CMSC 351 Summer 2025 Homework 9

Due Friday 15 August 2025 by 23:59EST on Gradescope.

Directions:

- Homework must be done on printouts of these sheets and then scanned properly, or via Latex, or by downloading, writing on the PDF, and uploading. If you use Latex please do not change the Latex formatting.
- Do not use your own blank paper!
- The reason for this is that Gradescope will be following this template to locate the answers to the problems so if your answers are organized differently they will not be recognized.
- Tagging is automatic, you will not be able to manually tag.

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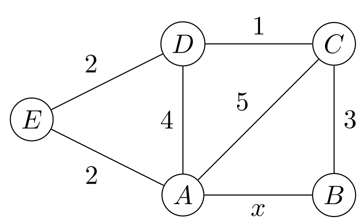
1. Suppose G is an unweighted, undirected, complete graph (every two vertices are adjacent) with four vertices indexed $0, 1, 2, 3$. [10 pts]

How many possible spanning trees are there total?	16
How many possible spanning trees are there that include edge $0-3$ or $1-2$?	12

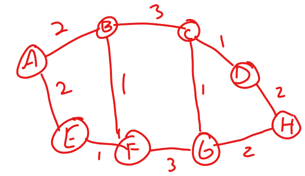
2. Consider the following graph where x is a positive integer:



If there is no minimum spanning tree that includes $A - B$, what must be true about x? [15 pts]

Answer	x>3

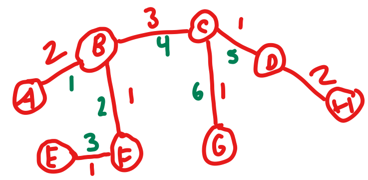
Scratch work; not graded:



3. A graph has the following adjacency matrix, where – means no edge:

PRIM

	A	B	C	D	E	F	G	H
A	–	2	–	–	2	–	–	–
B	2	–	3	–	–	1	–	–
C	–	3	–	1	–	–	1	–
D	–	–	1	–	–	–	–	2
E	2	–	–	–	–	1	–	–
F	–	1	–	–	1	–	3	–
G	–	–	1	–	–	3	–	2
H	–	–	–	2	–	–	2	–



w(MST) = 2+1+1+3+1+1+2=11

We denote edges by pairs of letters xy in alphabetical order, so for example the edge between C and G would be denoted CG (not GC). This way edges may be alphabetized, so for example $CG < HD$.

(a) Suppose we run Prim's algorithm on the graph, starting at A. When a choice of edges is possible choose the edge earlier in the alphabet. In what order will the edges be added? [10 pts]

AB	BF	EF	BC	CD	CG	DH

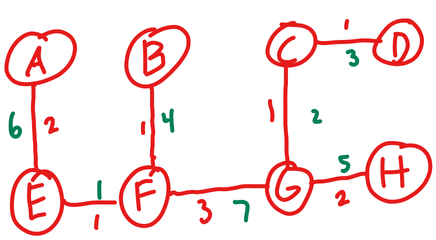
(b) Suppose we run Kruskal's algorithm on the graph. When a choice of edges is possible choose the edge later in the alphabet. In what order will the edges be added? [10 pts]

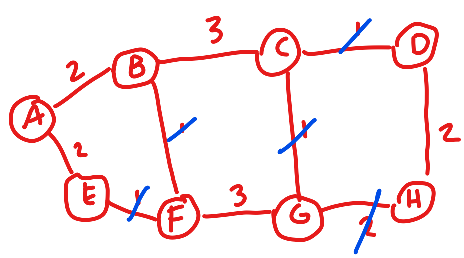
EF	CG	CD	BF	GH	AE	FG

(c) What will be the weight of any minimal spanning tree? [5 pts]

Weight =	11

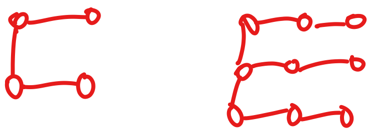
Kruskal:

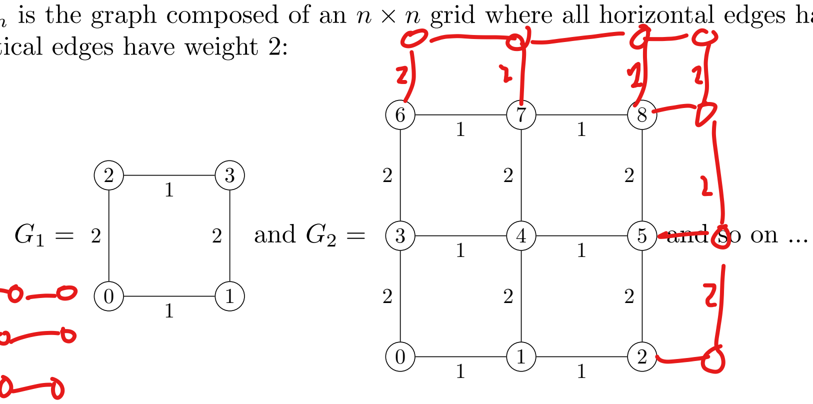




w(MST) = 2+1+1+3+1+1+2 = 11

4. Suppose G_n is the graph composed of an $n \times n$ grid where all horizontal edges have weight 1 and all vertical edges have weight 2: [25 pts]





Prim's Algorithm is run on G_n starting at $s = 0$. Suppose that when a horizontal edge is added it takes k seconds and when a vertical edge is added it takes $k+1$ seconds. Here k is the number of edges already included. Nothing else takes any time at all.

Write down an expression for the total time required. Your expression should be simplified to remove any $\sum$ or +...+ stuff.

Hint: Take some time to think about this - it's not overly difficult if you approach it carefully!

Solution:

For each Gn+1 graph, you add

Horizontal edge added takes k seconds
Vertical edge added takes k+1 seconds

G1 = 2x2 (4 vertices) $n \cdot 2 = 2$, $1 \cdot 2 = 2$
- 2V, 2H

G2 = 3x3 (9 vertices) $n \cdot 3 = 6$, $2 \cdot 3 = 6$
- 6V, 6H

G3 = 4x4 (16 vertices) $3 \cdot 4 = 12$, n
- 12V, 12H

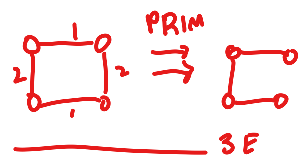
GN = (n+1)(n+1) vertices in a grid $= (n+1)^2$ *

so... for a graph Gn,
Vertical / Horizontal Edges: $n(n+1)$
total $E = 2(n(n+1))$

so final answer:
$$T = \frac{n^4 + 4n^3 + 3n^2}{2}$$

Prim's algorithm starts at s=0, and will always prefer moving horizontal rather than vertical because the cost will always be less at 1<2. After adding all of the horizontal edges possible, it will move up one vertical edge, then repeat until done.
so all the horizontal edges will be connected, which is (n+1) rows given this fact *
times n connecting edges (because there is n+1 vertices to connect in a row but for n vertices in a line there should be n-1 edges). so that = n(n+1) horizontal edges plus the vertical edges which should be the minimum amount because those edges will be prioritized the least so that is n edges by the same logic previously

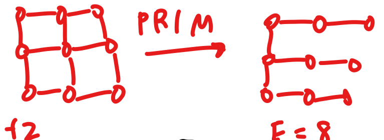
check: G(1) =
$1(2) + 1 = 3$



that means edges when using prims algorithm = n(n+1) + n
$= H + V$
$= n^2 + 2n$

with HorizE=n(n+1) and VertE = n
H V

G(2) =
$2(3)+2$
$= 6+2 = 8$



Consider sum of time as
$$\underbrace{\sum_{K=0}^{H-1} K + \sum_{K=0}^{H+V-1} (k+1)}$$

$$= \frac{H(H-1)}{2} + VH + \left(\frac{V(V-1)}{2}\right) + V \Big\} = \frac{H(H-1)(V(V-1)) + 2VH}{2} + V \quad \frac{(H+V)^2 - (H+V)}{2} + V$$

Combine $\frac{2VH}{2} = VH$

$$= \frac{(H^2-H)(V^2-V) + 2VH}{2} + V$$

$$= H^2 + 2VH + V^2 - (H+V) =$$

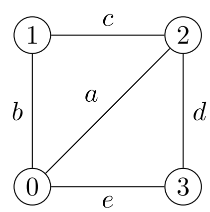
$$= (n^2+2n)^2 - (n^2+2n)$$

$$= n^4 + 4n^3 + 4n^2 - n^2 - 2n$$

$$= \frac{n^4 + 4n^3 + 3n^2 - \cancel{2n}}{2} + \cancel{\frac{2n}{2}}$$

$$\frac{n^4 + 4n^3 + 3n^2}{2}$$

5. Consider the following graph: [25 pts]



Assuming the weights are all different there are $5! = 120$ possible orderings, for example $a < b < c < d < e$ and $e < d < c < b < a$ and $a < c < b < e < d$ and so on. Assuming each of these is equally likely, if we run Kruskal's Algorithm on this graph, what is the probability that a is chosen?

Note: Do not do this by listing all possible orderings and explain your work with sentences, not just calculations!

Solution:

Kruskal's alg. always takes the minimal weight edge that doesn't create a cycle and uses that to patch the MST together.
In the graph, the edge a is 0-2.
There are 2 other ways/paths to get from 0-2: 0-1-2 AND 0-3-2.
0-b->1-c->2, 0-e->3-d->2
So, the edge a will not be added if the weight of a is greater than the weight of all the edges from either of the other paths that connect 0-2 which are 0-1-2 and 0-3-2.
which means one of the following must be true: (w meaning weight)
For 0-1-2: w(a) > w(b) && w(a) > (c)
For 0-1-2: w(a) > w(e) && w(a) > w(d)
Because of both of the weights on an alternate path from 0-2 are lighter than w(a), then the Kruskal Algorithm will add those edges before adding a, and at that point adding a would create a cycle from 0-1-2-0 or 0-3-2-0.

Now find P(a chosen).

All edge weights are different so there are 120 possible orders of weight.
Our "bad" condition is when a is "after" b and c OR when a is "after" d and e (in the equality chains above)
Our "good" conditions are here:
when a is first: chains of the form a<?<?<?<? (Count = 4! = 24)
when a is second: chains of the form ?<a<?<?<? (Count = another 24 same logic)
when a is third but the 2 BEFORE a are one of {b,c} OR {d,e} because otherwise a wouldn't connect
so that is {b or c} < {d or e} < a < ? < ?
2 (b or c) * 2(d or e) = 4 * 4(same process for the opposite side of the heavier ones) = 16.
24+24+16 = 64/120

so P[a chosen] = 64/120=8/15 = 0.533