Interest Rate Swaps

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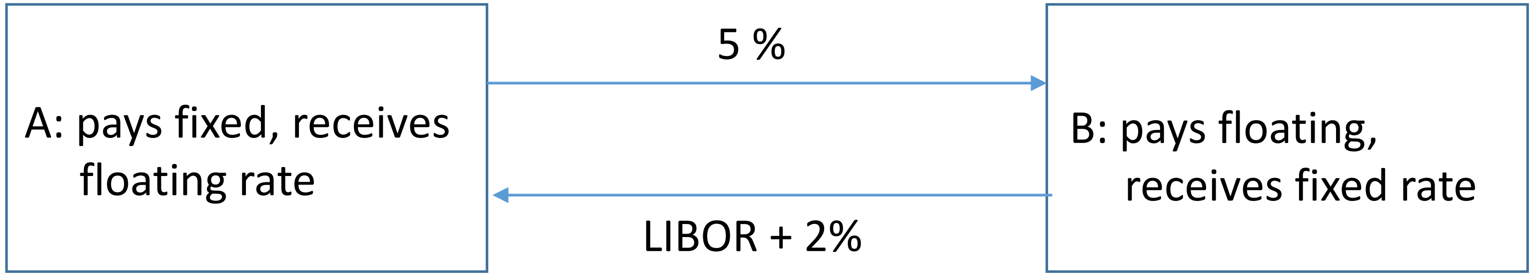
Interest Rate Swaps

- Interest rate swaps (IRS) are OTC (over-the-counter) derivatives
- Most interest rate swaps (70%) are centrally cleared (CME, ICE, LCH, HKEX…)
- This is the largest class of derivatives in terms of notional amount and risk exposures
- IRS are standardized in terms of payment date (IMM dates) and rates (3M LIBOR, 3M Euribor)
- Interest rate swaps are so dominant that many lawyers believe that `swaps' and `derivatives' are synonymous.

Definition of IRS

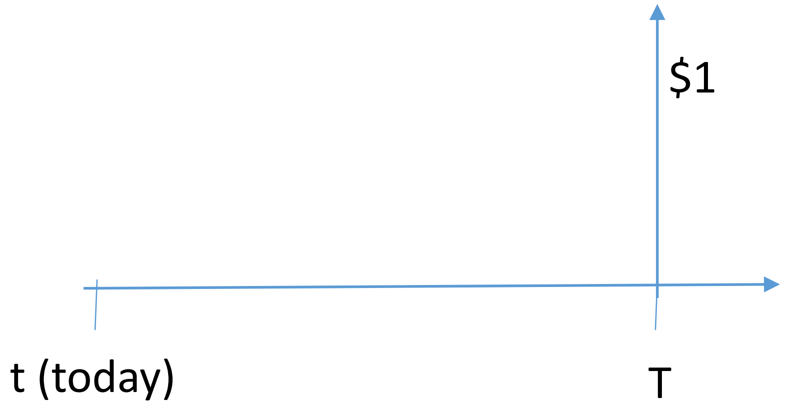


- Swap end-users (non-dealers) fund liabilities, such as bond issuances, using short-term money market or interbank rates (they pay floating)
- Bond investors can hedge the price risk of the bond by entering into a swap in which they pay fixed, receive floating.
- Dealers do both

Review of Bond Math: Zero-coupon bond

- Zero-coupon bond

pays a fixed amount at a future date, no coupon



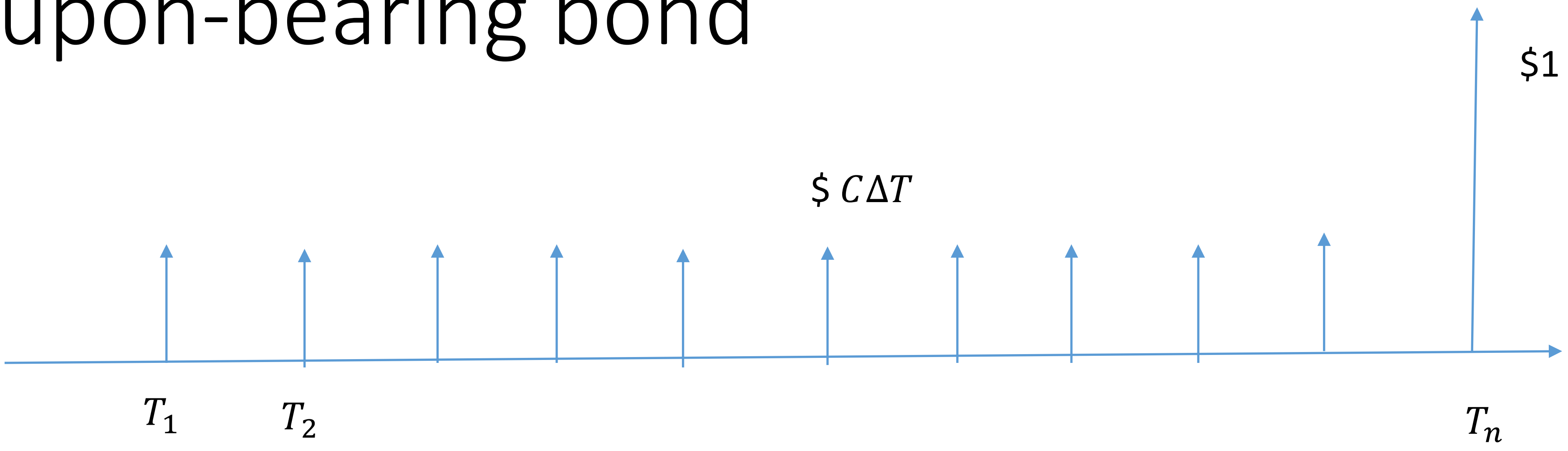
$$Z(t,T) = e^{-R(t,T)(T-t)}$$

$$R(t,T) = -\frac{1}{T-t}\log Z(t,T)$$

$$Z(t,T) = \frac{1}{1 + r(t,T)(T-t)}$$

$$r(t,T) = \frac{1}{T-t}\left(\frac{1}{Z(t,T)} - 1\right)$$

Coupon-bearing bond



$\Delta T = frequency \ (\frac{1}{2}, \frac{1}{12}, 1 \ldots)$

$C = coupon\ rate$

T_i =payment dates

Value of a bond

$$B(t) = \sum_{i=1}^{n} C\Delta T\, Z(t, T_i) + Z(t, T_n)$$

Premium, discount, yield

Discount bond: $B(t) < 1.0 \quad or \quad C < Y_t$

Premium bond: $B(t) > 1.0 \quad or \quad C > Y_t$

Par bond: $B(t) = 1.0 \quad or \quad C = Y_t$

(geometric sum calculation)

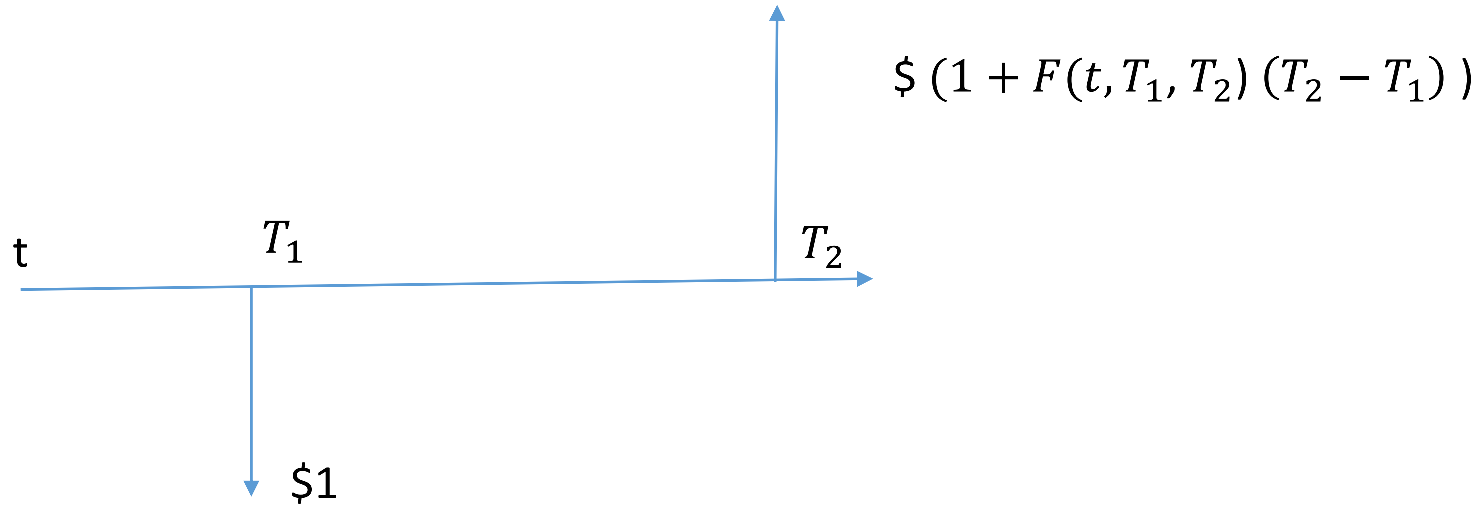
Yield: Y_t

$$B(t) = \sum_{i=1}^{n} \frac{C\, \Delta T}{(1 + Y_t \Delta T)^i} + \frac{1.0}{(1 + Y_t \Delta T)^n}$$

Yield is the internal rate of return. It is used to compare different bonds.

If you go long a bond with high yield and short a bond with low yield, and hold both to maturity, you make $$.
(no default)

Forward Rate Agreement



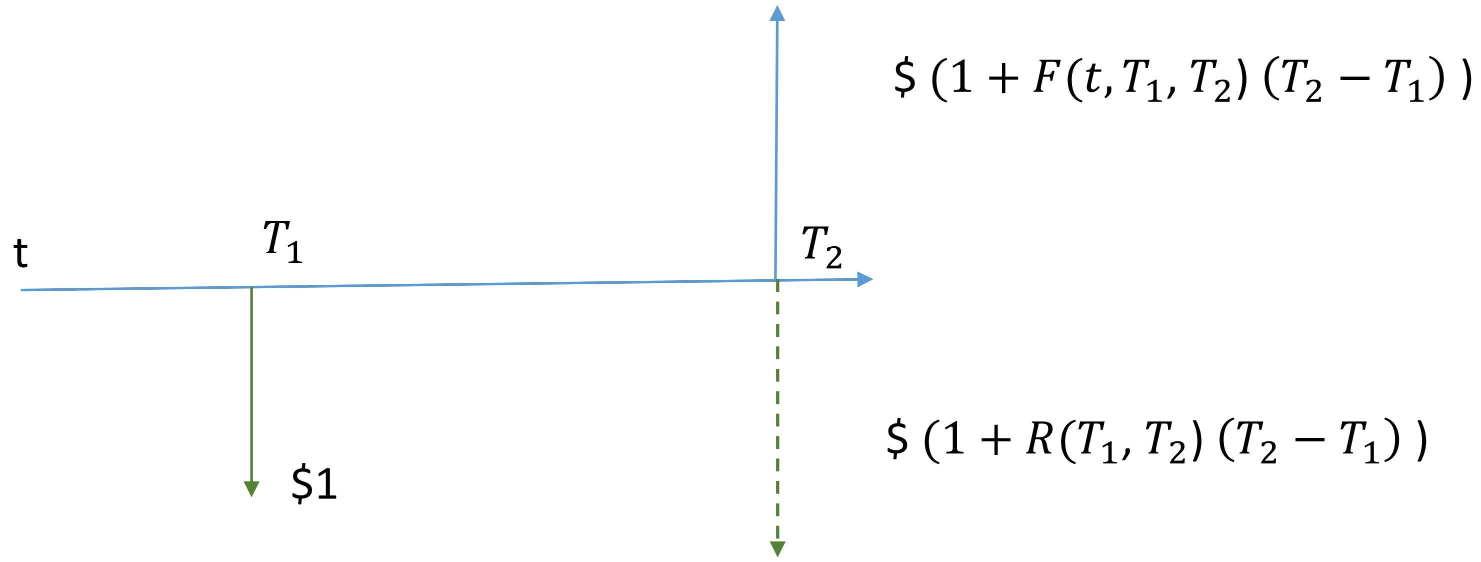
- Borrow $1 at future date T1 and return (with interest) on date T2

Forward rate

$$Z(t,T_2) = Z(t,T_1)\frac{1}{1 + F(t,T_1,T_2)\,(T_2 - T_1)}$$

$$\boxed{F(t,T_1,T_2) = \frac{1}{T_2 - T_1}\left(\frac{Z(t,T_1)}{Z(t,T_2)} - 1\right)}$$

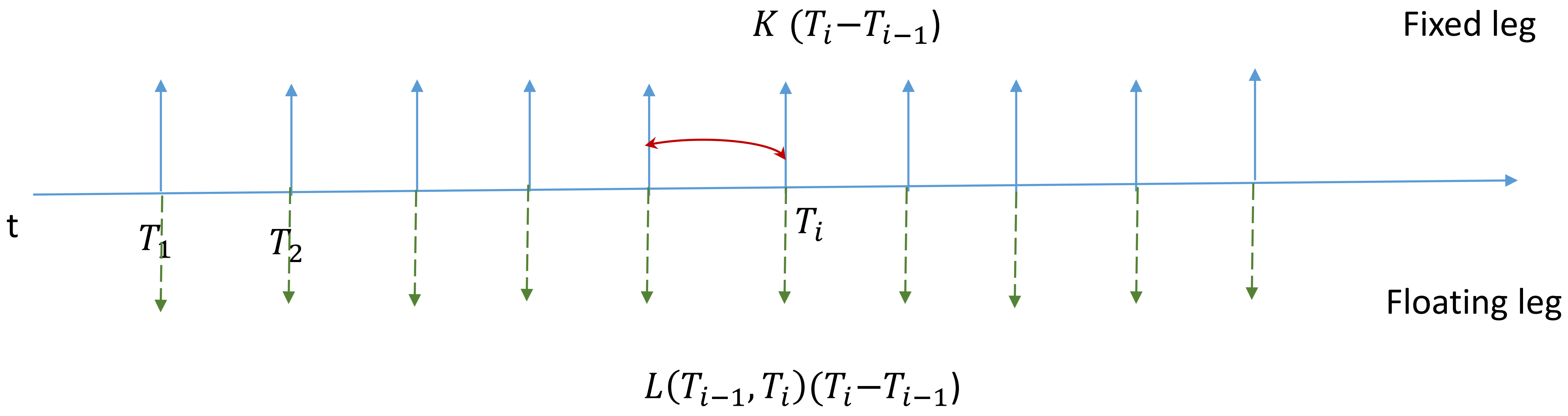
Forward Rate vs Floating Rate



The Forward Rate is the rate that can be swapped for a floating payment in arrears at zero cost

Valuation of IRS

K= swap fixed rate, floating rate = 3M LIBOR or other benchmark rate



Swap Value (fixed payer) = (Value of Floating Leg) – (Value of the fixed Leg)

Values of the Fixed and Floating Legs

Accrued interest

Fixed Leg Value =

$$\sum_{i=1}^{N} K\,(T_i - T_{i-1})\,Z(t,T_i) - \underbrace{K\,(t-T_0)}_{}\,Z(t,T_1)$$

Floating Leg Value= depends on future unknown LIBOR rates… BUT if reference rate is LIBOR for the same Frequency as the swap payments

Floating Leg Value= $\sum_{i=1}^{N} F\,(t,T_{i-1},T_i)(T_i - T_{i-1})\,Z(t,T_i) - L(T_0,T_1)(t-T_0)\,Z(t,T_1)$

Accrued interest

Mathematically, get rid of AI by setting T_0= t

Fixed Leg $\sum_{i=1}^{N} K\ (T_i - T_{i-1})\ Z(t, T_i\ )$

Floating Leg $$\sum_{i=1}^{N} F\ (t, T_{i-1}, T_i)(T_i - T_{i-1})\ Z(t, T_i\ )$$ $F(t, t, T_1)$=$r(t, T_0)$

Exact calculation when rates reset at the swap frequency

If the floating rate frequency is equal to the swap frequency, then

Value of Floating leg $= 1 - Z(t, T_n)$

Proof:

- Borrow \$1 from now to swap expiration date (cost= $Z(t, T_n)$)
- \$ 1 gets remunerated $\$(1 + L(T_{i-1}, T_i)\ (T_i - T_{i-1}))$, generating a cash-flow $L(T_{i-1}, T_i)(T_i - T_{i-1})$
- Reinvesting \$1.0 on each period generates the floating leg of the swap.
- Return the \$1.0 to the lender.

Swap Rate

- The swap rate (S_t) is defined as the rate that makes the fixed and the floating legs equal

$$\sum_{i=1}^{N} S_t\ (T_i - T_{i-1})\ Z(t, T_i) := \sum_{i=1}^{N} F\ (t, T_{i-1}, T_i)(T_i - T_{i-1})\ Z(t, T_i)$$

$$\boxed{S_t := \frac{\sum_{i=1}^{N} F\ (t, T_{i-1}, T_i)(T_i - T_{i-1})\ Z(t, T_i)}{\sum_{i=1}^{N} (T_i - T_{i-1})\ Z(t, T_i)}}$$

The swap rate depends on the curve of forward rates (FRAs) and the zero-coupon bonds: It is therefore observable if these quantities are known.

Compact formula for swap price

$$V(fixed\,payer) = (Floating - Fixed) = \ (S_t - K)\sum_{i=1}^{N} (T_i{-}T_{i-1})\, Z(t, T_i\ )$$

$$\boxed{V(fixed\,payer) = (S_t - K)\, A(t, T_1, \ldots.., T_n)}$$

$A(t, T_1, \ldots.., T_n)$= value of an annuity which pays $1 per year on each of the payment dates

Equal-frequency case

$$S_t := \frac{1.0 - Z(t, T_n)}{\sum_{i=1}^{N} (T_i - T_{i-1})\, Z(t, T_i)}$$

- Most common swaps are such that the floating rate is 3-month LIBOR (the most liquid rate) and the frequency is semi-annual

- The swap rate can be computed using the ``Annuity'' formula but the swap rate is not as above.

- As usual with derivatives, the swap rates matter more than the FRAs, so in practice the forward rate curve is implied from swap rates.

Zero Coupon Bonds & forwards

- Assume that the payment dates are fixed once and for all. Then, the zero coupon bonds can be expressed in terms of forward rates.

Since
$$Z(t,T_2) = Z(t,T_1)\frac{1}{1+F(t,T_1,T_2)\,(T_2-T_1)}$$

Recursively,

$$\boxed{Z(t,T_i) = \frac{1}{1+r(t,T_1)\,(T_1-t)} \times \prod_{j=2}^{i} \frac{1}{1+F(t,T_{j-1},T_j)\left(T_j-T_{j-1}\right)}}$$

Building the spot forward rate curve

- The forward rate curve observed today is the main tool for marking-to-market and risk-managing swap portfolios

- Institutions (banks, end-users, dealers, and CCPs) need to be able to value their swap portfolios and also calculate price risk by making scenarios for how the market can change (in terms of the forward rates).

- Instruments available to build swap curves are daily closing prices for:

  -- Money-market instuments for less than 3 months

  -- Eurodollar futures (CME) from 3 months to 5 years

  -- LIBOR swap spreads at standard maturities: 2,3,5,7,10, 15 and 20 years.

  -- some FRAs …

Mathematical formulation of curve construction problem

- Granularity: 1 month , i.e. 30 years corresponds to 360 points.

- Prepare the data using swap rates and futures. As an example, consider the following problem

ED SPOT
1-month 0.19
ED FUTURES
3-month 0.33
6-month 0.46
LIBOR SWAP SPREADS
1-year 0.64
2-year 0.97
3-year 1.26
4-year 1.48
5-year 1.66
7-year 1.94
10-year 2.21
30-year 2.67

``Brute force'' approach:

- Write all instruments in terms of unknown forward rate vector $\Phi = \{\varphi_1, \ldots, \varphi_{360}\}$

$$\varphi_i = F\left(0, \frac{(i-1)\times 30}{360}, \frac{i\times 30}{360}\right)$$

- Consider the penalization function

$$J(\Phi) = \sum_i a\,(\varphi_i - \varphi_{i-1})^2 + \sum_i b\,(\varphi_{i+1} + \varphi_{i-1} - 2\varphi_i)^2$$

Optimization approach

- Write all instruments in terms of unknown

forward rate vector $\Phi = \{\varphi_1, \dots, \varphi_{360}\}$

$$\varphi_i = F\left(0, \frac{(i-1)\times 30}{360}, \frac{i\times 30}{360}\right).$$

$P_j = P_j(\Phi)$, for j=1,…,M (number of price inputs)

- Consider the penalization function

$$J_{a,b}(\Phi) = \sum_i a\,(\varphi_i - \varphi_{i-1})^2 + \sum_i b\,(\varphi_{i+1} + \varphi_{i-1} - 2\varphi_i)^2$$

Problem to solve:

$$\boxed{\min_{\Phi}\left\{\sum_{j=1}^{M}\left(P_{j,mkt}-P_j(\Phi)\right)^2+J_{a,b}(\Phi)\right\}}$$

The parameters a, b determine the smoothness of the curve. Other penalty functions are also possible.

This problem can be input into an optimization package and an answer can be found in that way

Smoothing the swap rates – an alternative algorithm

- First consider all inputs as equivalent FRA (first ones) or swaps spreads (issue with the floating rate 3M/6M is ignored temporarily).

- Using a monthly ``grid'' compute a function that interpolates smoothly between the swap rates and has a granularity of 1 month (360 grid points)

- Deduce all forward rates and zero coupon rates by using relation between successive swaps. This is called ``bootstrapping''.

- If necessary, implement the $J_{ab}(\Phi)$ to match better the market.