigenvectors

Stock market fluctuations can be characterized as moves along the eigenvector directions. We seek to extract mathematical factors from the PCA analysis.

PCA: Explained variance from the viewpoint of eigenvalues



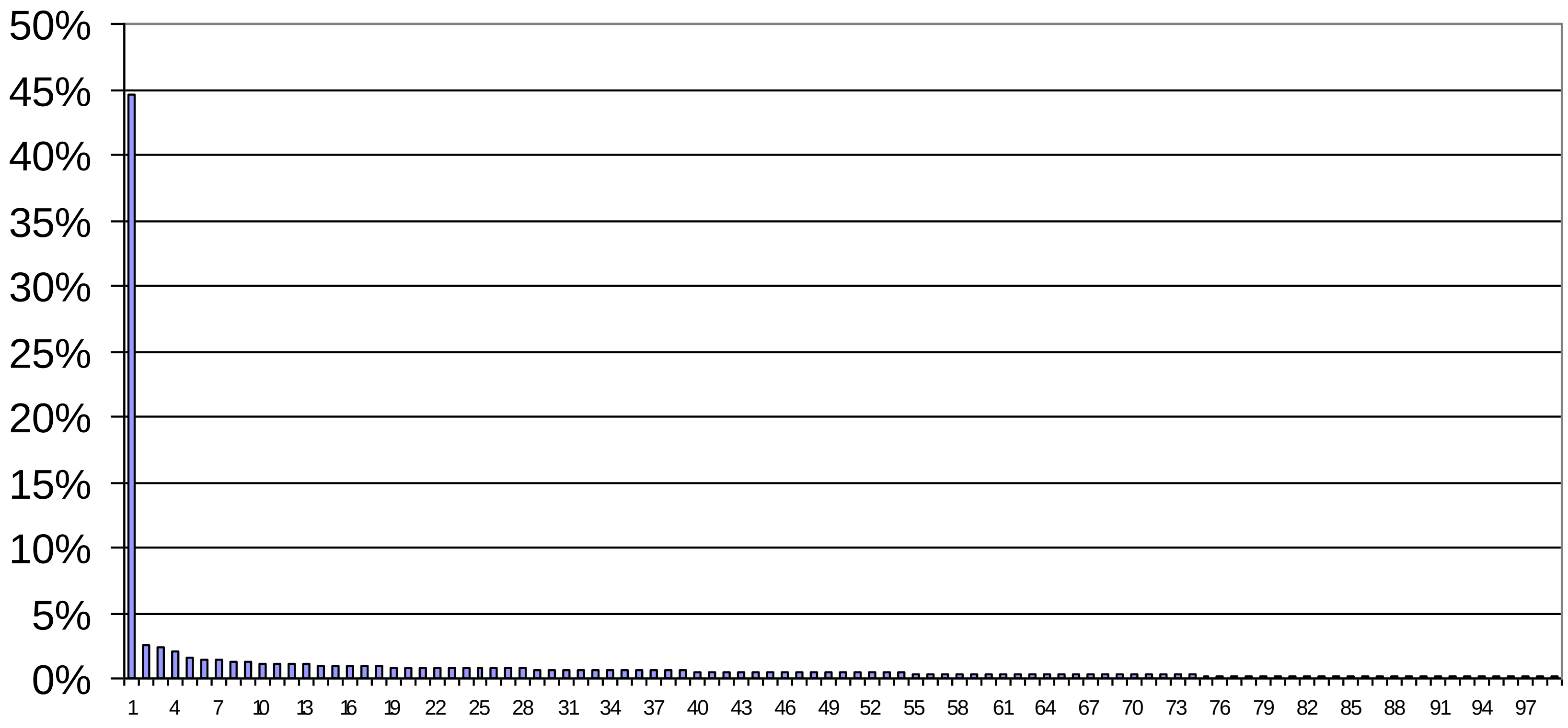
Big universe: Jan 2007-Dec 2007

Nasdaq-100
Components of NDX/QQQQ

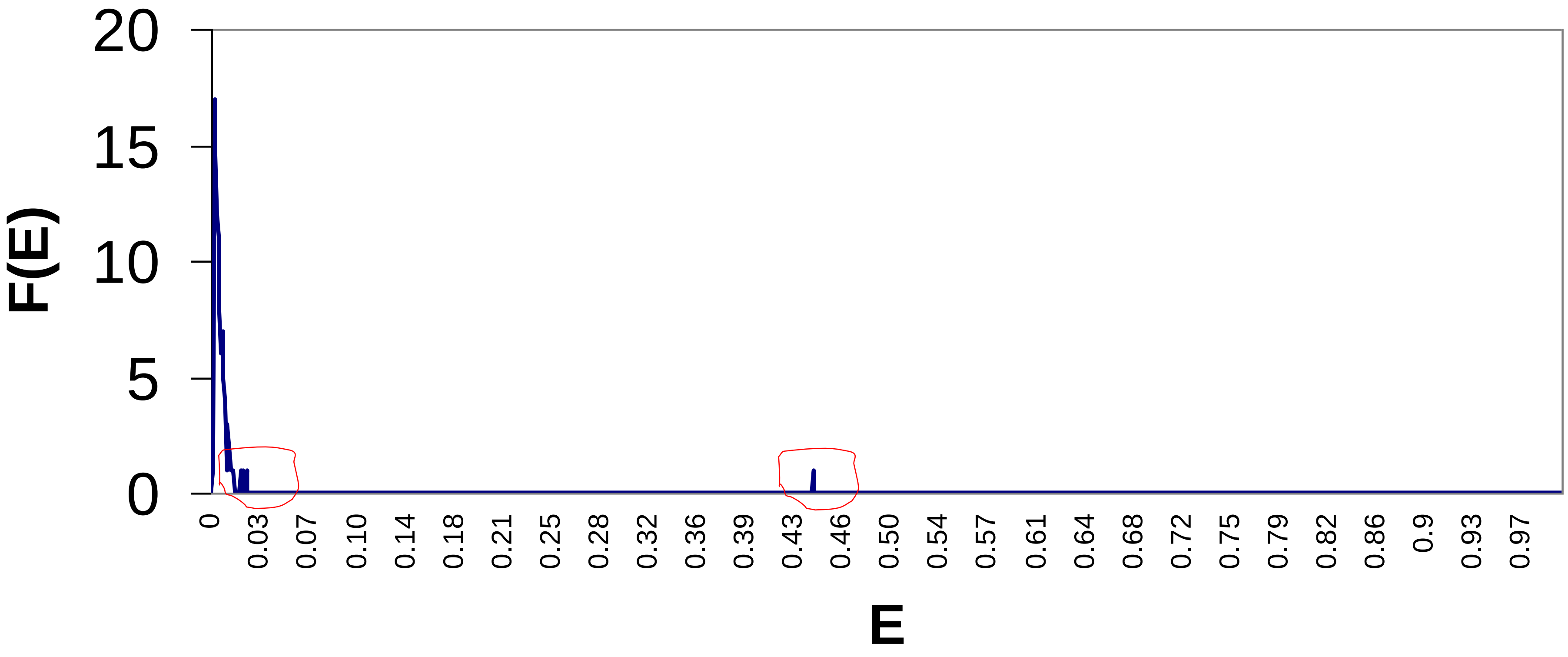
Data: Jan 30, 2007 to Jan 23, 2009

502 dates, 501 periods

99 Stocks (1 removed) MNST (Monster.com), now listed in NYSE

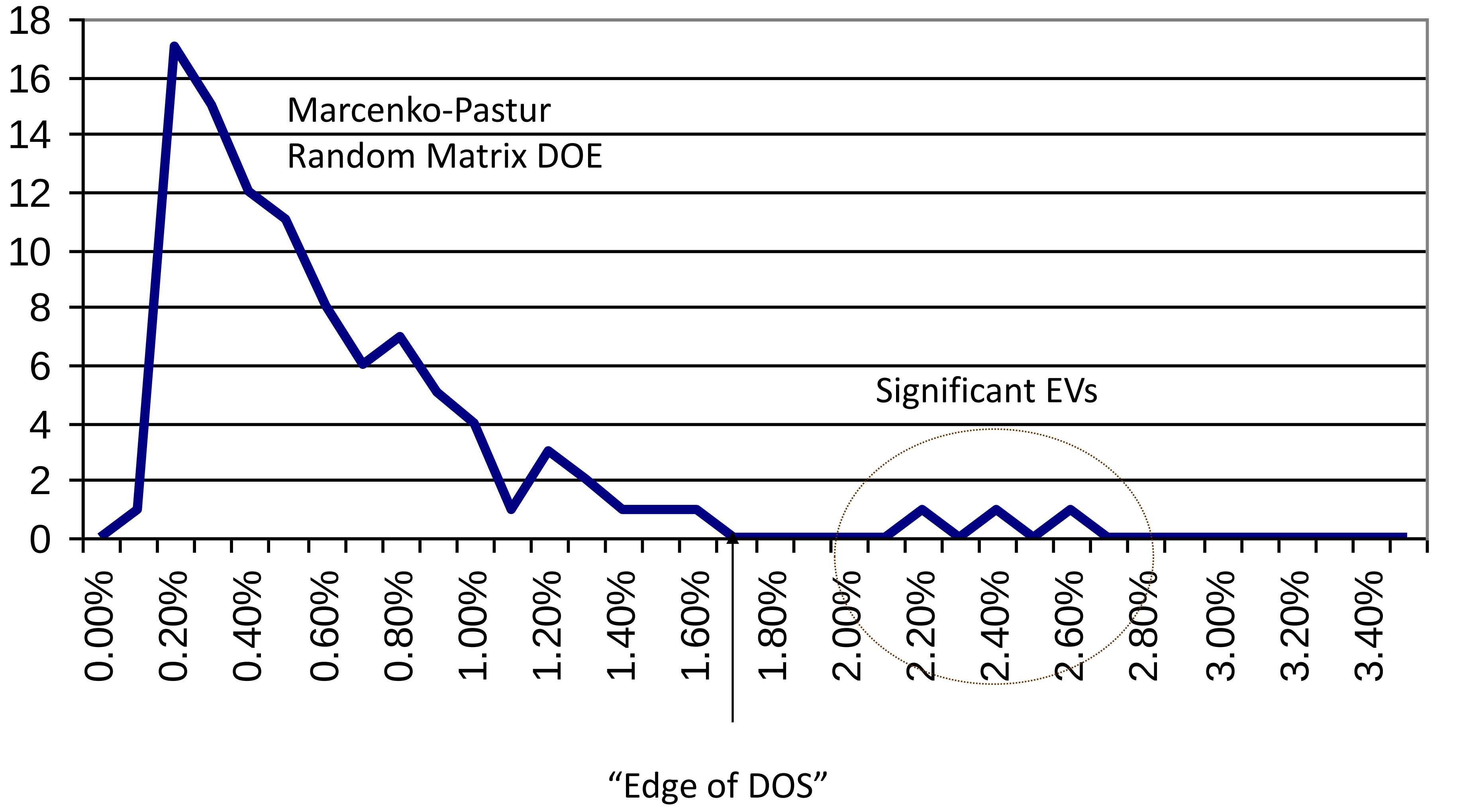


Density of States (from previous data)



One large mass at 0.44,
Some masses near 0.025
Nearly continuous density for lower levels

Zoom of the DOS for low eigenvalues



Factors & Eigenportfolios

For each eigenvector, build a portfolio which is weighted proportionally
to the coefficient of each stock and inversely proportionally to its volatility

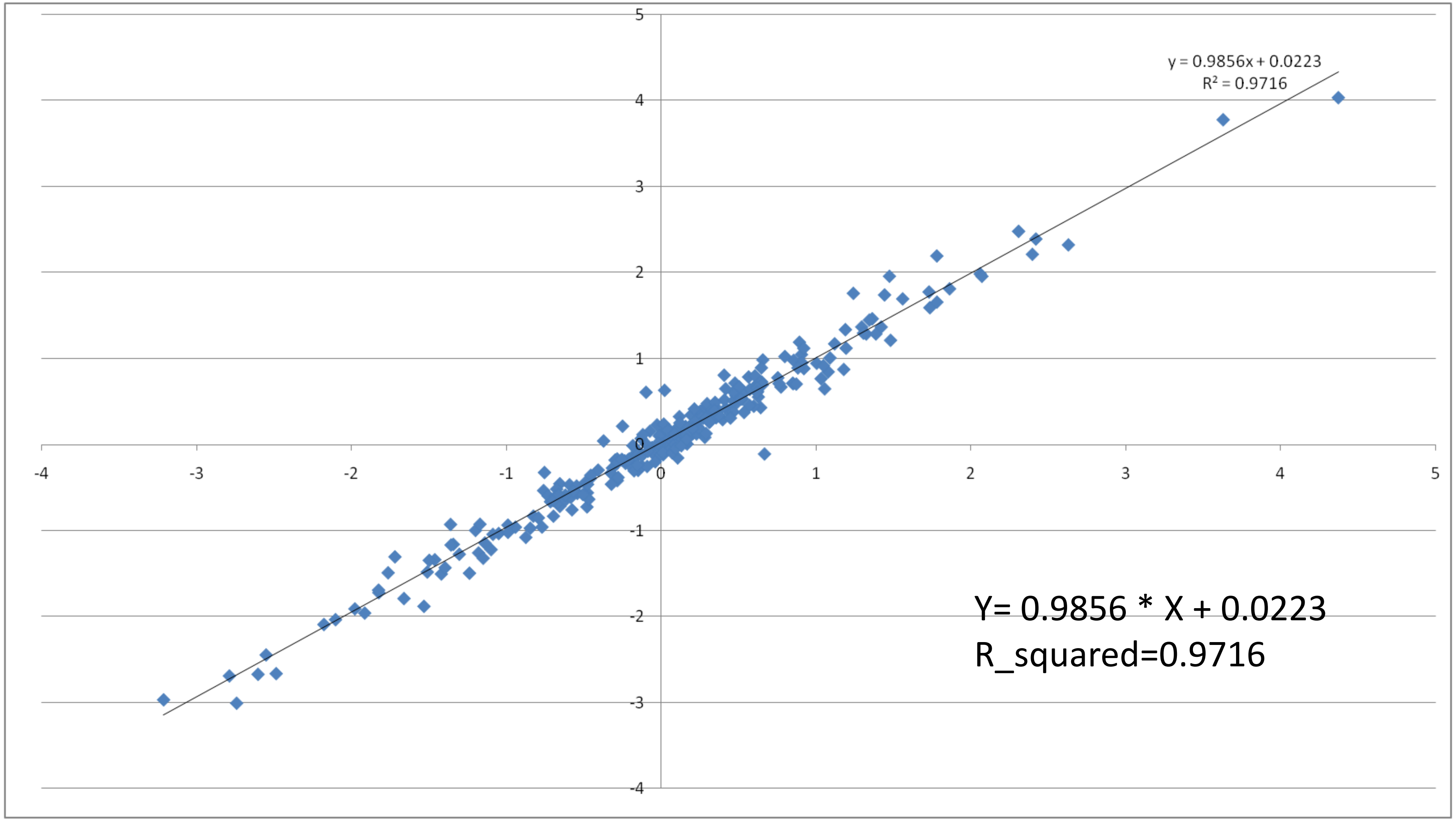
$$Q_i^{(j)} = \frac{v_i^{(j)}}{\overline{\sigma_i}}$$

Portfolio weight of j-th eigenportfolio

$$F_j = \sum_{i=1}^{N} Q_i^{(j)} R_i = \sum_{i=1}^{N} \left(\frac{v_i^{(j)}}{\overline{\sigma_i}} \right) R_i$$

J-th factor is the return of the
j-th eigenportfolio

First eigenportfolio returns compared with S&P 500 returns (1/5/2009 to 1/29/2010)



How many eigenportfolios are significant?

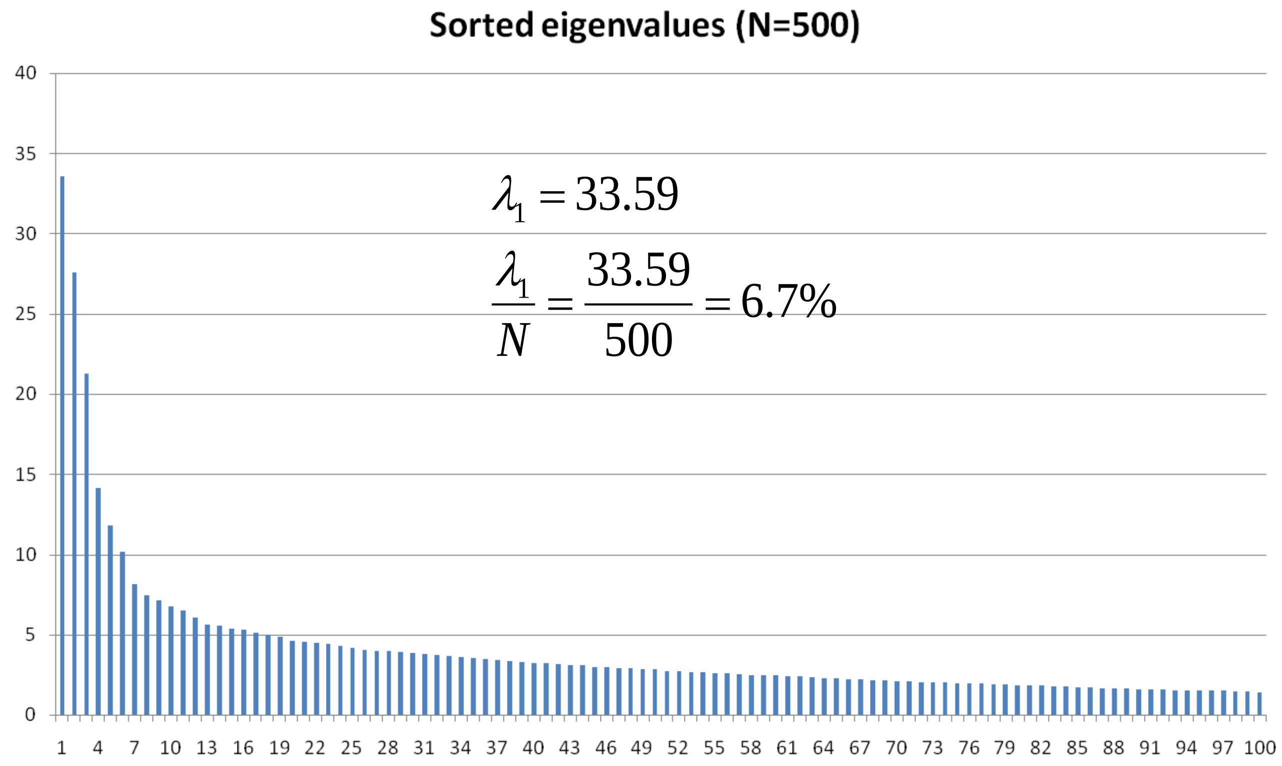
- Perform PCA on the empirical correlation matrix with 1 year moving window

- Consider the correlation matrix of the residuals after removing 1, 2 ,3… eigenportfolios

- Compare the DOS of the correlation of the residuals with the spectrum of the correlation matrix of pure noise (Marcenko-Pastur)

-The number of significant factors corresponds to the **first m for which the matrix of residuals is close to MP** (e.g. in the sense of hypothesis testing)

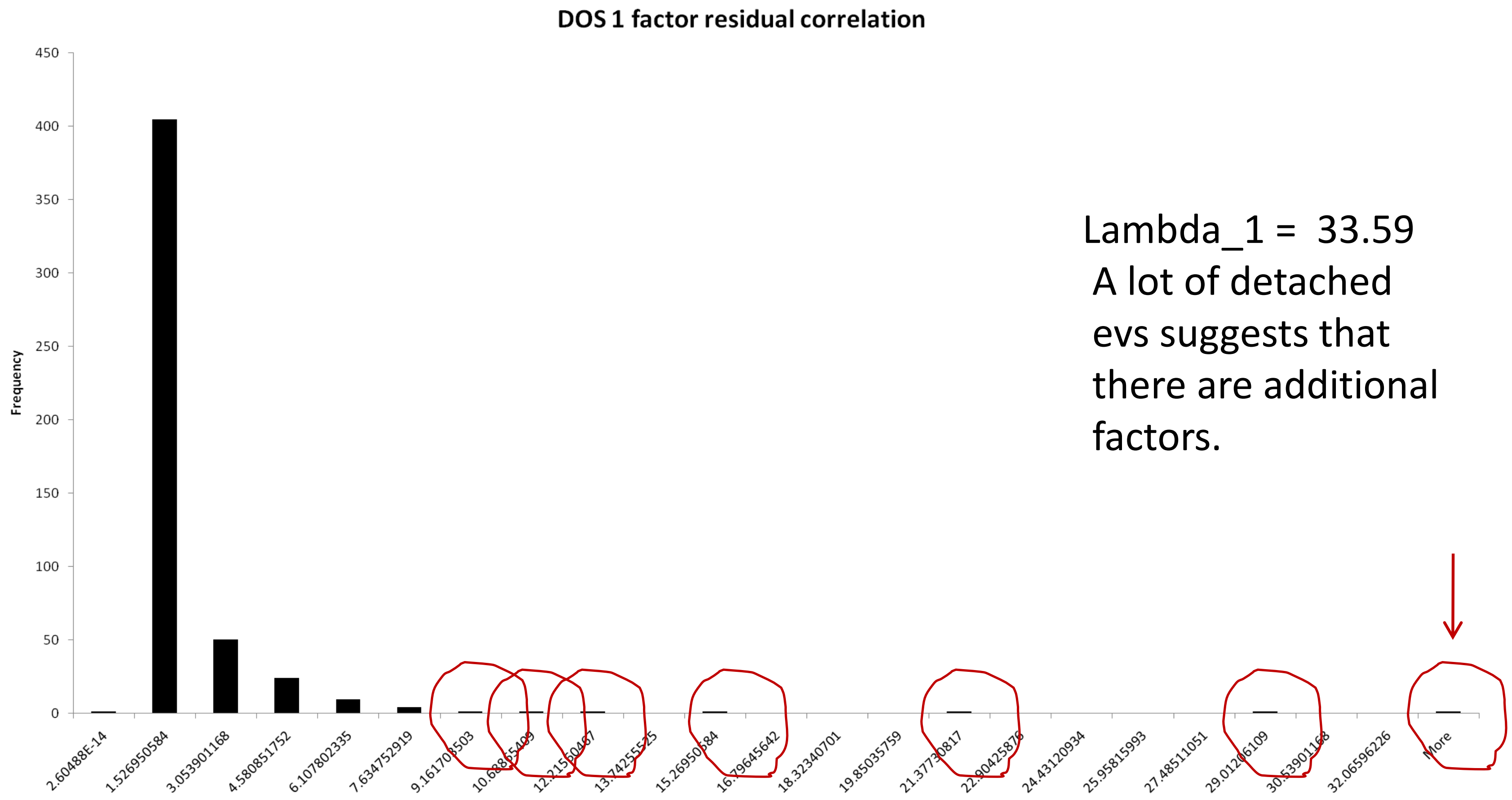
Eigenvalues of the correlation matrix of residuals (m=1)



Recall that lambda_1 for the original correlation matrix was ~ 220, so the residuals matrix has ``smaller'' correlations.

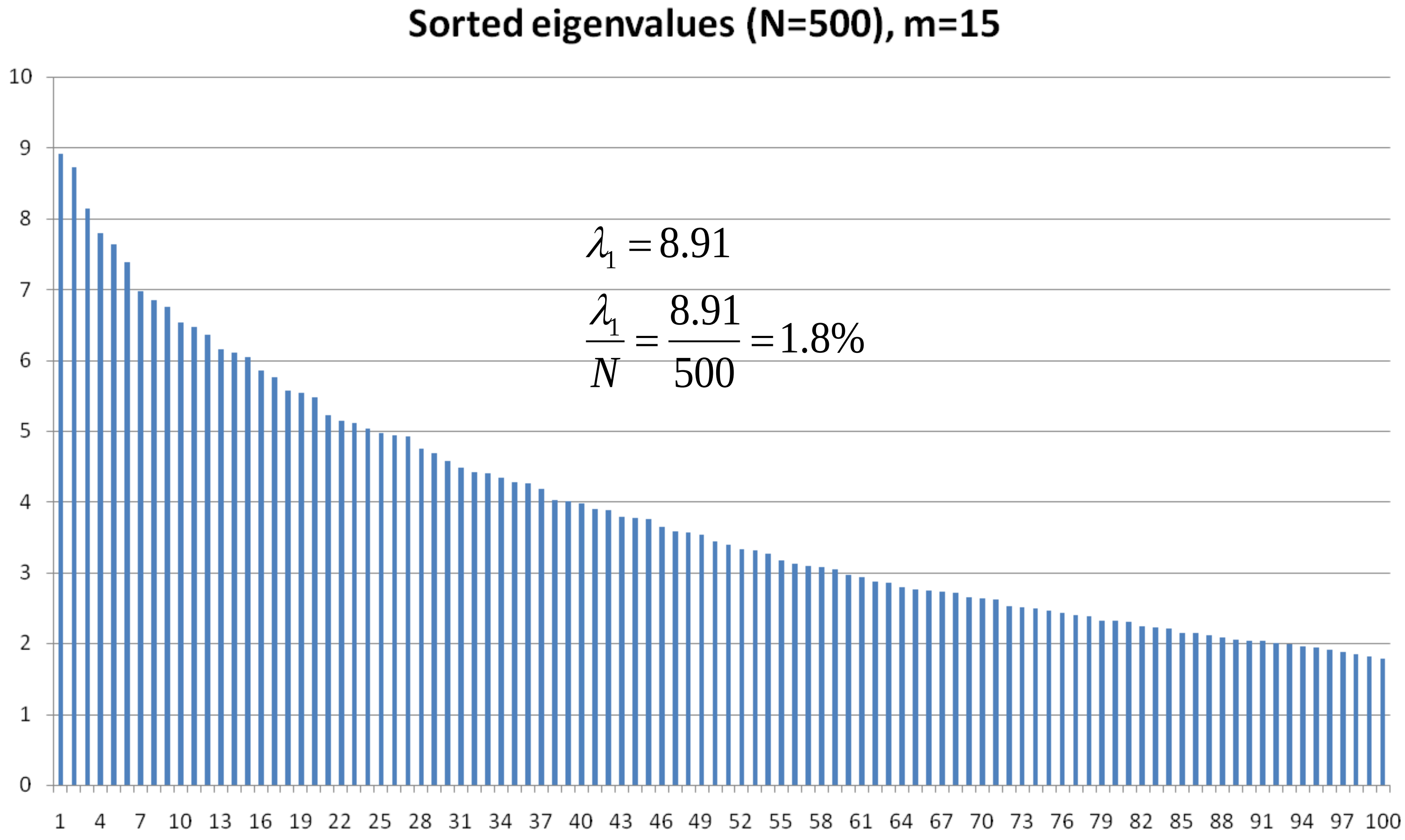
Ratio lambda/N is a proxy for the average correlation.

Density of States, or Histogram, of Eigenvalues for m=1 (correlation of returns with first factor removed)

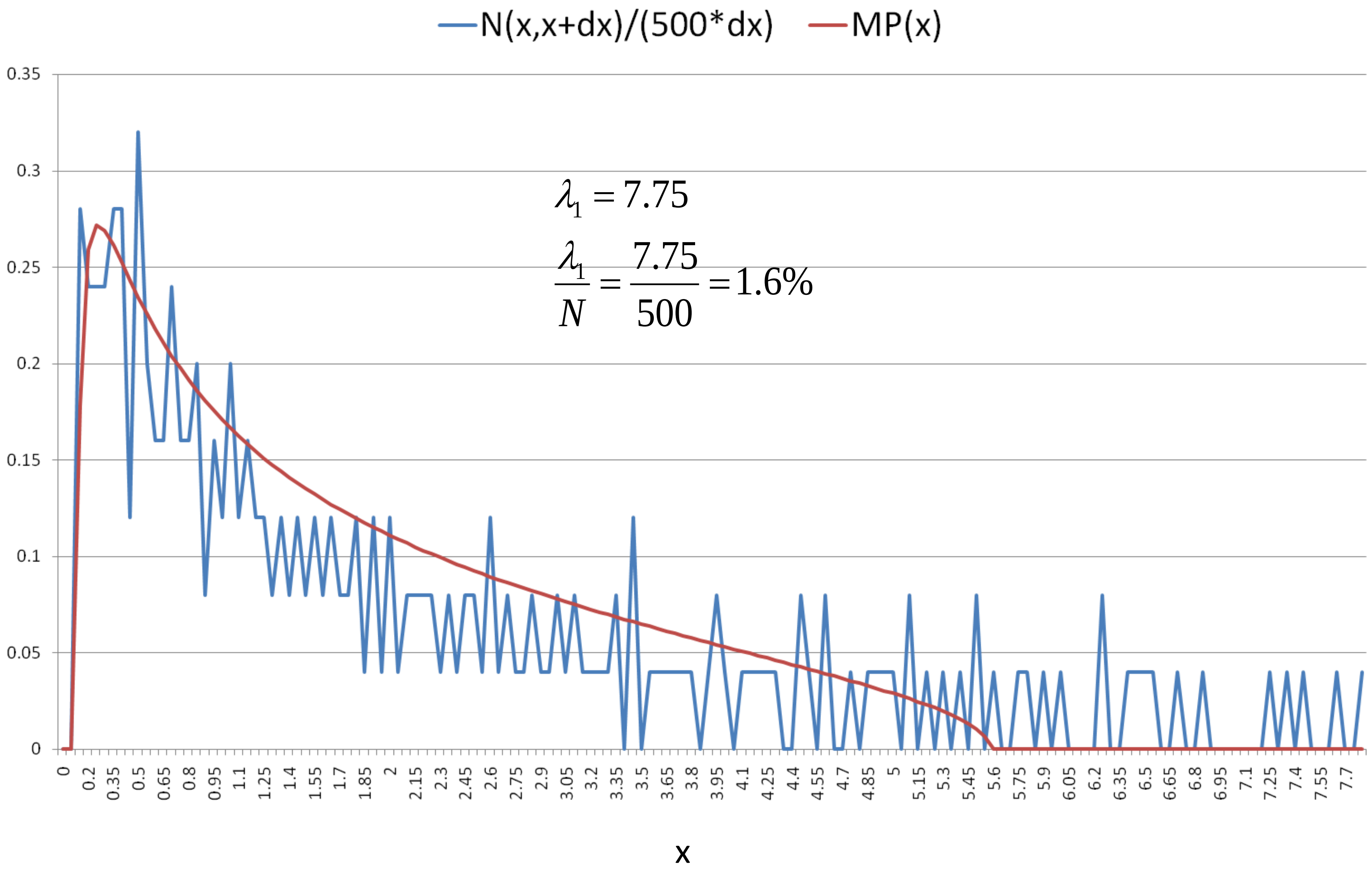


Lambda_1 = 33.59
A lot of detached evs suggests that there are additional factors.

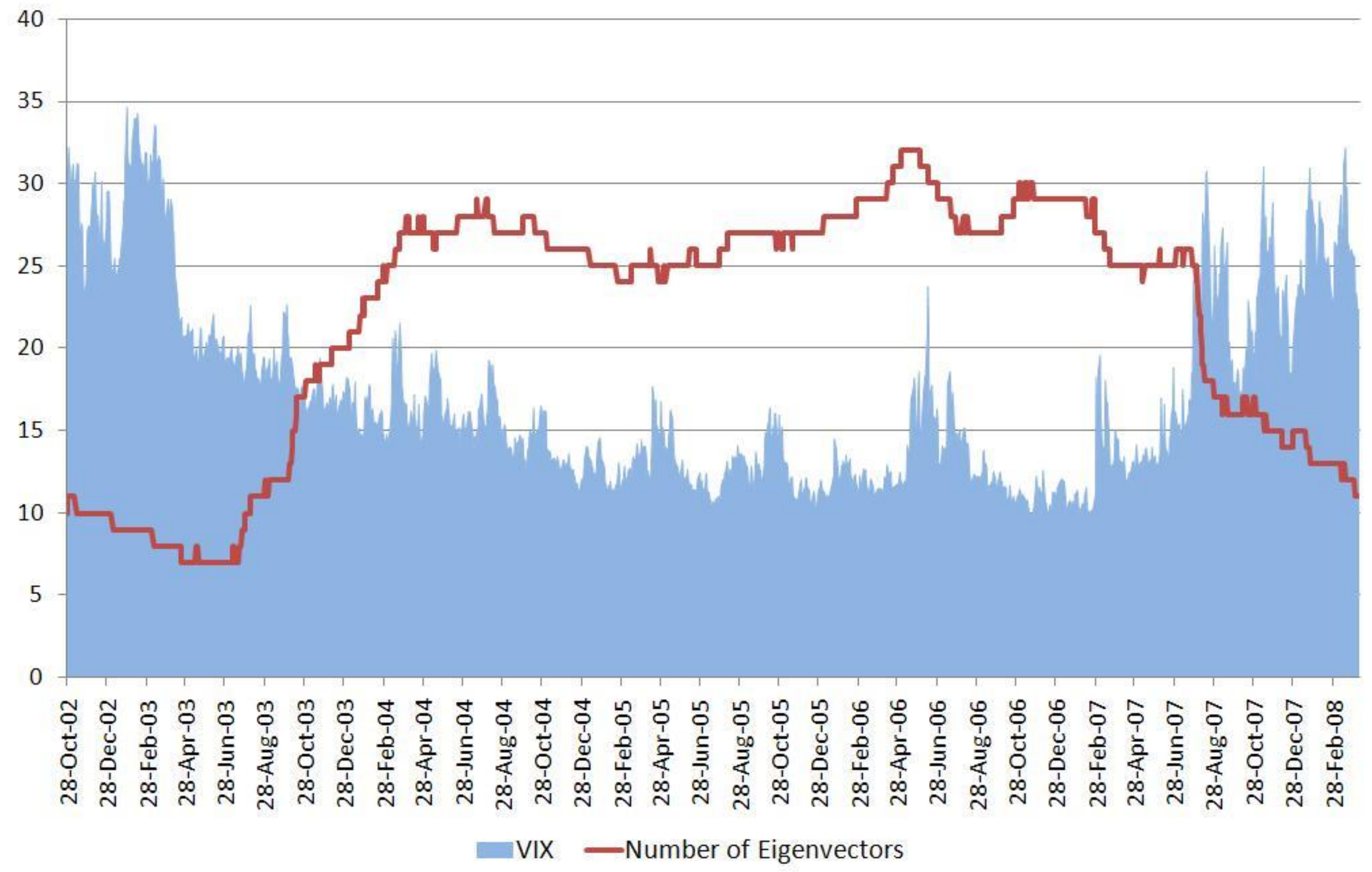
Top 100 eigenvalues of the correlation matrix of residuals (m=15)



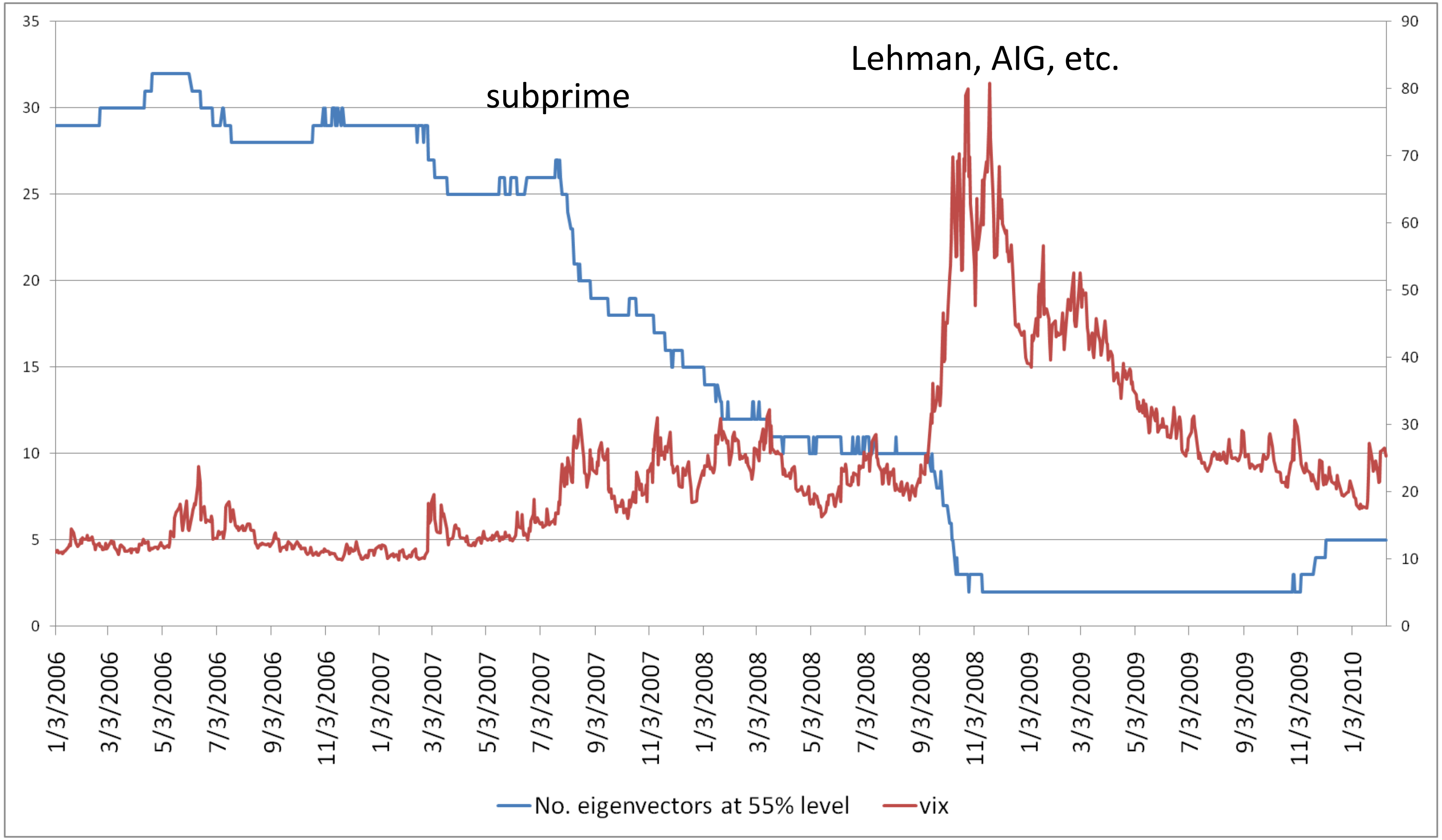
Marcenko Pastur compared to data with m=30



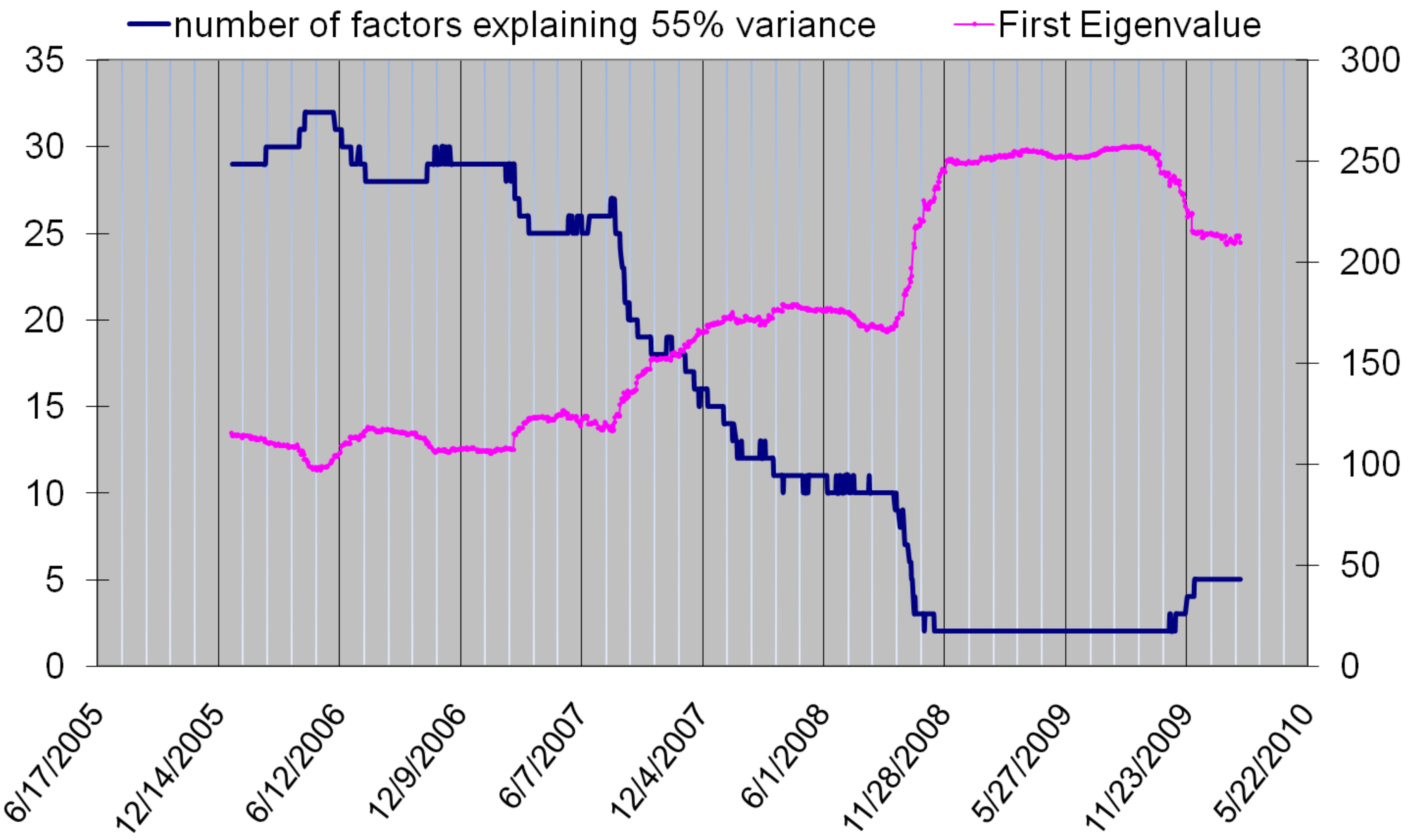
Number of factors explaining 55% variance vs. VIX
(2002-2008)



Number of significant EVs versus VIX (1/2006-2/2010)



Number of significant factors vs. first eigenvalue of correlation matrix



3. Focus on Leveraged ETFs & Options

Summary

- Leveraged ETFs, 3X, -3X,…
- Empirical facts about LETFs
- Path-dependence explained
- Empirical validation of the theoretical formula on 54 LETFs
- Rebalancing: replicating leveraged returns over long-term horizons
- Options

Leveraged ETFs

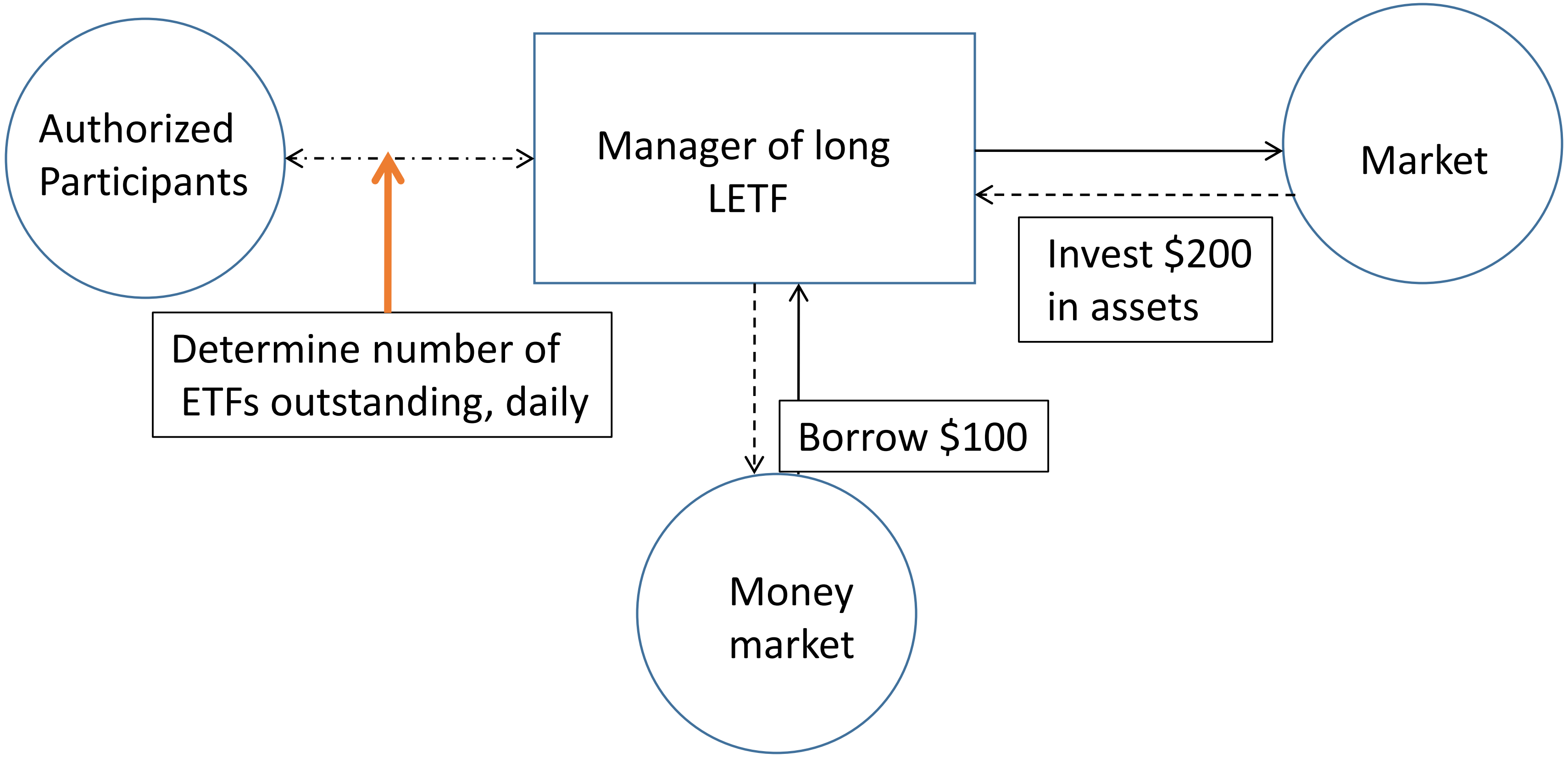
Products offer a multiple of the *daily* return of a reference index

Examples:

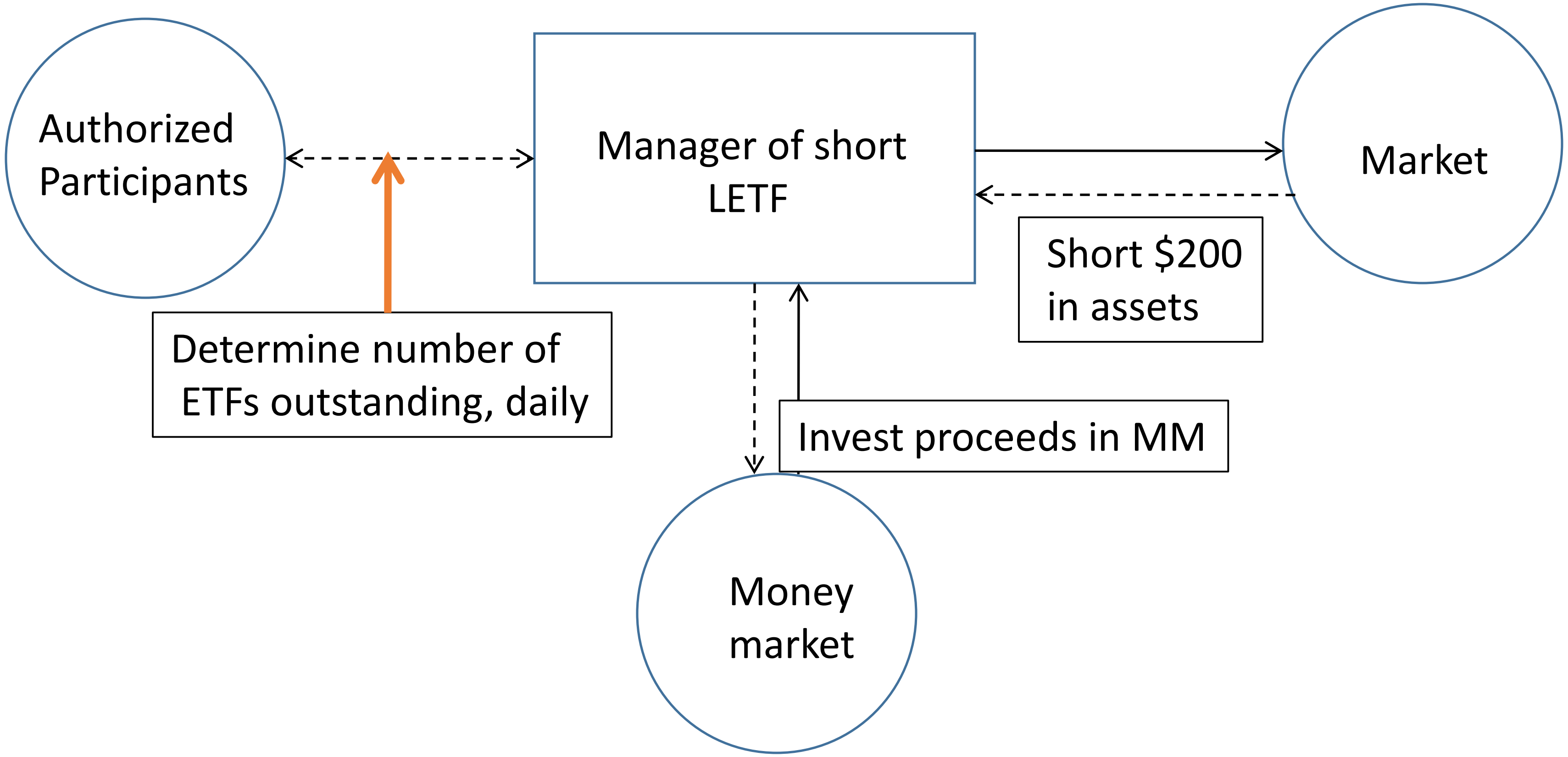
Proshares Ultra Financials ETF (UYG)
Offers a daily exposure to 2 times the Dow Jones Financial Index
(long 200% of underlying undex)

Proshares UltraShort Financials ETF (SKF)
Offers a daily exposure to -2 times the Dow Jones Financial Index
(short 200% of underlying index)

“Bullish” leveraged ETF

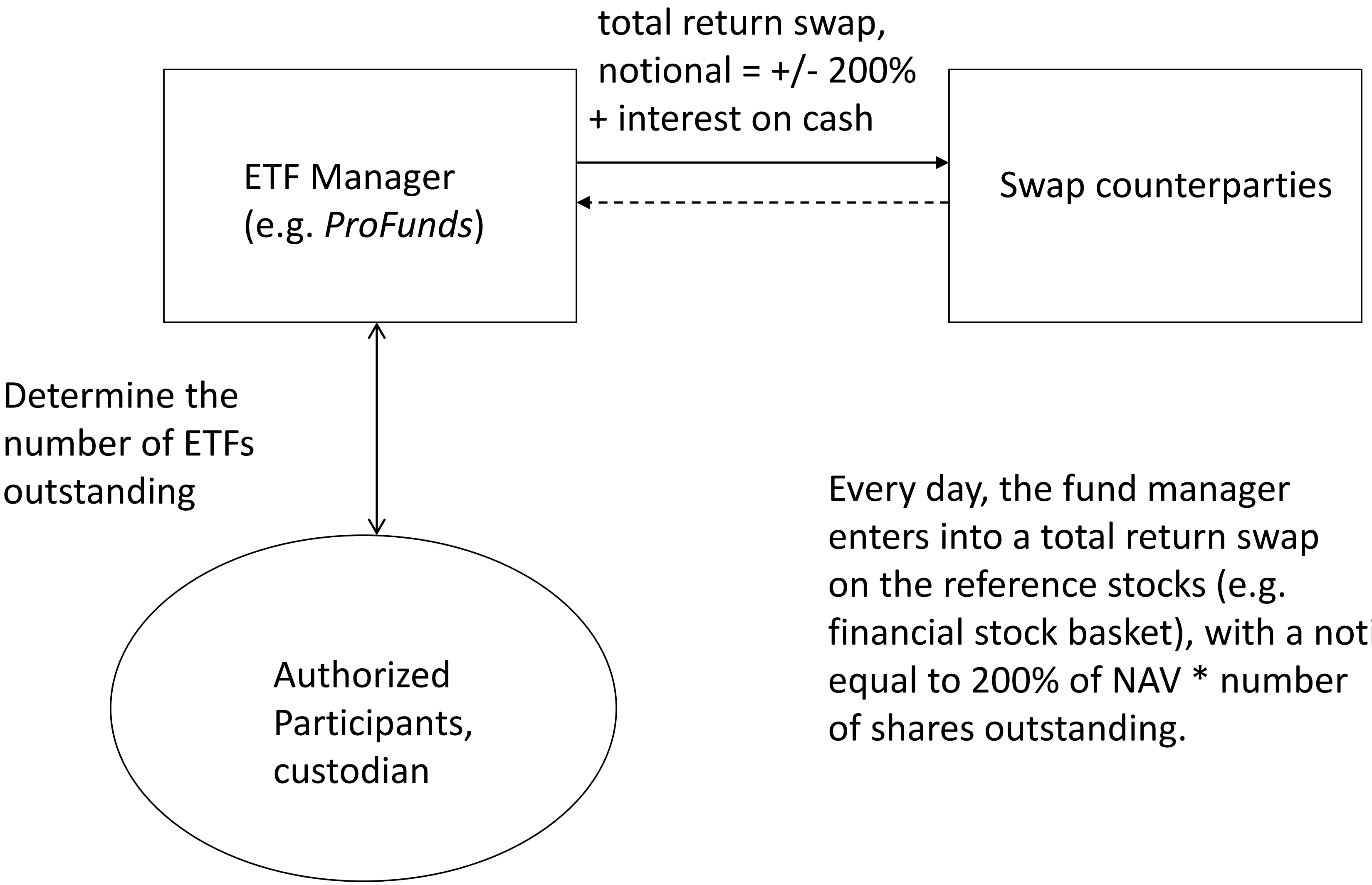


"Bearish" leveraged ETF



In reality this can be more complicated (use of swap counterparties).

Hedging with a Total Return Swap



Every day, the fund manager enters into a total return swap on the reference stocks (e.g. financial stock basket), with a notional equal to 200% of NAV * number of shares outstanding.

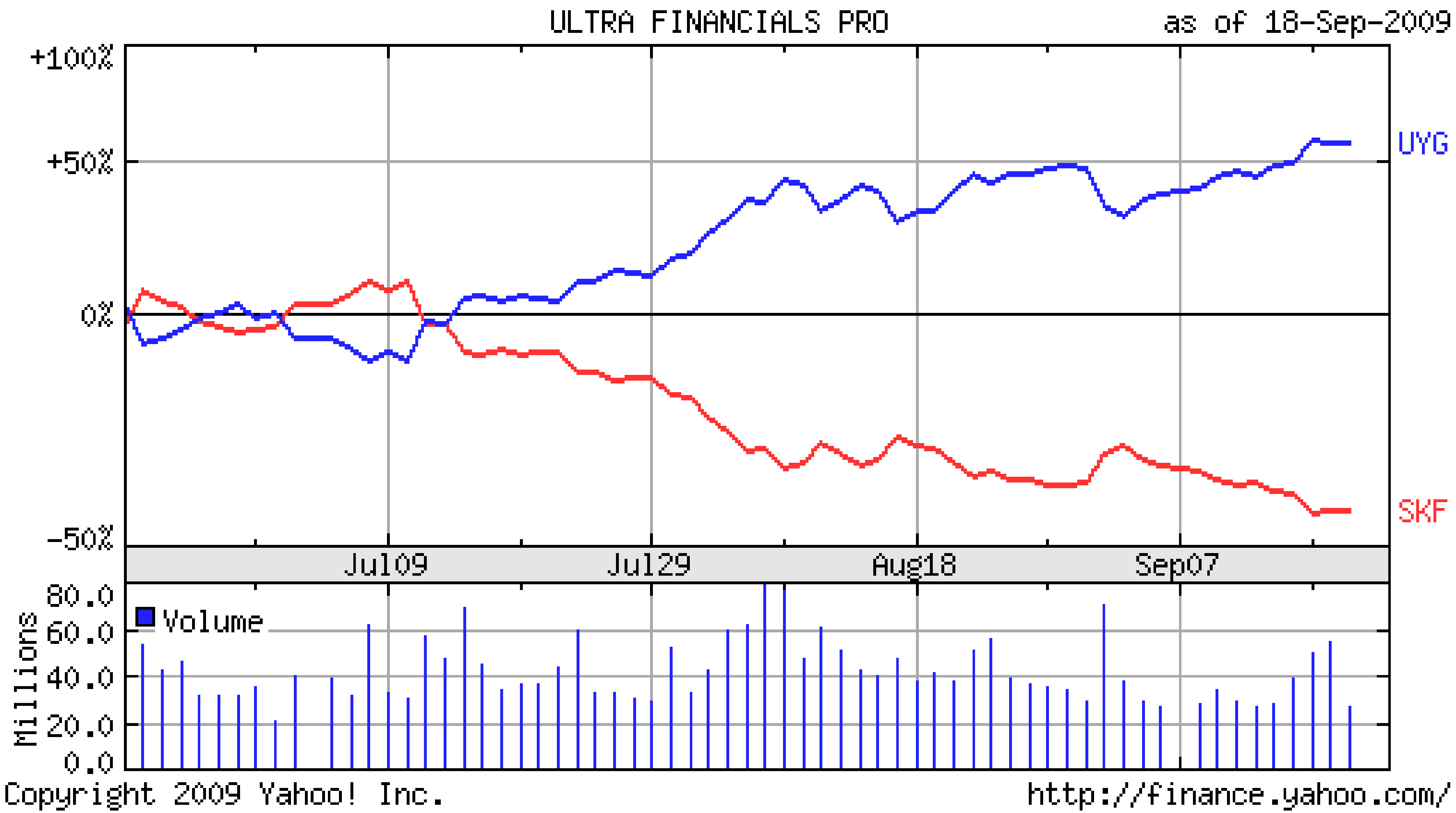
LETFs & buy-and hold investors: *Caveat Emptor*

- Issues have been raised in the marketplace pertaining to the suitability of leveraged ETFs for long-term investors seeking to replicate a multiple of an index performance

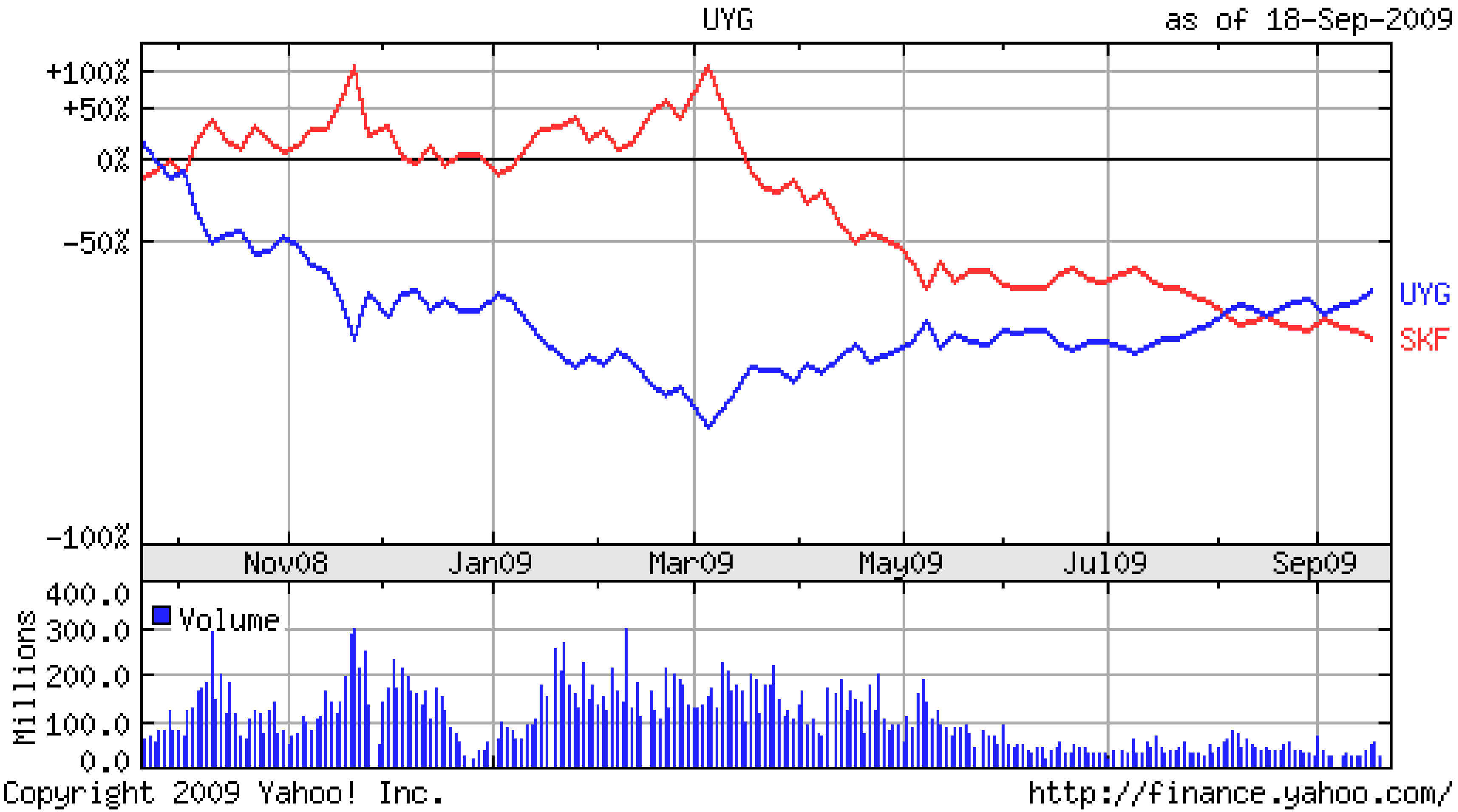
- ``UBS AG U.S. brokerage business stopped selling ETFs that use leverage because such products do not conform to its emphasis on long-term investing'' *Bloomberg News, July 27, 2009*

- `` Due to the effects of compounding, their performance over longer periods of time can differ significantly from their stated daily objective. Therefore, inverse and leveraged ETFs that are reset daily typically are unsuitable for retail investors who plan to hold them longer than one trading session, particularly in volatile markets'' *FINRA Regulatory Notice, June 31, 2009*

SKF/UYG Past 3 months

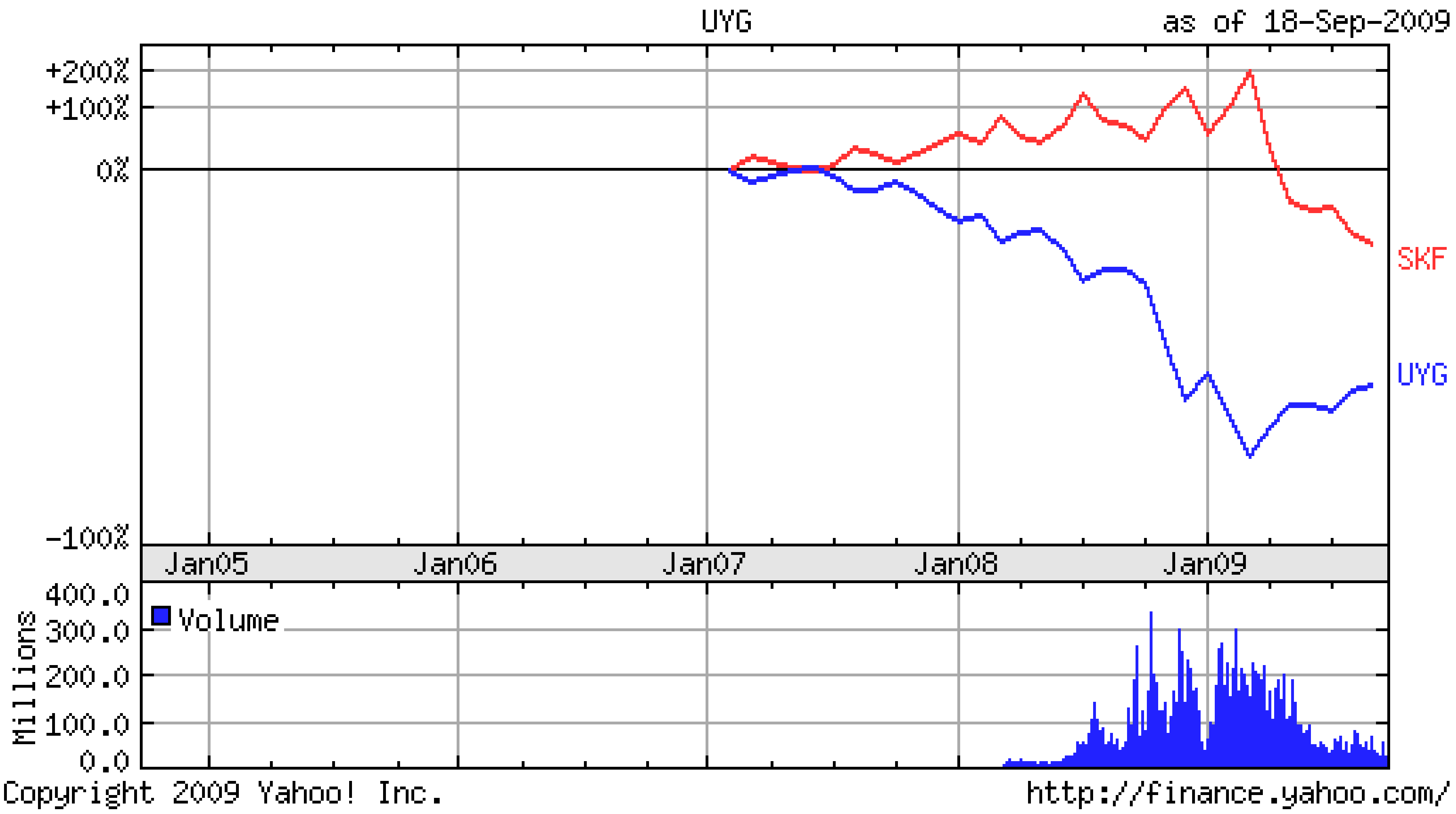


Past year

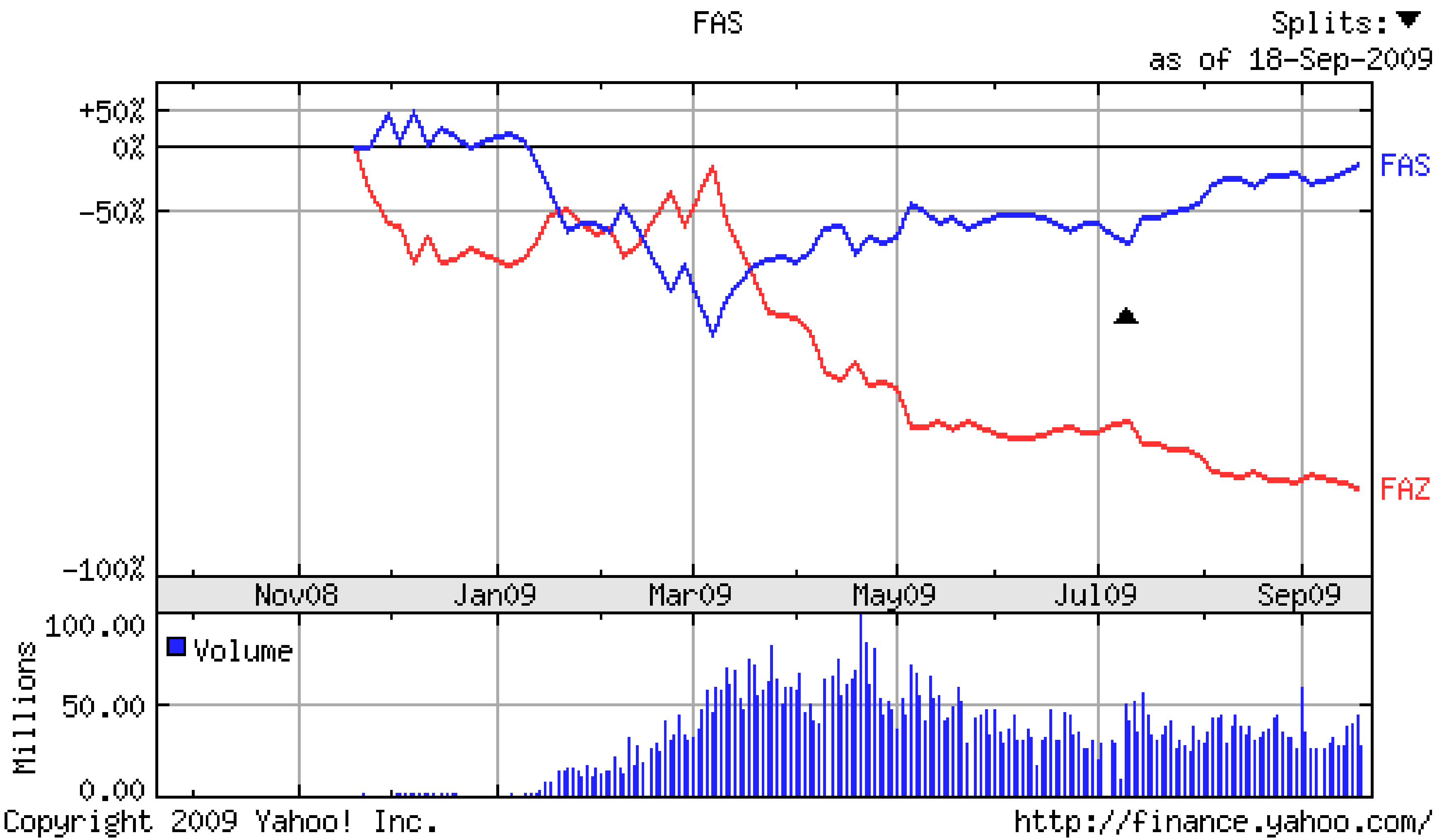


Notice that both returns are negative (big) over 1 year

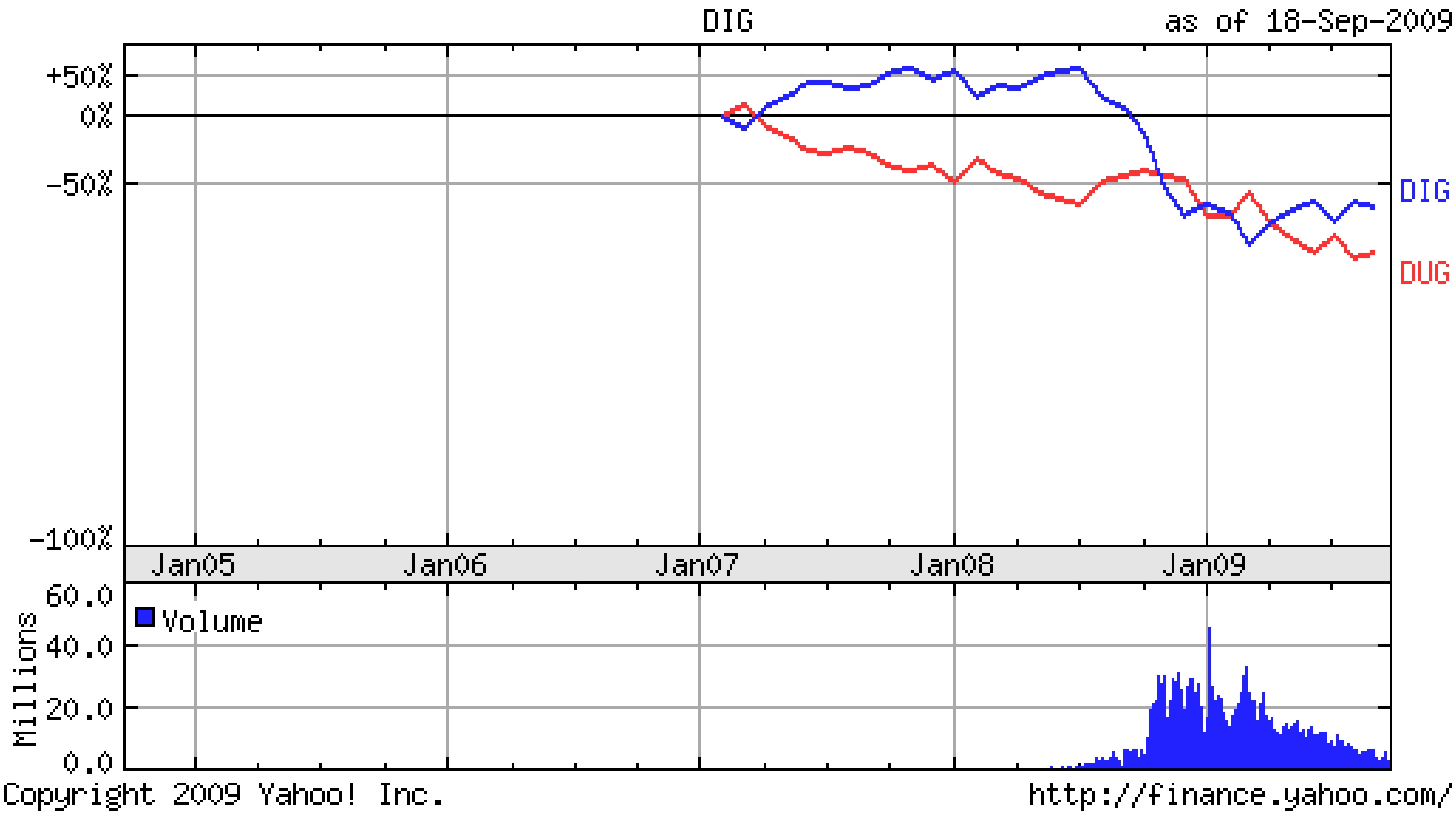
Since inception



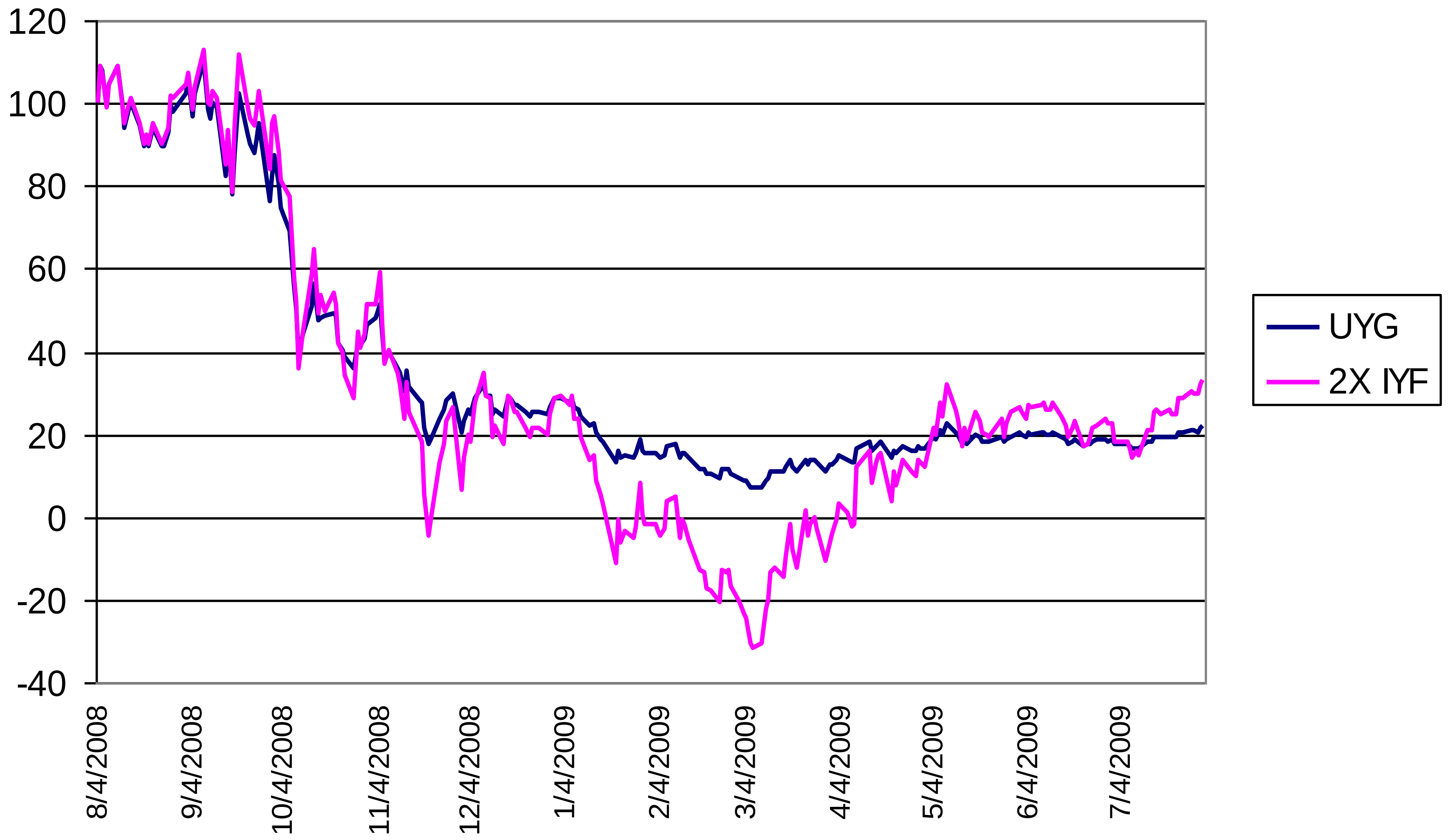
Another example: FAS/FAZ
Direxion 3X and -3X Financial ETF



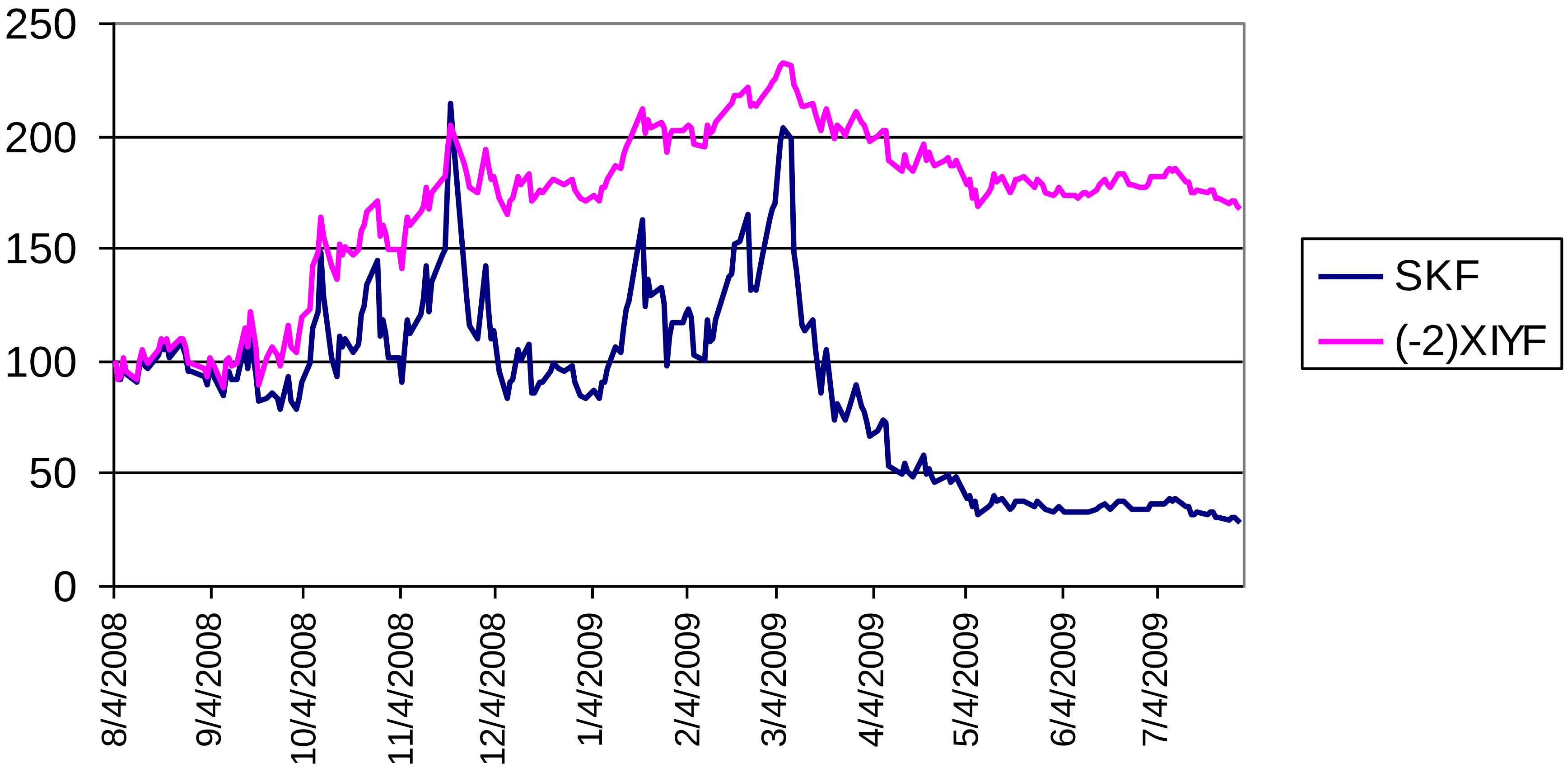
Oil & Gas Proshares DIG (long) DUG (short)



UYG vs. 2X IYF, 1 year



SKF vs. -2X IYF



250
200
150
100
50
0
8/4/2008
9/4/2008
10/4/2008
11/4/2008
12/4/2008
1/4/2009
2/4/2009
3/4/2009
4/4/2009
5/4/2009
6/4/2009
7/4/2009
SKF
(-2)XIYF

Correlation decay

If we assume $\frac{dL_t^+}{L_t^+} = -\frac{dL_t^-}{L_t^-}$ and $\frac{dL_t^+}{L_t^+} = \sigma_t dW_t$, we can derive that

$$\rho(t) = corr(\frac{L_t^+}{L_0^+}, \frac{L_t^-}{L_0^-}) = \frac{E\left[\left(\frac{L_t^+}{L_0^+}\right)\left(\frac{L_t^-}{L_0^-}\right)\right] - E\left[\frac{L_t^+}{L_0^+}\right]E\left[\frac{L_t^-}{L_0^-}\right]}{\sqrt{E\left[(\frac{L_t^+}{L_0^+} - E\left[\frac{L_t^+}{L_0^+}\right])^2\right]E\left[(\frac{L_t^-}{L_0^-} - E\left[\frac{L_t^-}{L_0^-}\right])^2\right]}} = \frac{E[e^{-\int_0^t \sigma_s^2}]-1}{E[e^{\int_0^t \sigma_s^2}]-1}$$

If volatility is deterministic,

$$\boxed{\rho(t) = \frac{E[e^{-\int_0^t \sigma_s^2}]-1}{E[e^{\int_0^t \sigma_s^2}]-1} = \frac{e^{-\int_0^t \sigma_s^2} - 1}{e^{\int_0^t \sigma_s^2} - 1} = -e^{-\int_0^t \sigma_s^2}}$$

Thus, the correlation goes from -1 to 0 as $t \to \infty$. It decays exponentially as a function of accumulated variance.

LETFs: The discrete model

$R_{S,n}$ = return of the underlying index over the nth period

$R_{L,n}$ = return of the leveraged ETF over the nth period

S_t = price of the underlying index or ETF

L_t = price of the leveraged ETF

f = expense ratio for leveraged ETF

$$R_{L,n} = \beta R_{S,n} + (1-\beta) r\Delta t - f\Delta t$$

$$\begin{aligned} L_t &= \prod_{n=1}^{N} \left(1 + R_{L,n}\right) \\ &= \prod_{n=1}^{N} \left(1 + \beta R_{S,n} + (1-\beta) r\Delta t - f\Delta t\right) \end{aligned}$$

$$\ln\left(1+\beta R_{S,n}+(1-\beta)r\Delta t-f\Delta t\right)\approx \beta R_{S,n}+(1-\beta)r\Delta t-f\Delta t-\frac{1}{2}\beta^2 R_{S,n}^2$$

$$\beta\ln\left(1+R_{S,n}\right)\approx \beta R_{S,n}-\frac{1}{2}\beta R_{S,n}^2 \qquad\qquad \therefore$$

$$\ln\left(1+\beta R_{S,n}+(1-\beta)r\Delta t-f\Delta t\right)\approx \beta\ln\left(1+R_{S,n}\right)+(1-\beta)r\Delta t-f\Delta t-\frac{1}{2}\left(\beta^2-\beta\right)R_{S,n}^2$$

$$\frac{L_t}{L_0}\approx\left(\frac{S_t}{S_0}\right)^{\beta}\exp\left[(1-\beta)rt-ft-\frac{1}{2}\left(\beta^2-\beta\right)\sum_{n=1}^{N}R_{S,n}^2\right]$$

$$\boxed{\frac{L_t}{L_0}\approx\left(\frac{S_t}{S_0}\right)^{\beta}\exp\left[(1-\beta)rt-ft-\frac{1}{2}\left(\beta^2-\beta\right)\int_0^t\sigma_s^2\,ds\right]}$$

Continuous-time model

$$\frac{dS_t}{S_t} = \sigma_t dZ_t + \mu_t dt$$

$$\frac{dL_t}{L_t} = \beta \frac{dS_t}{S_t} + (1-\beta) r dt - f dt$$

$$d \ln L_t = \frac{dL_t}{L_t} - \frac{1}{2}\left(\frac{dL_t}{L_t}\right)^2 = \frac{dL_t}{L_t} - \frac{1}{2}\beta^2 \sigma_t^2 dt$$

$$d \ln S_t = \frac{dS_t}{S_t} - \frac{1}{2}\left(\frac{dS_t}{S_t}\right)^2 = \frac{dS_t}{S_t} - \frac{1}{2}\sigma_t^2 dt$$

$$d \ln L_t - \beta d \ln S_t = (1-\beta) r dt - f dt - \frac{1}{2}\left(\beta^2 - \beta\right)\sigma_t^2 dt$$

$$\boxed{\frac{L_t}{L_0} = \left(\frac{S_t}{S_0}\right)^{\beta} \exp\left[(1-\beta) r t - f t - \frac{1}{2}\left(\beta^2 - \beta\right)\int_0^t \sigma_s^2 ds\right]}$$

Path-dependence of LETF returns has to do with realized volatility

$$\frac{L_t}{L_0} = \left(\frac{S_t}{S_0}\right)^{\beta} \exp\left[(1-\beta) rt - ft - \frac{1}{2}(\beta^2 - \beta)\int_0^t \sigma_s^2 ds \right]$$

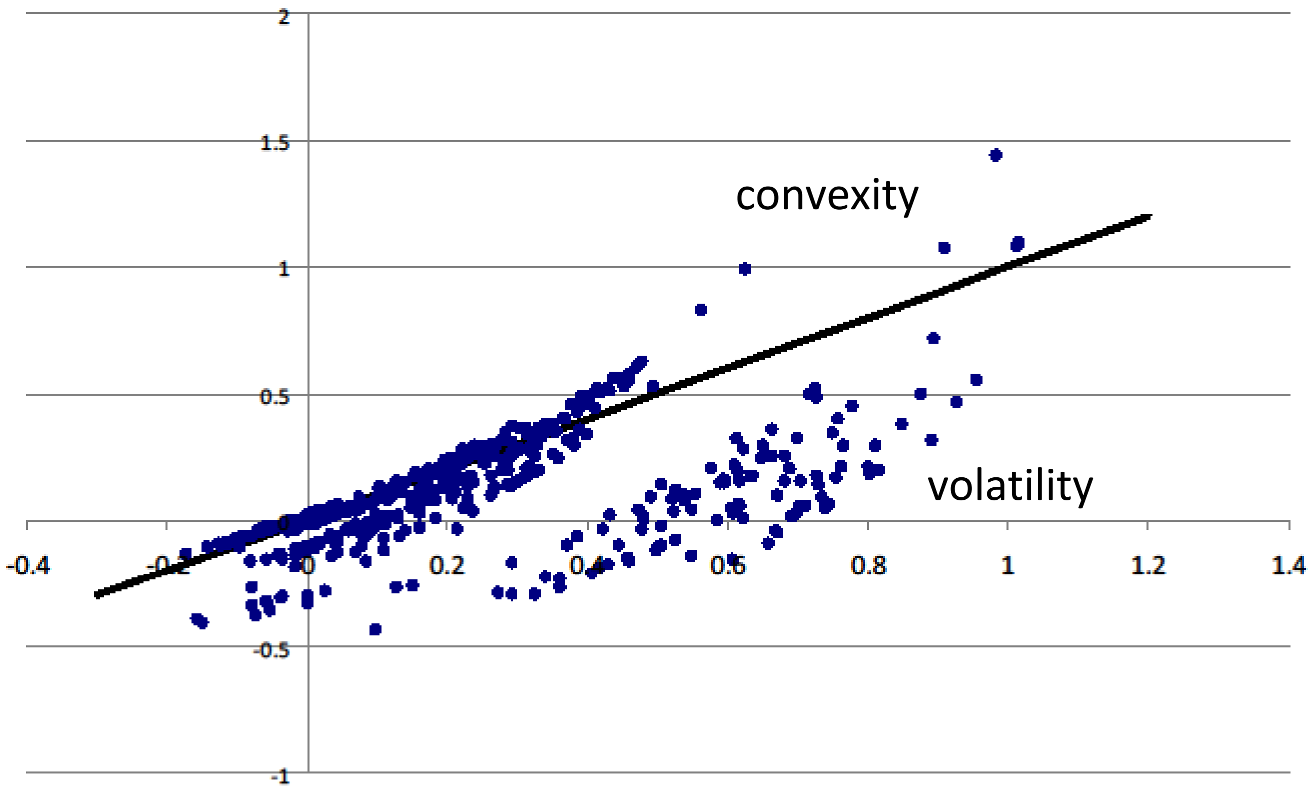
convexity due to daily compounding

financing & fees

volatility

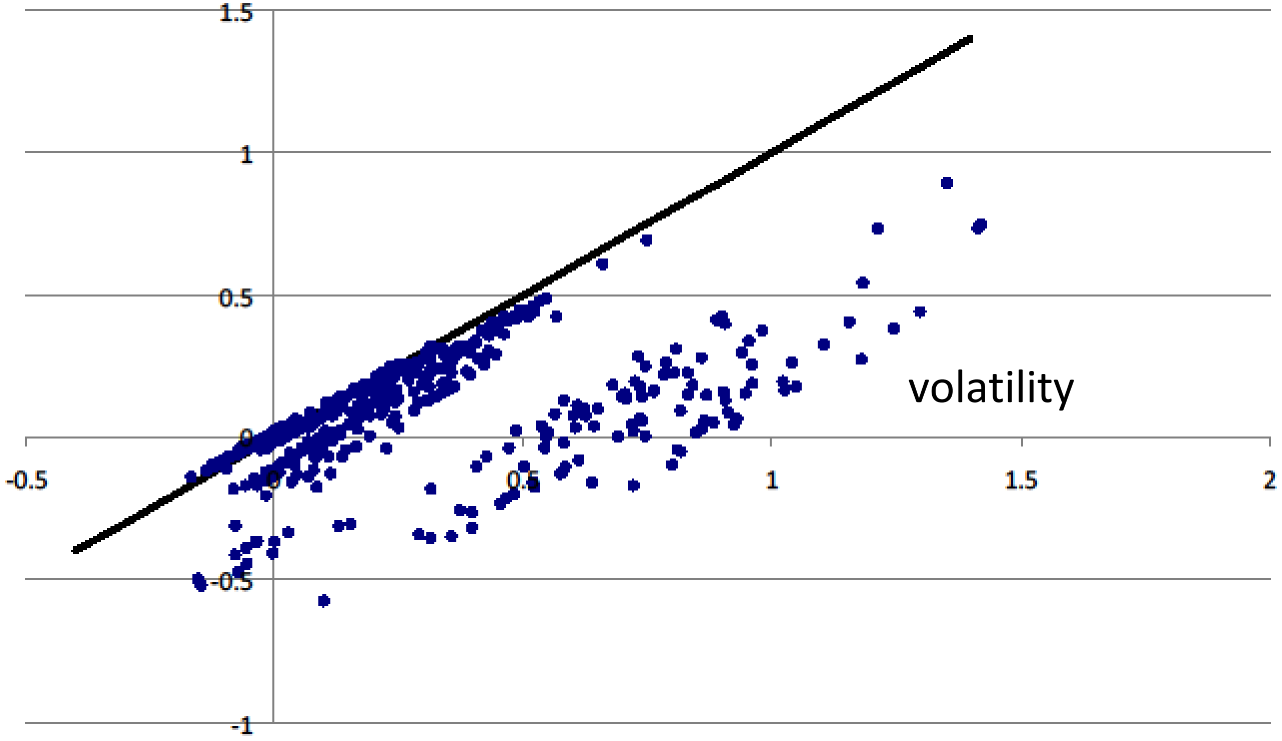
In practice, we will estimate the stochastic volatility as the 10-day standard deviation of the underlying ETF returns.

SKF vs. -2*IYF 3-month returns



Feb 2, 2008 to Mar 2, 2009, overlapping 3M returns

Ln SKF vs. -2*Ln IYF, 3 month returns



Feb 2, 2008 to Mar 2, 2009, overlapping 3M returns

Empirical validation of the model

Tracking error:

$$\varepsilon_t = \frac{L_t}{L_0} - \left(\frac{S_t}{S_0}\right)^{\beta} \exp\left[(1-\beta)rt - ft - \frac{1}{2}(\beta^2 - \beta)\int_0^t \sigma_s^2 ds\right]$$

Double leveraged ETF considered in the study

Table 2.1: Double-Leveraged ETFs considered in the study

Underlying ETF	ProShares Ultra ($\beta = 2$)	ProShares Ultra ($\beta = -2$)	Short Index/Sector
QQQQ	QLD	QID	Nasdaq 100
DIA	DDM	DXD	Dow 30
SPY	SSO	SDS	S&P500 Index
IJH	MVV	MZZ	S&P MidCap 400
IJR	SAA	SDD	S&P Small Cap 600
IWM	UWM	TWM	Russell 2000
IWD	UVG	SJF	Russell 1000
IWF	UKF	SFK	Russell 1000 Growth
IWS	UVU	SJL	Russell MidCap Value
IWP	UKW	SDK	Russell MidCap Growth
IWN	UVT	SJH	Russell 2000 Value
IWO	UKK	SKK	Russell 2000 Growth
IYM	UYM	SMN	Basic Materials
IYK	UGE	SZK	Consumer Goods
IYC	UCC	SCC	Consumer Services
IYF	UYG	SKF	Financials
IYH	RXL	RXD	Health Care
IYJ	UXI	SIJ	Industrials
IYE	DIG	DUG	Oil & Gas
IYR	URE	SRS	Real Estate
IYW	ROM	REW	Technology
IDU	UPW	SDP	Utilities

Triple leveraged ETF considered in the study

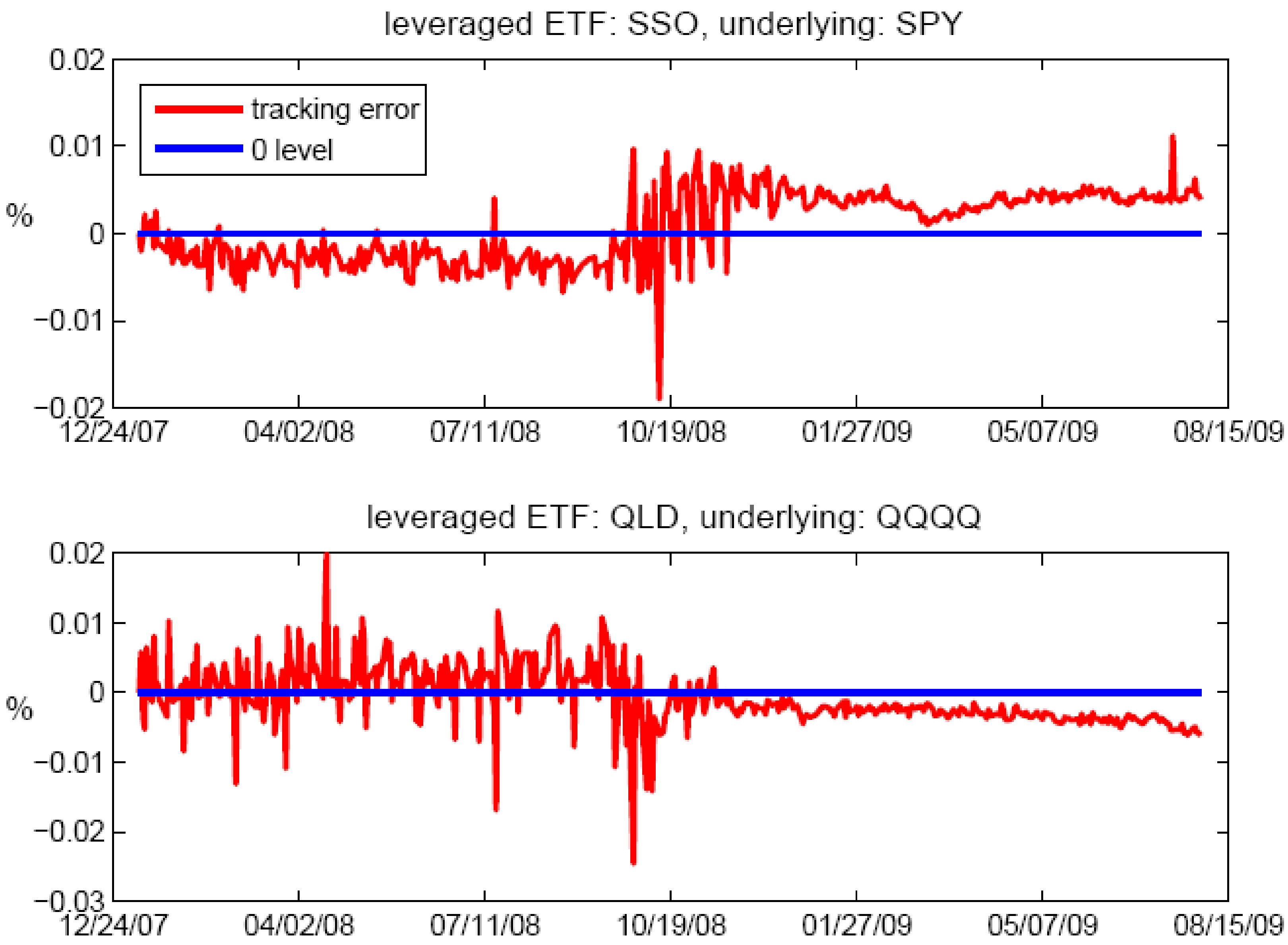
Table 2.2: Triple-Leveraged ETFs considered in the study

Underlying ETF or Index	Direxion 3X Bull ($\beta = 3$)	Direxion 3X Bear ($\beta = 3$)	Index/Sector
IWB	BGU	BGZ	Russell 1000
IWM	TNA	TZA	Russell 2000
RIFIN.X	FAS	FAZ	Russell 1000 Financial
RIENG.X	ERX	ERY	Russell 1000 Energy
EFA	DZK	DPK	MSCI EAFE Index
EEM	EDC	EDZ	MSCI Emerging Markets

Double leveraged bullish ETFs, 2/2008 to 3/2009

Double-Leveraged Bullish ETFs

Underlying ETF	Tracking Error average,%	Standard Deviation %	Leveraged ETF
QQQQ	0.04	0.47	QLD
DIA	0	0.78	DDM
SPY	-0.06	0.4	SSO
IJH	-0.06	0.38	MVV
IJR	1.26	0.71	SAA
IWM	1.26	0.88	UWM
IWD	1	0.98	UVG
IWF	0.5	0.59	UKF
IWS	-0.33	1.2	UVU
IWP	-0.02	0.61	UKW
IWN	2.15	1.29	UVT
IWO	0.5	0.74	UKK
IYM	1.44	1.21	UYM
IYK	1.2	0.75	UGE
IYC	1.56	1.04	UCC
IYF	-0.22	0.74	UYG
IYH	0.4	0.42	RXL
IYJ	1.05	0.74	UXI
IYE	-0.73	1.71	DIG
IYR	1.64	1.86	URE
IYW	0.51	0.55	ROM
IDU	0.25	0.55	UPW


leveraged ETF: SSO, underlying: SPY
tracking error
0 level
%
0.02
0.01
0
−0.01
−0.02
12/24/07
04/02/08
07/11/08
10/19/08
01/27/09
05/07/09
08/15/09
leveraged ETF: QLD, underlying: QQQQ
%
0.02
0.01
0
−0.01
−0.02
−0.03
12/24/07
04/02/08
07/11/08
10/19/08
01/27/09
05/07/09
08/15/09

Double leveraged bearish ETFs, 2/2008 to 3/2009

Double-Leveraged Bearish ETFs

Underlying ETF	Tracking Error average,%	Standard Deviation %	Leveraged ETF
QQQQ	0.22	0.8	QID
DIA	-2.01	3.24	DXD
SPY	-1.4	2.66	SDS
IJH	0.69	0.64	MZZ
IJR	-0.55	0.86	SDD
IWM	0.94	0.91	TWM
IWD	0.32	1.4	SJF
IWF	-0.3	1.34	SFK
IWS	-2.06	3.03	SJL
IWP	0.93	0.92	SDK
IWN	-2.21	1.8	SJH
IWO	-0.19	0.79	SKK
IYM	1.82	0.99	SMN
IYK	-0.76	1.98	SZK
IYC	0.79	0.92	SCC
IYF	3.3	3.03	SKF
IYH	1.04	0.91	RXD
IYJ	0.32	0.74	SIJ
IYE	0.43	3.09	DUG
IYR	2	2.07	SRS
IYW	0.01	0.8	REW
IDU	1.75	1.06	SDP

Triple leveraged ETFs, since inception (Nov 2008 – Mar 2009)

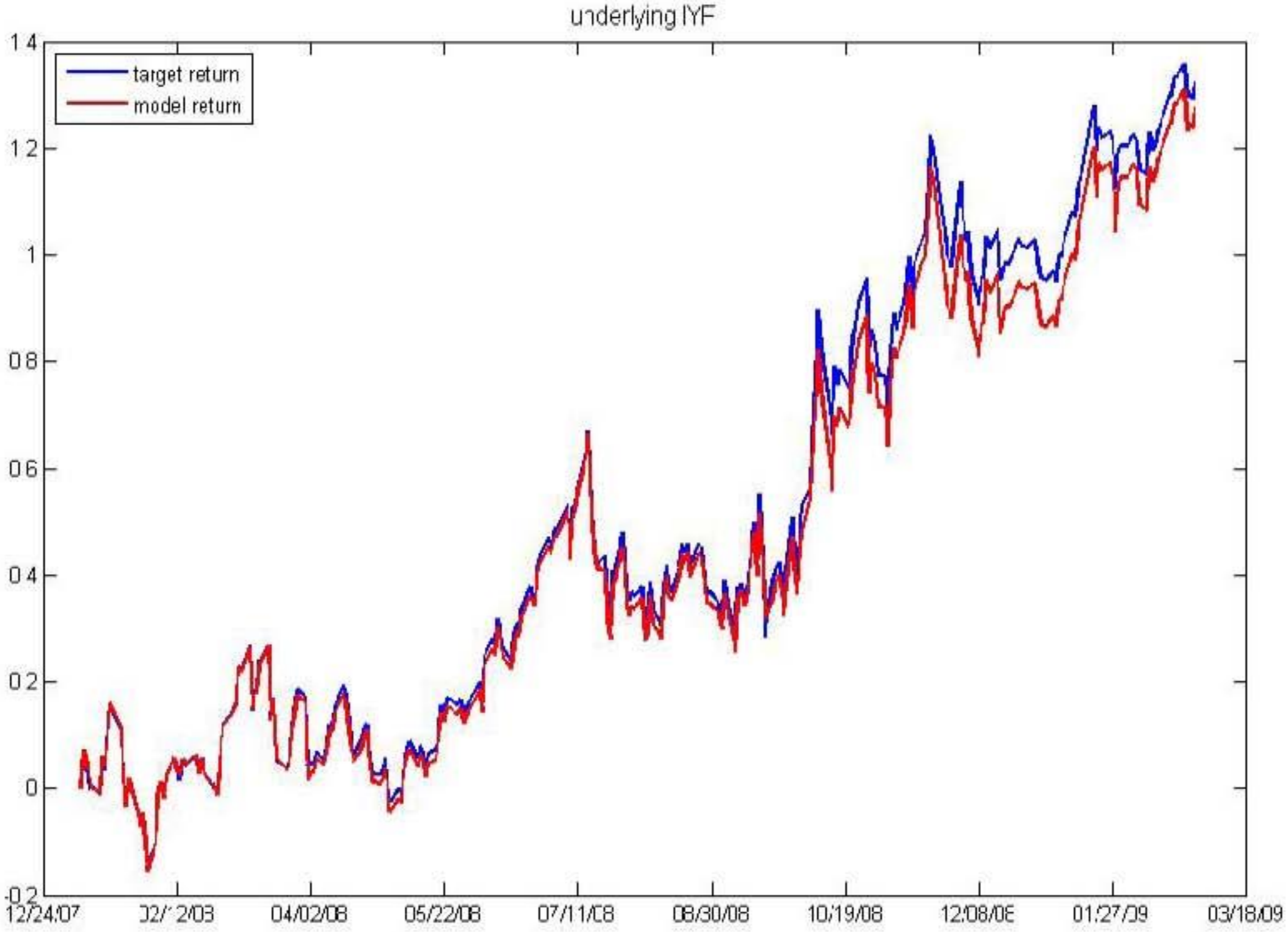
Triple-Leveraged Bullish ETFs

Underlying ETF/Index	Tracking Error average,%	Standard Deviation %	Leveraged ETF
IWB	0.44	0.55	BGU
IWM	0.81	0.75	TNA
RIFIN.X	3.67	2.08	FAS
RIENG.X	2.57	0.7	ERX
EFA	1.26	2.32	DZK
EEM	1.41	1.21	EDC

Triple-Leveraged Bearish ETFs

Underlying ETF/Index	Tracking Error average,%	Standard Deviation %	Leveraged ETF
IWB	-0.08	0.64	BGZ
IWM	0.65	0.76	TZA
RIFIN.X	-1.63	4.04	FAZ
RIENG.X	-1.41	1.01	ERY
EFA	-1.54	1.86	DPK
EEM	0.49	1.43	EDZ

Tra



Tracking SKF since December 2007 using the actual prices and the formula

A replication scheme: think VIX

Let's consider a double leveraged bullish ETF, that is $\beta = 2$

$$g(S_t) = \left(L_0 \left(\frac{S_t}{S_0} \right)^2 e^{-V_t} - k \right)^+ = L_0 e^{-V_t} \left((\frac{S_t}{S_0})^2 - \frac{k}{L_0} e^{V_t} \right)^+ = L_0 e^{-V_t} (x^2 - (k^*)^2)^+$$

where $x = \frac{S_t}{S_0}$ and $k^* = \left(\frac{k}{L_0} e^{V_t} \right)^{\frac{1}{2}}.$

It is well-known that any payoff function $g(S)$ and scalar $c \in \mathbb{R}$ (see Carr-Madan (2002))

$$g(S) = g(c) + g'(c)(S-c) + \int_{-\infty}^{c} g''(K)(K-S)^+ dK + \int_{c}^{\infty} g''(K)(S-K)dK$$

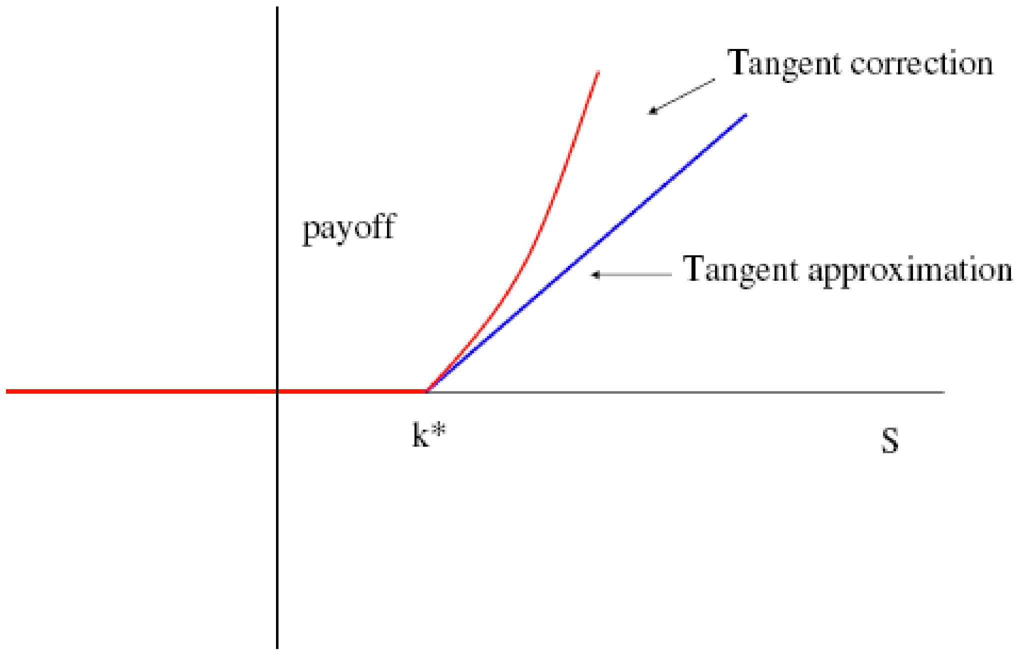
In our case

$$g'(x) = 2H(x-k^*)L_0 e^{-V_t} x \qquad g''(x) = \begin{cases} 2L_0 e^{-V_t} & \text{if} \quad x > k^* \\ \delta(x-k^*)2L_0 e^{-V_t} k^* & \text{if} \quad x = k^* \\ 0 & \text{if} \quad x < k^* \end{cases}$$

Approximate by a single option + residual

Substituting the first and second derivatives into Carr-Madan formula, we have

$$g(x) = \underbrace{2L_0 e^{-V_t} k^* (x-k^*)^+}_{\text{tangent approximation}} + \int_{k^*}^{\infty} \underbrace{2L_0 e^{-V_t} (x-K)^+}_{\text{tangent correction}} \, dK$$



Formulas, formulas, formulas!

Assuming deterministic volatility and using the no arbitrage condition, we have

$$C_L(k,t) = 2\frac{L_0}{S_0}e^{-V_t}\left(k^* C_S(S_0 k^*, t) + \int_{k^*}^{\infty} C_S(S_0 K, t)\, dK\right)$$

where $C_L(k,t)$ is a call option on L with strike k and maturity t , $C_S(S_0 k^*, t)$ is a call option on S with strike $S_0 k^*$. We call $S_0 k^*$ the `most likely'' strike.

Similarly, we can derive the following equations

$$C_{L^+}(k,t) = \frac{L_0}{S_0}e^{\frac{\beta-\beta^2}{2}V_t}\beta\left((k^*)^{\beta-1} C_S(S_0 k^*, t) + (\beta-1)\int_{k^*}^{\infty} K^{\beta-2} C_S(S_0 K, t)\, dK\right)$$

$$C_{L^-}(k,t) = \frac{L_0}{S_0}e^{\frac{\beta-\beta^2}{2}V_t}|\beta|\left((k^*)^{\beta-1} P_S(S_0 k^*, t) + (1-\beta)\int_{0}^{k^*} K^{\beta-2} P_S(S_0 K, t)\, dK\right)$$

$$P_{L^+}(k,t) = \frac{L_0}{S_0}e^{\frac{\beta-\beta^2}{2}V_t}\beta\left((k^*)^{\beta-1} P_S(S_0 k^*, t) + (\beta-1)\int_{0}^{k^*} K^{\beta-2} P_S(S_0 K, t)\, dK\right)$$

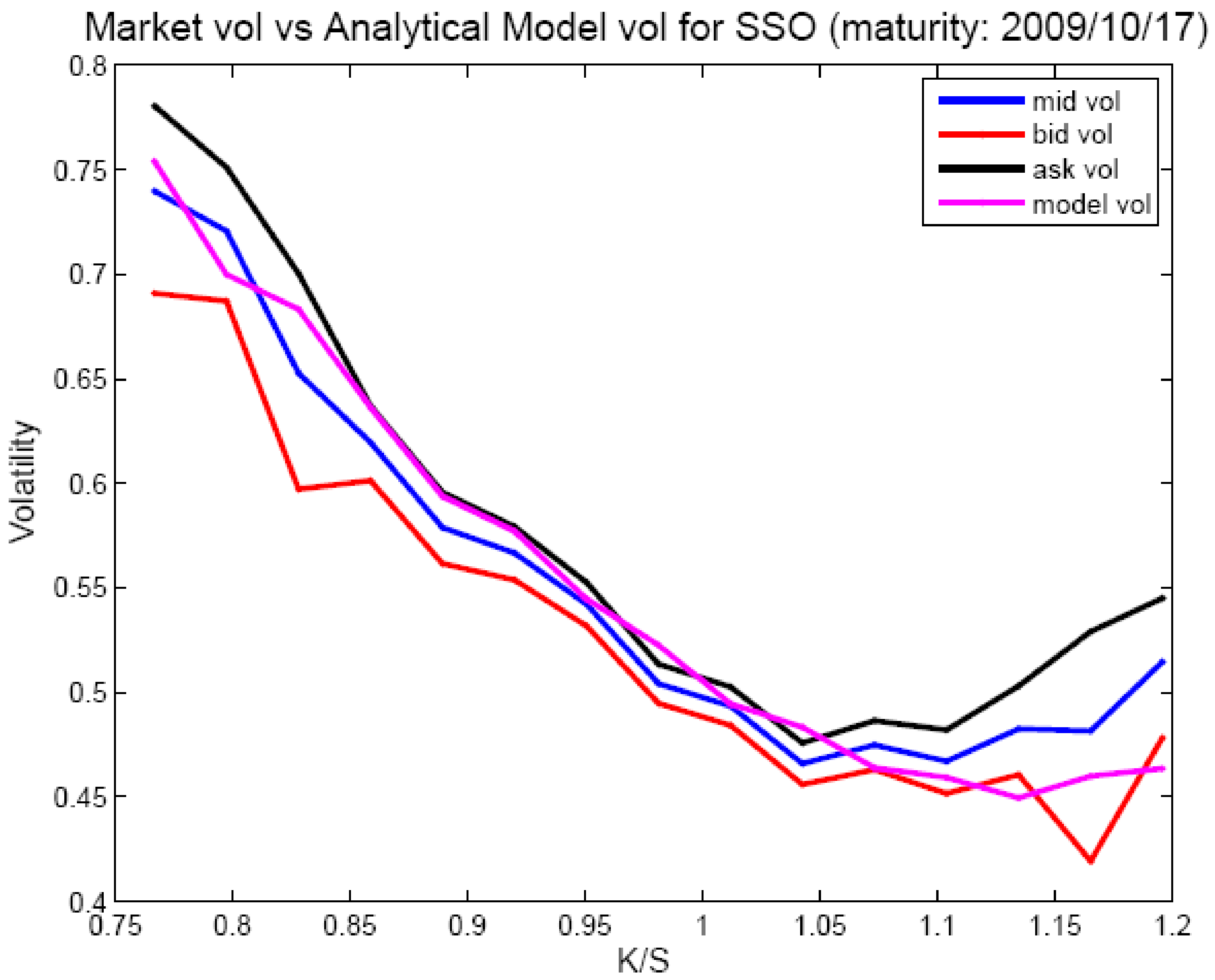
$$P_{L^-}(k,t) = \frac{L_0}{S_0}e^{\frac{\beta-\beta^2}{2}V_t}|\beta|\left((k^*)^{\beta-1} C_S(S_0 k^*, t) + (1-\beta)\int_{k^*}^{\infty} K^{\beta-2} C_S(S_0 K, t)\, dK\right)$$

General (stochastic) volatility environments: use the implied vols

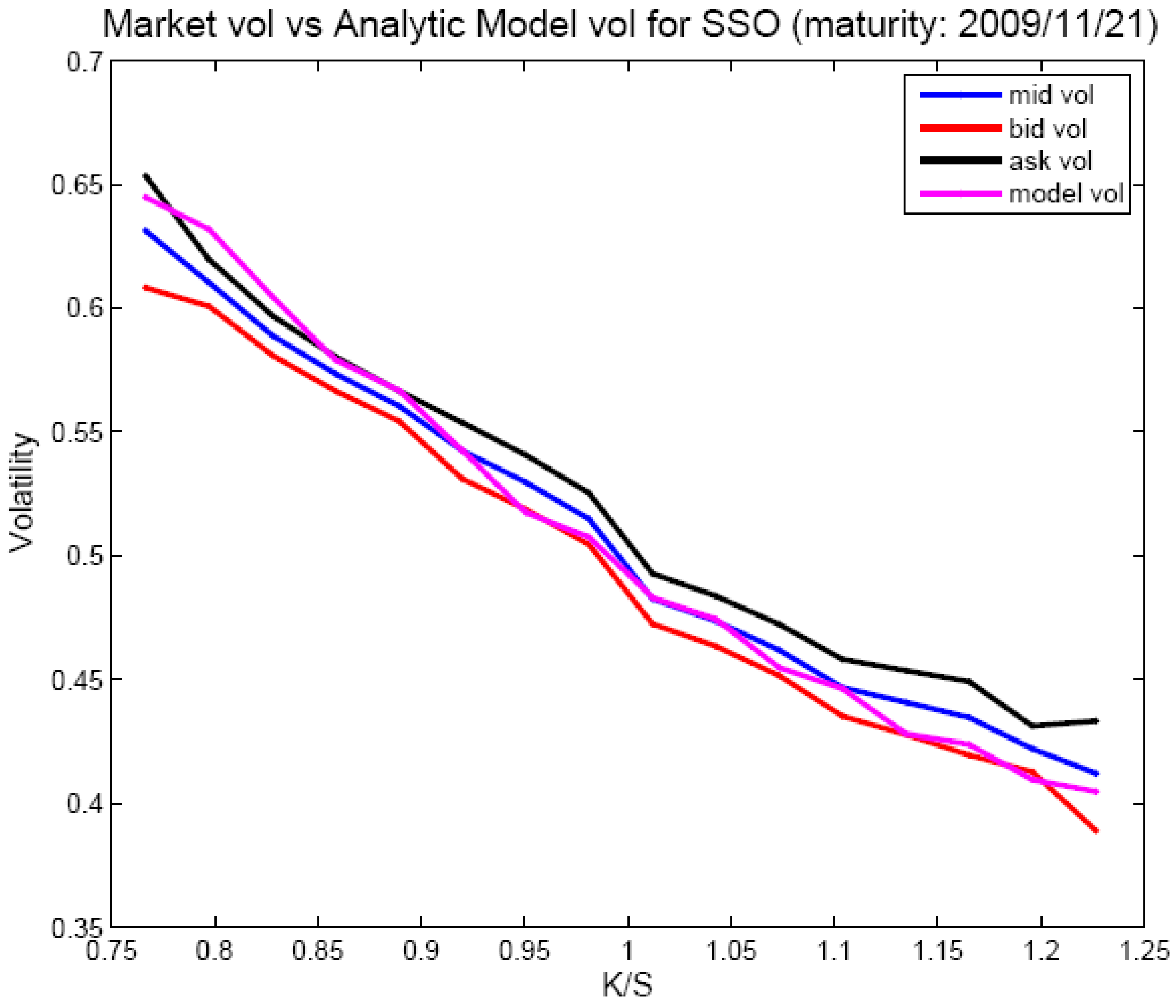
To extend the formula to a stochastic volatility environment, we consider replacing $V_t = \int_0^t \sigma_s^2 ds$ with $E[\int_0^t \sigma_s^2 ds]$. We can use (implied vol)^2 * t as the value of $E[\int_0^t \sigma_s^2 ds]$

We take the prices of the options on the underlying ETF from the market and use the formula discussed in the previous slide to price the options on leveraged ETFs.

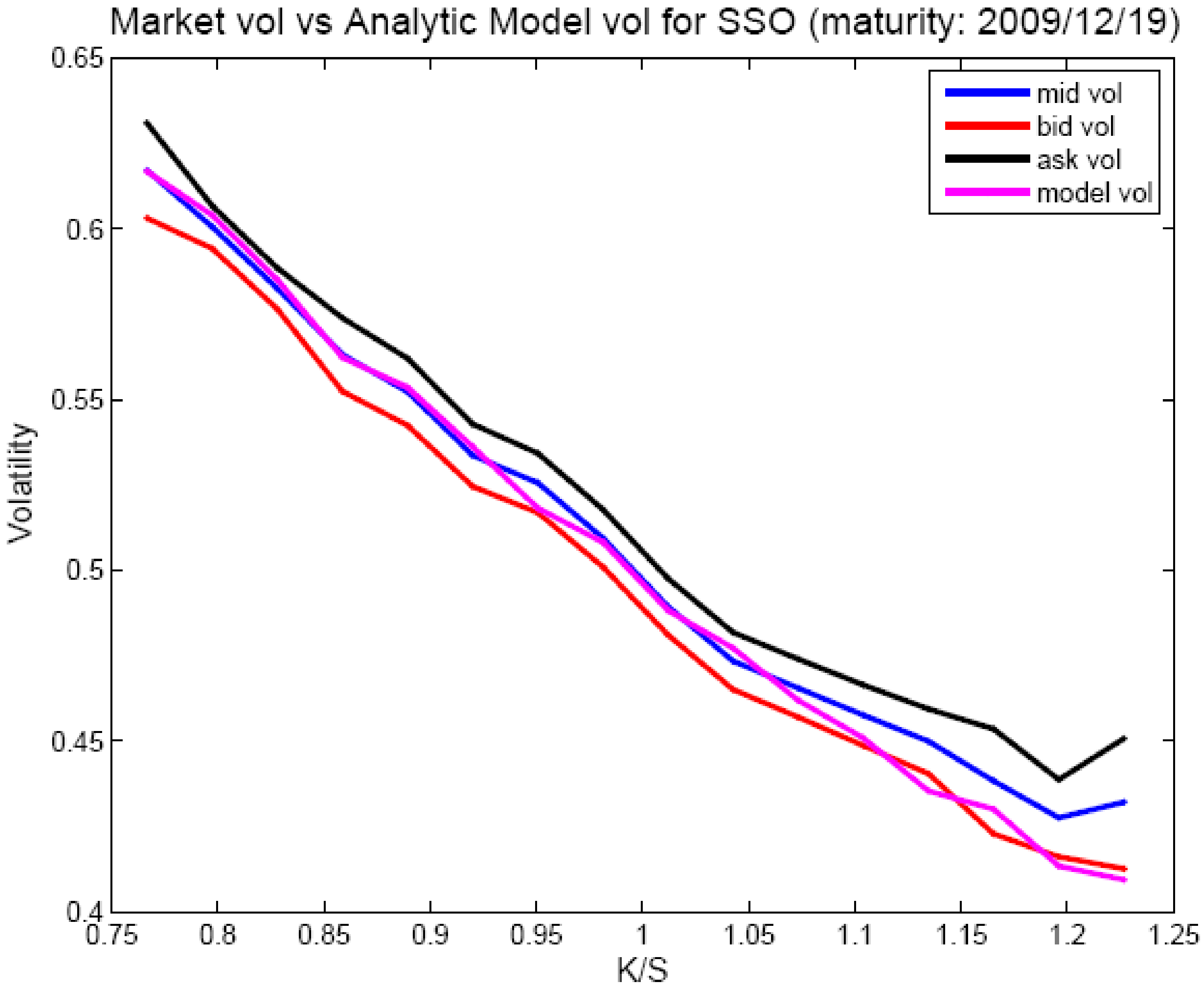
The correct implied vol to use is (implied vol of leveraged ETF at strike *k*)/beta



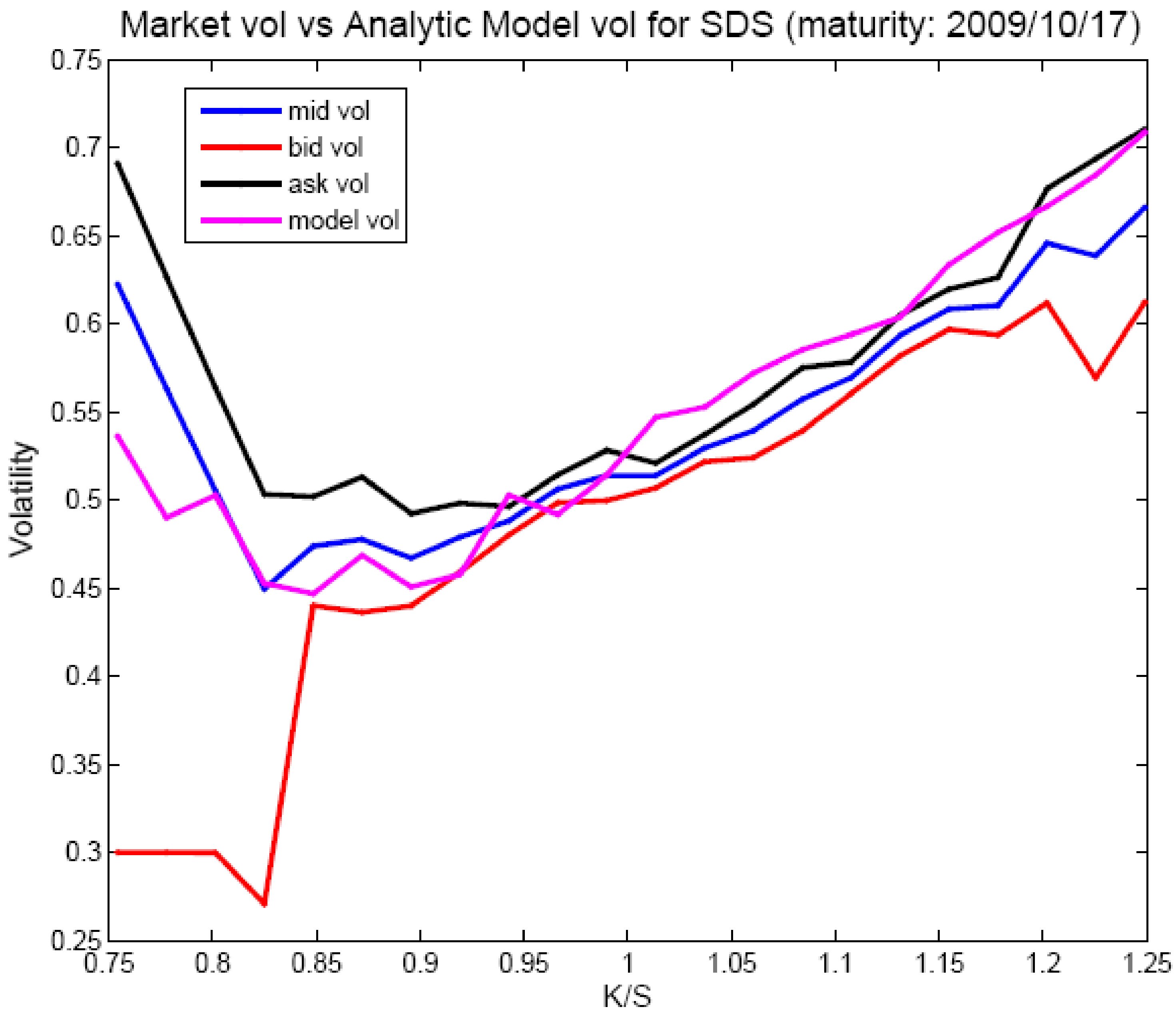
Market vol vs Analytical Model vol for SSO (maturity: 2009/10/17)
mid vol
bid vol
ask vol
model vol
Volatility
K/S
0.8
0.75
0.7
0.65
0.6
0.55
0.5
0.45
0.4
0.75
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2

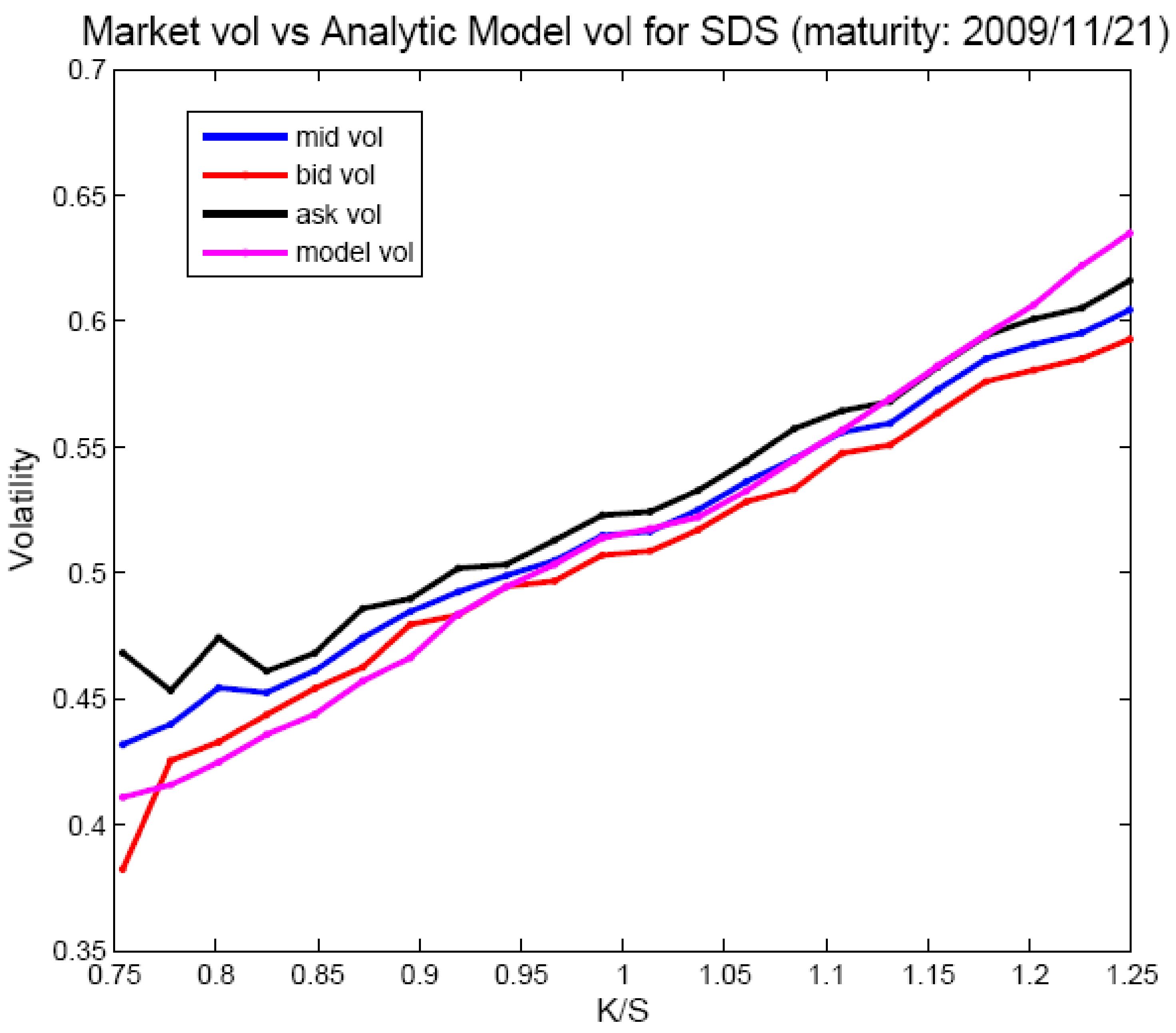


Market vol vs Analytic Model vol for SSO (maturity: 2009/11/21)
mid vol
bid vol
ask vol
model vol
Volatility
K/S
0.35
0.4
0.45
0.5
0.55
0.6
0.65
0.7
0.75
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2
1.25

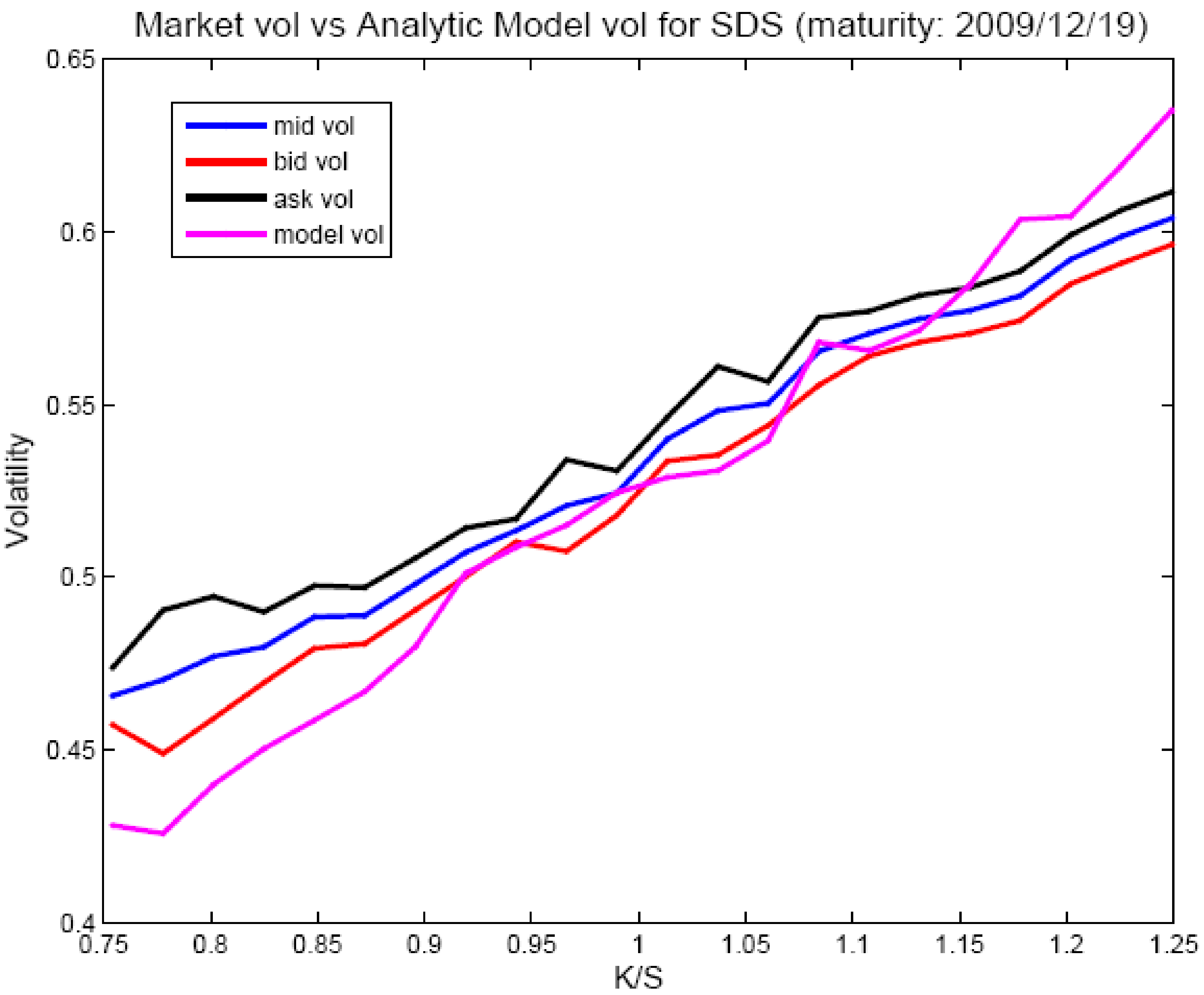


Market vol vs Analytic Model vol for SSO (maturity: 2009/12/19)
mid vol
bid vol
ask vol
model vol
Volatility
K/S
0.65
0.6
0.55
0.5
0.45
0.4
0.75
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2
1.25


Market vol vs Analytic Model vol for SDS (maturity: 2009/10/17)
mid vol
bid vol
ask vol
model vol
Volatility
K/S
0.25
0.3
0.35
0.4
0.45
0.5
0.55
0.6
0.65
0.7
0.75
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2
1.25



Market vol vs Analytic Model vol for SDS (maturity: 2009/11/21)
mid vol
bid vol
ask vol
model vol
Volatility
K/S
0.35
0.4
0.45
0.5
0.55
0.6
0.65
0.7
0.75
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2
1.25


Market vol vs Analytic Model vol for SDS (maturity: 2009/12/19)
mid vol
bid vol
ask vol
model vol
Volatility
K/S
0.4
0.45
0.5
0.55
0.6
0.65
0.75
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2
1.25

Non-parametric Approximation

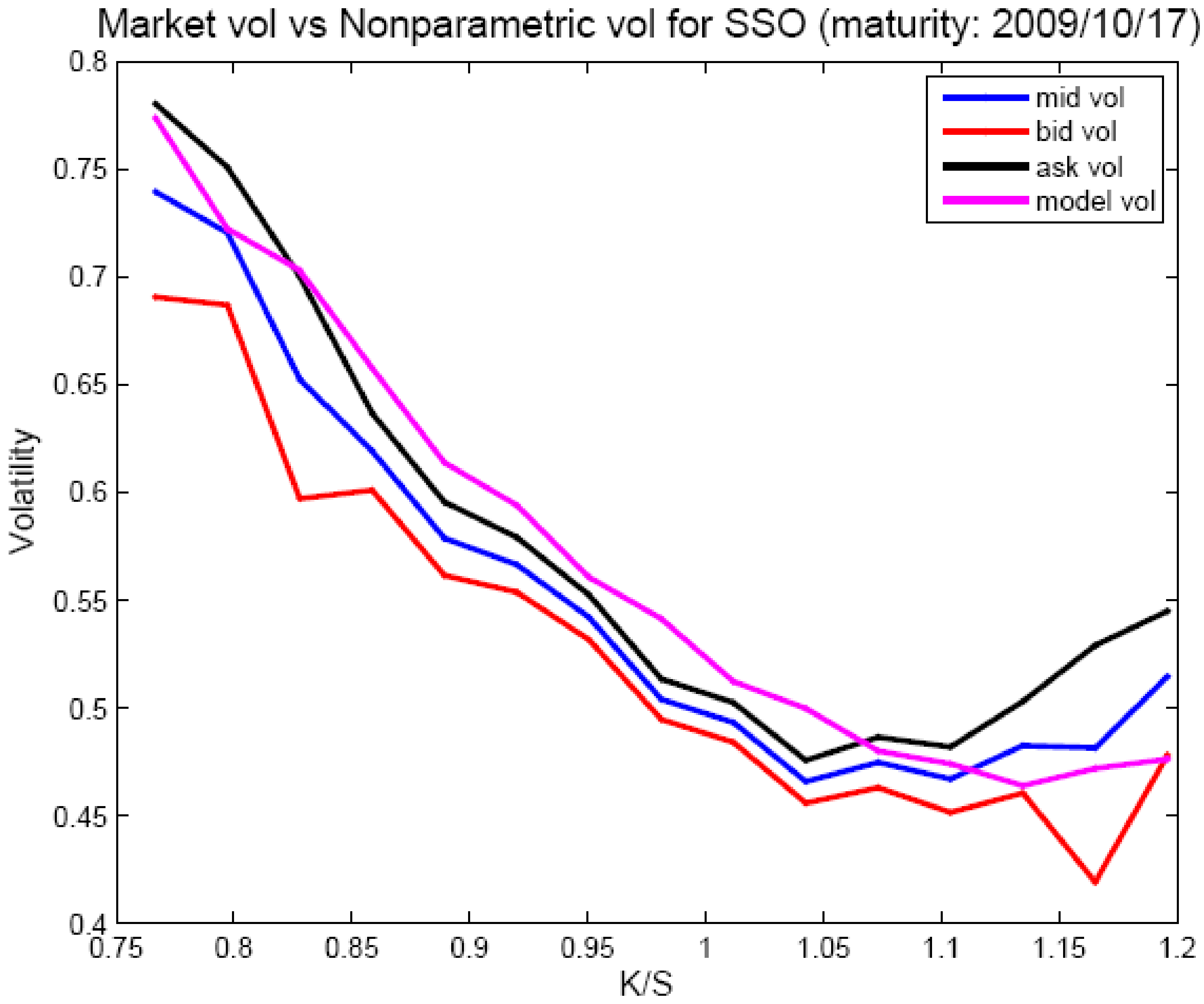
We propose the following model

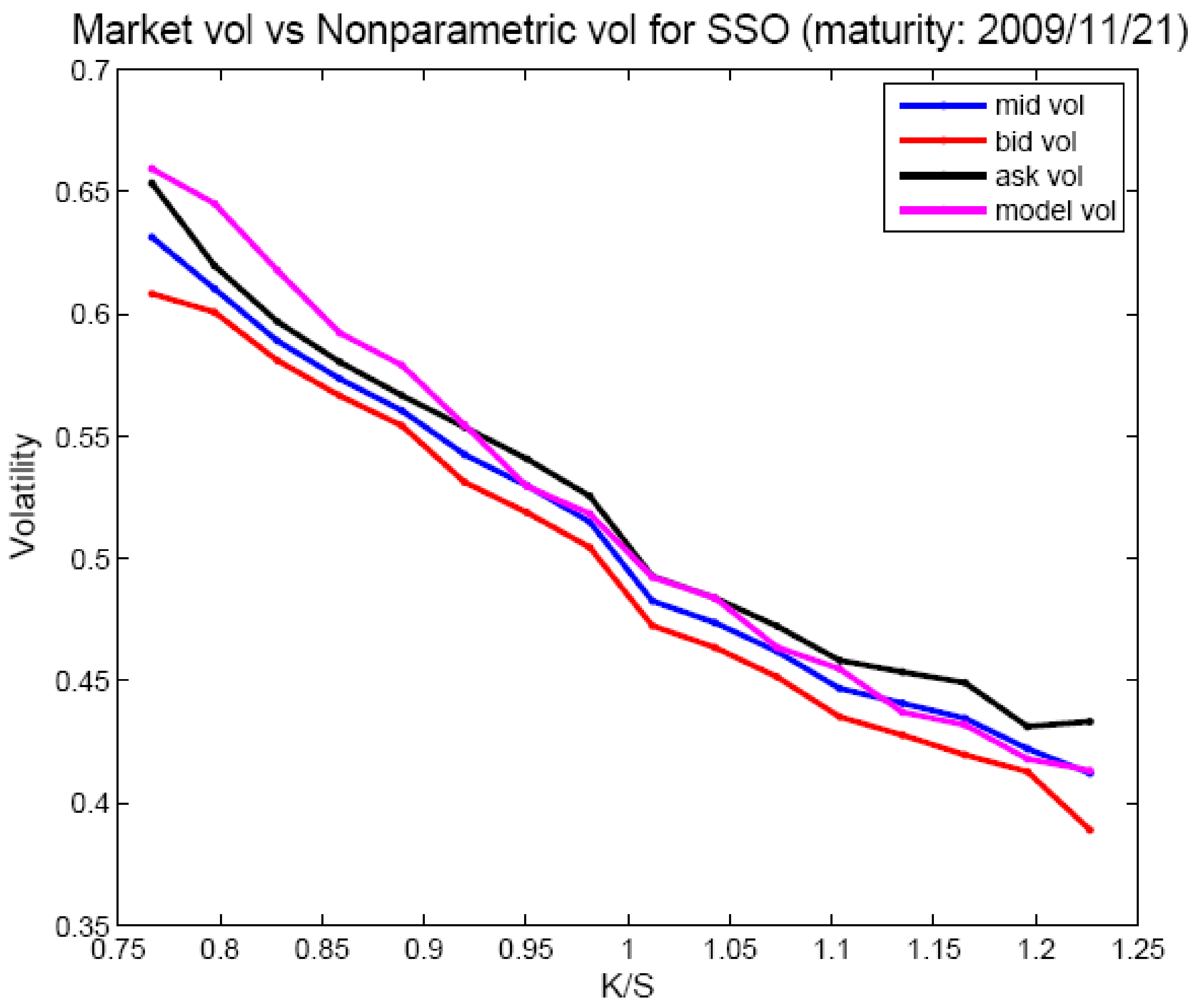
$$\sigma_L(k) = |\beta|\,\sigma_S(S_0 k^*)$$

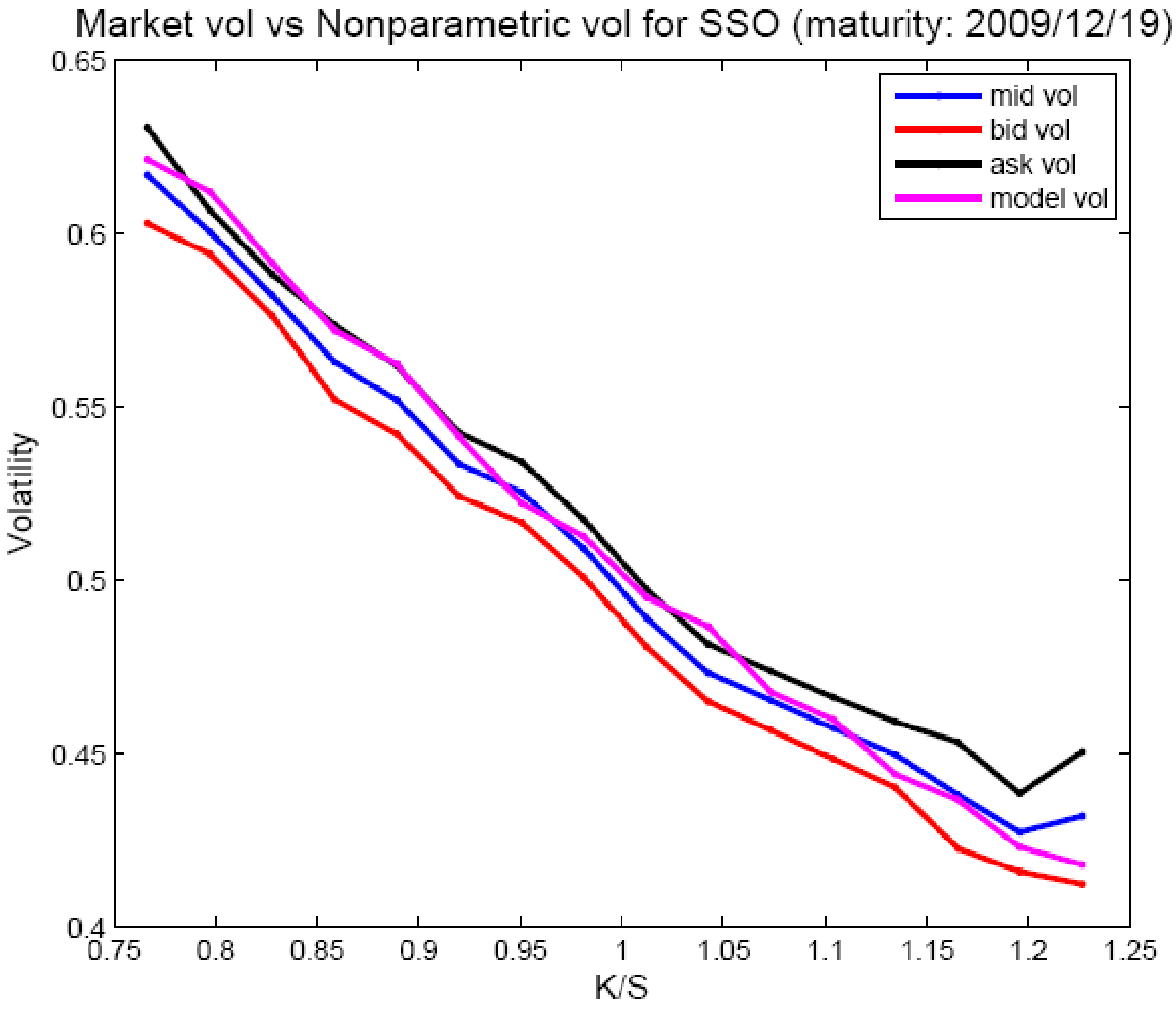
where $\sigma_L(k)$ is the implied vol of an option on L with strike k and $\sigma_S(S_0 k^*)$ is the implied vol of an option on S with strike $S_0 k^*$.

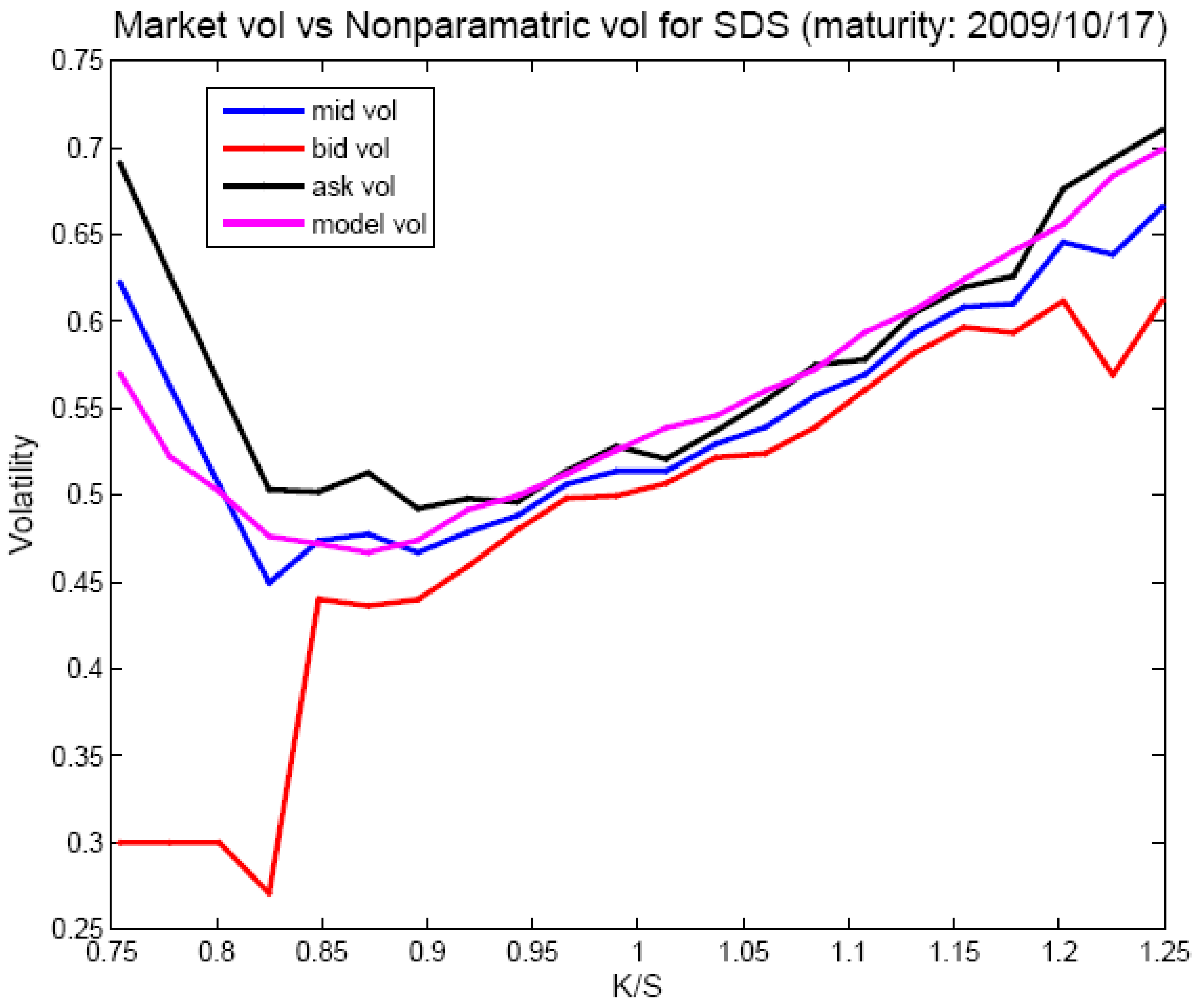
Again,

$$S_0 k^* = S_0 \left(e^{\frac{\beta^2-\beta}{2} V_t} \right)^{\frac{1}{\beta}} \quad \text{and} \quad V_t = E\left[\int_0^t \sigma_s^2 \, ds\right]$$

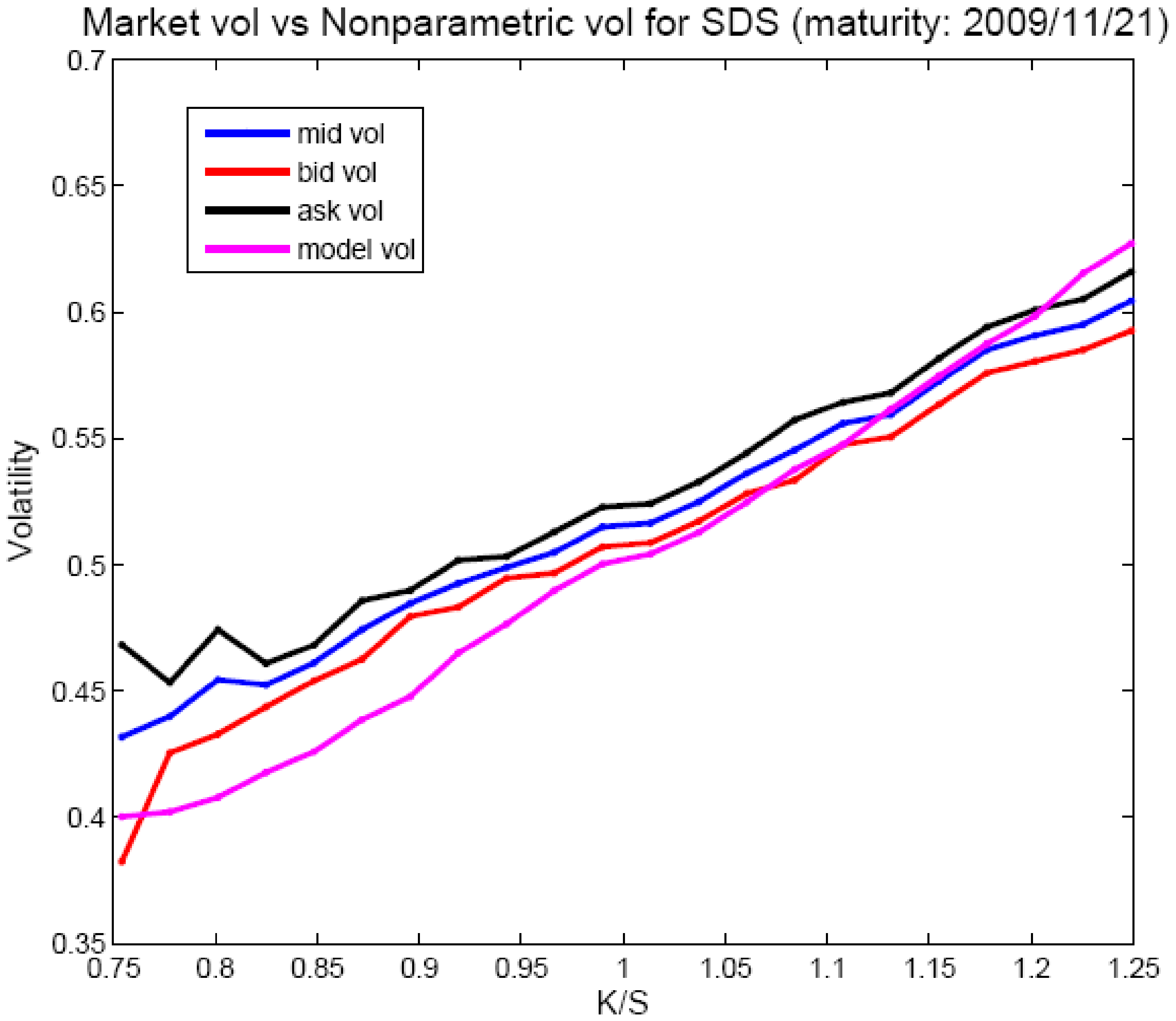

Market vol vs Nonparametric vol for SSO (maturity: 2009/10/17)
mid vol
bid vol
ask vol
model vol
Volatility
K/S
0.4
0.45
0.5
0.55
0.6
0.65
0.7
0.75
0.8
0.75
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2

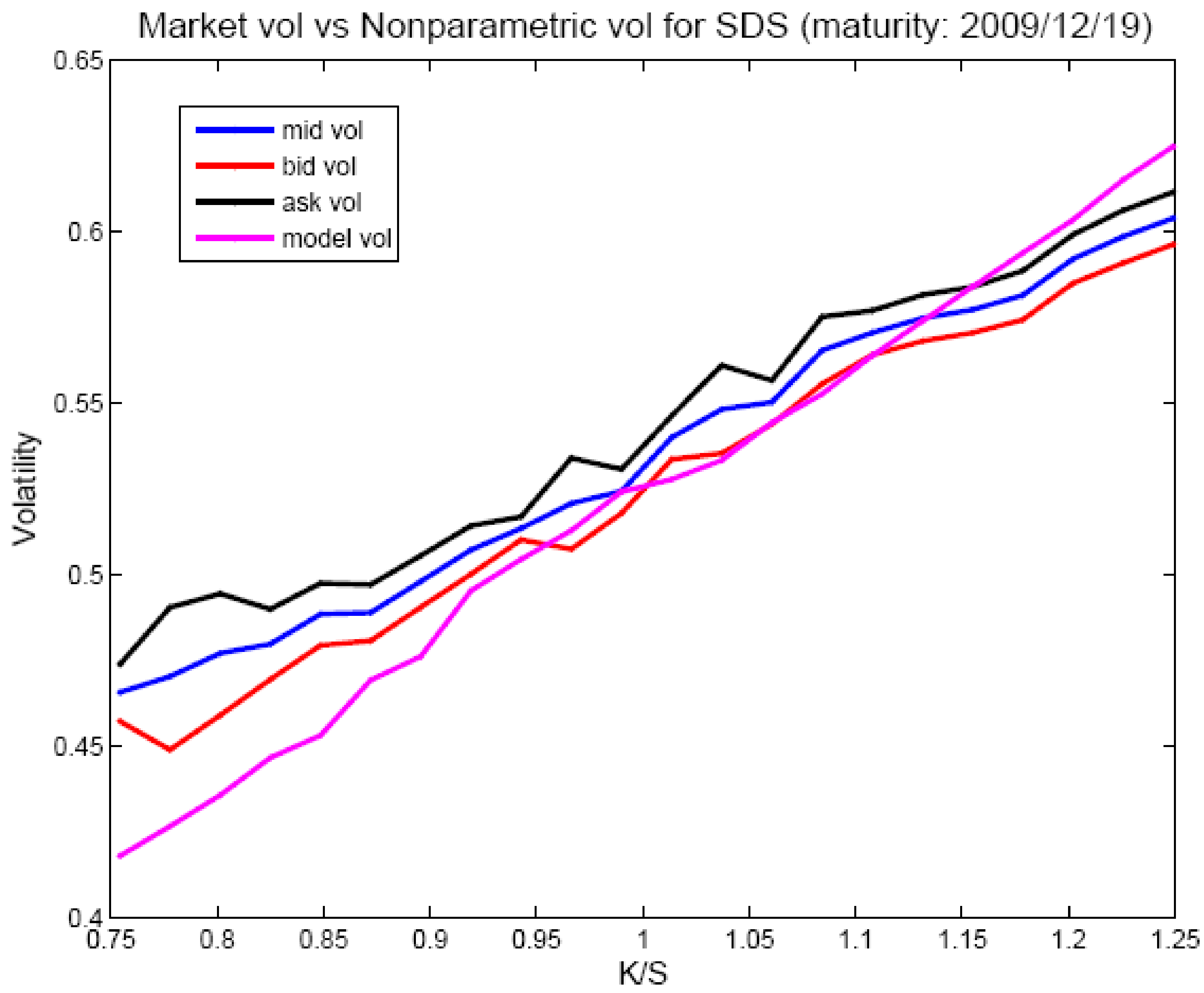

Market vol vs Nonparametric vol for SSO (maturity: 2009/11/21)
mid vol
bid vol
ask vol
model vol
Volatility
K/S
0.35
0.4
0.45
0.5
0.55
0.6
0.65
0.7
0.75
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2
1.25


Market vol vs Nonparametric vol for SSO (maturity: 2009/12/19)
mid vol
bid vol
ask vol
model vol
Volatility
K/S
0.65
0.6
0.55
0.5
0.45
0.4
0.75
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2
1.25



Market vol vs Nonparamatric vol for SDS (maturity: 2009/10/17)
mid vol
bid vol
ask vol
model vol
Volatility
K/S
0.75
0.7
0.65
0.6
0.55
0.5
0.45
0.4
0.35
0.3
0.25
0.75
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2
1.25


Market vol vs Nonparametric vol for SDS (maturity: 2009/11/21)
mid vol
bid vol
ask vol
model vol
Volatility
K/S
0.7
0.65
0.6
0.55
0.5
0.45
0.4
0.35
0.75
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2
1.25


Market vol vs Nonparametric vol for SDS (maturity: 2009/12/19)
mid vol
bid vol
ask vol
model vol
Volatility
K/S
0.4
0.45
0.5
0.55
0.6
0.65
0.75
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2
1.25

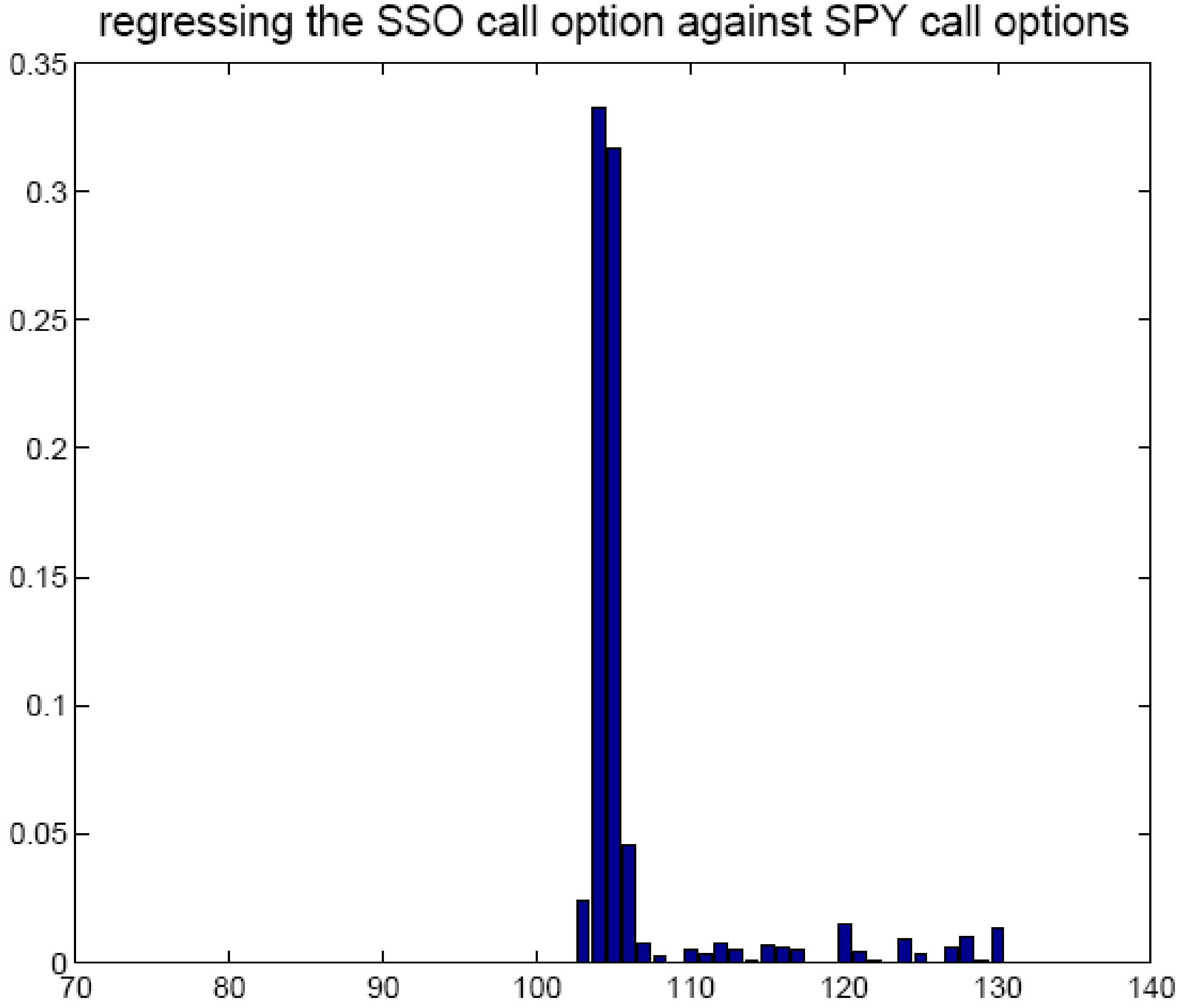
Hedging

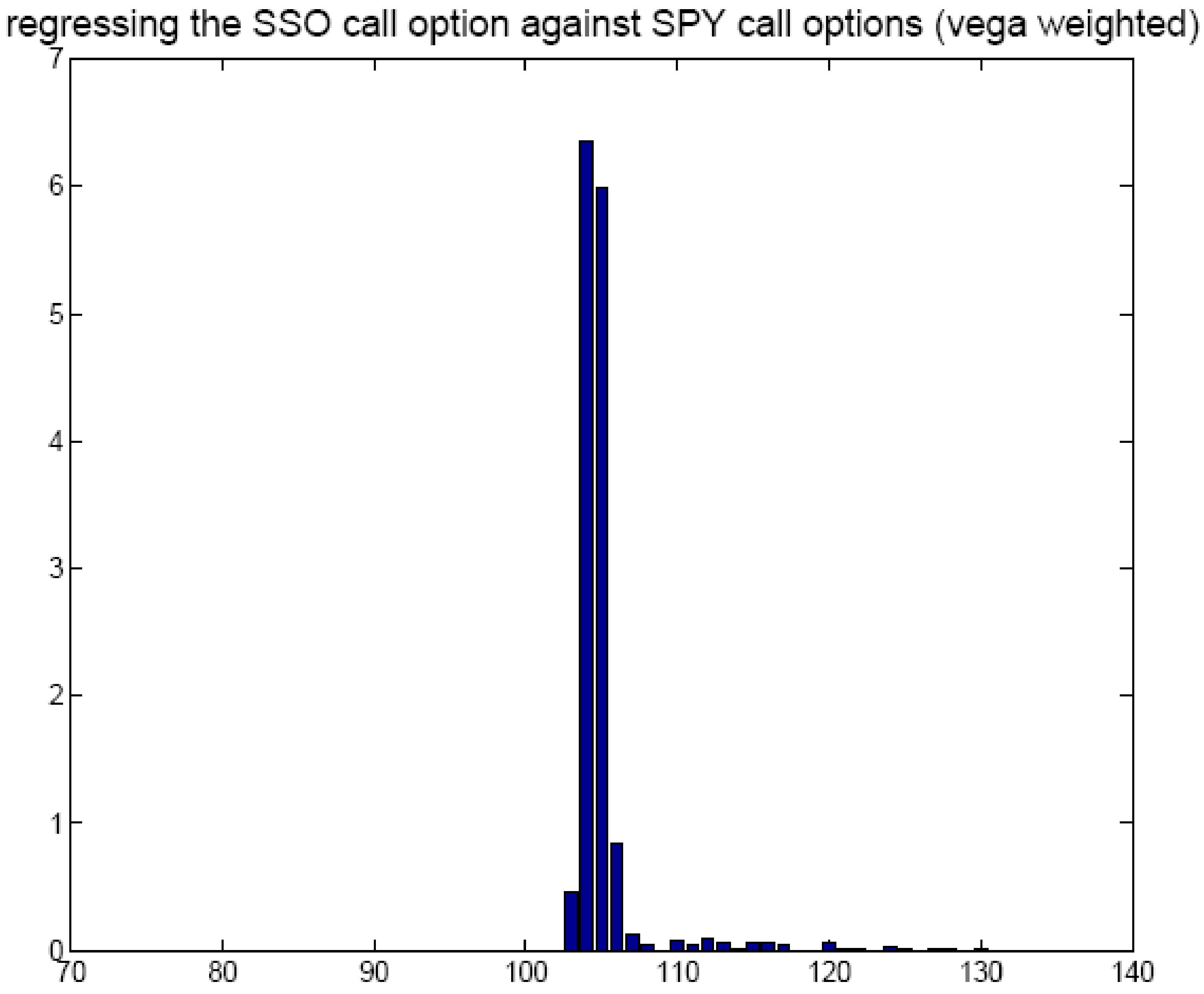
$$C_{L^+}(k,t) = \frac{L_0}{S_0} e^{\frac{\beta-\beta^2}{2}V_t} \beta\left((k^*)^{\beta-1} C_S(S_0 k^*, t) + (\beta-1)\int_{k^*}^{\infty} K^{\beta-2} C_S(S_0 K, t)\, dK \right)$$

If $\beta = 2$, we obtain

$$C_{L^+}(k,t) = 2\frac{L_0}{S_0} e^{-V_t} \left(k^* C_S(S_0 k^*, t) + \int_{k^*}^{\infty} C_S(S_0 K, t)\, dK \right)$$

We validate this formula by regressing the payoffs of C_L against the payoffs of C_S. The results show that replacing $V_t = \int_0^t \sigma_s^2 ds$ with $E[\int_0^t \sigma_s^2 ds]$ is a good approximation.


regressing the SSO call option against SPY call options
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
70
80
90
100
110
120
130
140



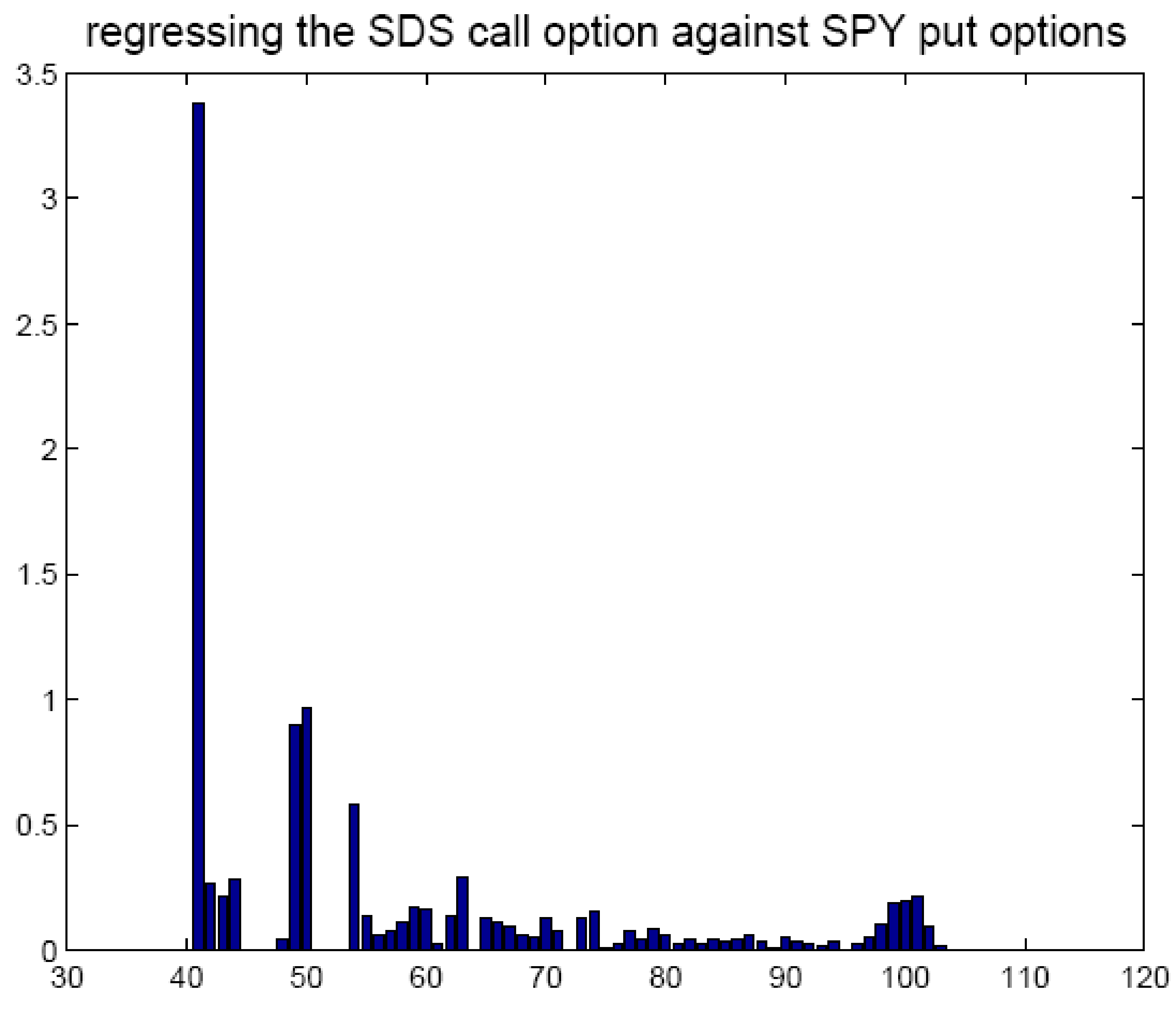
regressing the SSO call option against SPY call options (vega weighted)
7
6
5
4
3
2
1
0
70
80
90
100
110
120
130
140

Hedging options on LETFs (Cont'd)

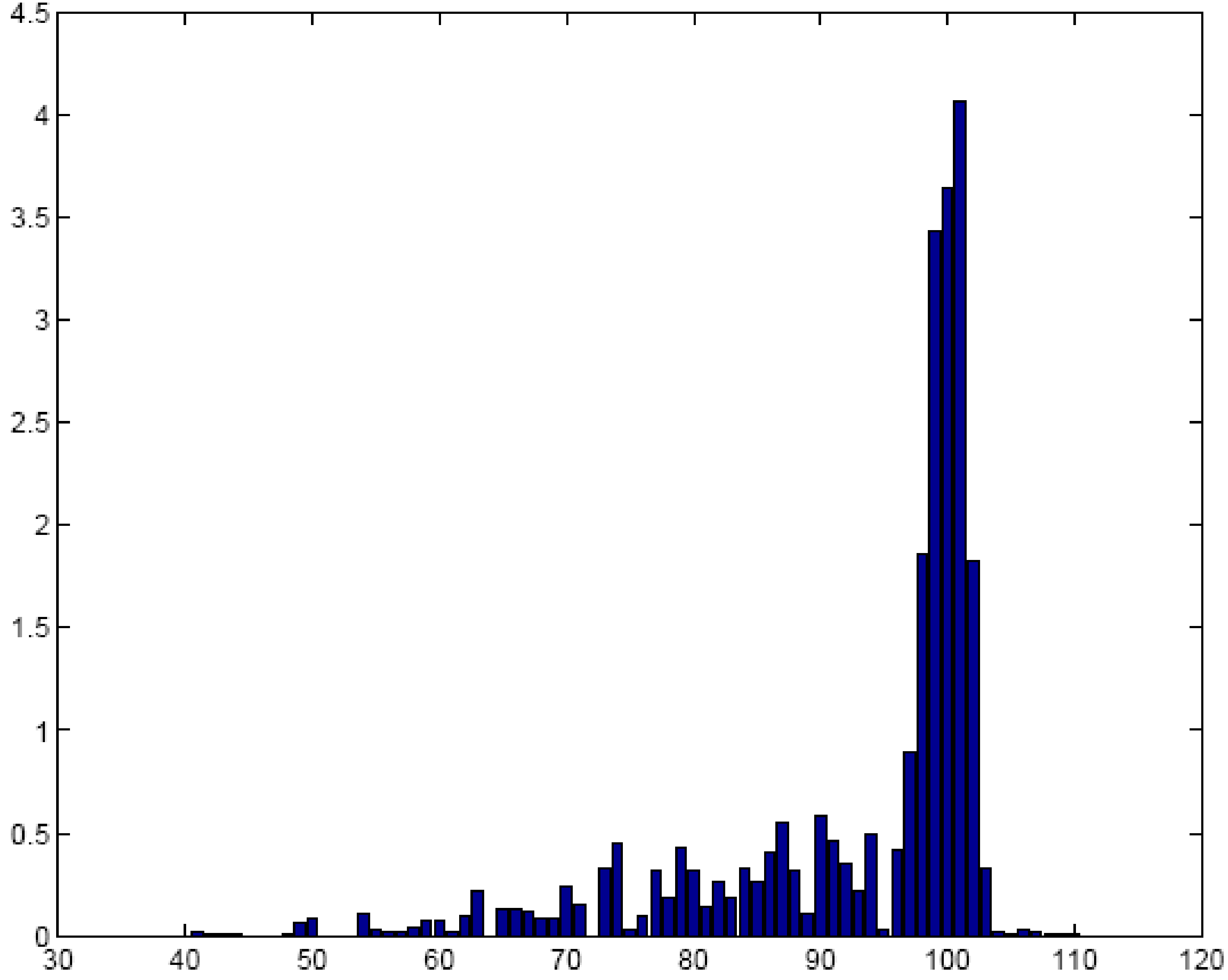
$$C_{L^-}(k,t) = \frac{L_0}{S_0} e^{\frac{\beta-\beta^2}{2} V_t} \, |\beta| \left((k^*)^{\beta-1} P_S(S_0 k^*, t) + (1-\beta) \int_0^{k^*} K^{\beta-2} P_S(S_0 K, t) dK \right)$$

If $\beta = -2$, we have

$$C_{L^-}(k,t) = 2 \frac{L_0}{S_0} e^{-3V_t} \left((k^*)^{-3} P_S(S_0 k^*, t) + 3 \int_0^{k^*} K^{-4} P_S(S_0 K, t) dK \right)$$


regressing the SDS call option against SPY put options
3.5
3
2.5
2
1.5
1
0.5
0
30
40
50
60
70
80
90
100
110
120

regressing the SDS call option against SPY put options (vega weighted)

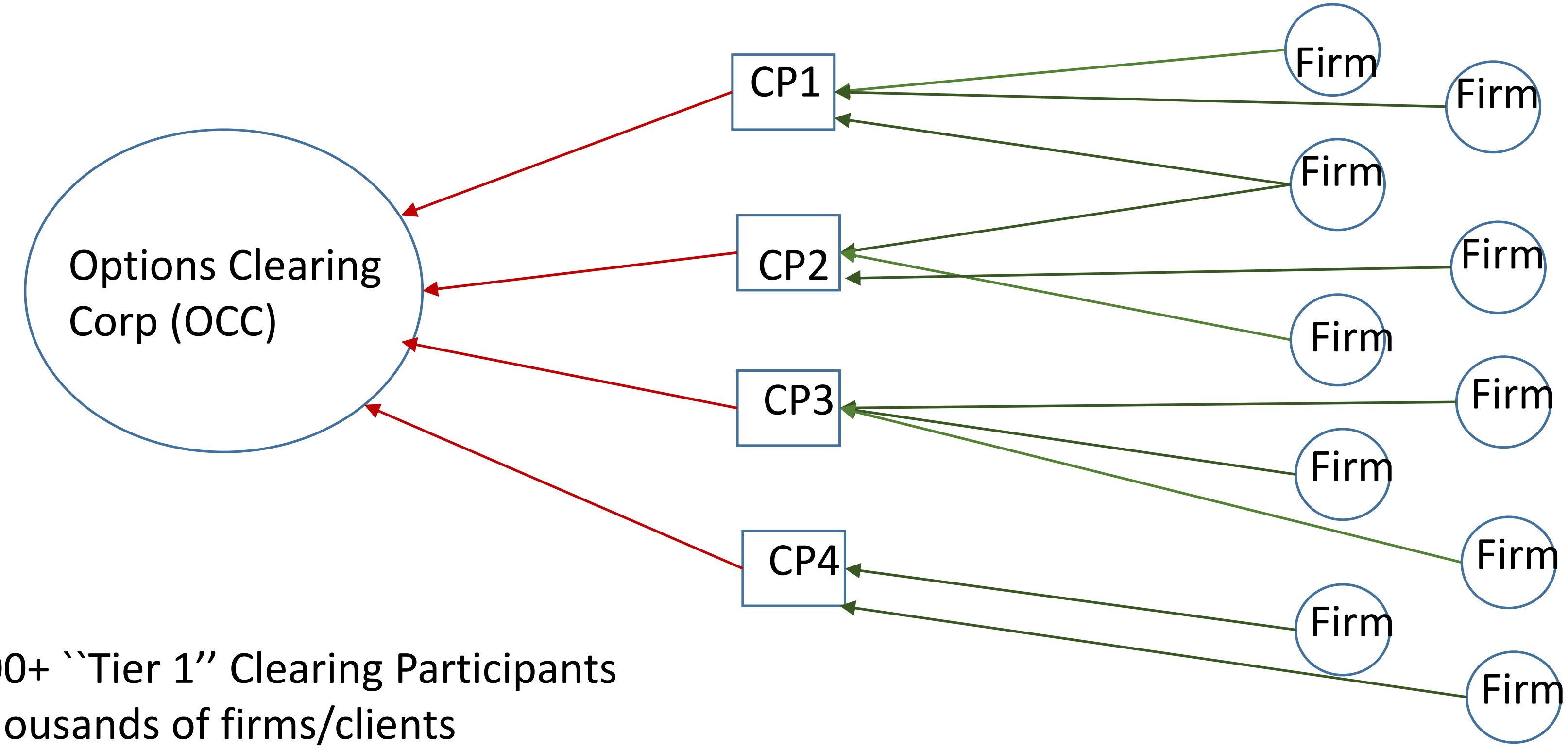


4. Focus on Risk Management

U.S. Equity Derivatives in Numbers

- Number of underlying securities (Stocks, Indices, ETFs) : ~ 9,000
- Number of Open contracts per underlying asset: ~ 100 (average)
- Total number of open contracts: ~ 1,000,000
- Professional trading firms position size ~ 25,000+ positions
- Size of Daily Mark-to-Market (including distributions, rates, Greeks) : 60 MB compressed zip file
- 5 Years historical MTM : 75 GB
- Commercial products: Hanweck Option Volatility Service, IVY OptionMetrics
- Intraday data: orders of magnitude larger!

The market Infrastructure: Clearing & Initial Margin



Firm: non-clearing organization or retail client, HF

- 200+ ``Tier 1'' Clearing Participants
- Thousands of firms/clients

Clearing Participants' Risk Management : STANS ``System for Theoretical Analysis and Numerical Simulations'' (2006)
Non-clearing firms' Risk Mgmt.: TIMS, CPM ``Customer Portfolio Margin'' (SEC approved IM for non-clearing firms)

Customer Portfolio Margin
(FINRA rule 4210)

- Apply stress tests or slides by using mathematical formulas to create new market values for positions based on theoretical movements of the underlying stock
- Move the price of the underlier by between **+6% and -8% at 10 equal intervals** for bb indexes & options
- Move by +15% and -15% for ETF, equities & options
- Sum worst losses for each separate underlying stock to obtain final CPM margin requirement
- CPs must use an SEC/FINRA approved model to margin their clients (at least)
- Currently, the Option Clearing Corporation's TIMS is the only approved model

Note: **very rigid,** does not recognize any correlations except for Broad-Based Indexes. (Basically, 1980's approach).

Reference: http://www.optionsclearing.com/risk-management/cpm/

STANS: Initial Margin For CPs (2006)

- Grids are replaced by one Monte Carlo Simulation for 2-day changes in all (correlated) underlying prices
- Amplitudes of moves based on estimated Standard Deviations, **correlations** between underlying stocks taken into account via MC
- Portfolio is re-priced using **10,000 theoretical changes** of the underlying stocks and Black-Scholes model based on MC.

$$Base\ Charge = ES_{99\%}$$

(Expected shortfall @ 99%)

$$Dependence\ Charge = 0.25 \times \left[\max\left(ES^{H}_{99.5\%}, ES^{\rho=1}_{99.5\%}, ES^{\rho=0}_{99.5\%}\right) - ES_{99\%}\right]$$

(Correlations scenarios)

$$Concentration\ Charge = 0.25\ \times \left[\ {}^{2}cES_{99.5\%} +^{2} rES_{99.5\%} - ES_{99\%}\right]$$

(Worst 2-asset portfolio)

$$STANS\ IM = Base\ Charge + Dependence\ Charge + Concentration\ Charge$$

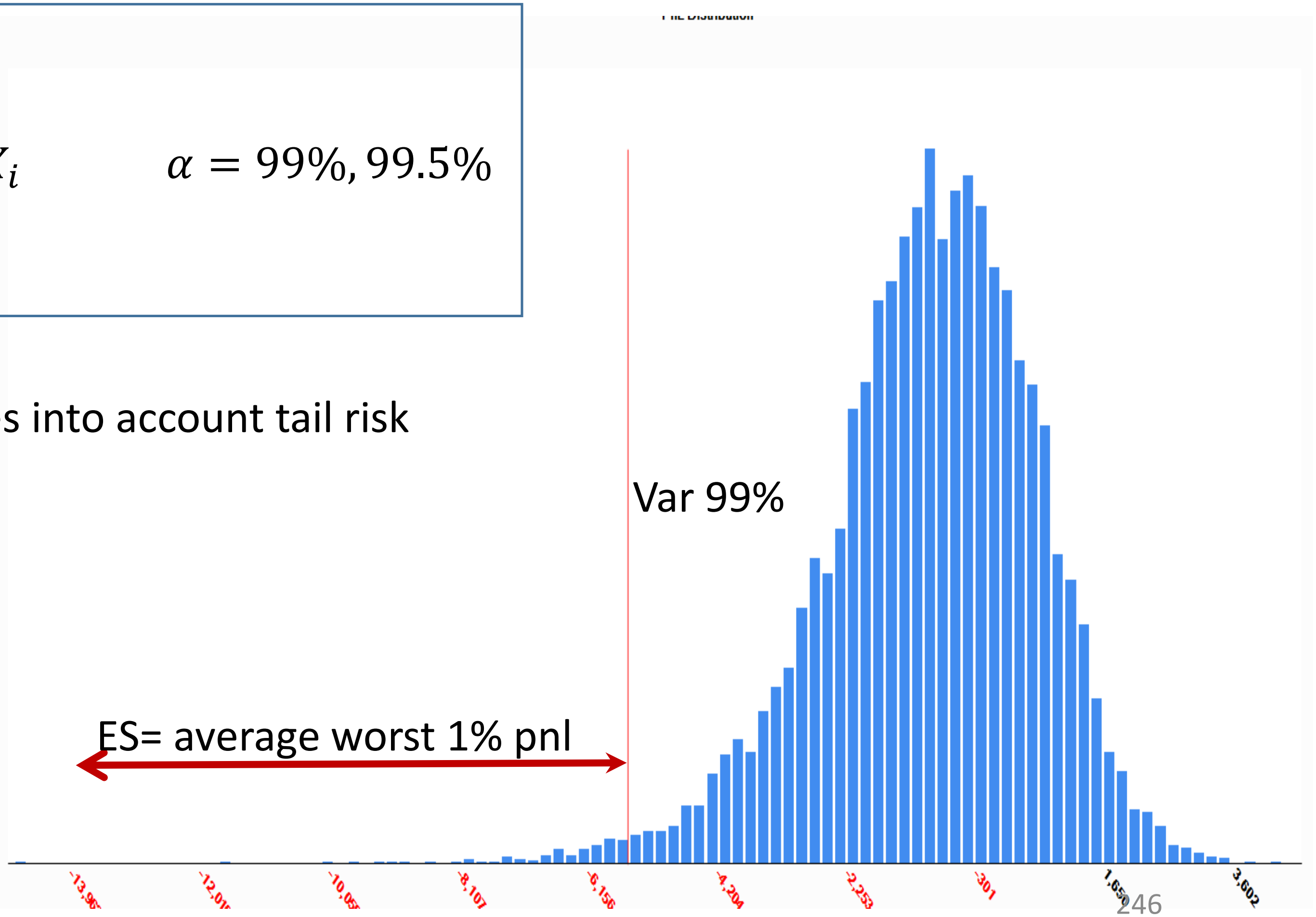
http://www.optionsclearing.com/risk-management/margins/

Review: Expected Shortfall (ES)

- Given $N = 10{,}000$ scenarios for theoretical portfolio changes: $X_1 < X_2 < X_3 < \cdots < X_N$

$$ES_\alpha = \frac{1}{N(1-\alpha)} \sum_{i=1}^{N(1-\alpha)} X_i \qquad \alpha = 99\%, 99.5\%$$

- ES is better than Value at Risk because it takes into account tail risk beyond VaR



Improving STANS (2013-2016)

- STANS ``scenarios'' only take into account changes in the underlying asset, but
- STANS does not move the implied volatility (IVOL)

STANS:	$BS(S,T,K,\sigma) \to BS(S+\Delta S,T,K,\sigma)$
NEW STANS:	$BS(S,T,K,\sigma) \to BS(S+\Delta S,T,K,\sigma+\Delta\sigma)$

- Motivation: IVOL risk is more important than underlying stock risk for long-dated options
- Futures and ETFs referencing the VIX volatility index blur the boundary between what is an underlying asset and implied volatility.
- M. A. and Finance Concepts LLC advised the OCC in creating the improved STANS
- Improved STANS was recently approved by SEC

The New STANS (SEC Filing)

SECURITIES AND EXCHANGE COMMISSION
Washington, D.C. 20549

Form 19b-4

Advance Notice
by

THE OPTIONS CLEARING CORPORATION

Pursuant to Rule 19b-4 under the
Securities Exchange Act of 1934

Item 1. **Text of the Advance Notice**

In accordance with Section 806(e)(1) of the Payment, Clearing, and Settlement Supervision Act of 2010 ("Payment, Clearing and Settlement Supervision Act")[1] and Rule 19b-4(n)(1)(i)[2] of the Securities Exchange Act of 1934 ("Act"),[3] this advance notice is filed by The Options Clearing Corporation ("OCC") in that would modify OCC's margin methodology by incorporating variations in implied volatility for "shorter tenor" options within the System for Theoretical Analysis and Numerical Simulations ("STANS").

Item 2. **Procedures of the Self-Regulatory Organization**

The proposed change was approved for filing with the Commission by the Board of Directors of OCC at a meeting held on May 20, 2015.

Questions should be addressed to Stephen Szarmack, Vice President and Associate General Counsel, at (312) 322-4802.

Item 3. **Self-Regulatory Organization's Statement of the Purpose of, and Statutory Basis for, the Advance Notice**

Not applicable.

Item 4. **Self-Regulatory Organization's Statement on Burden on Competition**

Not applicable.

Item 5. **Self-Regulatory Organization's Statement on Comments on the Advance Notice Received from Members, Participants or Others**

Written comments were not and are not intended to be solicited with respect to the

http://www.optionsclearing.com/components/docs/legal/rules_and_bylaws/sr_occ_15_804.pdf

Principles and construction of the market scenarios

1. Statistical Model

- Identify the risk factors (stocks, IVOLS) – data model
- Estimate the Volatility of the Risk Factors
- Estimate Correlations between Risk Factors

2. Numerical Implementation

- Perform Monte-Carlo Simulation of changes in RFs for 2-day horizon
- Using the N random scenarios, re-value all the listed options with open interest

3. Technological aspect

- Generate binary file with 10,000 theoretical price changes for each instrument and distribute via net

Statistical Model

- How can we parametrize the options market for a given underlying asset?

  Answer : Use the ``implied volatility surface''

- How can we parametrize the volatility surface with the ``right'' number of degrees of freedom

  Answer: Use principal components analysis on the correlation of IVOLs for each asset to find a minimal set of risk factors

Academic study (M.A., Doris Dobi Ph.D., Finance Concepts)

- Data source: IVY OptionMetrics (available at WRDS for colleges), which gives EOD prices from OPRA
- Size of the problem: ~ 9000 optionable securities with 130 delta-maturity points for each security: approximately 1MM variables
- This study: 4,000 optionable securities with 52 delta-maturity points per underlying asset + underlying asset
- Use smoothing of implied volatilities of options to generate a constant-maturity, constant-moneyness dataset for each day:

$$\underbrace{\delta = (20,25,30,\ldots,75,80,100)}_{\text{BS Delta (13 strikes)}}, \qquad \underbrace{\tau=(30,91,182,365)}_{\text{4 settlement dates}}$$

- Historical period: August 31, 2004 to August 31, 2013

http://www.math.nyu.edu/faculty/avellane/DorisDobiThesis.pdf

Correlation Analysis for Stock and IVOL surface

- For each underlying stock, ETF or index, we formed the matrix

$$X = \begin{bmatrix} X_{1,1} & \cdots & X_{1,53} \\ \vdots & \ddots & \vdots \\ X_{T,1} & \cdots & X_{T,53} \end{bmatrix}$$

T=1257 (5 years history)

$X_{t,i}$ = standardized returns of stock (i=1) or IVS surface point labeled i

- Perform an SVD of the volatility surface for each underlying asset in the dataset.

- Analyze eigenvectors and eigenvalues to find out how correlated the options are

Principal Component Analysis of the IVS Correlation matrix using RMT

- How many ``significant'' eigenvalues/eigenvectors do the correlation matrices of equity options have?

- **Random matrix theory:** if X is a random matrix of IID random variables with mean zero and variance 1, of dimensions $T \times N$, the histogram of the eigenvalues of the correlation matrix

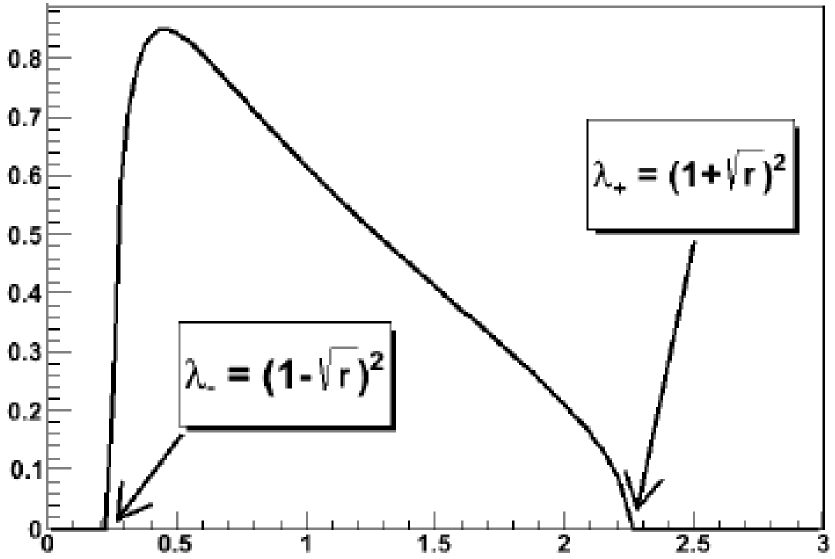
$$C = \frac{1}{T} X X'$$

approaches, as N and T tend to infinity with ratio N/T=γ, the Marcenko-Pastur distribution:

$$\frac{\#\{\lambda: \lambda \leq x\}}{N} \quad \to \quad MP(\gamma; x) = \int_0^x f(\gamma; y) dy$$

$$\mathrm{N} \to \infty, \frac{N}{T} \to \gamma$$

Marcenko-Pastur distribution & threshold



$$f(\gamma;x) = \left(1 - \frac{1}{\gamma}\right)^{+} \delta(x) + \frac{1}{2\pi\gamma} \frac{\sqrt{(x-\lambda_-)(\lambda_+ - x)}}{x} \qquad \lambda_- \le x \le \lambda_+$$

$$\lambda_- = (1-\sqrt{\gamma})^2 \quad \lambda_+ = (1+\sqrt{\gamma})^+$$

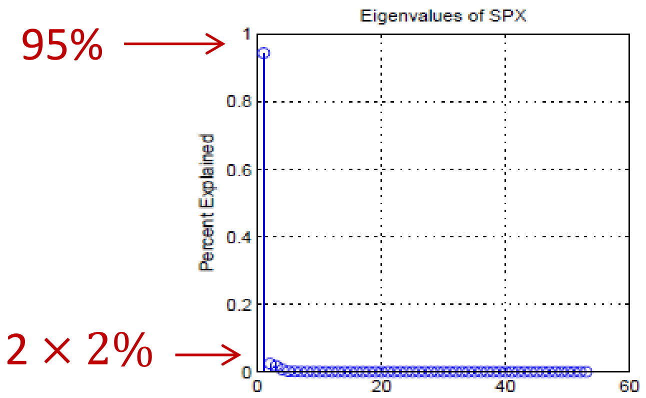
Marcenko-Pastur threshold

The theoretical top EV for N=53 and T=1250 is $\lambda_+ = \left(1 + \sqrt{\frac{53}{1257}}\right)^2 = 1.45$

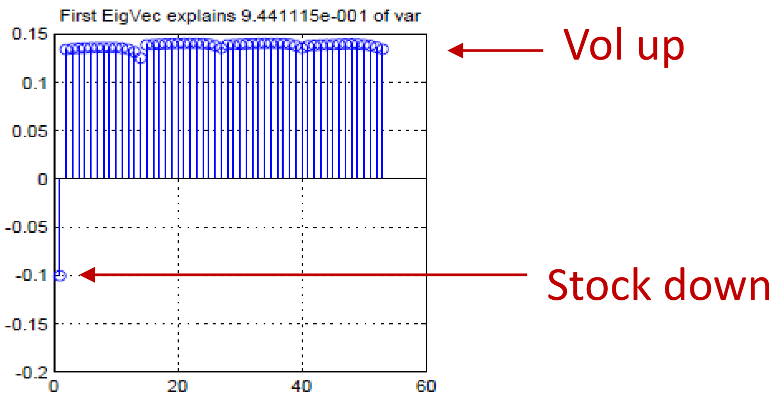
Eigenvalues of the correlation matrix associated with non-random features should lie **above the MP threshold** (within error; Laloux, et al (2000),Bouchaud and Potters (2000))

Analysis of SPX volatility surface

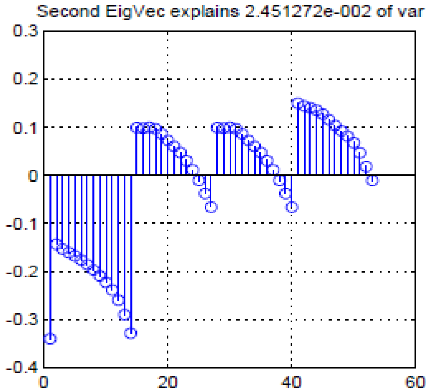
Spectrum



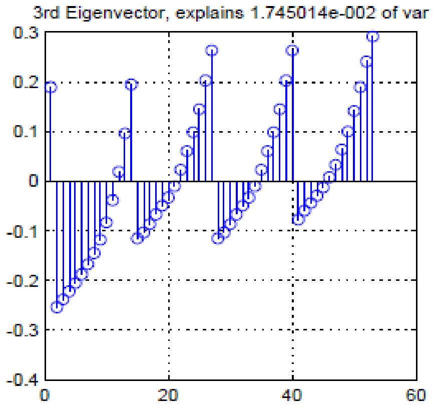
First eigenvector



Lambda_1	50.04
Lambda_2	1.3
Lambda_3	0.93



Second eigenvector

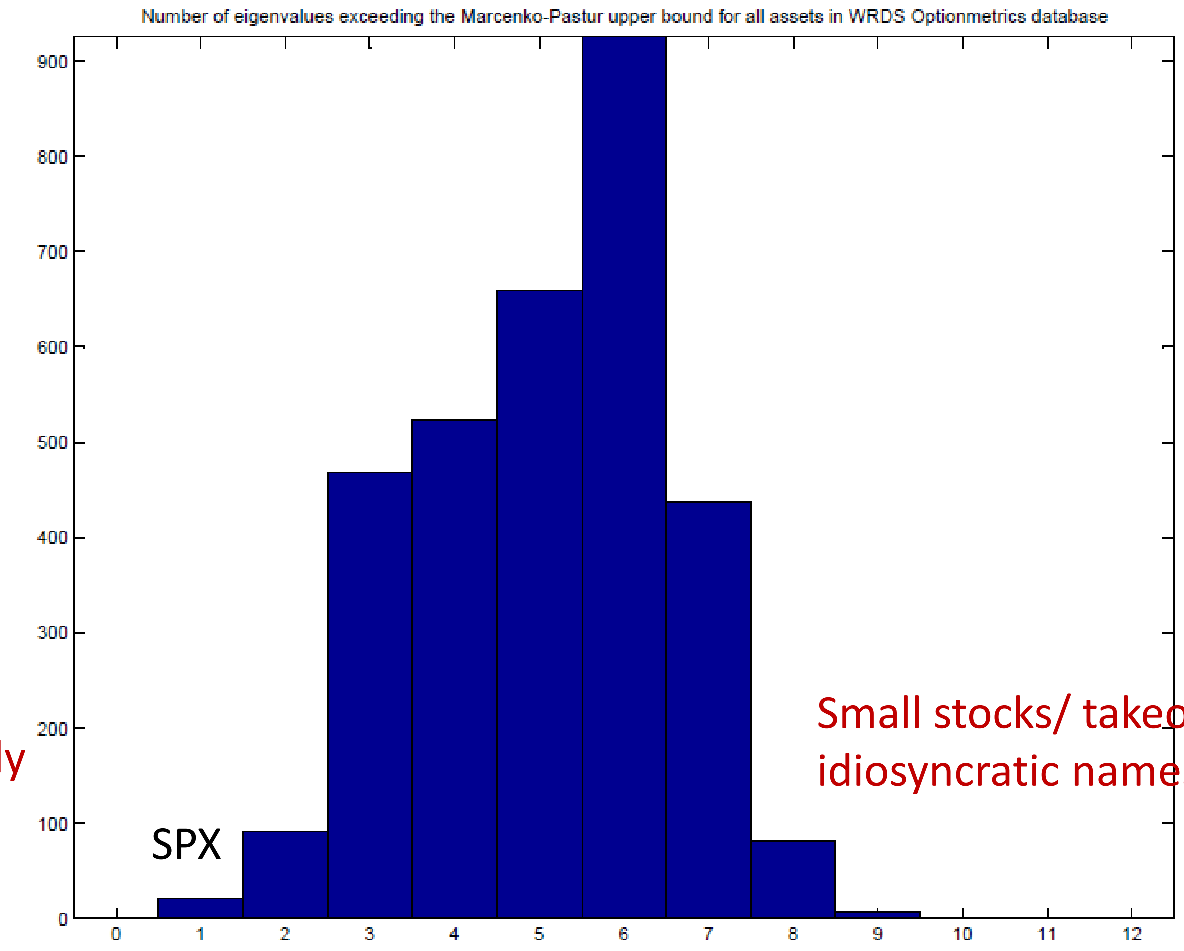


Third eigenvector

MP= 1.45

1 significant eigenvalue!
(out of 53)

Number of EVs above the MP threshold for all assets



Well-known/ broadly traded names

Small stocks/ takeovers/ idiosyncratic names

Dimension reduction

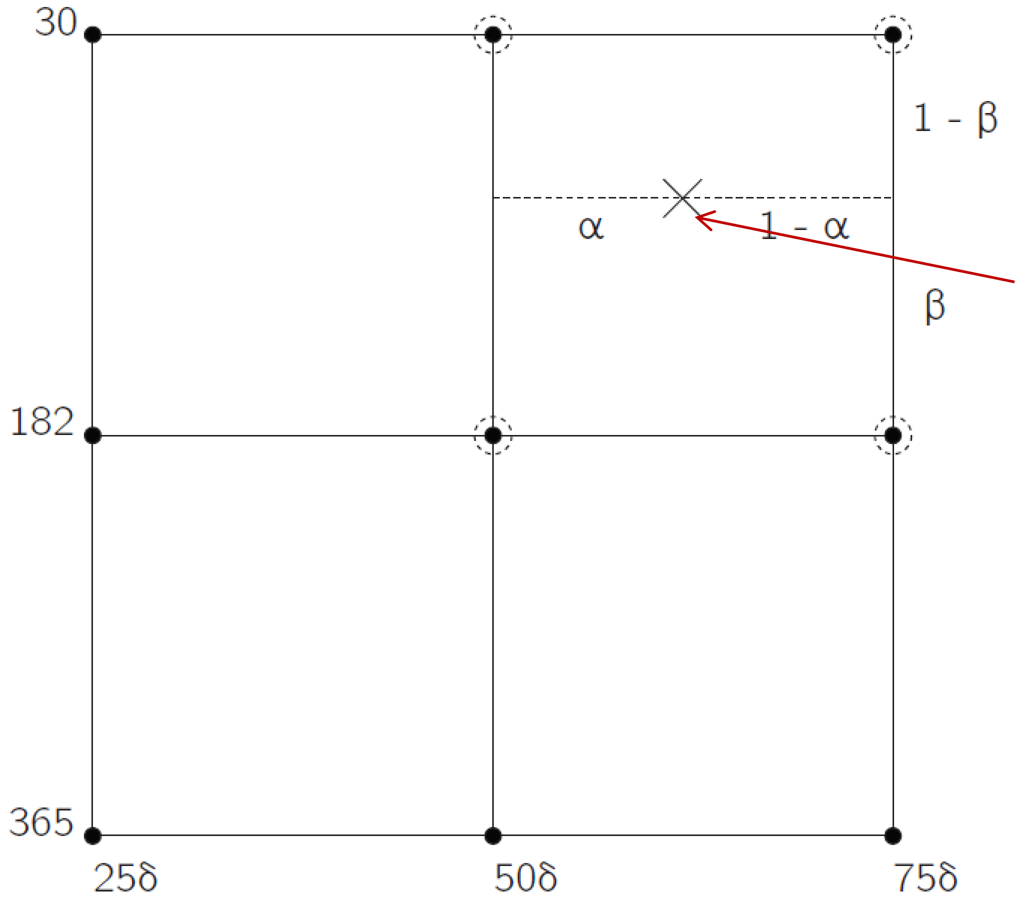
- Knowing that DF<=9, from PCA, choose a small set of points on the IVS and their fluctuations to model the changes in implied volatilities for that underlying.

- A ***pivot*** is a point on the delta/tenor surface used as a risk factor

- *A **pivot scheme**:* is a grid of pivots, which will be used to interpolate the implied volatility returns.

- Goal: find a pivot scheme that approximates well movements of the full volatility surface

9-pivot scheme



The change in the IVOL at this point
is the linear interpolation of the changes
for the 4 surrounding pivots

The pivot schemes that we tested

	2-Pivots	4-Pivots	5-Pivots	6-Pivots	7-Pivot	9-Pivots	12-Pivots
25δ 30				YES	YES	YES	YES
50δ 30	YES	YES	YES			YES	YES
75δ 30				YES	YES	YES	YES
25δ 91				YES	YES		YES
50δ 91					YES		YES
75δ 91				YES	YES		YES
25δ 182		YES	YES			YES	YES
50δ 182			YES			YES	YES
75δ 182		YES	YES			YES	YES
25δ 365				YES	YES	YES	YES
50δ 365	YES	YES	YES			YES	YES
75δ 365				YES	YES	YES	YES

The 9-pivot scheme is a good statistical approximation of the motion of the surface and produced reasonable portfolio Value at Risk in ``backtesting'' (tail risk for portfolios).

Modeling the Volatility of Risk-Factors (EWMA)

$$X_{n+1} = \sigma_n \epsilon_{n+1}$$

$$\sigma_{n+1}{}^2 = \sigma_n{}^2 + \alpha X_{n+1}{}^2 - \beta \sigma_n{}^2$$

GARCH 1-1 model
(Engle & Granger)

This model has ``persistence'' built in, in the sense that the change In volatility is affected by the contemporaneous squared-return, but with memory loss. This can be ``estimated from data'' or can be used as a paradigm for modeling volatility.

$$\boxed{\sigma_n{}^2 = \frac{\beta}{1-(1-\beta)^{T+1}} \sum_{j=0}^{T} (1-\beta)^j X_{n-j}{}^2}$$

Exponentially weighted
moving average of
past square returns

Putting the model together (inter-commodity correlations)

- We determined that for each equity and its listed options, the **9-pivot model** is sufficient to describe statistically the changes in the entire market

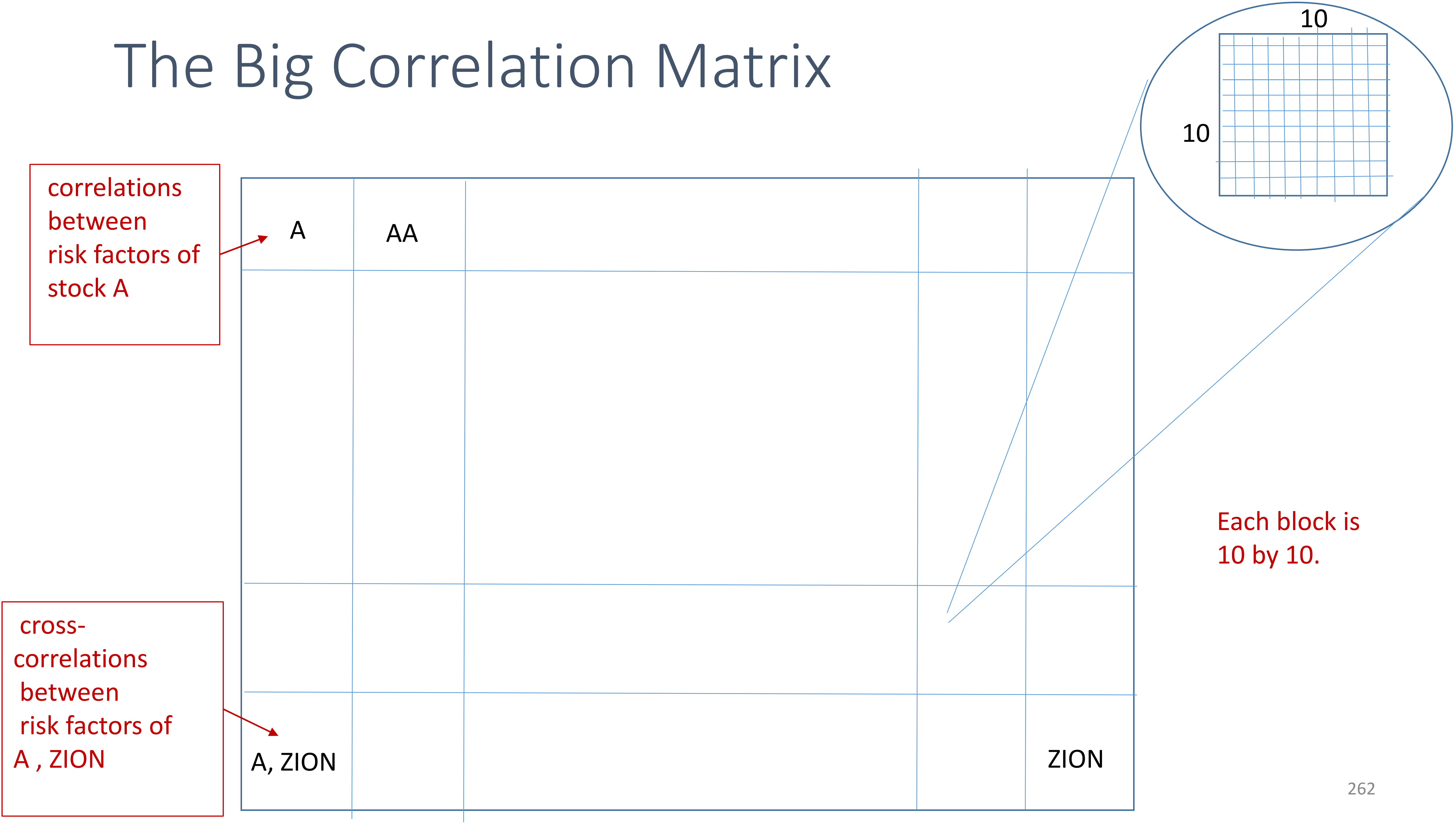
- Use this information to estimate **the joint correlation matrix** of all stocks/IVOLS in the DB.

- Experiment: We study ~ 3000 equities over 500 days. The dimensionality in column space (number of risk-factors) is N= $3{,}000 \times 10 = 30{,}000$. The number of rows is 500.

- We have to model a correlation matrix of roughly $30K \times 30K$.

- Idea: Perform PCA on the full correlation matrix of all ``pivot returns'' (30,000). Extract significant eigenvalues and eigenvectors

The Big Correlation Matrix



Each block is 10 by 10.

Marcenko-Pastur Threshold for Big Matrix

- The MP Threshold is

$$\lambda_+ \approx \left(1 + \sqrt{\frac{31410}{500}}\right)^2 = 79.67$$

- This suggests that we keep eigenvalues above 79.67 and declare that the rest is noise....

- Question : how many EVs exceed (significantly) the threshold level 79.67?

Answer: There are ~ 108 significant Evs in the options market

Top 110 Eigenvalues	s-value	$F_1(s)$
$\lambda_1 = 3742$	24843	1
$\lambda_5 = 209.27$	879.14	1
$\lambda_{10} = 143.5$	433.04	1
$\lambda_{20} = 118.19$	261.32	1
$\lambda_{40} = 102.62$	155.74	1
$\lambda_{50} = 97.40$	120.35	1
$\lambda_{70} = 90.48$	73.35	1
$\lambda_{90} = 84.56$	33.21	1
$\lambda_{107} = 80.21$	3.70	.9996
$\lambda_{108} = 80.04$	2.60	.996
$\lambda_{109} = 79.65$	-.10	.80
$\lambda_{110} = 79.41$	-1.71	.35

MP threshold

Numerically, this implies that we need to calculate only the top 108 eigenvalues/eigenvectors of the `raw' correlation matrix.

Monte Carlo Simulation*

$$X = \Sigma R^{1/2} \mathrm{Z}$$

Where

X = vector of changes in all risk-factors ($\mathrm{N} \times 10$)

$\Sigma =$ diagonal matrix of estimated EWMA standard deviations (2-day changes)

$R^{1/2}$= square-root of the estimated correlation matrix of X (SVD, 108 top eigenvalues)

Z = vector of standardized uncorrelated random variables with suitable probability distributions

10,000 random draws of Z give 10,000 scenarios for risk factors

* Simplified for the purposes of this presentation. The data inputs are the same as in the true model.

Numerical Linear Algebra

- Our first calculations of spectra and eigenvalues for the Big Correlation Matrix were hopelessly slow.
- Storage issues (get more RAM!)
- SVD calculations without care are $O(N^3)$ where N is the number of factors
- Fortunately, a series of techniques used by Data Mining and Big Data scientists can be applied to reduce computational times dramatically
- Idea: sample the column data and the row data randomly or pre-multiply data by a random matrix.

Fast SVD, low rank approximations

Let A be a ``data matrix'': m rows, n columns

$$\underset{m \times n}{\begin{pmatrix} \\ A \\ \\ \end{pmatrix}} = \underset{m \times m}{\begin{pmatrix} \\ U \\ \\ \end{pmatrix}} \cdot \underset{m \times n}{\begin{pmatrix} \\ \Sigma \\ \\ \end{pmatrix}} \cdot \underset{n \times n}{\begin{pmatrix} \\ V \\ \\ \end{pmatrix}}^T$$

We look for a good rank k approximation of A, where k<<n:

$$\underset{m \times n}{\begin{pmatrix} \\ A_k \\ \\ \end{pmatrix}} = \underset{m \times k}{\begin{pmatrix} \\ U_k \\ \\ \end{pmatrix}} \cdot \underset{k \times k}{\begin{pmatrix} \Sigma_k \end{pmatrix}} \cdot \underset{k \times n}{\begin{pmatrix} V_k^T \end{pmatrix}}$$

The best rank k approximation uses the top k eigenvectors of the matrix AA^T.
(The approximation is in the sense of the L2 norm for matrices.)

Rokhlin, Zlam, and Tygert, 2009

- For SVD, approximate the optimal rank-k approximation by multiplying the data by a random $n \times k$ matrix, G (i.i.d. Uniform(0,1)), and performing SVD

$$\Bigg(\qquad\qquad\Bigg) \rightarrow \Bigg(\qquad\qquad\Bigg)\Bigg(\qquad\qquad\Bigg)$$

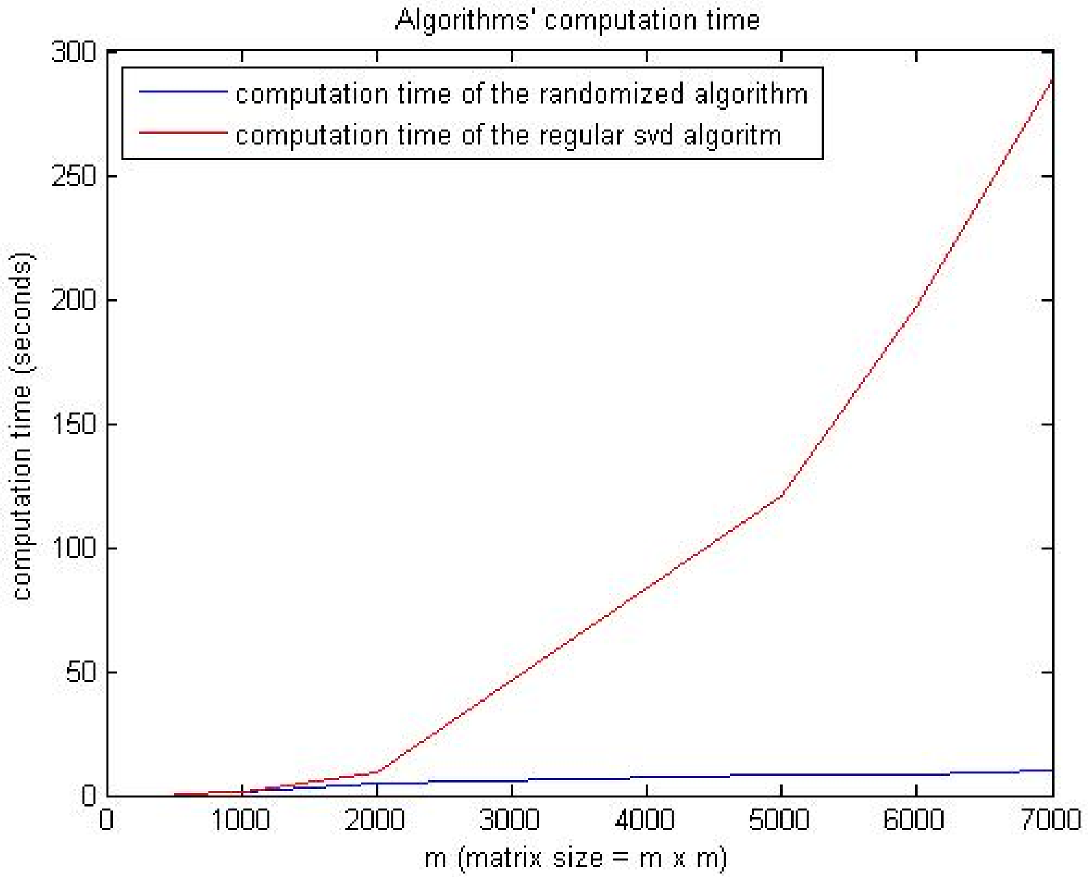
A $\qquad\qquad$ G $\times$ A

- Pre-multiplying has the effect of population sampling (our interpretation) and preserves the correlation matrix of the market. The advantage is that we work with a much smaller matrix.

- Using appropriate choice of k, according to the rank of m (108) , leads to very small errors in the spectrum. (Hence accurate reconstruction of true correlation). ``Offline'' PCA.

Rokhlin, Zlam, and Tygert, 2009 : Fast SVD

- For a data matrix, approximating the optimal rank k approximation (top EVECS /EVALS) by multiplying the data by a random rank k matrix G (i.i.d. Uniform(0,1)).

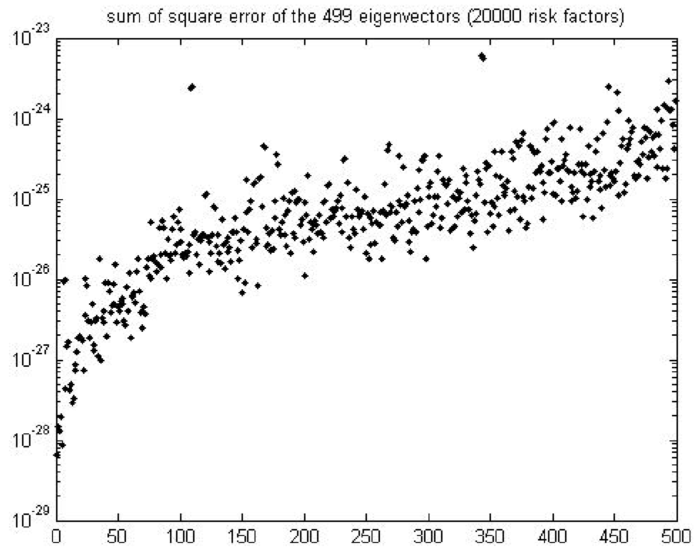


Picture source: Finance Concepts, 2014

All available stocks in OptionMetrics +pivots

Data size: N=31,837, k=500
Computational time, randomized SVD=41 secs
Computational time, regular SVD = too long to observe

For comparison purposes, we also did a 20,000 risk factor matrix.
Data size: N=20,000, k=500
Computational time, randomized SVD=17 secs
Computational time, regular SVD = 4520 secs



Comparison of approximate and actual evs for the top 500 Eigenvalues give errors of order $10^{-28} - 10^{-23}$

Source: Finance Concepts, 2014

Numerical implementation issues

- Due to fast SVD algorithm, the computation of the square-root of the correlation matrix is very fast.

The main bottlenecks that ICEBERG faces are:

- Computing the initial 9-pivots for each surface from closing data
- Repricing all the options with BS under the 10,000 risk scenarios (10 billion BS calculations)
- Getting appropriate servers to deliver the scenarios to users.