Derivative Securities: Lecture 4

Introduction to option pricing, rev. 2019

Sources:

J. Hull, 7th edition

Avellaneda and Laurence (2000)

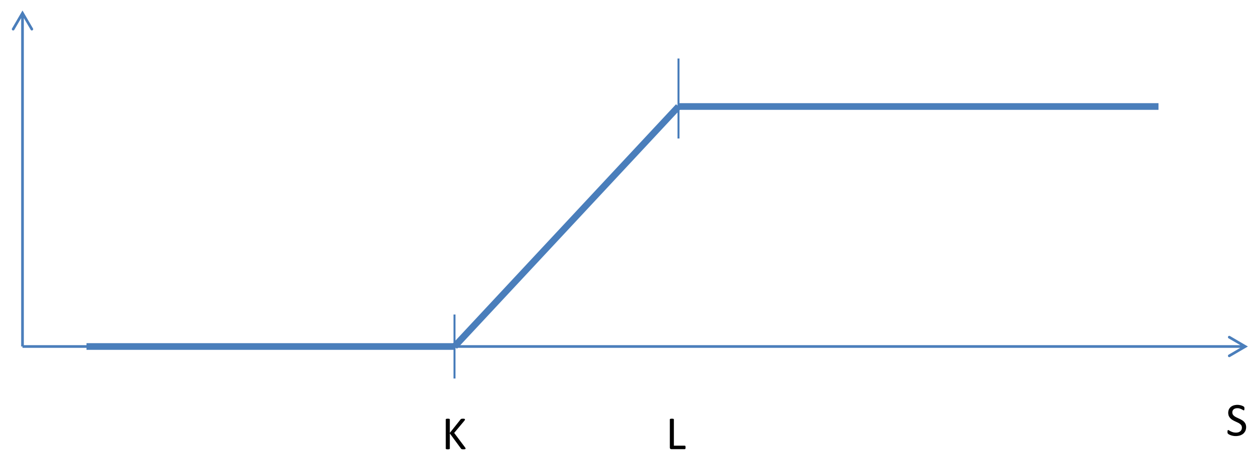
Yahoo!Finance & assorted websites

Option Pricing

- In previous lectures, we covered forward pricing and the importance of cost-of carry
- We also covered Put-Call Parity, which can be viewed as relation that should hold between European-style puts and calls with the same expiration
- Put-call parity can be seen as pricing conversions relative to forwards on the same underlying asset
- What other relations exist between options and spreads on the same underlying asset?

Call Spread

Call Spread: Long a call with strike K, short a call with strike L (L>K)



Since the payoff is non-negative, the value of the spread must be positive

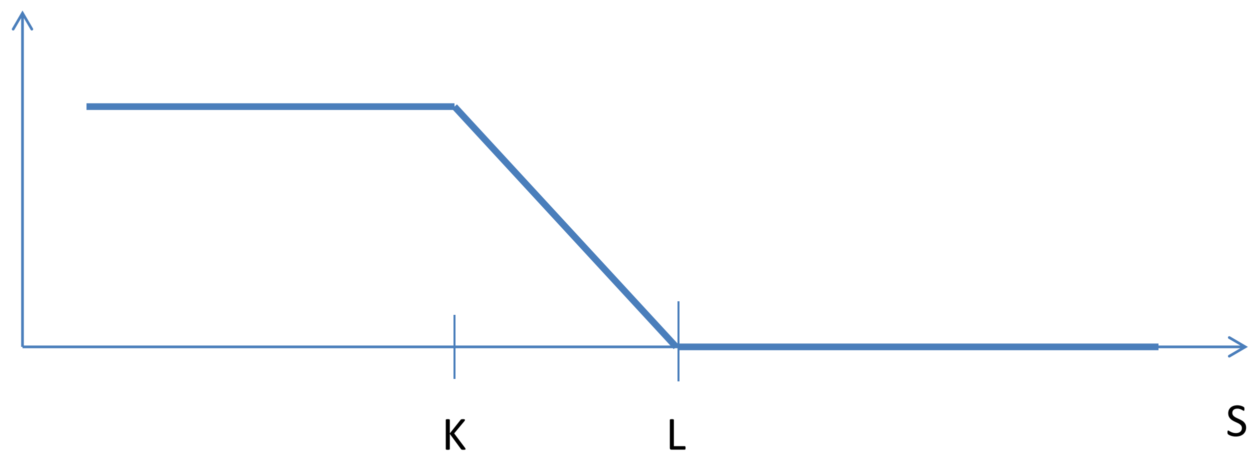
$$K < L \quad \Rightarrow \quad Call(K,T) > Call(L,T)$$

$$CS(K,L,T) = Call(K,T) - Call(L,T) > 0$$

Spread makes money if the price of the underlying goes up

Put Spread

Put Spread: Long a put with strike L, short a put with strike L (L>K)



Since the payoff is non-negative, the value of the spread must be positive

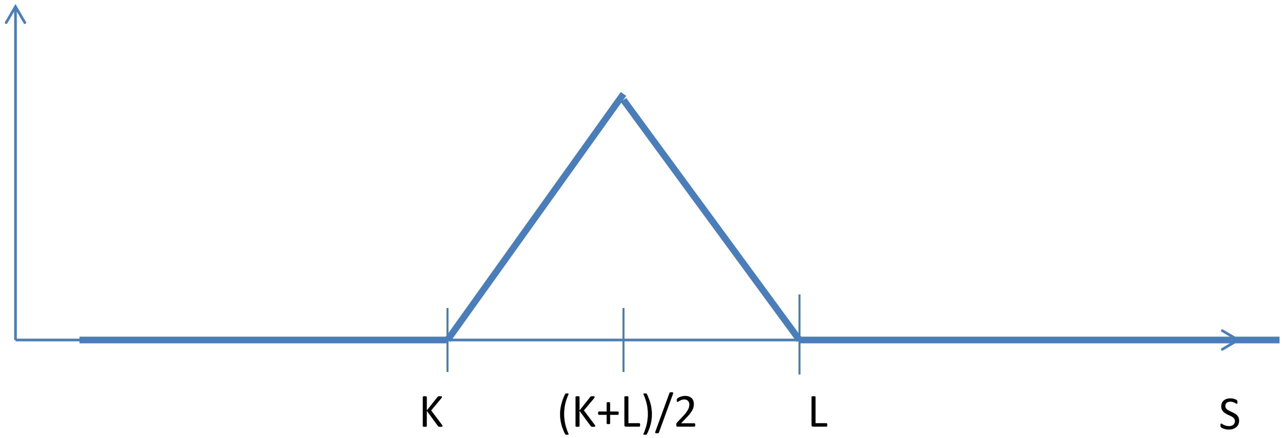
$$K < L \quad \Rightarrow \quad Put(K,T) < Put(L,T)$$

$$PS(K,L,T) = Put(L,T) - Put(K,T) > 0$$

Spread makes money if the price of the underlying goes down

Butterfly Spread

Butterfly spread: Long call with strike K, long call with strike L, short 2 calls with strike (K+L)/2



Since the spread has non-negative payoff, it must have positive value

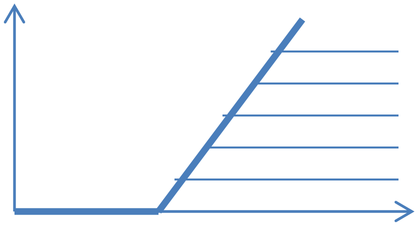
$$B\left(K,(K+L)/2,L,T\right)\equiv Call\left(K,T\right)+Call\left(L,T\right)-2Call\left(\frac{K+L}{2},T\right)>0$$

Butterflies make $ if the stock price is near (K+L)/2 at expiration.

Reconstructing Call prices from Butterfly Spreads

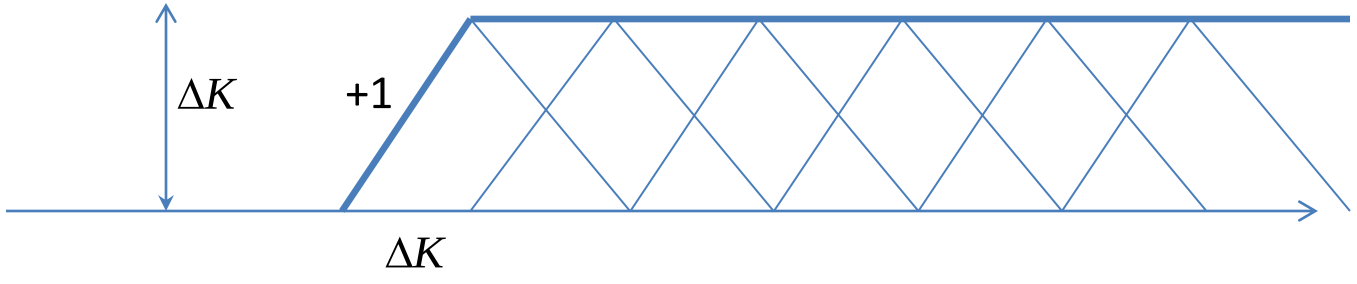
Assume for simplicity a countable number of strikes, $K_n = n\Delta K,\ \ n = 0,1,2,3,.....$
and that the stock price can only take values on the lattice $S_T = m\Delta K,\ \ m = 1,2,3,....$

$$Call(K_n, T) = \sum_{j \ge n} \left(Call(K_j, T) - Call(K_{j+1}, T) \right) = \sum_{j \ge n} CS(K_j, K_{j+1}, T)$$



- A call can be viewed as a portfolio of call spreads
- A call spread can be viewed as a portfolio of butterfly spreads

$$CS(K_j, K_{j+1}, T) = \sum_{i \ge j} B(K_i, K_{i+1}, K_{i+2}, T)$$



Calls as super-positions of butterfly spreads

$$Call(K_n, T) = \sum_{j=n}^{\infty} \sum_{i \geq j} B(K_i, K_{i+1}, K_{i+2}, T)$$

$$= \sum_{j=n}^{\infty} (j+1-n) B(K_j, K_{j+1}, K_{j+2}, T)$$

$$= \sum_{j=n}^{\infty} (j+1-n) \Delta K \cdot \left(\frac{B(K_j, K_{j+1}, K_{j+2}, T)}{\Delta K} \right)$$

$$= \sum_{j=n}^{\infty} (K_{j+1} - K_n) w(K_{j+1}, T) \qquad w(K_{j+1}, T) \equiv \frac{B(K_j, K_{j+1}, K_{j+2}, T)}{\Delta K}$$

$$= \sum_{j=0}^{\infty} (K_{j+1} - K_n)^{+} w(K_{j+1}, T)$$

The weights correspond to values of Butterfly spreads centered at each Kj. In particular, they are positive

From weights to probabilities

$w(K,T)>0$

$$\sum_{1}^{\infty} w(K_j,T)=\frac{1}{\Delta K}\sum_{1}^{\infty} B(K_{j-1},K_j,K_{j+1},T)=\frac{1}{\Delta K}CS(0,\Delta K,T)$$

$$=\mathrm{PV}(\$1)=e^{-rT}$$ (assuming that the stock can only take values $>\Delta K$)

$$w(K_j,T)=e^{-rT}p(K_j,T); \qquad \sum_j p(K_j,T)=1, \quad p(K_j,T)>0.$$

$$\boxed{\begin{aligned} Call(K,T) &= e^{-rT}\sum_j \max(K_j-K,0)p(K_j,T) \\ &= e^{-rT}E^p\{\max(S_T-K,0)\} \end{aligned}}$$

First moment of *p* is the forward price

$$
\begin{aligned}
E^{p}\{S\} &= e^{rT} Call(0,T) \\
&= e^{rT} e^{-qT} S_0 = S_0 e^{(r-q)T} \\
&= F_{0,T}
\end{aligned}
$$

- A call with strike 0 is the option to buy the stock at zero at time T
  Its value is therefore the present value of the forward price (pay now, get stock later).

- It follows that the first moment of p is the forward price.

- It also follows that put prices are given by a similar formula, namely

$$
\begin{aligned}
Put(K,T) &= Call(K,T) + Ke^{-rT} - Se^{-qT} \\
&= e^{-rT} E^{p}\left\{(S_T - K)^{+}\right\} + e^{-rT} K - e^{-rT} F \\
&= e^{-rT} E^{p}\left\{(S_T - K)^{+}\right\} + e^{-rT} E^{p}\{K - S_T\} \\
&= e^{-rT} E^{p}\left\{(K - S_T)^{+}\right\}
\end{aligned}
$$

General Payoffs

- Any twice differentiable function *f(S)* can be expressed as a combination
- of put and call payoffs, using the formula
- $$f(S) = f(0) + f'(0)(S-F) + \int_F^S (S-Y) f''(Y)\,dY$$ (this is just Taylor expansion)

$$= f(0) + f'(0)(S-F) + \int_0^F \underline{(Y-S)}^+ f''(Y)\,dY + \int_F^\infty \underline{(S-Y)}^+ f''(Y)\,dY$$

- Thus, a European-style payoff can be viewed as a spread of puts and calls. By linearity of pricing,

$$\text{Fair value of a claim with payoff } f(S_T) =$$

$$\begin{aligned}
&= e^{-rT} f(0) + 0 + \int_0^F Put(Y,T)\, f''(Y)\,dY + \int_F^\infty Call(Y,T)\, f''(Y)\,dY \\
&= e^{-rT} f(0) + e^{-rT} f'(0) E^p\{S_T - F\} + e^{-rT} E^p\left\{\int_0^F (Y-S_T)^+ f''(Y)\,dY + \int_F^\infty (S_T - Y)^+ f''(Y)\,dY\right\} \\
&= e^{-rT} E^p\{f(S_T)\}
\end{aligned}$$

Fundamental theorem of pricing (one period model)

- An **arbitrage opportunity** is a portfolio of derivative securities and cash which has the following properties:

  - The payoff is non-negative in all future states of the market
  - The price of the portfolio is zero or negative (a credit)

Assume that each security has a unique price (i.e. assume bid-offer).

- If there are no arbitrage opportunities, then there exists a probability distribution of future states of the market such that, for any function *f(S),* the price of a security with payoff $f(S_T)$ is

$$\boxed{P_f = e^{-rT} E^{p}\left\{ f\left(S_T\right)\right\}}$$

- Conversely, if such a probability exists there are no arbitrage opportunities

Practical Application to European Options

- A pricing measure is a probability of future prices of the underlying asset with the property that

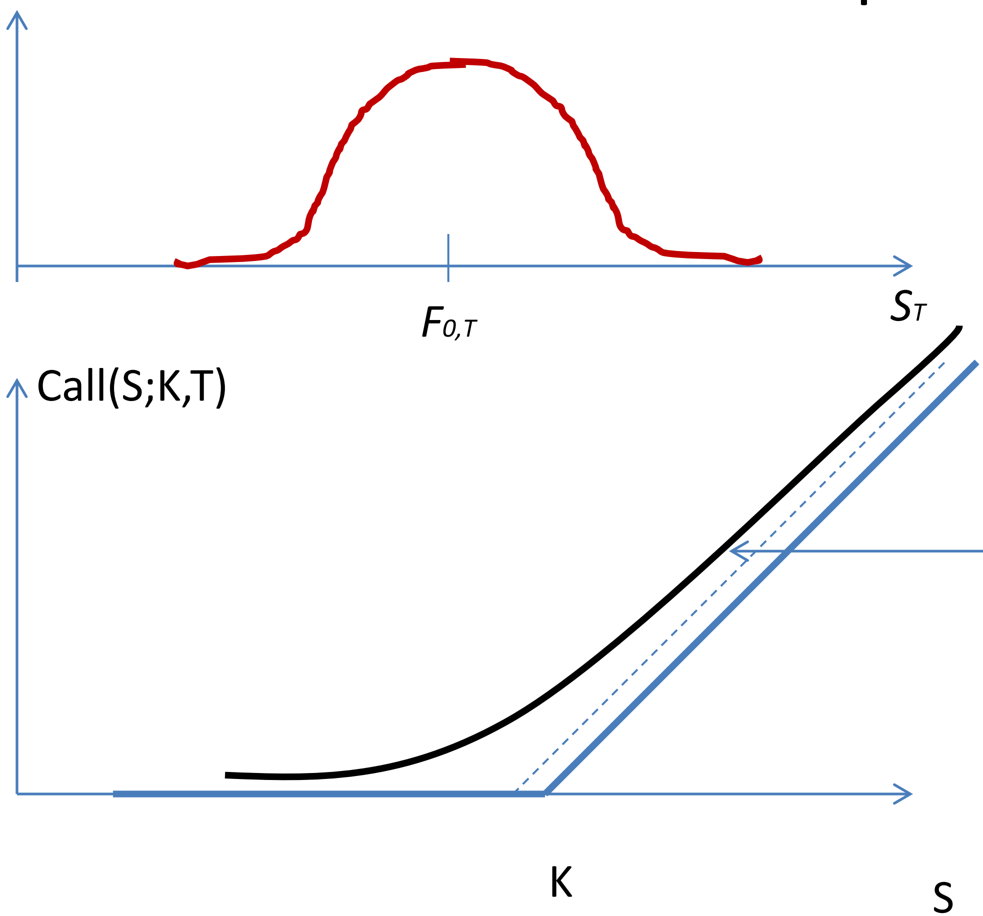
$$E^{p}\{S_T\} = F_{0,T}$$

- If we determine a suitable pricing measure, then all European options with expiration date *T* should have value given by

$$Call(K,T) = e^{-rT} E\{(S_T - K)^+\}, \quad Put(K,T) = e^{-rT} E\{(K - S_T)^+\}$$

- The main issue is then to determine a suitable pricing measure in the real, practical world.

What does a pricing measure achieve in the case of options?



Option fair value is a smoothed out version of the payoff

The pricing measure gives a model to compute the option's fair value as a function of the price of the underlying asset, the strike and the maturity

The Black-Scholes Model

Assume that the pricing measure is log-normal, i.e. **log-returns are normal**

$$S_T = S_0 e^X, \quad X \sim N\left(\mu T, \sigma^2 T\right)$$

$$E\{S_T\} = \int_{-\infty}^{+\infty} S_0 e^y e^{-\frac{(y-\mu T)^2}{2\sigma^2 T}} \frac{dy}{\sqrt{2\pi\sigma^2 T}} = S_0 e^{\mu T + \frac{\sigma^2 T}{2}} = S_0 e^{\left(\mu + \frac{\sigma^2}{2}\right)T}$$

$$\therefore \qquad \mu + \frac{\sigma^2}{2} = r - q \qquad \therefore \quad \mu = r - q - \frac{\sigma^2}{2}$$

$$X = Z\sigma\sqrt{T} - \frac{\sigma^2}{2}T + (r-q)T, \quad Z \sim N(0,1)$$

Call pricing with the Black-Scholes model

$$Call(S,K,T)=e^{-rT}E\left\{\left(S_T-K\right)^+\right\}=e^{-rT}\int_{-\infty}^{+\infty}\left(Se^{z\sigma\sqrt{T}-\sigma^2T/2+(r-q)T}-K\right)^+e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}}$$

$$=e^{-rT}\int_{A}^{+\infty}Se^{z\sigma\sqrt{T}-\sigma^2T/2+(r-q)T}e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}}-e^{-rT}K\int_{A}^{+\infty}e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}}\qquad\left(A=\frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{K}{S}\right)+\frac{\sigma^2T}{2}-(r-q)T\right)\right)$$

$$=e^{-qT}S\left(\int_{A}^{+\infty}e^{z\sigma\sqrt{T}-\sigma^2T/2}e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}}\right)-e^{-rT}K\left(\int_{A}^{+\infty}e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}}\right)$$

$$=e^{-qT}S\left(\int_{A}^{+\infty}e^{-\frac{\left(z-\sigma\sqrt{T}\right)^2}{2}}\frac{dz}{\sqrt{2\pi}}\right)-e^{-rT}K\left(\int_{A}^{+\infty}e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}}\right)$$

$$=e^{-qT}S\left(\int_{A-\sigma\sqrt{T}}^{+\infty}e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}}\right)-e^{-rT}K\left(\int_{A}^{+\infty}e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}}\right)$$

$$=e^{-qT}S\left(\int_{-\infty}^{-A+\sigma\sqrt{T}}e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}}\right)-e^{-rT}K\left(\int_{-\infty}^{-A}e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}}\right)$$

Black-Scholes Formula

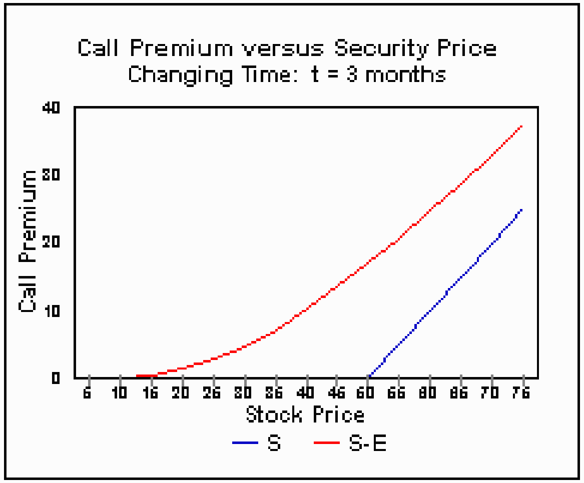
$$\boxed{BSCall(S,T,K,r,q,\sigma) = Se^{-qT}N(d_1) - Ke^{-rT}N(d_2)}$$

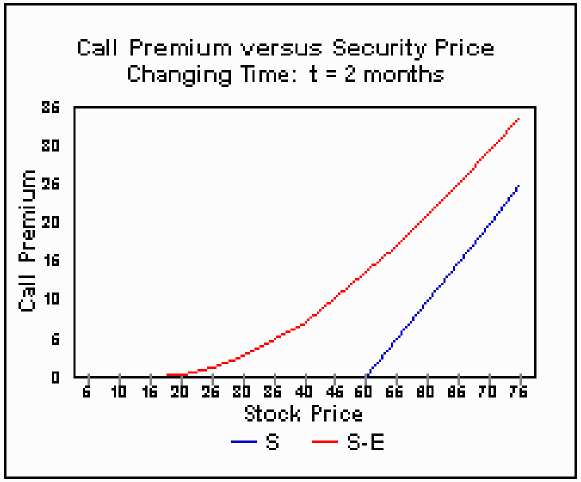
$$d_1 = \frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{F_{0,T}}{K}\right) + \frac{\sigma^2 T}{2}\right), \quad d_2 = \frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{F_{0,T}}{K}\right) - \frac{\sigma^2 T}{2}\right), \quad F_{0,T} = Se^{(r-q)T}$$

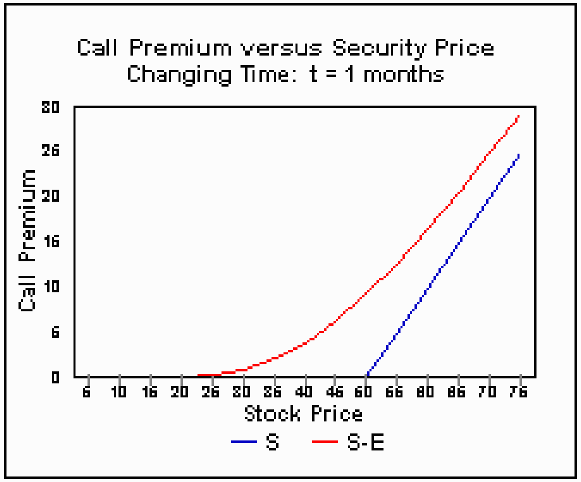
$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{z^2}{2}}\,dz$$

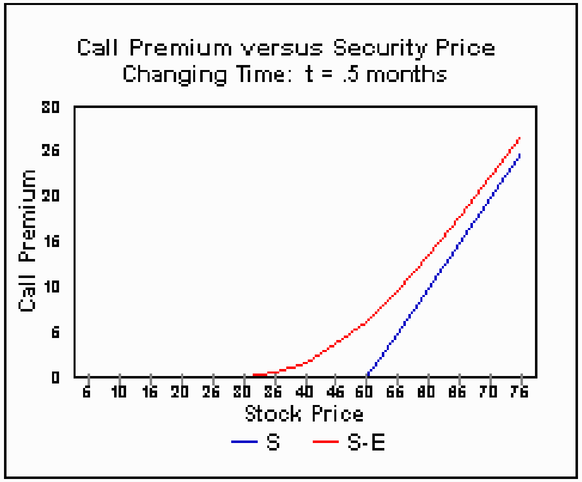
cumulative normal distribution

Black-Scholes Formula at work









S=$48, K=$50, r=6%, sigma=40%, q=0

Log-normality assumption: pros and cons

- The Black-Scholes has only one adjustable parameter: the volatility

- The implied forward and the interest rate can be derived from observable market data

- Simplicity is the most important feature of the B.-S. pricing formula.

- In applications – i.e. when applied to different asset classes -- we see that it may have some drawbacks.

Implied Volatility

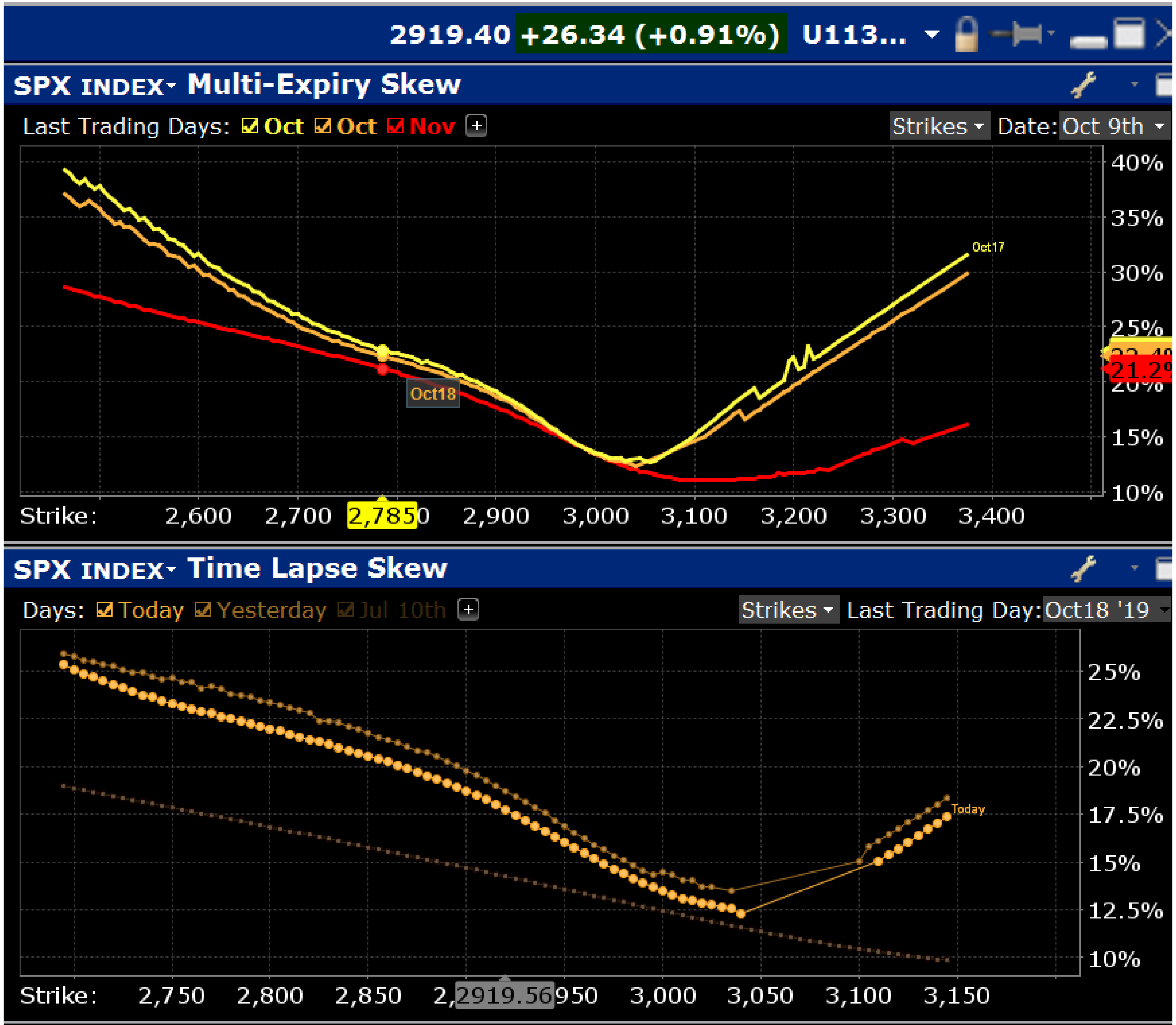
- Implied Volatility: the value of σ which makes the B.S. formula price equal to the market price.

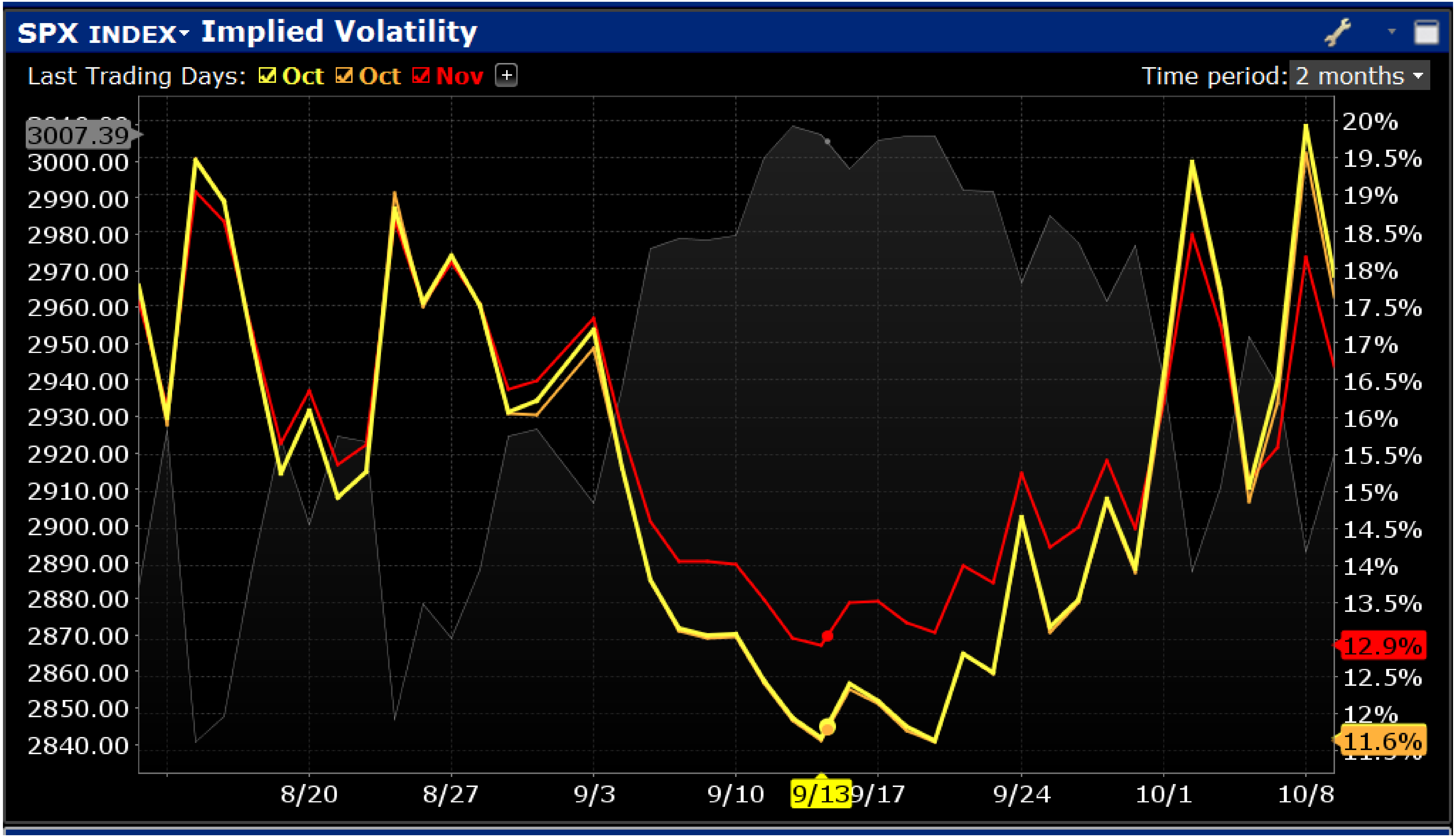
$$\bar{C}_{mkt} = BSCall(S, T, K, r, d, \sigma_{imp})$$

- $\bar{C}_{mkt}$ is the mid-market value (mid-point between bid & offered prices)

- Often, we focus on the dependence of σ_{imp} on strike and time-to-maturity.

$$\sigma_{imp} = \sigma(K, T)$$


2919.40 +26.34 (+0.91%) U113...
SPX INDEX Multi-Expiry Skew
Last Trading Days: Oct Oct Nov
Strikes
Date: Oct 9th
Oct17
Oct18
21.2%
40%
35%
30%
25%
20%
15%
10%
Strike: 2,600 2,700 2,785 2,900 3,000 3,100 3,200 3,300 3,400
SPX INDEX Time Lapse Skew
Days: Today Yesterday Jul 10th
Strikes
Last Trading Day: Oct18 '19
Today
25%
22.5%
20%
17.5%
15%
12.5%
10%
Strike: 2,750 2,800 2,850 2919.56 3,000 3,050 3,100 3,150



SPX INDEX Implied Volatility
Last Trading Days: Oct Oct Nov
Time period: 2 months
3007.39
3000.00
2990.00
2980.00
2970.00
2960.00
2950.00
2940.00
2930.00
2920.00
2910.00
2900.00
2890.00
2880.00
2870.00
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12.9%
12.5%
12%
11.6%
8/20
8/27
9/3
9/10
9/13
9/17
9/24
10/1
10/8

Multi-period asset model

- Derivative securities may depend on multiple expiration/cash-flow dates. Furthermore, the 1-period model described above is rigid in the sense that it cannot price American-style options.

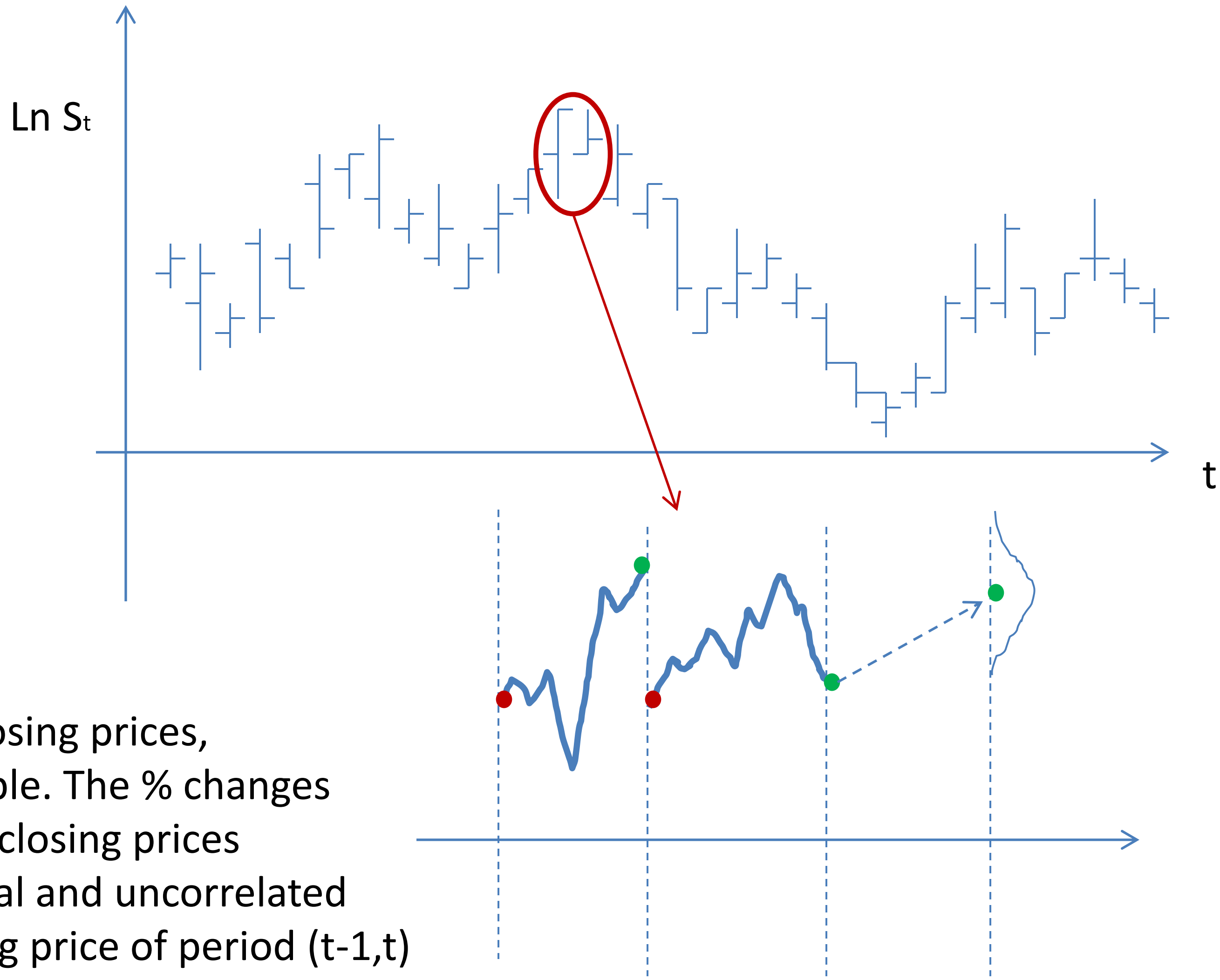
- We consider instead a more realistic approach to pricing based on the statistics of stock returns over short periods of time (e.g. 1 day).

- We assume that the underlying price has returns satisfying

$$\frac{S_{t+\Delta t} - S_t}{S_t} = \frac{\Delta S_t}{S_t} \sim N\left(\mu \Delta t, \sigma^2 \Delta t\right)$$

- We also assume that successive returns are uncorrelated.

Modeling the return of a price time-series (OHLC)



Model closing prices,
for example. The % changes
between closing prices
are normal and uncorrelated
St= closing price of period (t-1,t)

Parameterization

$$\mu\Delta t = E\left\{\frac{\Delta S_t}{S_t}\right\}, \qquad \sigma^2\Delta t = E\left\{\left(\frac{\Delta S_t}{S_t}\right)^2\right\} - \left(E\left\{\frac{\Delta S_t}{S_t}\right\}\right)^2$$

$\mu =$ annualized expected return

$\sigma =$ annualized standard deviation

1% daily standard deviation => 15.9% annualized standard deviation

$$\Delta t = \frac{1}{252}, \quad \sqrt{252} = 15.9$$

Pricing Derivatives

- Let us model the value of a derivative security as **a function of the underlying asset price and the time to expiration**

$$V_t = C(S_t, t) \qquad 0 < t < T$$

Change in market value over one period :

$$\Delta V_t = \Delta C(S_t, t)$$

$$= \frac{\partial C(S_t,t)}{\partial t}\Delta t + \frac{\partial C(S_t,t)}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 C(S_t,t)}{\partial t^2}\Delta t^2 + \frac{1}{2}\frac{\partial^2 C(S_t,t)}{\partial S^2}(\Delta S_t)^2 + \ldots.$$

$$= \frac{\partial C(S_t,t)}{\partial t}\Delta t + \frac{\partial C(S_t,t)}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 C(S_t,t)}{\partial S^2}S_t^{\,2}\left(\frac{\Delta S_t}{S_t}\right)^2 + o(\Delta t)$$

$$= \underline{\left(\frac{\partial C(S_t,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 C(S_t,t)}{\partial S^2}S_t^{\,2}\sigma^2\right)}\Delta t + \underline{\frac{\partial C(S_t,t)}{\partial S}}\Delta S_t + \underline{\frac{1}{2}\frac{\partial^2 C(S_t,t)}{\partial S^2}S_t^{\,2}\left[\left(\frac{\Delta S_t}{S_t}\right)^2 - \sigma^2\Delta t\right] + o(\Delta t)}$$

$$= \underline{\alpha}\Delta t + \underline{\beta}\Delta S_t + \underline{\varepsilon_t}$$

The hedging argument

- Consider a portfolio which is long 1 derivative and short β stocks.
- Assume derivative does not pay dividends

$$\boxed{\beta = \beta_t = \frac{\partial C(S_t, t)}{\partial S_t}}$$

Profit and loss, including financing and dividends:

$$PNL = -V_t \cdot r\Delta t + \Delta V_t - \beta\left(\Delta S_t - S_t r\Delta t + S_t q\Delta t\right)$$

$$= -V_t \cdot r\Delta t + \alpha\Delta t + \beta\Delta S_t + \varepsilon_t - \beta\left(\Delta S_t - S_t r\Delta t + S_t q\Delta t\right)$$

$$= -V_t \cdot r\Delta t + \alpha\Delta t + \beta S_t (r - q)\Delta t + \varepsilon_t$$

$$= \left(-C(S_t, t)r + \frac{\partial C(S_t, t)}{\partial t} + \frac{\partial C(S_t, t)}{\partial S} S(r - q) + \frac{1}{2}\frac{\partial^2 C(S_t, t)}{\partial S^2} S_t^2 \sigma^2\right)\Delta t + \varepsilon_t$$

Analyzing the residual term ε_t

$$\varepsilon_t = S_t^2 \frac{\partial^2 C(S_t,t)}{\partial S^2}\left[\left(\frac{\Delta S_t}{S_t}\right)^2 - \sigma^2 \Delta t\right] + o(\Delta t)$$

Conditional expectation of epsilon

$$E\{\varepsilon_t \mid S_t\} = S_t^2 \frac{\partial^2 C(S_t,t)}{\partial S^2} E\left\{\left(\frac{\Delta S_t}{S_t}\right)^2 - \sigma^2 \Delta t \,\middle|\, S_t\right\} + o(\Delta t)$$

$$= S_t^2 \frac{\partial^2 C(S_t,t)}{\partial S^2} \mu^2 \Delta t^2 + o(\Delta t)$$

$$= o(\Delta t)$$

- The residual term has essentially zero expected return (vanishing exp. return in the limit Dt_>0.)

The fair value of our derivative security is…

- The PNL for the long short portfolio of **1 derivative and – beta shares** has expected value

$$E\{PNL\} = \alpha\Delta t + o(\Delta t)$$

$$= \left(-C(S_t,t)r + \frac{\partial C(S_t,t)}{\partial t} + \frac{\partial C(S_t,t)}{\partial S} S(r-q) + \frac{1}{2}\frac{\partial^2 C(S_t,t)}{\partial S^2} S_t^2 \sigma^2 \right)\Delta t + o(\Delta t)$$

- This portfolio has no exposure to the stock price changes. Therefore, if *$C(S_t,t)$* represents the ``fair value'' of the derivative, the portfolio should have zero rate of return (we already took into acct its financing). Thus:

$$\boxed{\frac{\partial C(S\ \ ,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S_t,t)}{\partial S^2} + (r-q)S\frac{\partial C(S\ \ ,t)}{\partial S} - C(S,t)r = 0}$$

- This is the Black-Scholes partial differential equation (PDE).

American-style calls & puts

- Consider a call option on an underlying asset paying dividends continuously. Since the option can be exercised anytime, we have

$$C(S,t) \geq \max(S-K,0), \qquad t < T. \qquad (1)$$

- The terminal condition at *t=T* corresponds to the final payoff

$$C(S,T) = \max(S-K,0).$$

- Thus, the function *C(S,t)* should satisfy the **Black-Scholes PDE** in the region of the (S,t)-plane for which strict inequality holds in (1), and it should be equal to *max(S-K,0)* otherwise.

- The solution of this problem is done numerically and will be addressed in the next lecture.