Topic-11-recurrence-trees

Tuesday, July 16, 2024 1:01 PM

RECURRENCE TREES

① **Intro:** Recurrence trees are another way to understand recurrence relations:

(A) To calculate specific values

(B) To construct closed fmlas in (perhaps) more organized manner than digging down.

(C) Serves as background for proof of Master Theorem!

Note: used for RR of the form

not things like $T(n-1)$, $T(n-2)$, etc.

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

yes, like $T(n) = 2T\left(\frac{n}{3}\right) + 6n$

Let's look at: $\boxed{T(n) = 2T\left(\frac{n}{3}\right) + 5n+1 \quad \text{w/ } T(1) = 7}$

② Calc. Specific Values

Sps we want $T(9)$

we'll construct the tree in stages!

$n=9$ $\qquad$ $n=3$

start $\quad T(9)$ ⊛

we obs. that $T(9) = 2T(3) + 5(9)+1$

here's a tree which diagrams ↑

$$5(9)+1$$
$$T(3) \qquad T(3)$$

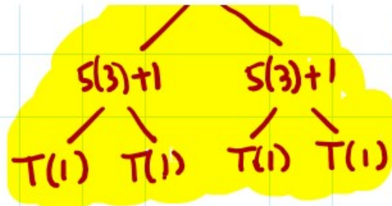
⊛ identical total sum!

repeat(ish) using the fact that $T(3) = 2T(1) + 5(3)+1$

replace the two $T(3)$ by subtrees:

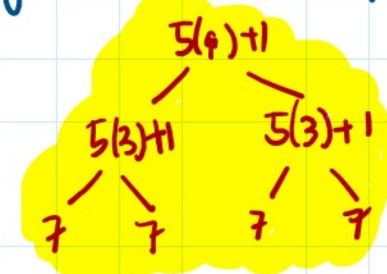
$$5(9)+1$$
$$5(3)+1 \qquad 5(3)+1$$

⊛ and again,



* and again, same total sum!

b/c we get to T(1) in leaves, and T(1) = 7:



* and again!

This total is T(9)! Thus

$T(9) = 5(9)+1$
$\quad + 2(5(3)+1)$
$\quad + 4(7) = 106$

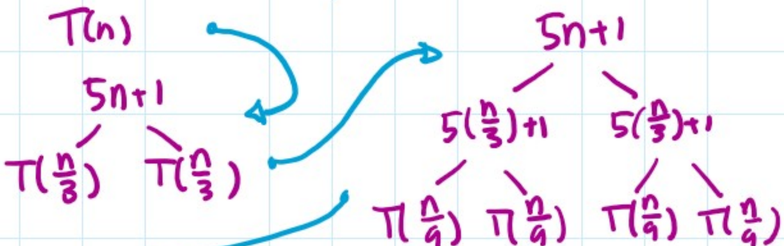
Level by Level

③ Finding a closed fmla!

same as ↑ but w/out specific n! $T(n) = 2T(\frac{n}{3}) + 5n + 1$ w/ $T(1) = 7$

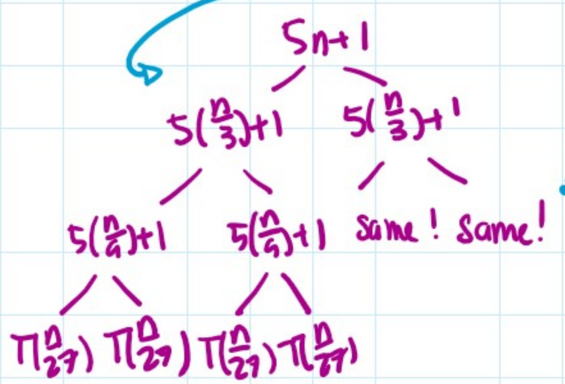
start

next



$\hookrightarrow T(\frac{n}{3}) = 2T(\frac{n}{9}) + 5(\frac{n}{3}) + 1$

$\hookrightarrow T(\frac{n}{9}) = 2T(\frac{n}{27}) + 5(\frac{n}{9}) + 1$



does this go forever?!
technically it stops at the base case which happens when the leaves are T(1)
(idealistic - assume n is nice, and we reach T(1)!)

Say level 0 = root level, next level is level 1, etc.
At level k, as we built it, we had $T(\frac{n}{3^k})$ before proceeding!
We get to T(1) when $\frac{n}{3^k} = 1$ which is when $k = \log_3 n$

- non-leaf levels are $0, 1, 2, \ldots, (\log_3 n) - 1$

We get to T(1) when $3^k = 1$ which is when $k = \log_3 n$

- non-leaf levels are $0, 1, 2, \ldots, (\log_3 n) - 1$
- leaf level is $\log_3 n$

This next table yields the total in each level!

Level	# nodes	value/node	total = # nodes · value/node
0	1	$5n+1$	$(1)(5n+1)$
1	2	$5(\frac{n}{3})+1$	$(2)(5(\frac{n}{3})+1)$
2	4	$5(\frac{n}{3^2})+1$	$(4)(5(\frac{n}{3^2})+1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-1$	2^{k-1}	$5(\frac{n}{3^{k-1}})+1$	$2^{k-1}(5(\frac{n}{3^{k-1}})+1)$
k	2^k	7	$2^k(7)$

b/c $T(1) = 7$

Add 'em up!

Find the pattern!

$$T(n) = 7 \cdot 2^k + \sum_{i=0}^{k-1} (2^i)\left(5\left(\frac{n}{3^i}\right) + 1\right)$$

Let's evaluate! Eventually we'll sub. $k = \log_3 n$

$$\sum_{i=0}^{J} r^i = \frac{r^{J+1} - 1}{r - 1} = \frac{1 - r^{J+1}}{1 - r}$$

$$T(n) = 7 \cdot 2^k + \sum_{i=0}^{k-1} \left[5n\left(\frac{2}{3}\right)^i + 2^i\right]$$

$$= 7 \cdot 2^k + 5n \sum_{i=0}^{k-1} \left(\frac{2}{3}\right)^i + \sum_{i=0}^{k-1} 2^i$$

$$= 7 \cdot 2^k + 5n \left[\frac{\left(\frac{2}{3}\right)^k - 1}{\frac{2}{3} - 1}\right] + 2^k - 1$$

$$= 7 \cdot 2^k - 15n \left[\left(\frac{2}{3}\right)^k - 1\right] + 2^k - 1$$

now sub $k = \log_3 n$

$$T(n) = 8 \cdot 2^{\log_3 n} - 15n\left[\left(\frac{2}{3}\right)^{\log_3 n} - 1\right] - 1$$

technically we're done. However, note that (using C.O.B)

Could sub in to get a given value!

technically we're done. However, note that (using 2.8.8)

$$2^{\log_3 n} = 2^{\frac{\lg n}{\lg 3}} = \left(2^{\lg n}\right)^{\frac{1}{\lg 3}} = n^{1/\lg 3} = n^{\log_3 2}$$

and

$$\left(\tfrac{2}{3}\right)^{\log_3 n} = \frac{2^{\log_3 n}}{3^{\log_3 n}} = \frac{n^{\log_3 2}}{n} = n^{\log_3 2 - 1}$$

to get a nicer value!