Derivative Securities Fall 2019, Homework 4

1. Options on Futures

An option on a futures contract (call/put) gives the holder to buy/sell a futures contract at a price K on the option expiration date. The option is cash-settled, with payoff $(f_T - K)^+$ (call) or $(K - f_T)^+$ (put). Here f_T denotes the futures price on the expiration date T. In general, options on futures are European-style. The futures contract settlement date may or may not coincide with the option expiration date.

1. If we ignore margin requirements, the profit-loss for holding β futures contracts over the period t, $t + \Delta t$ is $\beta\ (f_{t+\Delta t} - f_t)$. Show that European-style futures options satisfy a Put-Call parity relation and write it down.

2. Show that, under standard Black-Scholes assumptions, the values of options on futures satisfy the PDE

$$\frac{\partial C}{\partial t} + \frac{\sigma^2 f^2}{2}\frac{\partial^2 C}{\partial f^2} - rC = 0, \tag{1}$$

where σ is the volatility and r is the interest rate. Conclude that this is a special case of the Black-Scholes PDE.

3. Verify that, in the case of a European call option, the solution of the PDE and option value corresponds to the expectation

$$C(f, T, K, r, \sigma) = E\Big\{(f\, e^{\sigma\, Z\, \sqrt{T} - \frac{\sigma^2 T}{2}} - K)^+\Big\}, \tag{2}$$

where Z is $N(0, 1)$. Show that it gives rise to a closed-form solution which resembles the Black-Scholes formula. This model is often called the *Black 1976 model*, named after Fisher Black, its inventor.

4. Build a trinomial tree corresponding to Equation (1). Price a put option on the E-mini S&P futures with 95 days to go and strike price $K = \$2900$. The futures trades today at \$3120. Assume an interest rate of 1.5% and a volatility of 17.5%. Compare your price with the one obtained by the Black-76 explicit formula with the same inputs.

2. Options on Indices

An option (put/call) on an index I is a cash-settled contract which allows the holder to sell/buy the index and receive/pay the strike price. Examples of indices could be (a) the S&P 500 index (U.S.), the VIX Volatility Index (U.S.), the CSI index (China) or the Eurostoxx 50 index (Eurozone).

1. What is the difference between an *option on an index* and an *option on a futures contract which settles on the index*? (Hint: look at the expiration date).

2. Assume that the index corresponds to the value of a basket of stocks, e.g, the S&P 500 index. Using the data from Figure 1 and $SPX = \$3120$, derive the implied forward price, the implied dividend yield for the S&P500 index (SPX) for delivery in 90 days.

DEC 19 '19 (31 DAYS) | JAN 16 '20 (59 DAYS) | FEB 20 '20 (94 DAYS) | MAR 19 '20 (122 DAYS) | MORE

TABBED VIEW | All STRIKES | SMART | SPX | 100 | Load My Chains | IV: 12.6%

CALLS							STRIKE	PUTS						
LAST	CHANGE	BID x ASK	VOLUME	IMPLD VL...	DELTA	VEGA		LAST	CHANGE	BID x ASK	VOLUME	IMPLD VL...	DELTA	VEGA
c106.85		107.50 x 108.10		13.6%	0.584	6.183	3080	67.10	+0.50	65.20 x 65.60	54	13.5%	-0.416	6.183
c103.35		104.00 x 104.60		13.4%	0.575	6.211	3085	c68.05		66.60 x 67.00		13.4%	-0.425	6.211
c99.90		100.50 x 101.10		13.3%	0.566	6.236	3090	c69.55		68.10 x 68.50	3	13.3%	-0.434	6.236
c96.45		97.10 x 97.70		13.2%	0.557	6.259	3095	c71.10		69.60 x 70.00	1	13.2%	-0.443	6.259
89.90	-3.20	93.70 x 94.30	99	13.1%	0.548	6.278	3100	c72.70		71.20 x 71.60	15	13.1%	-0.452	6.278
c89.70		90.30 x 90.90		13%	0.538	6.295	3105	c74.30		72.80 x 73.20	4	12.9%	-0.462	6.295
c86.40		87.00 x 87.60	3	12.8%	0.529	6.308	3110	c75.95		74.40 x 74.90	7	12.8%	-0.471	6.308
c83.10		83.70 x 84.30		12.7%	0.519	6.317	3115	77.90	+0.25	76.10 x 76.60	8	12.7%	-0.481	6.317
80.60	+0.70	80.50 x 81.10	24	12.6%	0.509	6.322	3120	c79.40		77.80 x 78.40		12.6%	-0.491	6.322
76.50	-0.20	77.40 x 78.00	6.55K	12.5%	0.499	6.324	3125	c81.20		79.60 x 80.20	53	12.5%	-0.501	6.324
74.00	+0.40	74.30 x 74.90	10	12.4%	0.489	6.321	3130	82.00	-1.05	81.50 x 82.10	9	12.3%	-0.511	6.321
c70.60		71.20 x 71.80	2	12.3%	0.478	6.314	3135	c85.05		83.40 x 84.00		12.2%	-0.522	6.314
c67.60		68.30 x 68.80		12.2%	0.468	6.303	3140	88.80	+1.80	85.40 x 86.00	4	12.1%	-0.532	6.303
c64.70		65.30 x 65.90		12%	0.457	6.287	3145	c89.10		87.40 x 88.00		12%	-0.543	6.287
62.20	+0.30	62.50 x 63.00	3.58K	11.9%	0.446	6.266	3150	c91.25		89.50 x 90.10		11.9%	-0.554	6.266
c59.10		59.70 x 60.20		11.8%	0.435	6.240	3155	c93.45		91.70 x 92.30		11.8%	-0.565	6.240
c56.40		57.00 x 57.50		11.7%	0.424	6.209	3160	c95.70		94.00 x 94.60		11.7%	-0.576	6.209
c53.75		54.30 x 54.90		11.6%	0.413	6.173	3165	c98.05		96.30 x 96.90		11.6%	-0.587	6.173
c51.15		51.70 x 52.30		11.5%	0.402	6.131	3170	c100.40		98.60 x 99.30		11.5%	-0.598	6.131
47.70	-0.95	49.20 x 49.70	205	11.4%	0.390	6.084	3175	c102.90		101.10 x 101.70	3	11.4%	-0.610	6.084
c46.25		46.70 x 47.30	10	11.3%	0.379	6.031	3180	c105.45		103.60 x 104.20	3	11.3%	-0.621	6.031
c43.85		44.40 x 44.90		11.2%	0.368	5.973	3185	c108.05		106.20 x 106.80		11.2%	-0.632	5.973

Figure 1: Snapshot of SPX index options (European-style). Spot price is \$3120. This data can be used to compute the implied forward and implied rate, as well as the dividend yield. Use mid-market call and put prices to estimate the quantities of interest.

3. Compute the implied volatilities of the options with strikes 3100 through 3140 and compare with screen implied volatilities. Do you get similar answers?

3. Options on ETFs

Options on ETFs work like options on stocks: they are listed, American-style. The most traded name is SPY, which tracks the S&P 500 capitalization-weighted basket. Dividends are accumulated and paid out quarterly (see Yahoo!Finance for details).

1. Use the data in Figure 2 to calculate the implied dividend yield, implied forward associated with SPY for expiration March 31 2020. Compute the implied forward rate, the implied discount factor, the implied dividend.

2. Using a trinomial tree scheme, compute the implied volatilities of options with strikes 310 through 320.

Spot	311.97		Date : Nov 19, 2019		Expiry: March 31, 2020	
Calls				Puts		
Price	Volume	Open Interest	strike	Price	Volume	Open Interest
17.96	610	1,178	301	7.2	2	1,277
17.18	30	289	302	7.59	32	1,472
14.48	1	513	303	7.88	2	1,836
15.8	1	869	304	7.96	72	704
15.02	72	906	305	8.17	14	807
14.28	103	169	306	8.54	343	931
13.23	42	8,069	307	9	4	617
12.53	11	4,130	308	8.86	50	651
12.04	43	476	309	8.96	1	532
11.38	63	2,267	310	9.58	18	4,716
10.85	2,000	2,917	311	10.06	97	3,261
10.21	27	2,670	312	10.52	10	2,472
9.57	17	1,720	313	10.91	243	1,303
8.91	1	3,662	314	11.13	31	1,135
8.28	114	5,436	315	11.79	161	1,208
7.55	3	648	316	12.28	44	411
7.13	80	2,355	317	12.52	2	466
6.53	3	1,386	318	15.4	433	935
5.91	9	3,348	319	15.91	439	1,436
5.65	80	4,960	320	14.04	5	1,391
5.22	3	16,382	321	14.69	484	383
4.67	84	2,861	322	15.05	1,542	155
4.42	4	4,254	323	15.57	36	129

Figure 2: Option screen for SPY options expiring on March 31, 2020.

4. Lightning round

Give as much examples as you can.

1. How do you borrow stocks? (From whom, etc.) What is the borrow rate?
2. Give an example of a stock which is currently very expensive to borrow? Can the borrow rate be equal to 300% per year? Explain.
3. Explain the notion of implied cost-of-carry for futures. Give examples with WTI and E-mini S&P contracts.
4. Can you borrow an S&P futures contract? How so?
5. Can you borrow options? Swaps?
6. Are exotic options centrally-cleared?
7. Can you short ETFs? How?
8. How could the term-structure of volatility (as a function of maturity) affect the value of down-and-out barrier options?
9. What is the option-implied forward price of a stock/index/ETF?
10. What is the implied dividend? Describe the concept of implied dividend for hard-to-borrow stocks.