Derivatives Homework 4.

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I. Options on Futures

Q1. Let's prove put-call parity of European future option by exploring 2 sets of portfolios.

p.A: 1 European Call future option + cash Ke^{-rT}

p.B: 1 European Put future option + 1 long future contract + cash $F_0 e^{-rT}$, where F_0 is the future price

For portfolio A, cash grows to K at time T. ⇒

Let F_T be the future price at maturity of the option.

If $F_T > K$, the option is exercised, Value(pA) $= F_T$

If $F_T \le K$, the call is not exercised. Value(pA) $= K$.

⇒ Value(p.A) @ time T $= \max(F_T, K)$

For portfolio B, the cash grow to F_0 at time $= T$

The put option gives payoff $\max(K - F_T, 0)$

The future contract gives $F_T - F_0$ $\Rightarrow$.

The value of portfolio B at time T is

$$F_0 + (F_T - F_0) + \max(K - F_T, 0) = \max(F_T, K)$$

Since portfolio A and B worth the same at T,

They must worth the same today.

$$\Rightarrow \quad c + Ke^{-rT} = p + F_0 e^{-rT}$$

put-call parity proved for European future option

Q2.

Suppose T is the maturity date, time to maturity is $T-t$. Suppose C is the price of the call option contingent on f.

Assume stock price process is:

$$df = \mu f dt + \sigma f dz. \quad (\text{Itô Process}) \quad ①$$

Then by Itô Lemma.

$$\Rightarrow dC = \left(\frac{\partial C}{\partial f}\mu f + \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial f^2}\sigma^2 f^2\right)dt + \frac{\partial C}{\partial f}\sigma f dz \quad ②$$

The Wiener processes underlying f and C are the same. $\Rightarrow$ $dz = \varepsilon\sqrt{dt}$ in ① and ② are the same.

Therefore, we can form a portfolio Π to cancel the Wiener process.:

$$\Pi = -C + \frac{\partial C}{\partial f} f.$$

$$\Rightarrow d\Pi = -dC + \frac{\partial C}{\partial f} df \quad ③$$

Replace dC, df in ③ with ① and ②

$$\Rightarrow \partial\Pi = \left(-\frac{\partial C}{\partial t} - \frac{1}{2}\frac{\partial^2 C}{\partial f^2}\sigma^2 f^2\right) dt \quad ④.$$

Since this portfolio Π does not depend on z, it must be riskless during dt. so that no arbitrage is allowed.

$$\Rightarrow \quad d\Pi = r\Pi\, dt. \quad ⑤.$$

Bridge ④ and ⑤ we have.

$$\left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial f^2}\sigma^2 f^2\right) dt = r\left(C - \frac{\partial C}{\partial f} f\right) dt.$$

$$\Rightarrow \frac{\partial C}{\partial t} + rf\frac{\partial C}{\partial f} + \frac{1}{2}\sigma^2 f^2 \frac{\partial^2 C}{\partial f^2} = rC$$

Since the drift of the future price in a risk neutral world is zero, we eliminat $\frac{\partial C}{\partial f}$

$$\Rightarrow \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 f^2 \frac{\partial^2 C}{\partial f^2} - rC = 0.$$

Q3.

Based on the results we got previously, now we substitute C for f, f for S.

$$\frac{\partial S}{\partial t} + (r-q) S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$$

$$dS = (r-q) S\, dt + \sigma S\, dz.$$

$$E[\max(V-K,0)] = \int_K^\infty (V-K) g(V)\, dV.$$

$$m = \ln[E(V)] - w^2/2$$

$$Q = \frac{\ln V - m}{w}$$

$$h(Q) = \frac{1}{\sqrt{2\pi}} e^{-Q^2/2}.$$

$$\Rightarrow E[\max(V-K),0)] = \int_{\ln K - m/w}^\infty (e^{Qw+m} - K) h(Q)\, dQ$$

$$\Rightarrow 1 - N[(\ln K - m)/w - w] = N[(-\ln K + m)/w + w].$$

$$\Rightarrow N\left(\frac{\ln[E(V)/K] + w^2/2}{w}\right) = N(d_1).$$

$$\Rightarrow E[\max(V-K,0)] = e^{m+w^2/2} N(d_1) - KN(d_2).$$

$\Rightarrow e^{(r-q)T} S_0 N(d_1) - KN(d_2)$ is the price of call option in risk neutral world.

Then. $C = e^{-rT}[F_0 N(d_1) - KN(d_2)]$

$$p = e^{-rT}[KN(-d_2) - F_0 N(-d_1)].$$

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(F_0/K) - \sigma^2 T/2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Q4.

See the attached trinomial tree sheet.

The results are relatively close to the Black-76 formulas in terms of the shape.

II. Options on Indices.

Q1.

Futures options are referred by the delivery month of the underlying futures contract - not by the expiration month of the option. Indices options, similar to equity options, are refered by expiration month.

Q2.

The implied forward price is \$3112.80.

The implied dividen yield is 1.8%.

Q3.

Yes, I got similar implied volatilities in the range with min 12.1% and max of 13.5%.

III Options on ETPs.

Q1.

$$q = -\frac{1}{T} \ln \frac{c - p + Ke^{-rT}}{S_0}.$$

Results see attached sheet.

Q2.

The implied violatility is 11.44%.

IV. Lighting round.

See typed doc.