Lecture 8: Pricing Measures and Applications to First-Generation Exotic Options

Pricing Measures

- **Pricing measures**, a.k.a. **pricing kernels**, or **pricing models**, are probability measures of future market scenarios which are used to pricing derivatives by discounting expected cash-flows.

- Example: Consider an index, e.g. the S&P500. We wish to price derivatives based on the index. The pricing measure will be such that the index satisfies

$$\frac{\Delta S_t}{S_t} = \sigma_t \nu_t \sqrt{\Delta t} + r_t \Delta t - q_t \Delta t,$$

where nu_t are i.i.d. N(0,1) deviates, and sigma_t, r_t, q_t are respectively the volatility, the funding rate and the dividend yield over the period (t,t+ Δt).

Justification of the previous formula

Assume that the volatility, funding rate and the dividend yield of the index are and that we know the price of the index as well.

The previous formula implies that

$$S_{t+\Delta t} = S_t + S_t \sigma_t \nu_t \sqrt{\Delta t} + S_t r_t \Delta t - S_t q_t \Delta t$$

so

$$\begin{aligned} E\{S_{t+\Delta t} \mid S_t, \sigma_t, r_t, q_t\} &= S_t + S r_t \Delta t - S_t q_t \Delta t \\ &= S_t (1 + r_t \Delta t - q_t \Delta t) \\ &= F_{t,t+\Delta t} \end{aligned}$$

The conditional mean of the spot price is the forward price for the corresponding interval, ensuring that put-call parity will hold at any future time period.

Term-structures of interest rates, volatilities and dividends

- How do we compute σ_t, r_t, q_t in practice?

- Derivative markets contain information about forward interest rates, dividends and volatilities.

  For example, consider the following hypothetical table of quantities extracted from market data:

Maturity	30 days	60 days	90 days	180 days	360 days	
Interest rate	0.1	0.2	0.4	0.4	0.5	Zero-coupon bond rates
Dividend yield	2.1	2.2	2.2	1.9	1.8	Implied divs. from options or actual divs.
Implied Volatility	25	27	29	30	32	ATM implied vols.

Finding forward rates from term rates

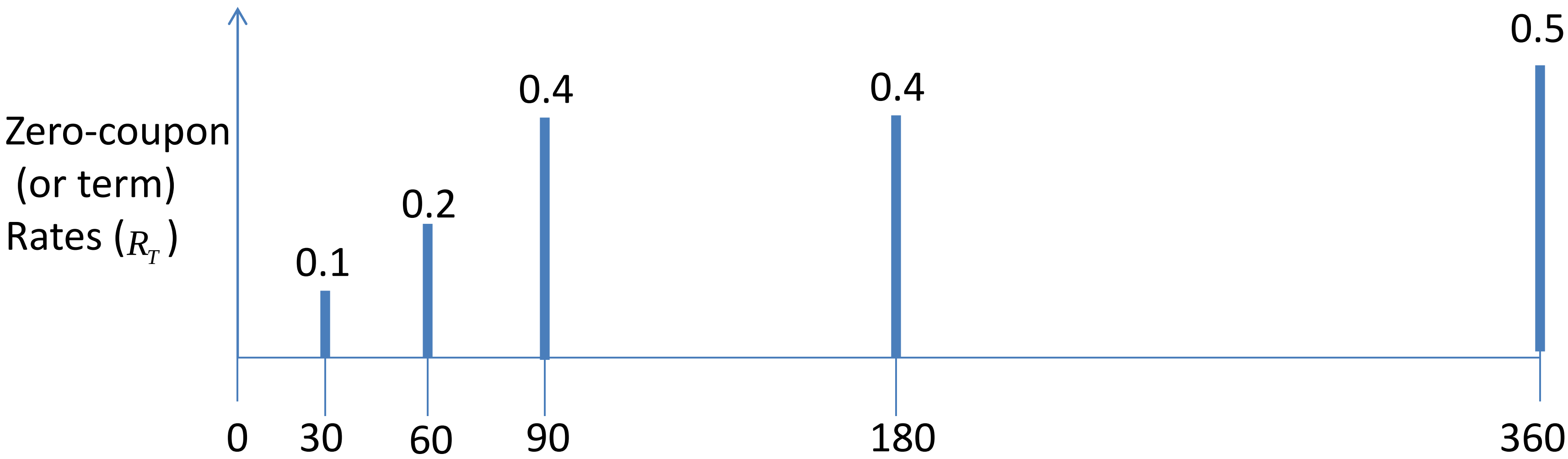
$$R_T = \frac{1}{T}\int_0^T r_t dt = \frac{1}{n\Delta t}\sum_{j=1}^{n} r_{t_j}\left(t_j - t_{j-1}\right)$$

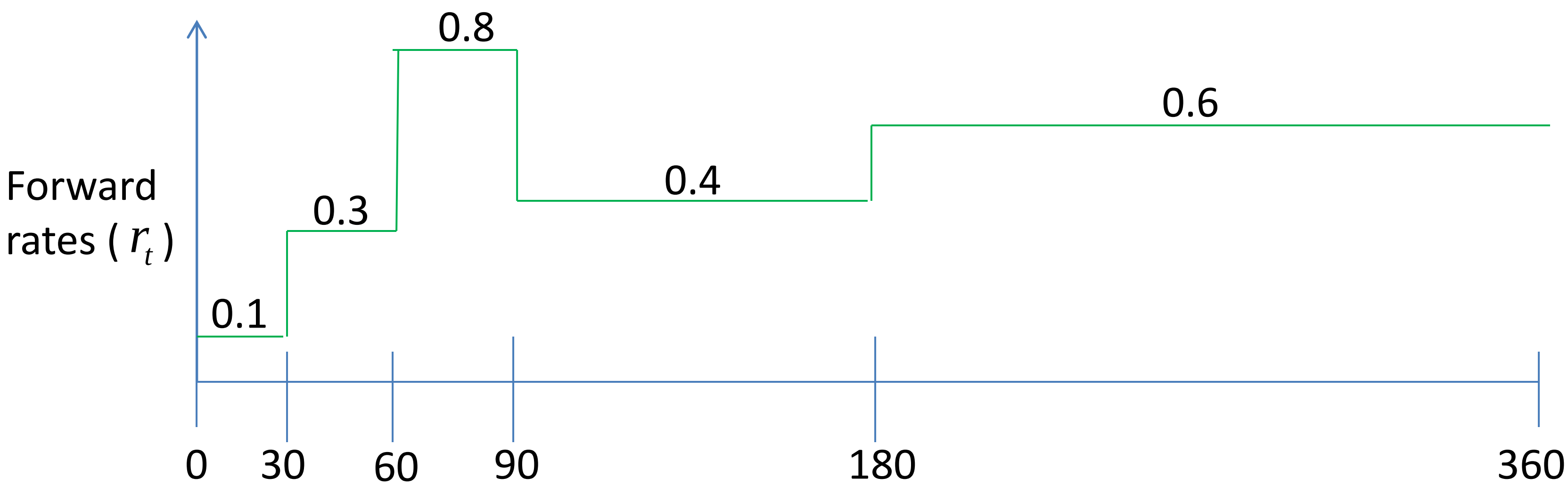
$$r_t = \frac{d}{dT}\left(TR_T\right)\Big]_{T=t}$$

The formula for backing out forward rates from term rates is

$$\boxed{r_{t,t+\Delta t} = \frac{\left(t+\Delta t\right)R_{t+\Delta t} - tR_t}{\Delta t}}$$

Concretely…


Zero-coupon
(or term)
Rates (R_T)
0.1
0.2
0.4
0.4
0.5
0
30
60
90
180
360


Forward
rates (r_t)
0.1
0.3
0.8
0.4
0.6
0
30
60
90
180
360

Finding forward dividends from (implied) term dividends

- Dividends scale like rates, so

$$\bar{q}_T = \frac{1}{T}\int_0^T q_t dt = \frac{1}{n\Delta t}\sum_{j=0}^{n-1} q_{t_i}\left(t_{i+1} - t_i\right)$$

$$t \in \left[T_i, T_{i+1}\right] \quad \Rightarrow \qquad q_t = \frac{T_{i+1}\bar{q}_{T_{i+1}} - T_i\bar{q}_{T_i}}{T_{i+1} - T_i}$$

Finding forward volatilities from (implied) term volatilities

- **Variances** scale linearly, like rates, so

$$\overline{\sigma}_T^2 = \frac{1}{T}\int_0^T \sigma_t^2 dt = \frac{1}{n\Delta t}\sum_{j=0}^{n-1} \sigma_{t_i}^2 \left(t_{i+1} - t_i\right)$$

$$t \in \left[T_i, T_{i+1}\right] \quad \Rightarrow \qquad \sigma_t^2 = \frac{T_{i+1}\overline{\sigma}_{T_{i+1}}^2 - T_i\overline{\sigma}_{T_i}^2}{T_{i+1} - T_i}$$

These three formulas, for rates, dividends and volatilities allow us to generate forward quantities to use in the model.

Passing from term-structure to forward quantities

MAT	0.0	30.0	60.0	90.0	180.0	360.0
RATE	0	0.1	0.2	0.4	0.4	0.5
DIV	0	2.1	2.2	2.2	1.9	1.8
VOL	0	25	27	29	30	32
F RATE	0	0.1	0.3	0.8	0.4	0.6
F DIV	0	2.1	2.3	2.2	1.6	1.7
F VOL	0	25	28.9	32.6	30.97	33.88

Data (term rates): RATE, DIV, VOL

Computed (forward rates): F RATE, F DIV, F VOL

Term rates & forward rates (formal calculation)

$$\frac{S_T}{S_0} = \prod_{j=0}^{n-1}\left(1+\sigma_j \nu_j \sqrt{\Delta t} + r_j \Delta t - q_j \Delta t\right) \qquad T = n\Delta t$$

$$= \exp\left[\sum_{j=0}^{n-1} \ln\left(1+\sigma_j \nu_j \sqrt{\Delta t} + r_j \Delta t - q_j \Delta t\right)\right] \qquad \left(\ln(1+\varepsilon) = \varepsilon - \frac{\varepsilon^2}{2} + o(\varepsilon)\right)$$

$$\cong \exp\left[\sum_{j=0}^{n-1} \sigma_j \nu_j \sqrt{\Delta t} + \sum_{j=0}^{n-1} r_j \Delta t - \sum_{j=0}^{n-1} q_j \Delta t - \frac{1}{2}\sum_{j=0}^{n-1} \sigma_j^2 \nu_j^2 \Delta t + o(1)\right]$$

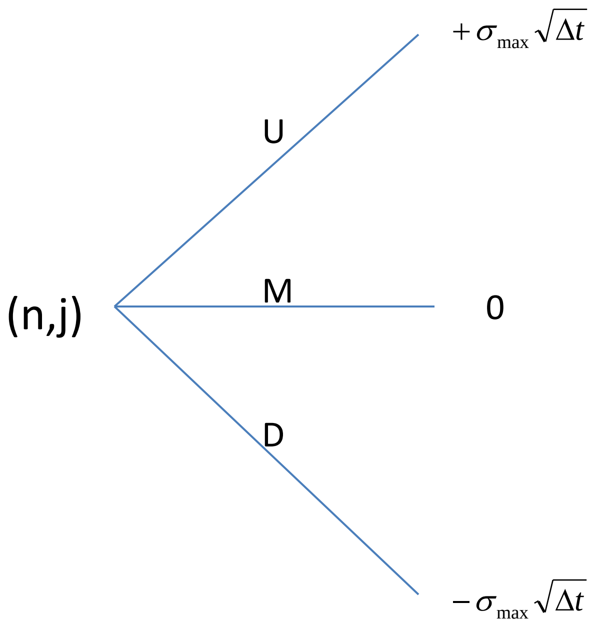
$$\cong \exp\left[\sum_{j=0}^{n-1} \sigma_j \nu_j \sqrt{\Delta t} + \sum_{j=0}^{n-1} r_j \Delta t - \sum_{j=0}^{n-1} q_j \Delta t - \frac{1}{2}\sum_{j=0}^{n-1} \sigma_j^2 \Delta t + o(1)\right] \qquad \text{(LLN)}$$

$$\cong \exp\left[\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1} \sigma_j \nu_j \sqrt{T} + \frac{1}{n}\sum_{j=0}^{n-1} r_j T - \frac{1}{n}\sum_{j=0}^{n-1} q_j T - \frac{1}{2n}\sum_{j=0}^{n-1} \sigma_j^2 T\right]$$

$$= \exp\left[\bar{\sigma}_T \nu \sqrt{T} + (R_T - \bar{q}_T)T - \frac{1}{2}\bar{\sigma}_T^2 T\right] \qquad \nu \sim \mathrm{N}(0,1) \qquad \text{(CLT)}$$

The construction gives rise to lognormal random variables with the appropriate term rates and volatilities.

The pricing model can be implemented in a trinomial tree

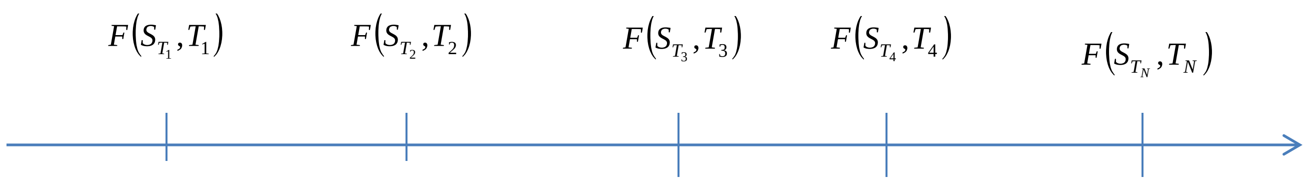


$$P_{U,n} = \frac{1}{2} f_{n,j}\left(1 - \frac{\sigma_{\max}\sqrt{\Delta t}}{2}\right) + \frac{\mu_n\sqrt{\Delta t}}{2\sigma_{\max}}$$

$$P_{M,n} = 1 - f_{n,j}, \qquad f_{n,j} = \frac{\sigma_n^2}{\sigma_{\max}^2}, \quad \mu_n = r_n - q_n$$

$$P_{D,n} = \frac{1}{2} f_{n,j}\left(1 + \frac{\sigma_{\max}\sqrt{\Delta t}}{2}\right) - \frac{\mu_n\sqrt{\Delta t}}{2\sigma_{\max}}$$

Pricing models as probabilities on future price paths



$$V = E\left\{ \sum_{j=1}^{N} e^{-R_{T_j} T_j} F\left(S_{T_j}, T_j\right) \right\}$$

Once a pricing measure has been specified, the value of a derivative security is the **expected value of its discounted cash-flows**.

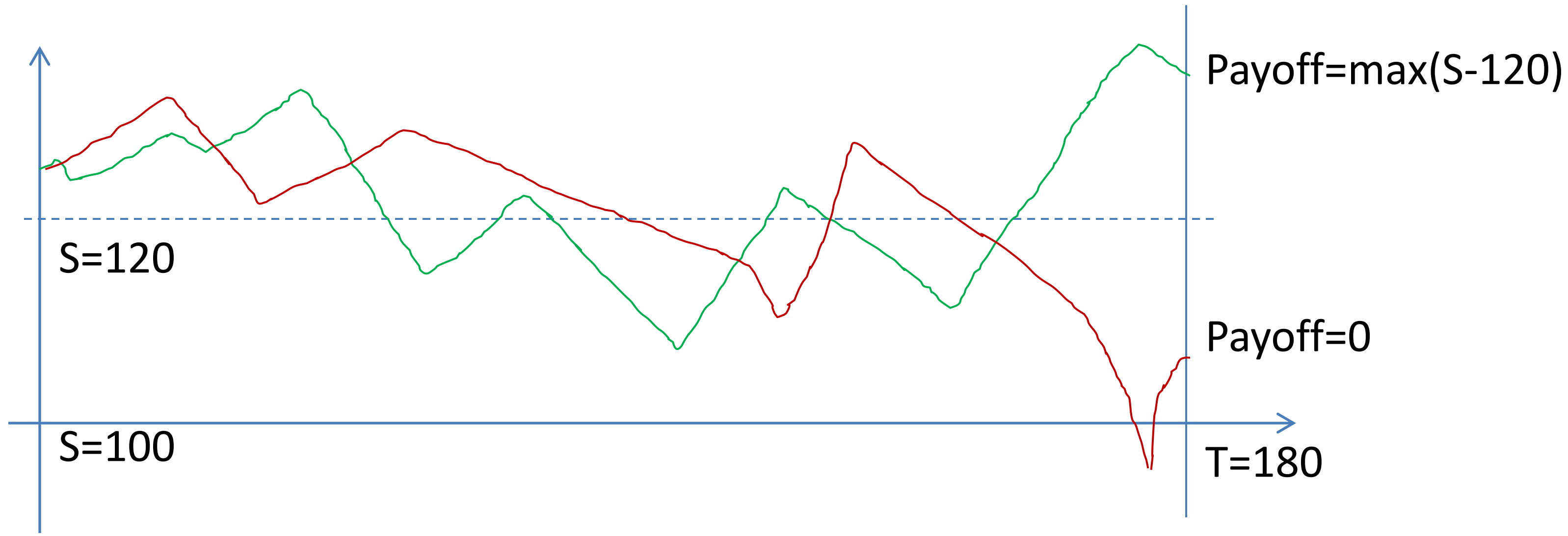
Barrier options

- Barrier options are puts and calls which are activated/deactivated as the underling asset crosses a given level
- These are sometimes called ``first generation exotic options''. They are OTC contracts which are tailored to end-user expectations and usually are cheaper than regular options.
- From a risk-management perspective, barrier options are more difficult than plain-vanilla options because their value is not monotone in the underlying price and the deltas, gammas and vegas can change sign.
- Barrier options are popular in FX, Equities and Fixed-income derivatives

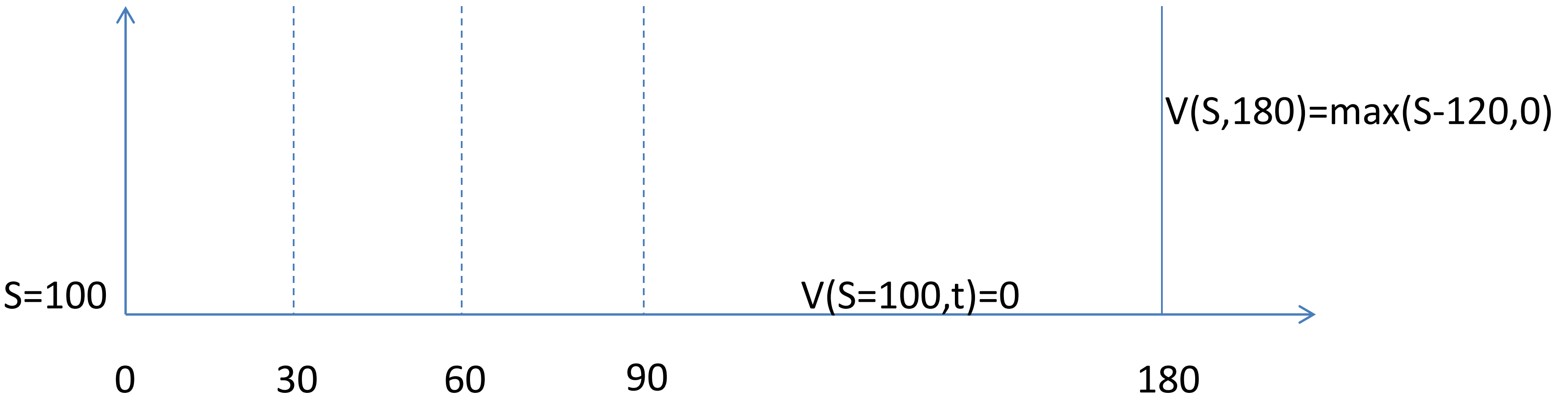
Down-and Out Calls

A **down-and-call** is a standard call with the additional provision that the contract is void if the underlying asset price goes below some level.

Example: 180 Day European SPY Call with strike 120 and knockout barrier 100.
Last price=123.12



Pricing a down-and-out call



Piecewise-constant interest, dividend, volatility to fit term-structures!

Recursive algorithm:

$$V_n^j = e^{-r_n \Delta t}\left(P_{U,n} V_{n+1}^{j+1} + P_{M,n} V_{n+1}^{j} + P_{D,n} V_{n+1}^{j-1}\right), \quad j > j_0, n < N$$

$$V_N^j = \max(S_N^j - 120, 0), \qquad V_n^{j_0} = 0 \quad \text{if} \quad S_n^{j_0} = 100$$

Closed-form solution for DO Call (from Hull)

If the parameters r, q, sigma are constant, there is a closed-from solution:

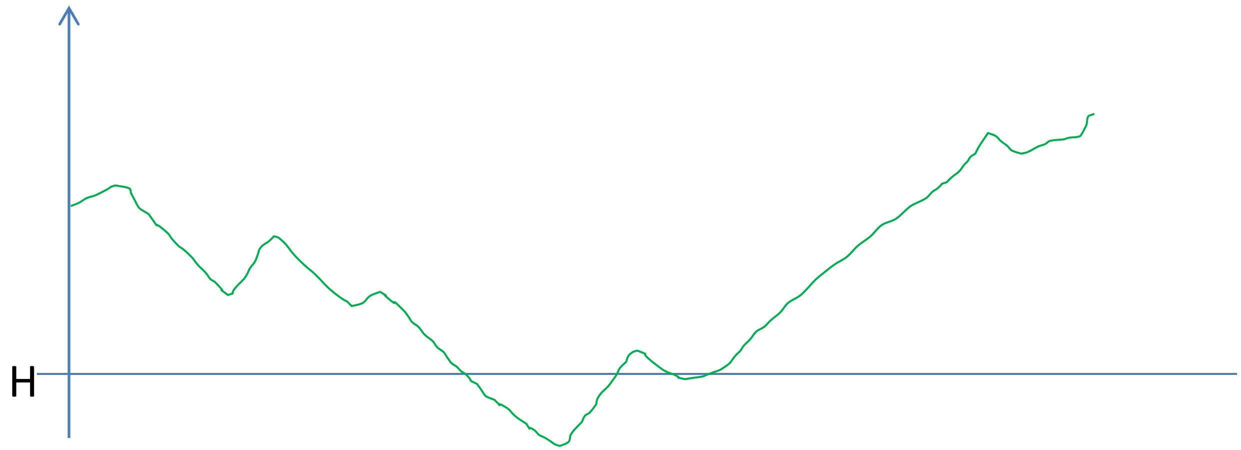
$$\boxed{\begin{aligned}&C_{do}(S,T,K,H,r,q,\sigma)=\\&Se^{-qT}N(d_1)-Ke^{-rT}N(d_2)-Se^{-qT}\left(\frac{H}{S}\right)^{2\lambda}N(e_1)+Ke^{-rT}\left(\frac{H}{S}\right)^{2\lambda-2}N(e_2)\end{aligned}}$$

$$d_1=\frac{1}{\sigma\sqrt{T}}\ln\left(\frac{F}{K}\right)+\frac{\sigma\sqrt{T}}{2},\quad d_2=d_1-\sigma\sqrt{T}$$

$$e_1=\frac{1}{\sigma\sqrt{T}}\ln\left(\frac{H^2}{SK}\right)+\lambda\sigma\sqrt{T},\quad e_2=e_1-\sigma\sqrt{T}$$

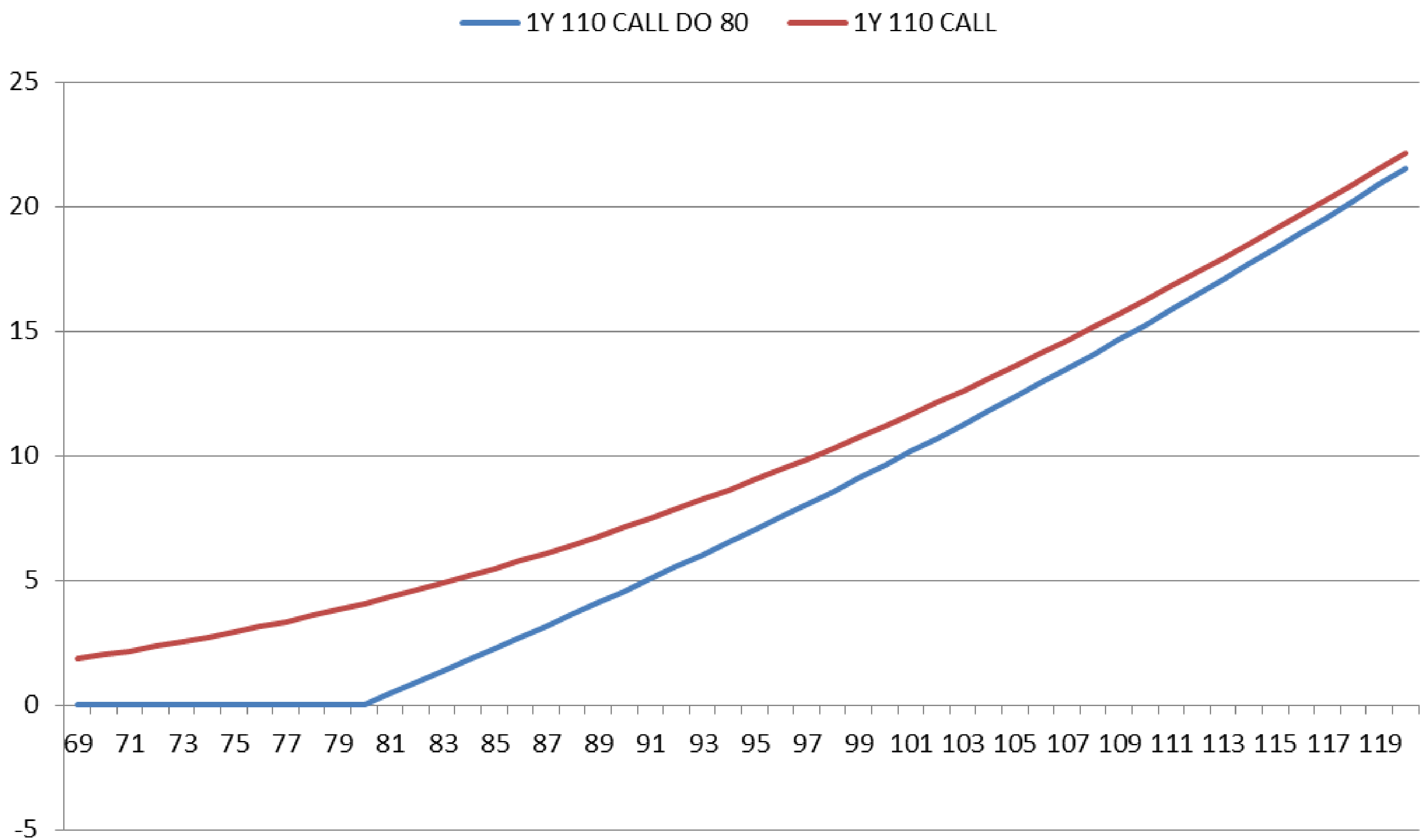
$$\lambda=\frac{r-q+\sigma^2/2}{\sigma^2}$$

Intuition for the pricing formula



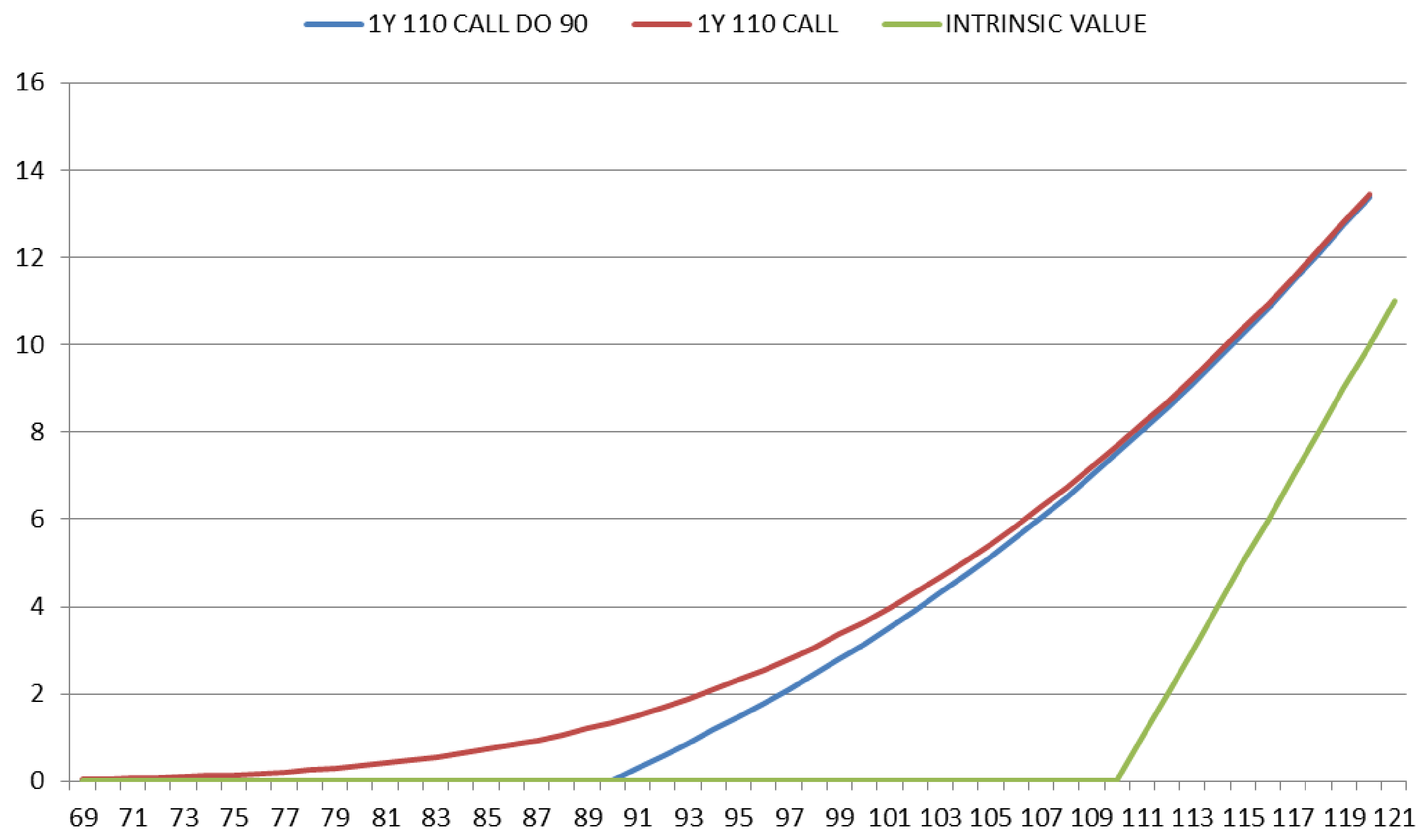
Down and Out = long Vanilla Call, short Down-and-In Call

D&O Call versus plain vanilla Call



K=110, T=1, H=80, r=10bps, q=200 bps, vol= 40%

1y 110 Call KO 90



K=110, T=1, H90, r=10bps, q=200 bps, vol=20%

Reverse Knock-outs

- Up and out calls are sometimes called **reverse knock-outs**. They have a discontinuity at the KO barrier which renders the Delta increasingly large as maturity approaches.

Formula (Hull) for a RKO Call: K=strike, H=barrier, constant rates and vol.

$$\boxed{C_{uo}(S,T,K,H,r,q,\sigma) = BSCall(S,T,K,r,q,\sigma) - C_{ui}(S,T,K,H,r,q,\sigma)}$$

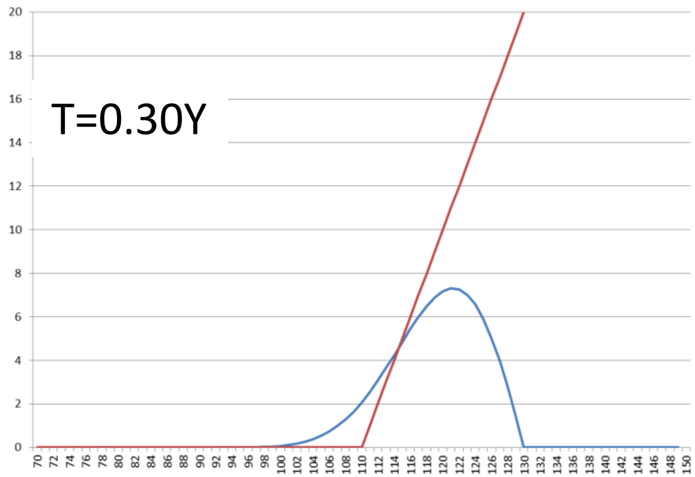
$$C_{ui}(S,T,K,H,r,q,\sigma) = Se^{-qT}N(f_1) - Ke^{-rT}N(f_2)$$
$$+ Se^{-qT}\left(\frac{S}{H}\right)^{2\lambda}\left[N(-e_1) - N(-g)\right] - Ke^{-rT}\left(\frac{S}{H}\right)^{2\lambda-2}\left[N(-e_1+\sigma\sqrt{T}) - N(-g+\sigma\sqrt{T})\right]$$

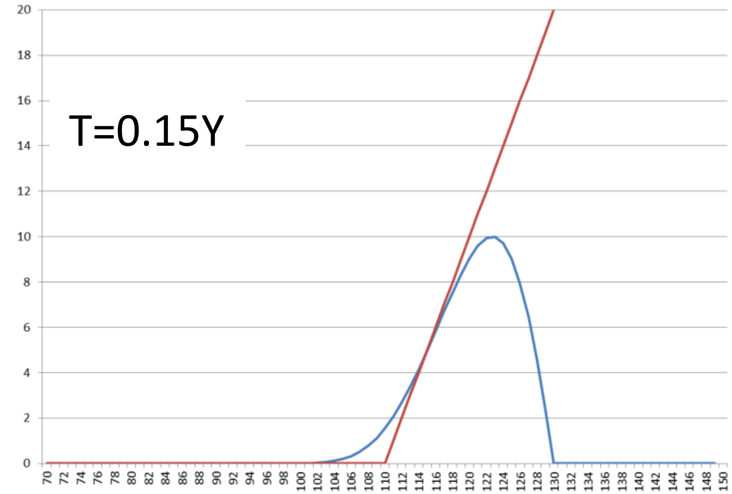
$$f_1 = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{S}{H}\right) + \lambda\sigma\sqrt{T}, \qquad f_2 = f_1 - \sigma\sqrt{T} \quad , \quad \lambda = \frac{r-q+\sigma^2/2}{\sigma^2}$$

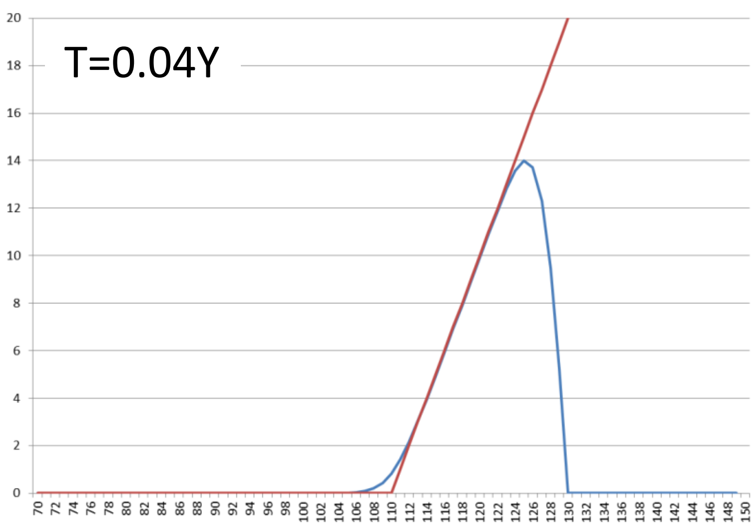
$$e_1 = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{H^2}{SK}\right) + \lambda\sigma\sqrt{T},$$

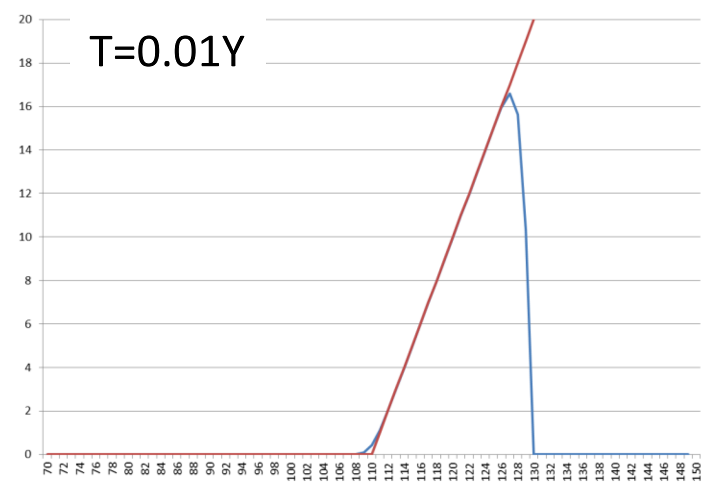
$$g = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{H}{S}\right) + \lambda\sigma\sqrt{T}$$

110 CALL KO 130



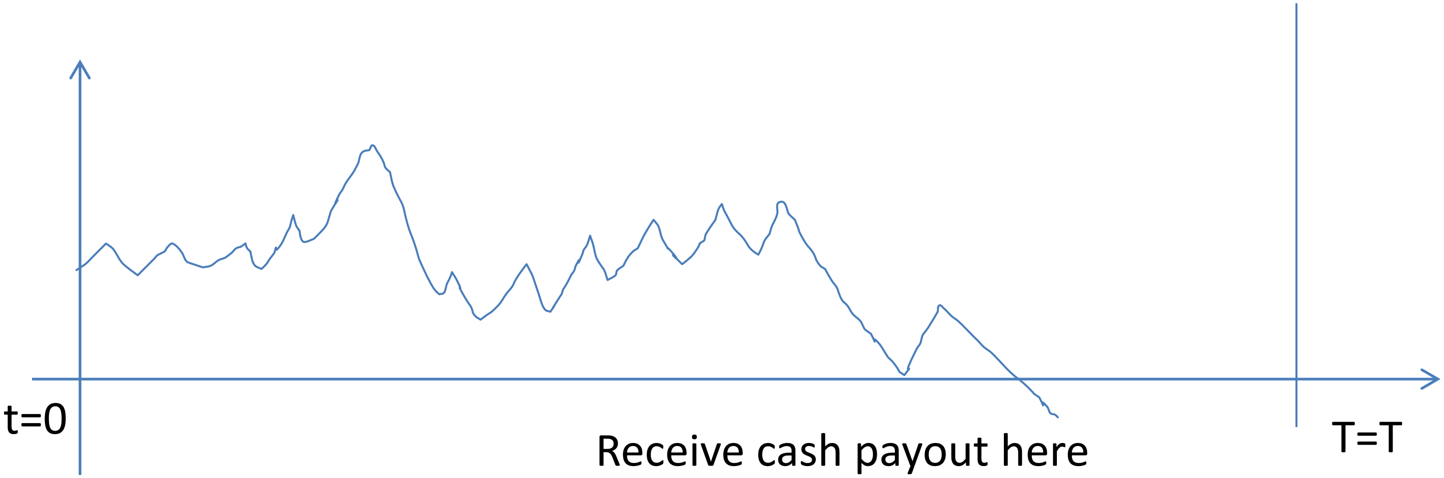






One-touch

- Contract delivers a cash payoff (or share payoff) if a barrier is hit within a certain time-period.



- A two-touch receives a payoff if either a lower level or an upper level is attained by the price of the underlying asset.