

CMSC 330: Organization of Programming Languages

Regular Expressions and Finite Automata

How do regular expressions work?

- ▶ What we've learned
 - What regular expressions are
 - What they can express, and cannot
 - Programming with them
- ▶ What's next: how they work
 - A great computer science result

A Few Questions About REs

- ▶ How are REs implemented?
 - Given an arbitrary RE and a string, how to decide whether the RE matches the string?
- ▶ What are the basic components of REs?
 - Can implement some features in terms of others
 - E.g., e^+ is the same as ee^*
- ▶ What does a regular expression represent?
 - Just a set of strings
 - This observation provides insight on how we go about our implementation

Definition: Alphabet

- ▶ An **alphabet** is a finite set of symbols
 - Usually denoted Σ
- ▶ Example alphabets:
 - Binary: $\Sigma = \{0, 1\}$
 - Decimal: $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
 - Alphanumeric: $\Sigma = \{0-9, a-z, A-Z\}$

Definition: String

- ▶ A **string** is a finite sequence of symbols from Σ
 - ϵ is the empty string (" in OCaml)
 - $|s|$ is the length of string s
 - $|Hello| = 5$, $|\epsilon| = 0$
 - Note
 - \emptyset is the empty set (with 0 elements)
 - $\emptyset \neq \{ \epsilon \}$ (and $\emptyset \neq \epsilon$)
- ▶ Example strings over alphabet $\Sigma = \{0,1\}$ (binary):
 - 0101
 - 0101110
 - ϵ

Definition: Language

- ▶ A **language** L is a set of strings over an alphabet
- ▶ Example: All strings of length 1 or 2 over alphabet $\Sigma = \{a, b, c\}$ that begin with a
 - $L = \{a, aa, ab, ac\}$
- ▶ Example: All strings over $\Sigma = \{a, b\}$
 - $L = \{\epsilon, a, b, aa, bb, ab, ba, aaa, bba, aba, baa, \dots\}$
 - Language of all strings written Σ^*

Definition: Language (cont.)

- ▶ Example: The set of phone numbers over the alphabet $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, (,), -\}$
 - Give an example element of this language (123) 456-7890
 - Are all strings over the alphabet in the language? No
 - Is there a regular expression for this language?
 $\backslash (\backslash d\{3\} \backslash) \backslash d\{3\} - \backslash d\{4\}$
- ▶ Example: The set of all valid (runnable) OCaml programs
 - Later we'll see how we can specify this language
 - (Regular expressions are useful, but not sufficient)

Operations on Languages

- ▶ Let Σ be an alphabet and let L, L_1, L_2 be languages over Σ
- ▶ **Concatenation** L_1L_2 creates a language defined as
 - $L_1L_2 = \{xy \mid x \in L_1 \text{ and } y \in L_2\}$
- ▶ **Union** creates a language defined as
 - $L_1 \cup L_2 = \{x \mid x \in L_1 \text{ or } x \in L_2\}$
- ▶ **Kleene closure** creates a language is defined as
 - $L^* = \{x \mid x = \varepsilon \text{ or } x \in L \text{ or } x \in LL \text{ or } x \in LLL \text{ or } \dots\}$

Operations Examples

Let $L_1 = \{ a, b \}$, $L_2 = \{ 1, 2, 3 \}$ (and $\Sigma = \{a,b,1,2,3\}$)

- ▶ What is $L_1 L_2$?
 - $\{ a1, a2, a3, b1, b2, b3 \}$
- ▶ What is $L_1 \cup L_2$?
 - $\{ a, b, 1, 2, 3 \}$
- ▶ What is L_1^* ?
 - $\{ \epsilon, a, b, aa, bb, ab, ba, aaa, aab, bba, bbb, aba, abb, baa, bab, \dots \}$

Quiz 1: Which string is **not** in L_3

$L_1 = \{a, ab, c, d, \varepsilon\}$ where $\Sigma =$
 $\{a, b, c, d\}$

$L_2 = \{d\}$

$L_3 = L_1 \cup L_2$

A. cd

B. c

C. ε

D. d

Quiz 1: Which string is **not** in L_3

$L_1 = \{a, ab, c, d, \varepsilon\}$ where $\Sigma =$
 $\{a, b, c, d\}$

$L_2 = \{d\}$

$L_3 = L_1 \cup L_2$

A. cd

B. c

C. ε

D. d

Quiz 2: Which string is **not** in L_3

$L_1 = \{a, ab, c, d, \varepsilon\}$ where $\Sigma = \{a, b, c, d\}$

$L_2 = \{d\}$

$L_3 = L_1(L_2^*)$

- A. a
- B. abd
- C. abdd
- D. adad

Quiz 2: Which string is **not** in L_3

$L_1 = \{a, ab, c, d, \varepsilon\}$ where $\Sigma = \{a, b, c, d\}$
 $L_2 = \{d\}$
 $L_3 = L_1(L_2^*)$

- A. a
- B. abd
- C. abdd
- D. adad

Regular Expressions: Grammar

- ▶ We can define a grammar for regular expressions R

$R ::= \emptyset$	The empty language
$ \varepsilon$	The empty string
$ \sigma$	A symbol from alphabet Σ
$ R_1 R_2$	The concatenation of two regexps
$ R_1 R_2$	The union of two regexps
$ R^*$	The Kleene closure of a regexp

Regular Languages

- ▶ Regular expressions denote **regular languages**
- ▶ Not all languages are regular
 - Examples (without proof):
 - The set of palindromes over Σ
 - $\{a^n b^n \mid n > 0\}$ (a^n = sequence of n a 's)
- ▶ Almost all programming languages are not regular
 - But aspects of them sometimes are (e.g., identifiers)
 - Regular expressions are commonly used in parsing tools

Semantics: Regular Expressions (1)

- Given an alphabet Σ , the **regular expressions** over Σ are defined inductively as follows

Constants

regular expression	denotes language
\emptyset	\emptyset
ϵ	$\{\epsilon\}$
each symbol $\sigma \in \Sigma$	$\{\sigma\}$

Ex: with $\Sigma = \{a, b\}$, regex a denotes language $\{a\}$
regex b denotes language $\{b\}$

Semantics: Regular Expressions (2)

- ▶ Let A and B be regular expressions denoting languages L_A and L_B , respectively. Then:

Operations

regular expression	denotes language
AB	$L_A L_B$
$A B$	$L_A \cup L_B$
A^*	L_A^*

- ▶ There are no other regular expressions over Σ

Terminology etc.

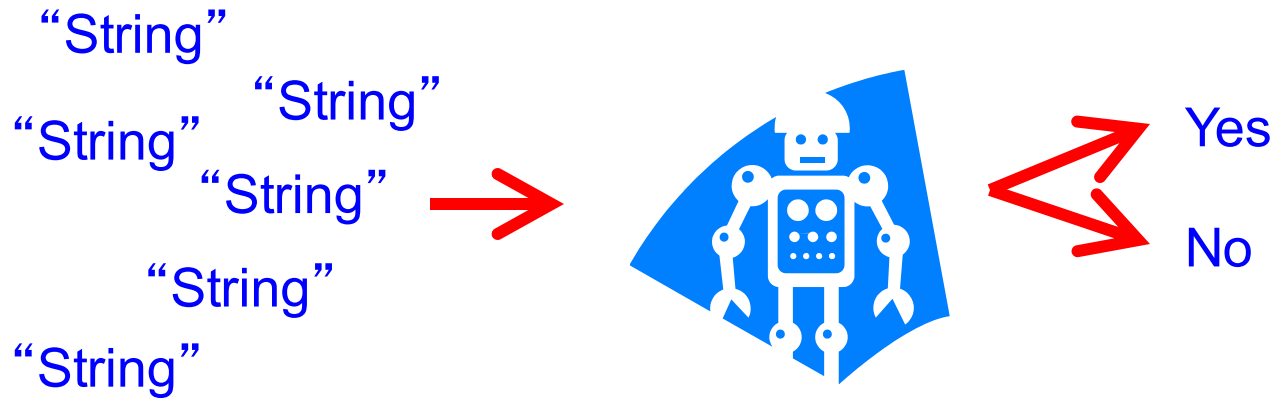
- ▶ Regexps apply operations to symbols
 - Generates a set of strings (i.e., a language)
 - (Formal definition shortly)
 - Examples
 - a generates language $\{a\}$
 - $a|b$ generates language $\{a\} \cup \{b\} = \{a, b\}$
 - a^* generates language $\{\epsilon\} \cup \{a\} \cup \{aa\} \cup \dots = \{\epsilon, a, aa, \dots\}$
- ▶ If $s \in$ language L generated by a RE r , we say that r accepts, describes, or recognizes string s

Regular Expressions

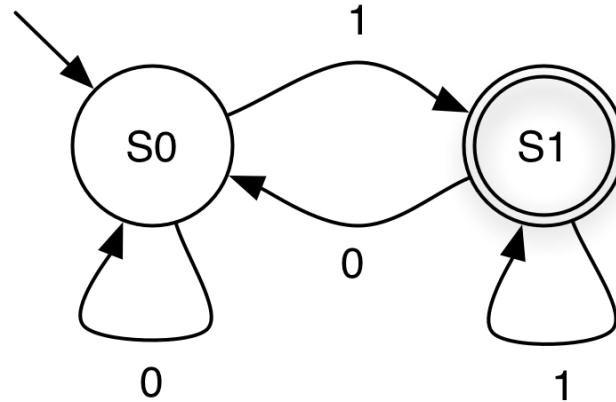
- ▶ Almost all of the features we've seen for REs can be reduced to this formal definition
 - OCaml – concatenation of single-symbol REs
 - /(OCaml|Rust)/ – union
 - /(OCaml)*/ – Kleene closure
 - /(OCaml)+/ – same as (OCaml)(OCaml)*
 - /(Ocaml)?/ – same as (ϵ |(OCaml))
 - /[a-z]/ – same as (a|b|c|...|z)
 - /[^0-9]/ – same as (a|b|c|...) for $a, b, c, \dots \in \Sigma - \{0..9\}$
 - ^, \$ – correspond to extra symbols in alphabet
 - Think of every string containing a distinct, hidden symbol at its start and at its end – these are written ^ and \$

Implementing Regular Expressions

- ▶ We can implement a regular expression by turning it into a **finite automaton**
 - A “machine” for recognizing a regular language



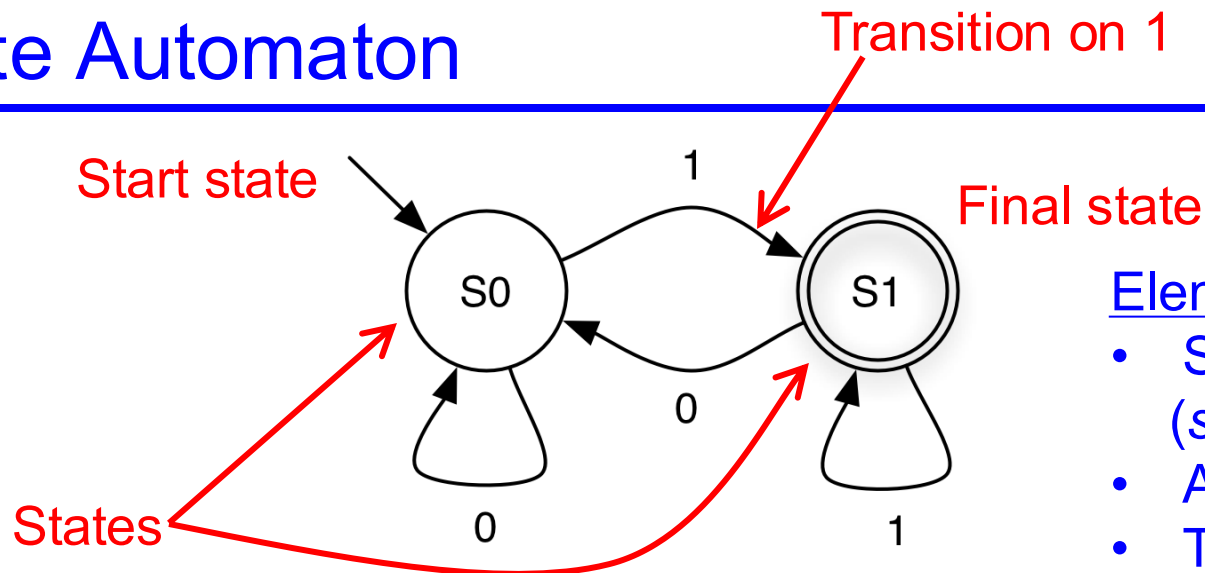
Finite Automaton



Elements

- States S
(*start, final*)
- Alphabet Σ
- Transition
edges δ

Finite Automaton



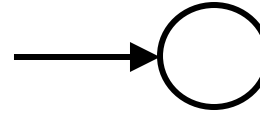
Elements

- States S (*start, final*)
 - Alphabet Σ
 - Transition edges δ
- ▶ Machine starts in **start** or **initial** state
 - ▶ Repeat until the end of the string s is reached
 - Scan the next symbol $\sigma \in \Sigma$ of the string s
 - Take **transition** edge labeled with σ
 - ▶ String s is **accepted** if automaton is in **final** state when end of string s is reached

Finite Automaton: States

► Start state

- State with incoming transition from no other state
- Can have only one start state

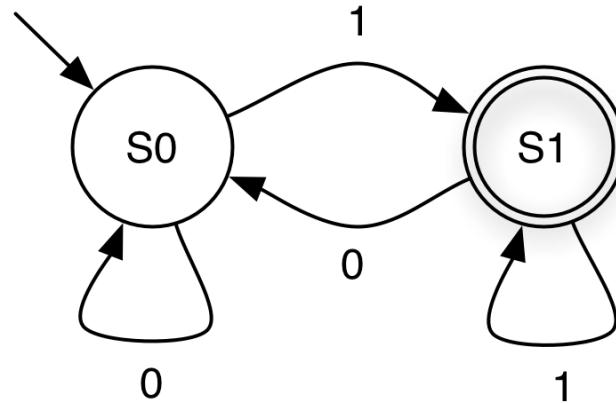


► Final states

- States with double circle
- Can have zero or more final states
- Any state, including the start state, can be final



Finite Automaton: Example 1

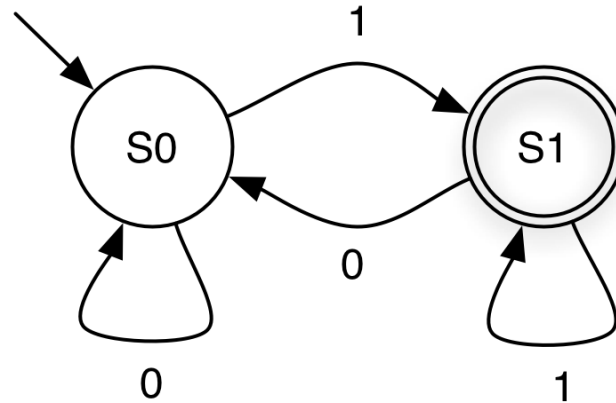


0 0 1 0 1 1

Accepted?

Yes

Finite Automaton: Example 2

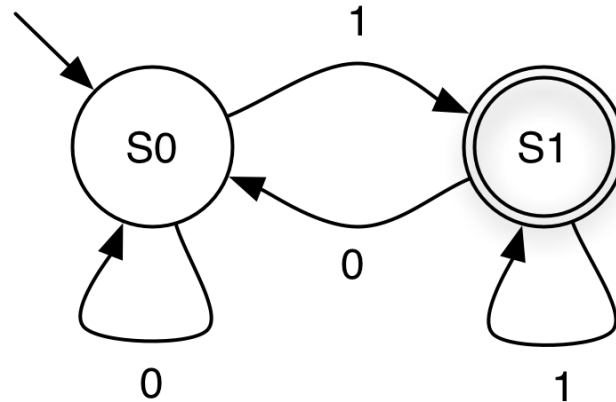


0 0 1 0 1 0

Accepted?

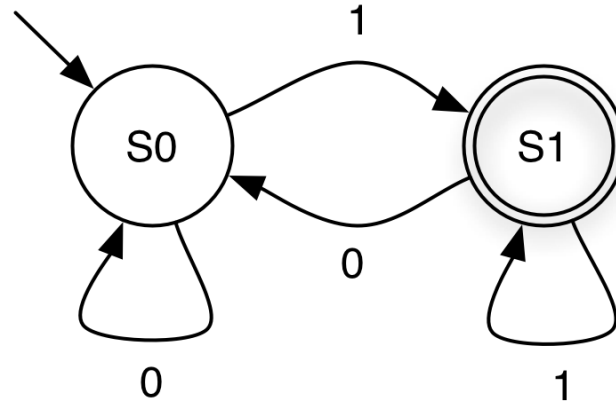
No

Quiz 3: What Language is This?



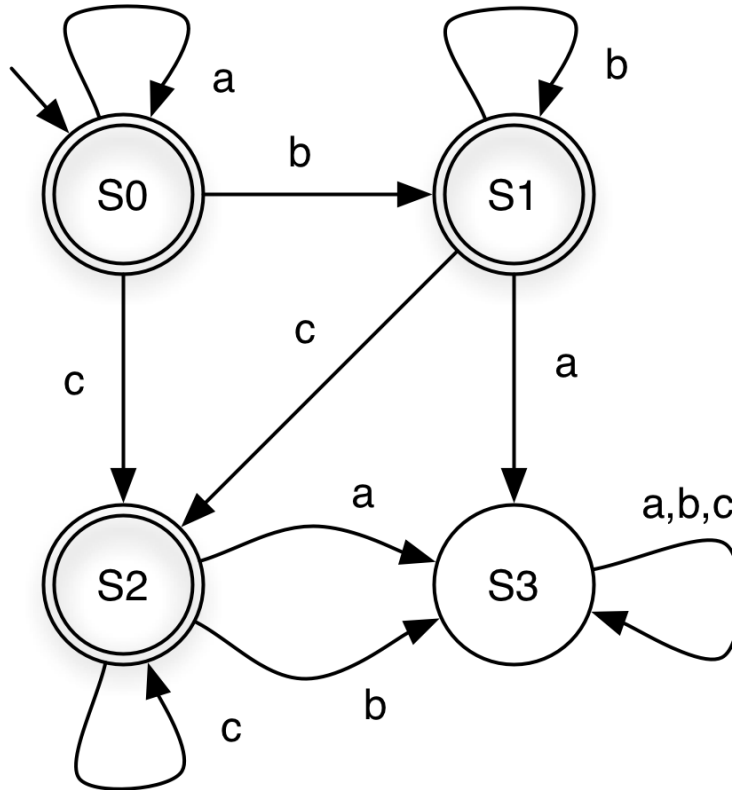
- A. All strings over $\{0, 1\}$
- B. All strings over $\{1\}$
- C. All strings over $\{0, 1\}$ of length 1
- D. All strings over $\{0, 1\}$ that end in 1

Quiz 3: What Language is This?



- A. All strings over $\{0, 1\}$
- B. All strings over $\{1\}$
- C. All strings over $\{0, 1\}$ of length 1
- D. All strings over $\{0, 1\}$ that end in 1
regular expression for this language is $(0|1)^*1$

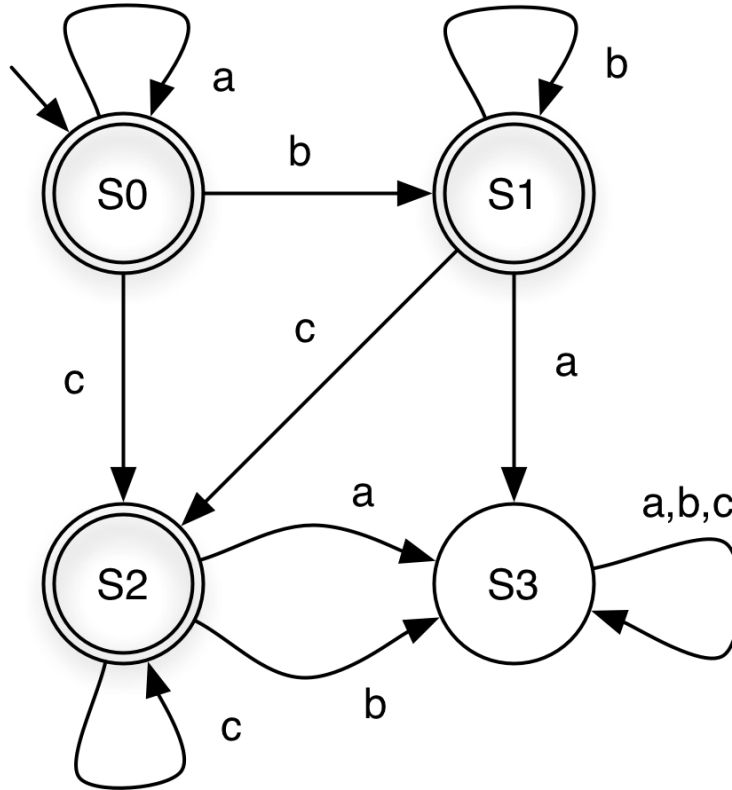
Finite Automaton: Example 3



string	state at end	accept s?
aabcc		

(a,b,c notation shorthand for three self loops)

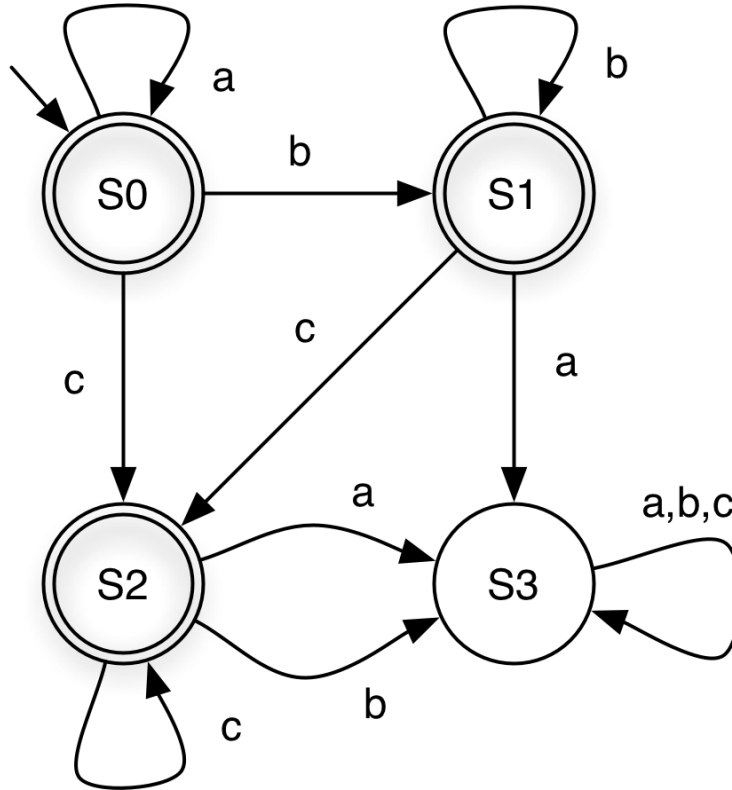
Finite Automaton: Example 3



string	state at end	accept s?
aabcc	S2	Y

(a,b,c notation shorthand for three self loops)

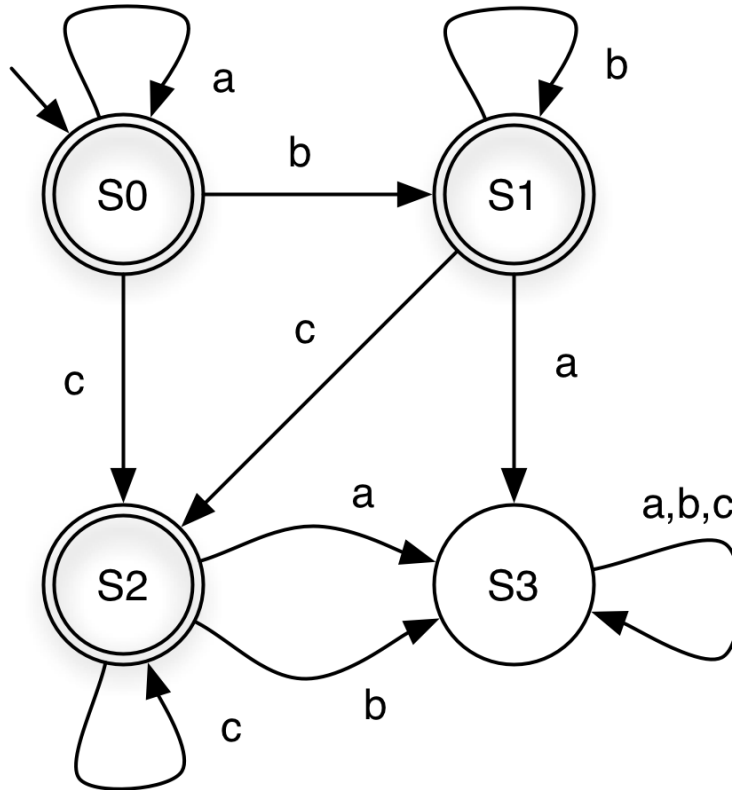
Finite Automaton: Example 3



string	state at end	accept s?
acca		

(a,b,c notation shorthand for three self loops)

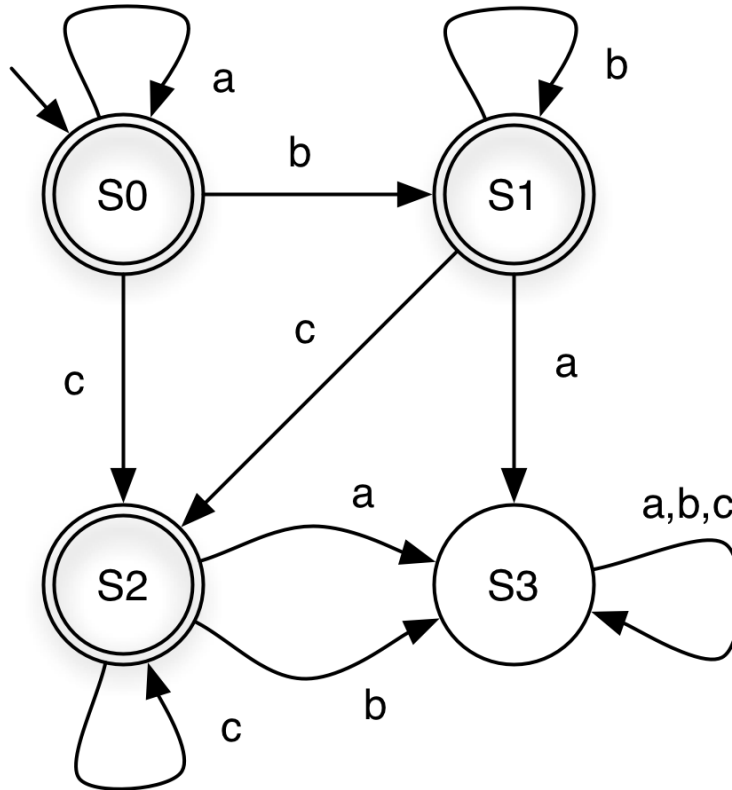
Finite Automaton: Example 3



string	state at end	accept s?
acca	S3	N

(a,b,c notation shorthand for three self loops)

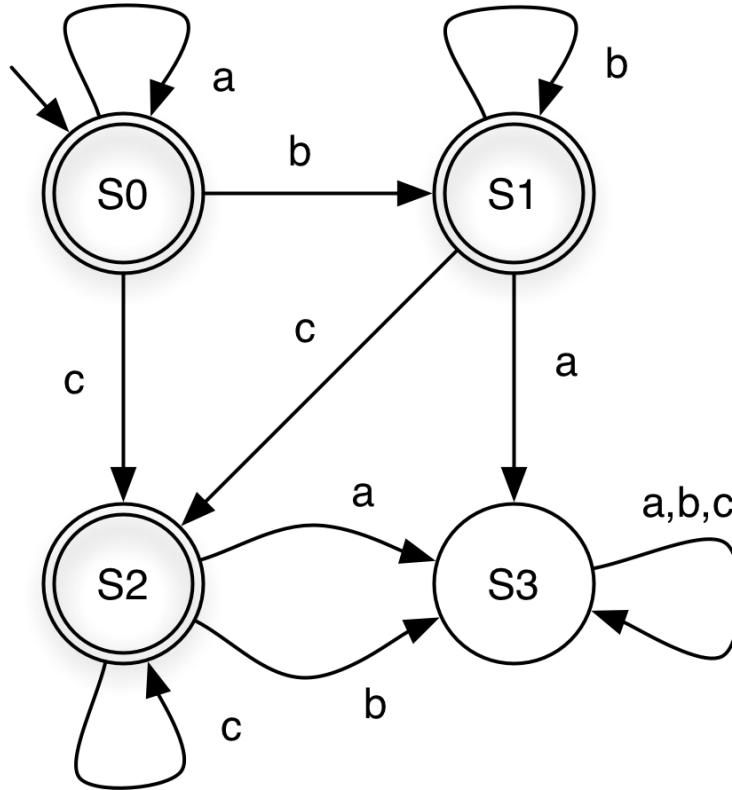
Finite Automaton: Example 3



string	state at end	accept s?
aacbbb		

(a,b,c notation shorthand for three self loops)

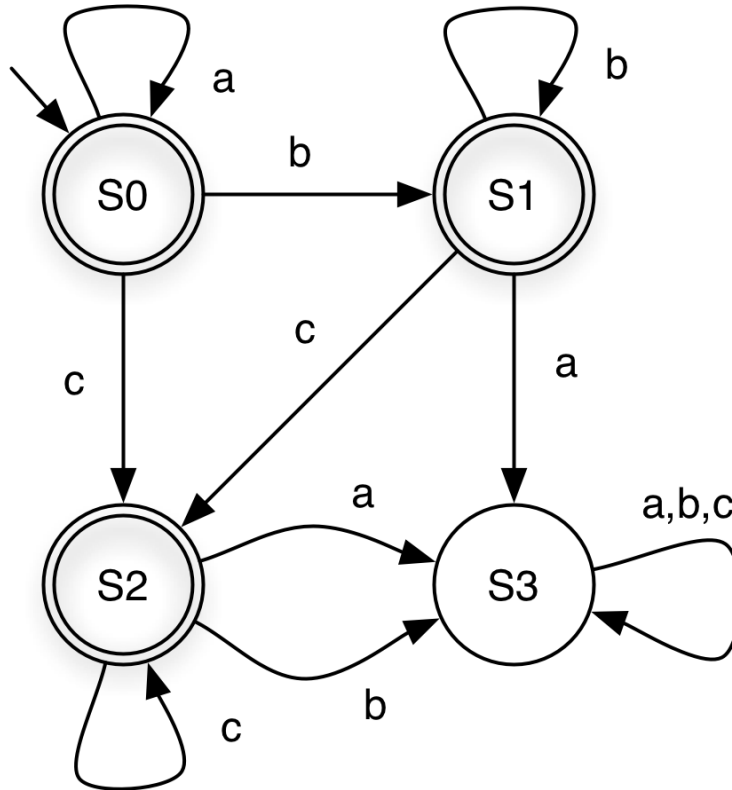
Finite Automaton: Example 3



string	state at end	accept s?
aacbbb	S3	N

(a,b,c notation shorthand for three self loops)

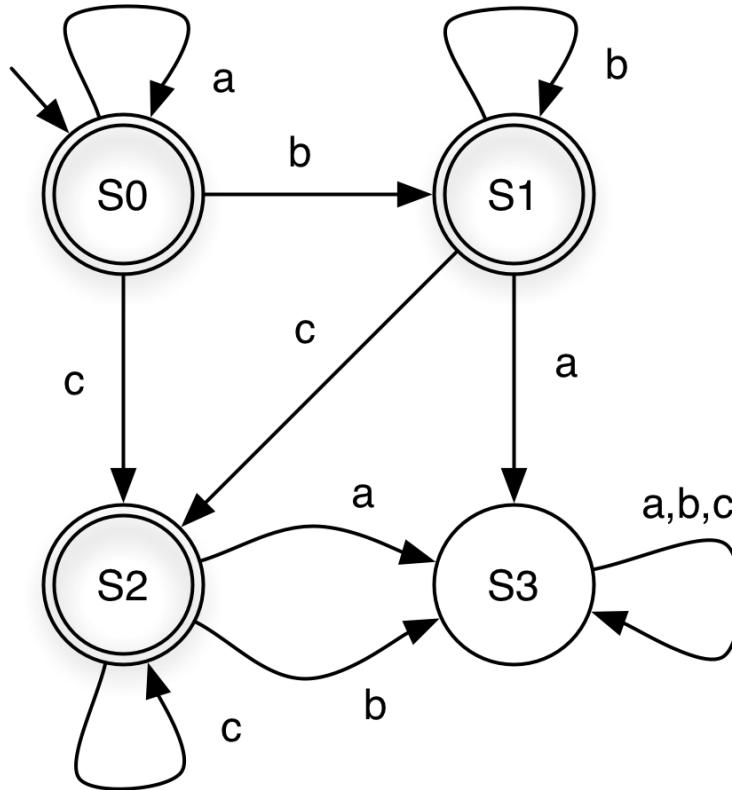
Finite Automaton: Example 3



string	state at end	accept s?
ϵ		

(a,b,c notation shorthand for three self loops)

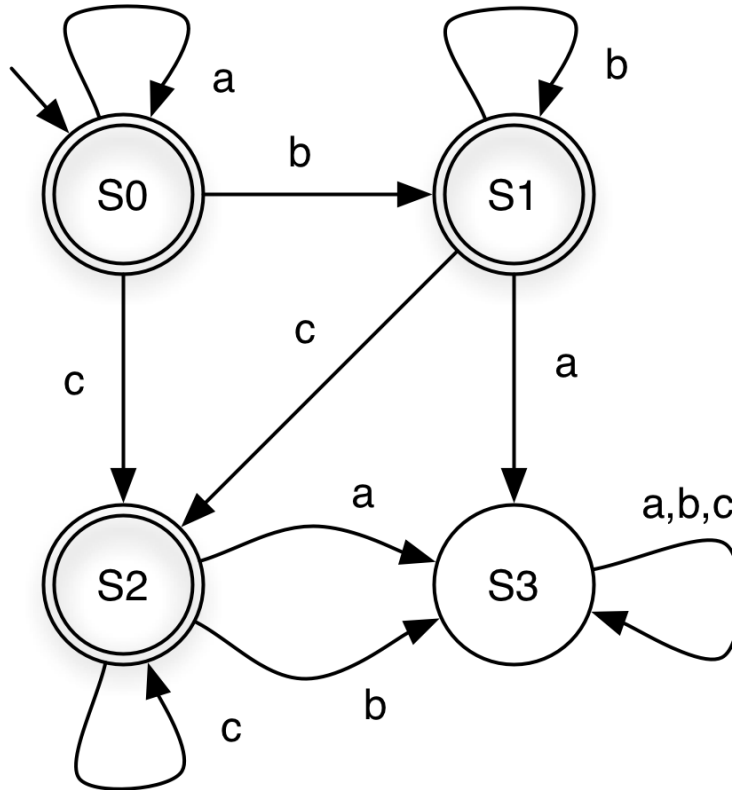
Finite Automaton: Example 3



string	state at end	accept s?
ϵ	S0	Y

(a,b,c notation shorthand for three self loops)

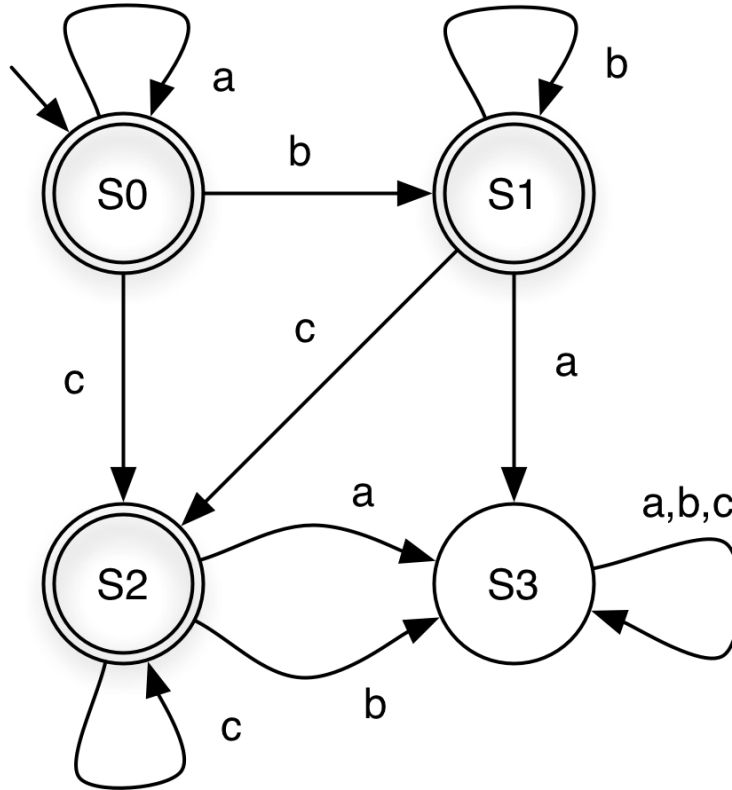
Finite Automaton: Example 3



string	state at end	accept s?
acba		

(a,b,c notation shorthand for three self loops)

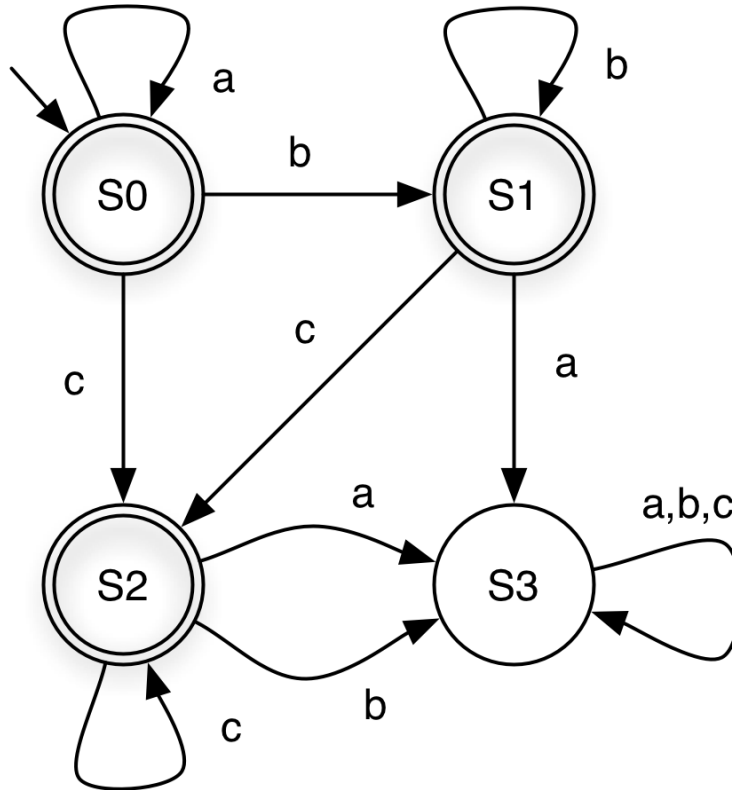
Finite Automaton: Example 3



string	state at end	accept s?
acba	S3	N

(a,b,c notation shorthand for three self loops)

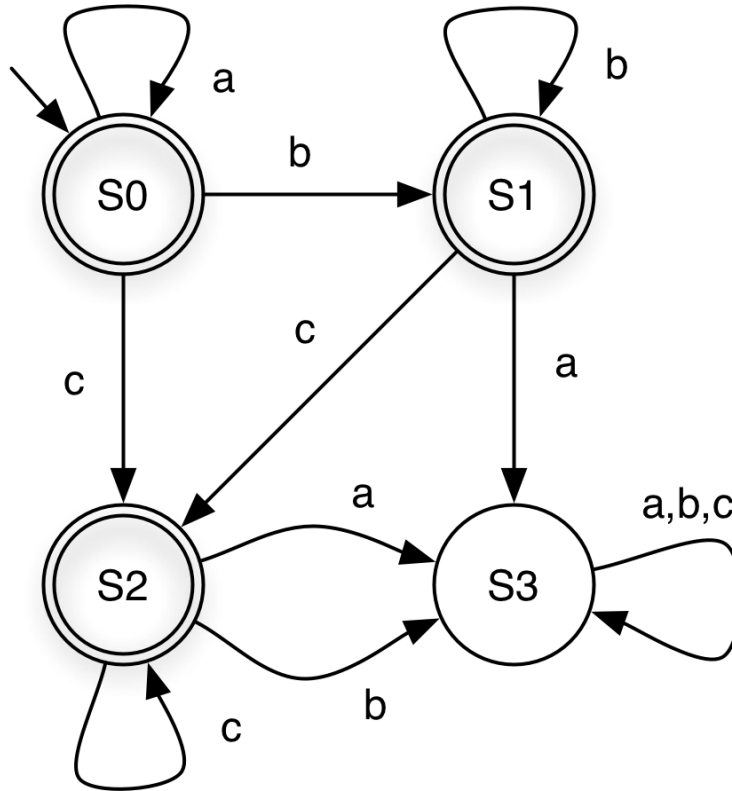
Quiz 4: Which string is **not** accepted?



- A. bcca
- B. abbbbc
- C. ccc
- D. ϵ

(a,b,c notation shorthand for three self loops)

Quiz 4: Which string is **not** accepted?



A. **bcca**

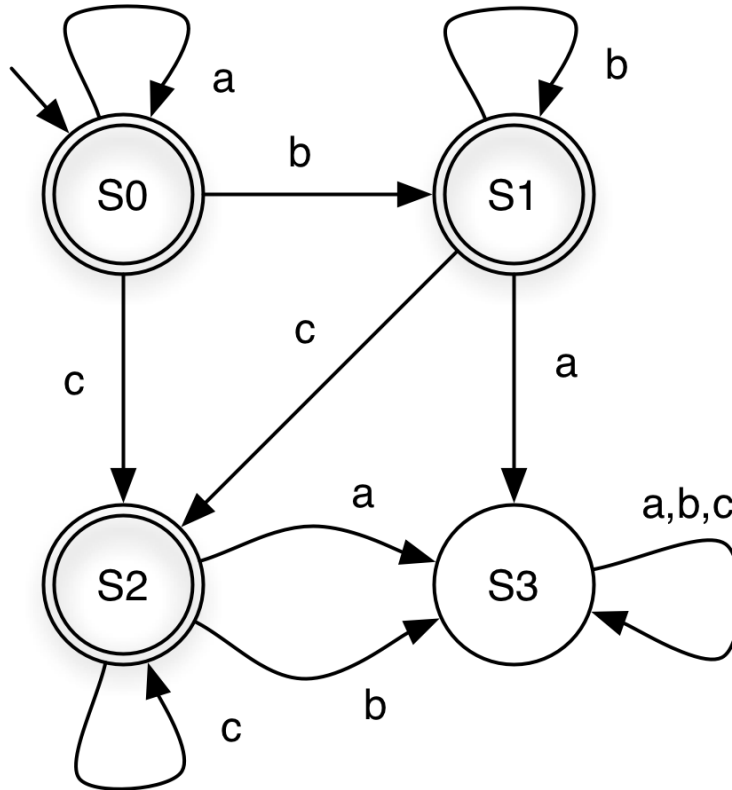
B. abbbbc

C. ccc

D. ϵ

(a,b,c notation shorthand for three self loops)

Finite Automaton: Example 3

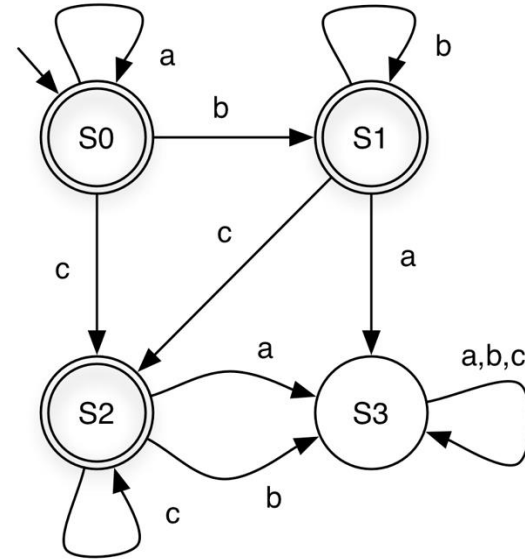
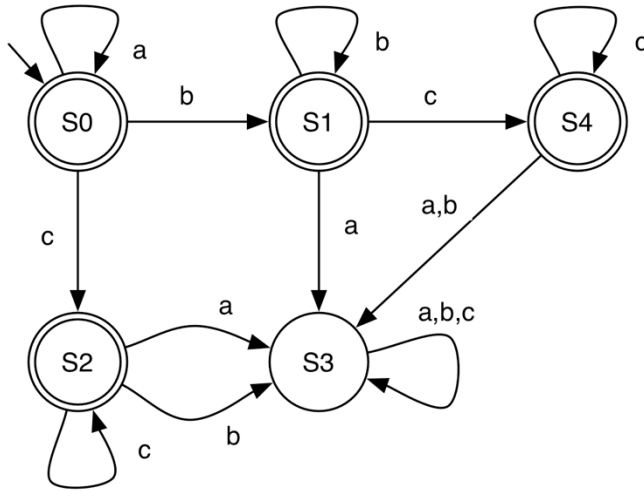


What language does this FA accept?

$a^*b^*c^*$

S3 is a **dead state** – a nonfinal state with **no** transition to another state
- aka a **trap state**

Finite Automaton: Example 4

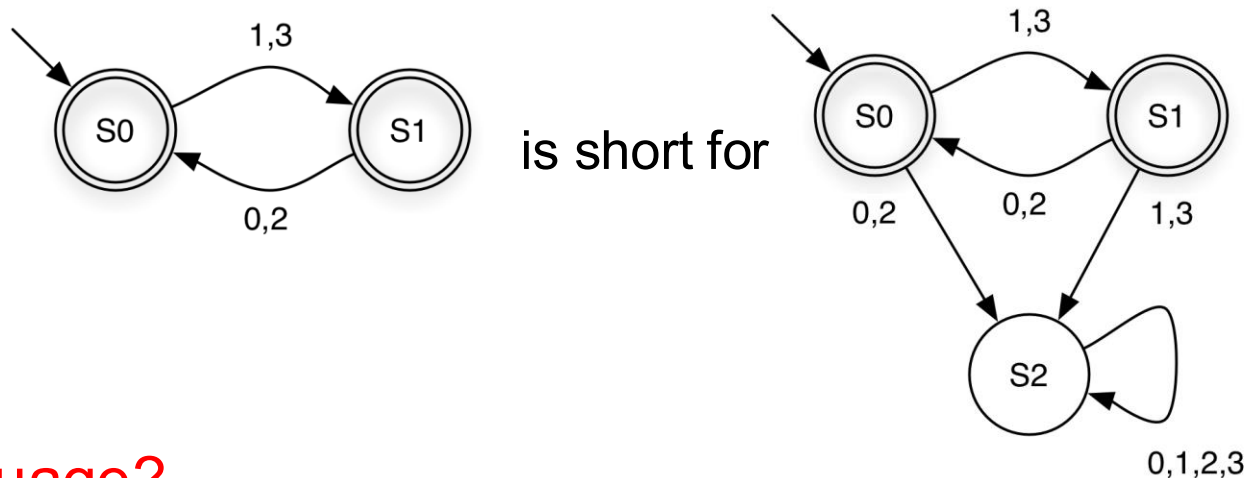


Language?

$a^*b^*c^*$ again, so FAs are not unique

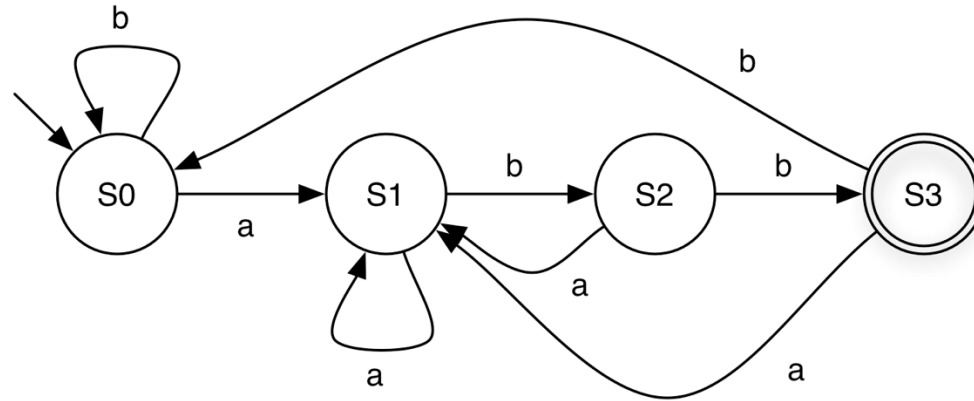
Dead State: Shorthand Notation

- ▶ If a transition is omitted, assume it goes to a dead state that is not shown



- ▶ **Language?**
 - Strings over $\{0,1,2,3\}$ with alternating even and odd digits, beginning with odd digit

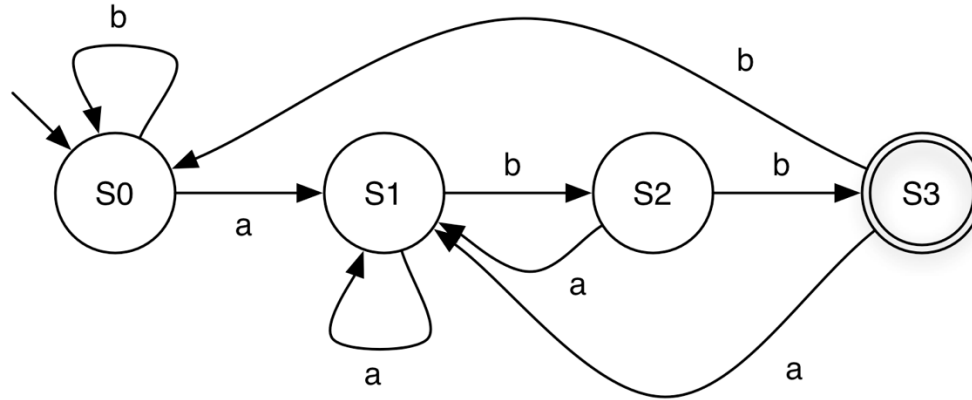
Finite Automaton: Example 5



► Description for each state

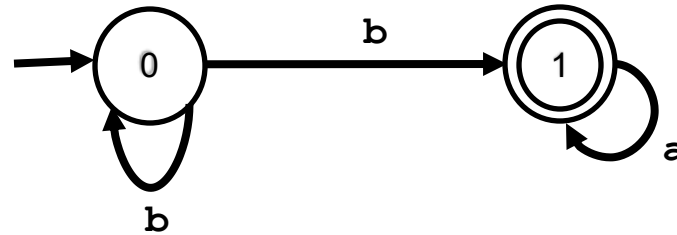
- S0 = “Haven't seen anything yet” OR “Last symbol seen was a b”
- S1 = “Last symbol seen was an a”
- S2 = “Last two symbols seen were ab”
- S3 = “Last three symbols seen were abb”

Finite Automaton: Example 5



- ▶ **Language** as a regular expression?
 - ▶ $(a|b)^*abb$

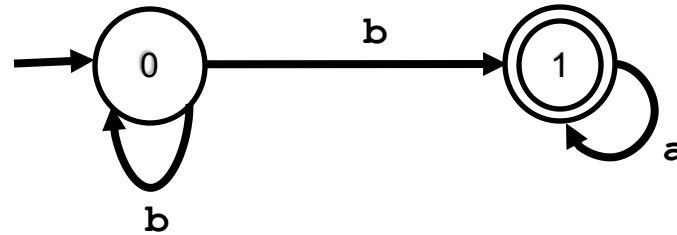
Quiz 5



Over $\Sigma=\{a,b\}$, this FA accepts only:

- A. A string that contains a single b.
- B. Any string in $\{a,b\}$.
- C. A string that starts with b followed by a's.
- D. One or more b's, followed by zero or more a's.

Quiz 5



Over $\Sigma=\{a,b\}$, this FA accepts only:

- A. A string that contains a single b.
- B. Any string in $\{a,b\}$.
- C. A string that starts with b followed by a's.
- D. One or more b's, followed by zero or more a's.

Exercises: Define an FA over $\Sigma = \{0,1\}$

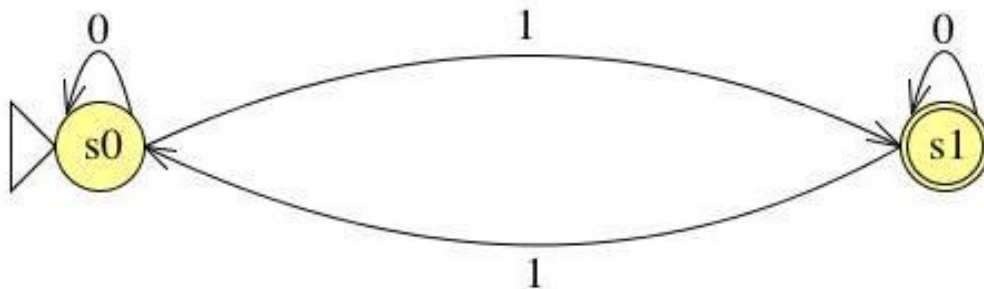
- ▶ That accepts strings containing two consecutive 0s followed by two consecutive 1s
- ▶ That accepts strings with an odd number of 1s
- ▶ That accepts strings containing an even number of 0s and any number of 1s
- ▶ That accepts strings containing an odd number of 0s and odd number of 1s
- ▶ That accepts strings that **DO NOT** contain odd number of 0s and an odd number of 1s

Exercises: Define an FA over $\Sigma = \{0,1\}$

- ▶ That accepts strings with an odd number of 1s

Exercises: Define an FA over $\Sigma = \{0,1\}$

- ▶ That accepts strings with an odd number of **1s**

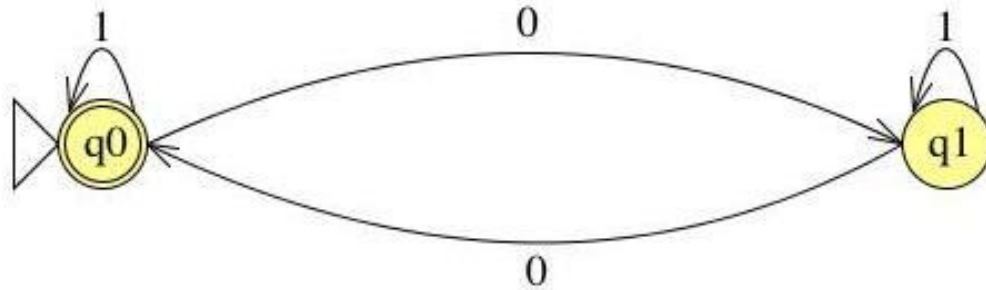


Exercises: Define an FA over $\Sigma = \{a,b\}$

- ▶ That accepts strings containing an even number of **a's** and any number of b's

Exercises: Define an FA over $\Sigma = \{0,1\}$

- ▶ That accepts strings containing an even number of 0s and any number of 1s

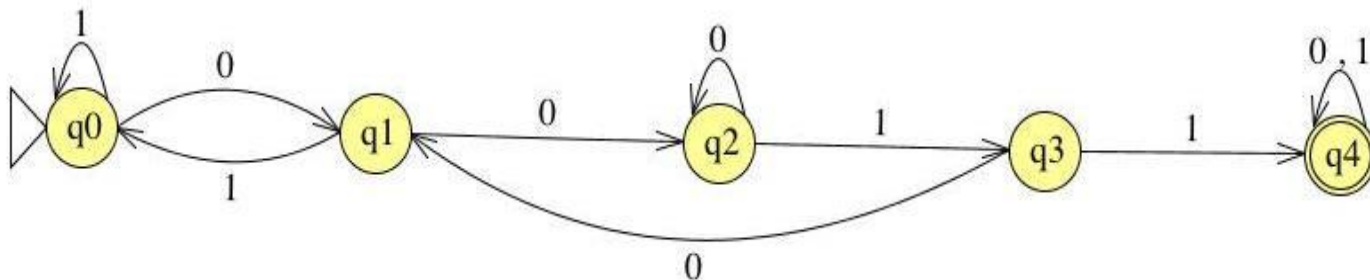


Exercises: Define an FA over $\Sigma = \{0,1\}$

- ▶ That accepts strings **containing** two consecutive **0s** followed by two consecutive **1s**

Exercises: Define an FA over $\Sigma = \{0,1\}$

- ▶ That accepts strings **containing** two consecutive **0s** very immediately (right after, no other things in between) followed by two consecutive **1s**

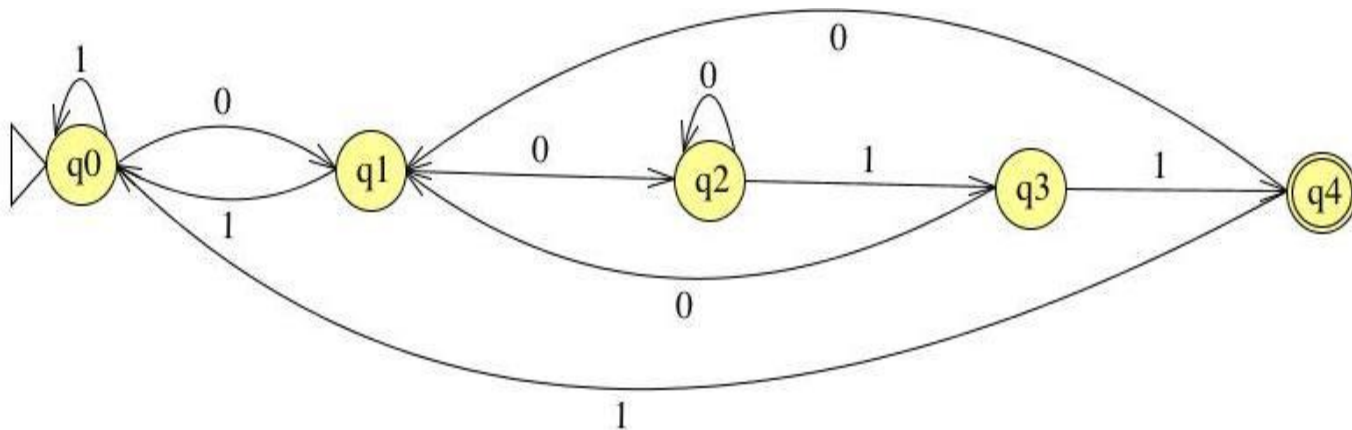


Exercises: Define an FA over $\Sigma = \{0,1\}$

- ▶ That accepts strings **end with** two consecutive **0s** followed by two consecutive **1s**

Exercises: Define an FA over $\Sigma = \{0,1\}$

- ▶ That accepts strings **end with** two consecutive **0s** followed by two consecutive **1s**



Exercises: Define an FA over $\Sigma = \{0,1\}$

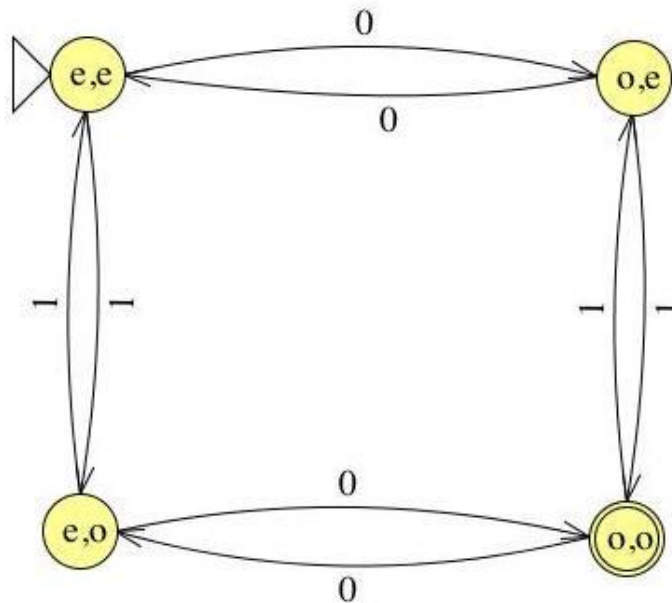
- ▶ That accepts strings containing an odd number of 0s and odd number of 1s

Exercises: Define an FA over $\Sigma = \{0,1\}$

- That accepts strings containing an **odd** number of **0s** and **odd** number of **1s**

4 states:

0s	1s
e	e
o	e
e	o
o	o



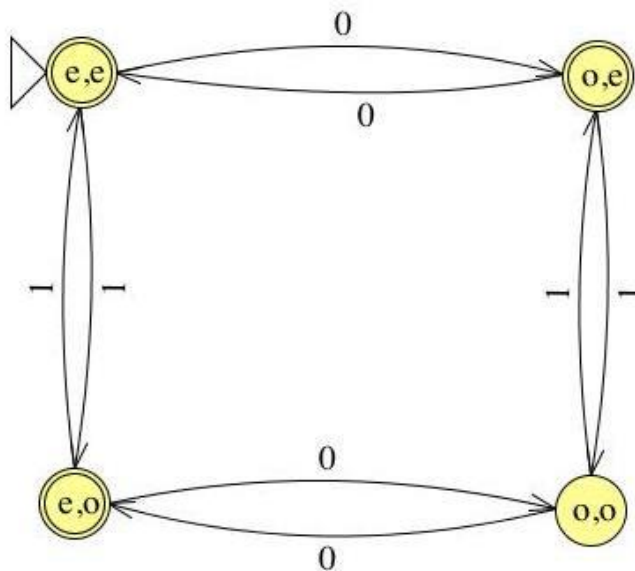
Exercises: Define an FA over $\Sigma = \{0,1\}$

- ▶ That accepts strings that **DO NOT** contain odd number of 0s and an odd number of 1s

Exercises: Define an FA over $\Sigma = \{0,1\}$

- ▶ That accepts strings that **DO NOT** contain odd number of 0s and an odd number of 1s

Flip each state



Languages and Machines

A formal language is a **set of strings** of **symbols** drawn from a **finite alphabet**.

Can be specified either by

- a set of rules (such as regular expressions or a CFG) that generates the language
- a *formal machine* that *accepts* (*recognizes*) the language.

