Exploring the Quantum Singular Value Transformation

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Abstract

One of the major pain points in the field of quantum computing is the lack of new algorithms besides the three main types: quantum simulation, unstructured search (Grover's algorithm), and the hidden subgroup problem (underlying factoring, Simon's algorithm, etc.). Quantum Singular Value Transformation (QSVT)[3], a relatively new, promising framework offers a new perspective on quantum algorithms. Isaac Chuang calls it the "Grand unification of quantum algorithms" [9], as all the primordial quantum algorithms can be expressed using this framework. The hope is that QSVT will inspire new avenues and tools of algorithm research. In this project we explore QSVT, through numerical simulations and report on the results and difficulties along the way.

I Introduction

If we look at the quantum algorithms with proven speedups compared to best known classical algorithms on the Quantum Algorithm Zoo [6], we can see that the majority of them belong to the following main categories:

- Abelian Hidden Subgroup Problem - this is where Shor's algorithm for factoring falls
- Quantum Search - Grover's search and derivations
- Quantum Simulation - Hamiltonian simulation
- Linear Algebra problems - for example matrix inversion, HHL

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Quantum Singular Value Transformation (QSVT) [3] is a framework that can encompass all of these algorithms. This exciting development is likely to create new avenues in algorithmic research. Martyn et al [9] wrote a pedagogical review of all these aspects. Here we report simulation results and challenges with parts of this paper.

I.1 Quantum singular value transformation

The Quantum Singular Value Transformation [3] is an elegant framework to provide transformations of unitaries.

The *singular value decomposition* of a matrix $A \in \mathbb{C}^{m\times n}$ is defined as the product of two unitary matrices $W \in \mathbb{C}^{m\times m}, V \in \mathbb{C}^{n\times n}$, and a diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$, so that $A = W\Sigma V^\dagger$. The diagonal elements Σ_{ii} are the singular values of A. QSVT provides a way to embed certain polynomial transformations of the singular values of a matrix in a quantum computer. Informally, we'll embed the matrix A of interest in a unitary by "block encoding" and transform the unitary by interleaving it with certain processing operations to get the transformed version of $Poly(A)$, which has the same singular vectors but some polynomial transformation of its singular values:

$$\begin{pmatrix} \mathrm{A} & \cdot \\ \cdot & \cdot \end{pmatrix} \rightarrow \begin{pmatrix} \mathrm{Poly(A)} & \cdot \\ \cdot & \cdot \end{pmatrix} \tag{1}$$

This is surprisingly powerful, as a wide range of quantum algorithms can be expressed in this framework.

QSVT was abstracted out from many ideas and formalisms over the last decade. The final goal towards we'll build is to explore some of these fundamental building blocks, namely single qubit Quantum Signal Processing and Qubitization of multiqubit systems, which also can be thought as special cases of QSVT.

II Single qubit QSP

II.1 Theory

We start with Quantum Signal Processing (QSP) introduced by Low, Yoder and Chuang [8]. QSP's origins are rooted in the field of quantum control, where the experimentalist can mitigate certain types of control errors via interleaving the control pulses with sequence with pre-defined pulse sequences (spin echo [4], BB1 [10], SK1 [1]). One particular example we'll look at here is the BB1 [10] sequence, which is of great interest in NMR spectroscopy as it leads to better image contrast.

We can gauge the most important features of QSP from QSVT point of view in a single qubit setting. QSP introduces the following main tenants: we define the signal unitary $W(a), a \in [1, -1]$ (typically a rotation around the X axis), a signal processing unitary $S(\phi), \phi \in \mathbb{R}$ (typically a rotation around the Z axis), a set of $d+1$ phases $\vec{\phi} = (\phi_0, \ldots, \phi_d) \in \mathbb{R}^{d+1}$. The final interleaved unitary, the *alternating phase modulation sequence* $U_{\vec{\phi}}$ is defined as:

$$U_{\vec{\phi}} = S(\phi_0)W(a)S(\phi_1)W(a)\ldots S(\phi_d) \tag{2}$$

This unitary will have the curious transformative properties we are interested in. Note that $W(a)$ is constant throughout, and only the $S(\phi)$ members are changing. We also define the "signal basis" to be either X or Z-basis, which determines the parts of the signal unitary to be transformed per our specification, which simply just means that we'll assume that our final unitary is applied to $|0\rangle$ in case of Z (or $|+\rangle$ for X basis), and we'll look at the overlap of the final state with $|0\rangle$ (or $|+\rangle$ for X basis).

The key message of QSP is that this structure of unitaries is surprisingly rich, and affords an interesting design space, as one can pick an *almost* arbitrary polynomial (with certain properties) $Poly$ and then it is possible to find angles $\vec{\phi}$, so that $Poly(a) = \langle 0| U_{\vec{\phi}} |0\rangle$ is satisfied, or $Poly(a) = \langle +| U_{\vec{\phi}} |+\rangle$ in case of the X notation.

QSP is used in multiple equivalent "conventions", which are important to know and recognize which particular algorithm is expressed in which convention, as they can influence the results significantly. The table below summarizes the QSP conventions covered in [9].

Convention	Signal	Signal Processor
W_x convention with Z signal basis: QSP(W_x, $\langle 0| \cdot |0\rangle$)	$W_x(a) = e^{i\theta/2X} = \begin{pmatrix} a & i\sqrt{1-a^2} \\ i\sqrt{1-a^2} & a \end{pmatrix}$, where $\theta = -2cos^{-1}(a)$	$S_z(\phi) = e^{i\phi Z}$
W_x convention with X signal basis: QSP(W_x, $\langle +| \cdot |+\rangle$)	$W_x(a) = e^{i\theta/2X} = \begin{pmatrix} a & i\sqrt{1-a^2} \\ i\sqrt{1-a^2} & a \end{pmatrix}$, where $\theta = -2cos^{-1}(a)$	$S_z(\phi) = e^{i\phi Z}$
Reflection convention: QSP(R, $\langle 0| \cdot |0\rangle$)	$R(a) = Xe^{i\theta/2Y} = \begin{pmatrix} a & \sqrt{1-a^2} \\ \sqrt{1-a^2} & -a \end{pmatrix}$, where $\theta = -2sin^{-1}(a)$	$S_z(\phi) = e^{i\phi Z}$
used when needed, the W_z convention	$W_z(w) = e^{i\theta/2Z} = \begin{pmatrix} w & 0 \\ 0 & w^{-1} \end{pmatrix}$, where $w = e^{i\theta/2}$	$S_x(\phi) = e^{i\theta X}$

We will utilize in this project QSP(W_x, $\langle 0| \cdot |0\rangle$), which can be thought of the single-qubit version of (1), however it has limitations on the polynomials that are possible to implement within it. We will explore QSP(W_x, $\langle +| \cdot |+\rangle$) which supposed to be more expressive. Finally, the reflection convention will be valuable for the multi-qubit amplitude amplification example, and we will study the conversion between phases of W_x and R because the former is provided by the PyQSP package, and the latter will be crucial in understanding the multi-qubit amplitude amplification example.

To formally state the QSP theorem:

Theorem II.1. *Quantum Signal Processing. A QSP phase sequence* $\vec{\phi} \in \mathbb{R}^{d+1}$ *exists such that*

$$e^{i\phi_0 Z} \prod_{k=1}^{d} W_x(a) e^{i\phi_k Z} = \begin{bmatrix} P(a) & iQ(a)\sqrt{1-a^2} \\ iQ^*(a)\sqrt{1-a^2} & P^*(a) \end{bmatrix} \tag{3}$$

for $a \in [-1,1]$ *and for any polynomials* $P, Q \in \mathbb{C}[a]$ *if and only if the following conditions hold:*

(i) $\deg(P) \leq d, \deg(Q) \leq d-1$

(ii) P *has parity* $d \mod 2$ *and* Q *has parity* $(d-1) \mod 2$

(iii) $\forall a \in [-1,1], |P|^2 + (1-a^2)|Q|^2 = 1$

Where the parity of a polynomial $P(x)$ is defined as *even* when it only has non-zero coefficients for even powers of x, and *odd*, when it only has non-zero coefficients for even powers of x. The product is exactly the product we discussed in equation (2).

The Z convention just picks $Poly(a) = P(a)$, which is a bit limiting in terms of polynomials, as only polynomials P for which a corresponding Q exists can be picked. This doesn't stop us from using it, for example in the amplitude amplification example later. With the X convention we are moving a bit away from the "block-encoding" picture, the element of the unitary that is transformed is the sum of all four elements as can be seen from this simple equation:

$$\langle +| \begin{pmatrix} a & b \\ c & d \end{pmatrix} |+\rangle = \frac{a+b+c+d}{2} \tag{4}$$

$$\tag{5}$$

Thus $Poly(a) = \langle +| U_{\vec{\phi}} |+\rangle = P(a) + P^*(a) + i(Q^{(}a)\sqrt{1-a^2} + Q^*(a)\sqrt{1-a^2}) = Re(P(a)) + iRe(Q(a)\sqrt{1-a^2})$.

As we would like to transform Wx phases into R phases, we'll need a precise way to do that, which we will discuss explicitly in the following two theorems, as they are not fully covered or explained thoroughly in the papers.

Theorem II.2. *The reflection convention is equivalent to the W_x conventions and for a given phase sequence $\vec{\phi} \in \mathbb{R}^{d+1}$ in the W_x convention, we can find the corresponding $\vec{\phi}' \in \mathbb{R}^{d+1}$ by setting $\phi'_0 = \phi_0 - \frac{\pi}{4}, \phi'_d = \phi_d - \frac{\pi}{4}, \phi'_k = \phi_k - \frac{\pi}{2}, k \in [1, d-1]$, and then the corresponding alternating phase modulation sequence $U_{\vec{\phi}'}$, is related as:*

$$U_{\vec{\phi}} = i^d U_{\vec{\phi}'}$$

Proof. The reflection convention is equivalent to the W_x conventions, by the following relationship:

$$W_x(a) = ie^{-i\frac{\pi}{4}Z} R(a) e^{-i\frac{\pi}{4}Z}. \tag{6}$$

Thus, given a phase sequence $\vec{\phi} \in \mathbb{R}^{d+1}$ in the W_x convention, we can find $\vec{\phi}' \in \mathbb{R}^{d+1}$ to bring our transformation to the same format. Using Eq.(6), it is easy to see that we'll need to tweak the two phases at the end by by the $\phi/4$ phase, and all the others by $\phi/2$, i.e. Then we have:

$$e^{i\phi_0 Z} \prod_{k=1}^{d} W_x(a) e^{i\phi_k Z} = e^{i\phi_0 Z} \prod_{k=1}^{d} ie^{-i\frac{\pi}{4}Z} R(a) e^{-i\frac{\pi}{4}Z} e^{i\phi_k Z} = \tag{7}$$

$$= i^d e^{i(\phi_0 - \frac{\pi}{4})Z} \left(\prod_{k=1}^{d-1} R(a) e^{-i\frac{\pi}{4}Z} e^{i\phi_k Z} e^{-i\frac{\pi}{4}Z} \right) R(a) e^{-i\frac{\pi}{4}Z} e^{i\phi_d Z} = \tag{8}$$

$$= i^d e^{i\phi'_0 Z} \prod_{k=1}^{d} R(a) e^{i\phi'_k Z}. \tag{9}$$

□

Note that, seemingly, there are miscalculations in both papers we are reviewing in this context with regards to this result: equation (A5) in [9], and the proof of Corollary 8 in the Arxiv pre-print [3], which is missing from the published version in [2]. They both account for the i phase using $ie^{i\phi_k Z} = e^{i\phi_k + \frac{\pi}{2}Z}$ which is not true in general. However, this argument works when we are in the Z basis and only looking at the $\langle 0| U |0\rangle$ element's transformation. This is what the next theorem is for, which will be very useful for the amplitude amplification case.

Theorem II.3. *Using the same definitions as Theorem II.2, in case of Z-basis quantum signal processing, we can find $\vec{\phi}' \in \mathbb{R}^d$ (i.e. one less element) reflection phases, that can be expressed as $\phi'_0 = \phi_0 + \phi_d + (d-1)\frac{\pi}{2}, \phi'_k = \phi_k - \frac{\pi}{2}, k \in [1, d-1]$, and then the corresponding alternating phase modulation sequence $U_{\vec{\phi}'}$, is related through the final projection as:*

$$\langle 0| U_{\vec{\phi}} |0\rangle = \langle 0| e^{i(\phi'_0)} R(a) \prod_{k=1}^{d-1} e^{i\phi'_k Z} R(a) |0\rangle = Poly(a)$$

Proof. The main trick to recognize here is that $e^{i\phi Z} |0\rangle = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} |0\rangle = e^{i\phi} |0\rangle$, and similarly $\langle 0| e^{i\phi Z} = e^{i\phi} \langle 0|$, thus the first and the last signal processing terms can be brought to the front as global phases. Then from Theorem II.2 and Theorem II.1, with some rearranging we'll get the stated result:

$$Poly(a) = \langle 0| i^d e^{i\phi'_0 Z} \prod_{k=1}^{d} R(a) e^{i\phi'_k Z} |0\rangle = \tag{10}$$

$$= \langle 0| i^d e^{i(\phi_0 - \frac{\pi}{4})} R(a) \left(\prod_{k=1}^{d-1} e^{i\phi'_k Z} R(a) \right) e^{i(\phi_d - \frac{\pi}{4})} |0\rangle \tag{11}$$

$$= \langle 0| e^{id\frac{\pi}{2}} e^{i(\phi_0 - \frac{\pi}{4})} R(a) \left(\prod_{k=1}^{d-1} e^{i\phi'_k Z} R(a) \right) e^{i(\phi_d - \frac{\pi}{4})} |0\rangle \tag{12}$$

$$= \langle 0| e^{i(\phi_0 + \phi_d + (d-1)\frac{\pi}{2})} R(a) \left(\prod_{k=1}^{d-1} e^{i\phi'_k Z} R(a) \right) |0\rangle . \tag{13}$$

□

This will be useful for us, as the phases provided by the PyQSP [5] package we'll use, the W_x convention is the default, however, we'll need to transform them for the amplitude amplification example later to the reflection convention.

II.2 Numerical experiments

We are interested in reproducing the following results:

- To demonstrate the original QSP effect, we'll reproduce the effect of the BB1 sequence on a simple X rotation
- We will ensure that the Chebyshev polynomials T_1, T_2, T_3, T_4, T_5 are produced correctly. These have well defined phases for W_x: $T_1 : [0, 0], T_2 : [0, 0, 0], ...$, and as such provide great test cases.

- We will explore some more complex polynomials used later in the report, namely the polynomial approximation of the sign function for amplitude amplification and the fixed point search function polynomial
- Some polynomials will be produced in all the conventions we care about, namely QSP(W_x, $\langle +| \cdot |+\rangle$), QSP($W_x$, $\langle 0| \cdot |0\rangle$), QSP($R$, $\langle 0| \cdot |0\rangle$). This will help verifying the conversion logic between the reflection convention and the QSP(W_x, $\langle +| \cdot |+\rangle$) convention.

The code for the BB1 experiment can be found in the `bb1.py`file, and for the QSP experiments in the `qsp.py` file. The simulations are straightforward, they build a Cirq Circuit object based on the alternating phase sequence and do the necessary projections on the unitary to extract the signal.

The circuits for the T_1 Chebyshev polynomial for example, i.e. for coefficients [0,0] in the W_x and QSP(R, $\langle 0| \cdot |0\rangle$) convention, are here for example:

```
QSP circuit for QSP([0, 0], conv=wx, basis=x) response
q: ───Rz(0)───Rx(-0.667π)───Rz(0)───
QSP circuit for QSP([0, 0], conv=r, basis=x) response
q: ───Rz(0.5π)───R(0.5)───Rz(0.5π)───
```

Conversion from W_x to R conversion is done in the `to_r_z_from_wx` and `to_r_x_from_wx` functions in `qsp.py`. All code is available at https://github.com/balopat/qsvt$_{experiments}$.

II.2.1 Results

We successfully recreated the functions and the plots match that of the paper's and more. In Figure 1 we show the results of the BB1 sequence, where the signal unitary is implemented with the $cirq.Rx(rads = theta)$ gate, the signal is $\theta \in [-\pi, \pi]$. It is clear how the BB1 transformation "increases the stability" of the qubit $|0\rangle$ state and the sensitivity for the signal values around $\theta \approx 2\pi/3$.

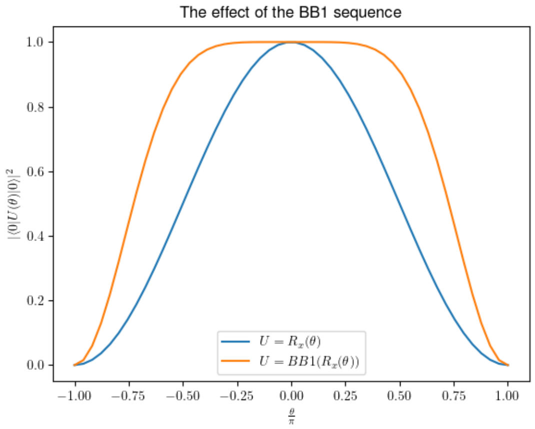


Figure 1: The effect of BB1 on R_x. Every point is the result of a simulation of a single BB1 sequence interleaved with a fixed θ rotation of $R_x(\theta)$.

Figures 2 and 3 show some of the resulting plots from the Chebyshev polynomial tests and that the real part of the QSP signal completely agrees with the target polynomials. Chebyshev polynomials have the great property that there are analytical proofs for them to work in QSP, and as such they serve as great test cases. They indicate that the logic in our single qubit QSP implementation is sound and in future development can be converted to test cases if the software package will get used in a library.

Finally, Figure 4 demonstrates the results for the application specific functions based on the angles defined in Appendix D.1. and D.2. in [9]. One aspect that was interesting is that the PyQSP package seems random for producing the phase angles for the sign function approximating polynomial. An issue[1] on the Gihub repository has been opened to clarify the behavior.

III Qubitization example

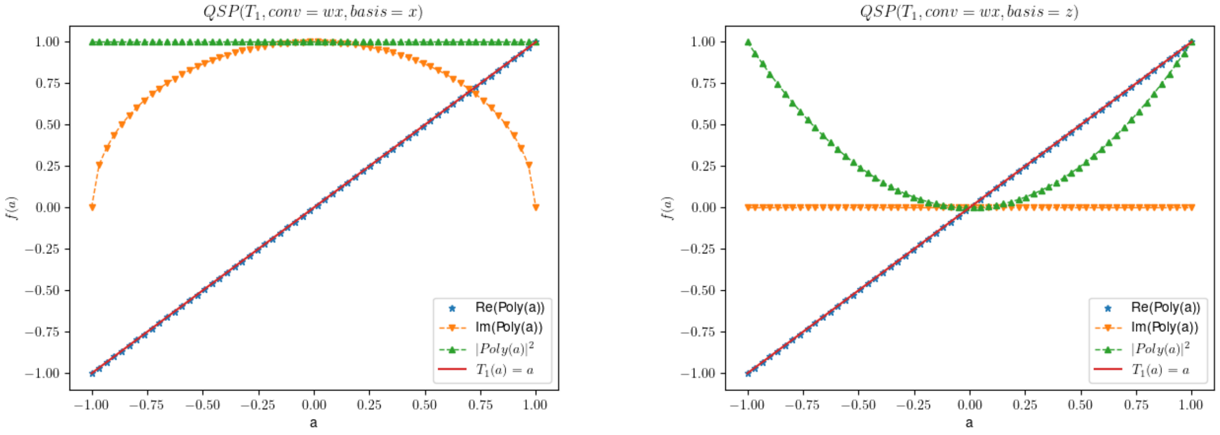
III.1 Theory

A natural way to generalize QSP to multi-qubit systems is by way of "qubitization" [7]. In problems where it is possible to identify one-qubit-like subsystems in a larger hilbert space, we can apply the QSP principles to create polynomial transformations of certain values. Specifically in this section we'll look at amplitude amplification - or "amplitude conversion", we'll pick an amplitude $a \in \mathbb{R}$ (we can assume it is real, as we can merge a potential phase into the starting state) of a unitary matrix and produce $Poly(a)$.

Problem input:

- a unitary U (which may act on some larger Hilbert space) with $a := \langle A_0| U |B_0\rangle \in \mathbb{R}$ initial value for two specific states, $|A_0\rangle$ and $|B_0\rangle$

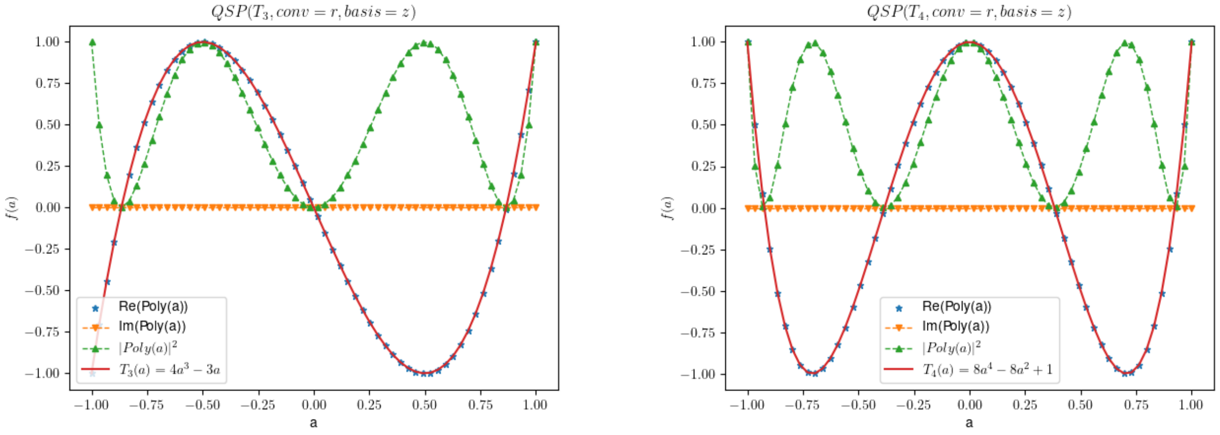
[1] https://github.com/ichuang/pyqsp/issues/1



(a) Poly(a)=a for the coefficients [0,0] for sanity checking in the X basis. The simulated points agree completely with the target polynomial.

(b) Poly(a)=a for the coefficients [0,0] for sanity checking in the Z basis. The simulated points agree completely with the target polynomial.

Figure 2: Sanity checking with the $T_1(x) = x$ Chebyshev polynomial



(a) $T_3 = 4x^3 - 3x$ Chebyshev polynomial, for the W_x signal $[0, 0, 0, 0]$, transformed to $QSP(R, \langle 0| \cdot |0\rangle)$ convention. The simulated points agree completely with the target polynomial.

(b) $T_4 = 8x^4 - 8x^2 + 1$ Chebyshev polynomial, for the W_x signal $[0, 0, 0, 0, 0]$, transformed to $QSP(R, \langle 0| \cdot |0\rangle)$ convention. The simulated points agree completely with the target polynomial.

Figure 3: Sanity checking with the higher order Chebyshev polynomials

- the inverse of our operation, $U^\dagger$
- $A_\phi = e^{i\phi\Pi_a}$, $B_\phi = e^{i\phi\Pi_b}$, where Π_a, Π_b are projectors to the subspace of $|A_0\rangle$ and $|B_0\rangle$ respectively. We choose these to be $\Pi_a = (2\,|A_0\rangle\langle A_0| - 1), \Pi_b = (2\,|B_0\rangle\langle B_0| - 1)$

Task: create a circuit Q using the elements above to create a final unitary that is $Poly(a)$.

This task, when $Poly(a) \rightarrow 1$, is known as *amplitude amplification*. It can be solved without knowing the specific initial value, a. The quantum algorithm we'll implement to solve amplitude amplification is called *fixed point amplitude amplification*. Using the idea of qubitization we'll show how this algorithm can be implemented using Theorem II.1 and the conversion theorems between W_x and R conventions.

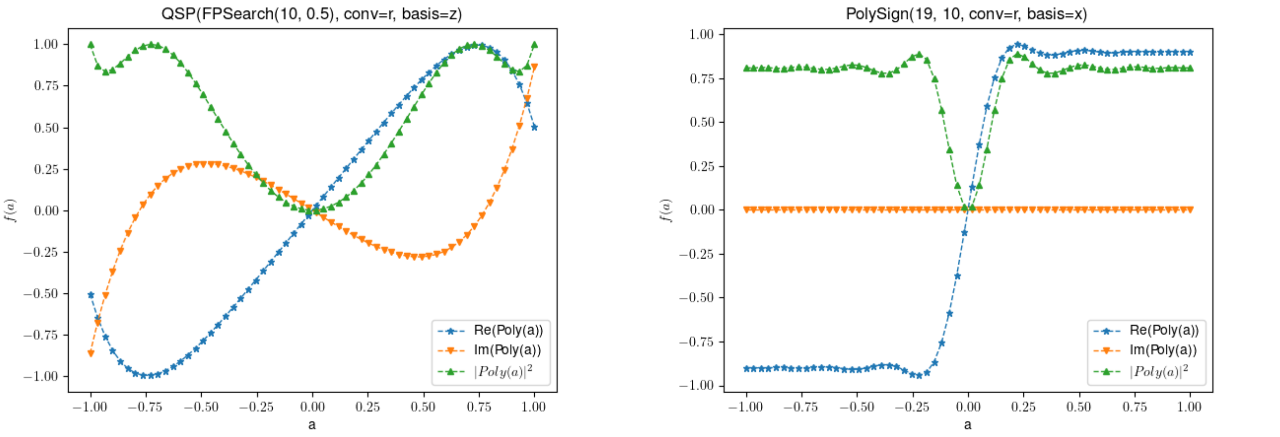
The key insight is that U in the bases of $|A_0\rangle$ and $U\,|B_0\rangle$'s orthogonal component $|A_\perp\rangle$, and $|B_0\rangle$, $|B_\perp\rangle$ becomes a two qubit matrix:

$$U = \begin{array}{c c} & \begin{array}{cc} |B_0\rangle & |B_\perp\rangle \end{array} \\ \begin{array}{c} |A_0\rangle \\ |A_\perp\rangle \end{array} & \begin{bmatrix} a & \sqrt{1-a^2} \\ \sqrt{1-a^2} & -a \end{bmatrix} \end{array} \tag{14}$$

, where the following are the definitions for the perpendicular states:

$$|A_\perp\rangle = \frac{(I - |A_0\rangle\langle A_0|)\,U\,\langle B_0|}{\|\sqrt{(I - |A_0\rangle\langle A_0|)\,U\,\langle B_0|}\|} \tag{15}$$

$$|B_\perp\rangle = U^\dagger \left(a\,|A_0\rangle + \sqrt{1-a^2}\,|A_\perp\rangle\right) \tag{16}$$



(a) Oblivious amplitude amplification QSP response for fpsearch with degree 10 and $\delta = 0.5$. All points are simulated numerically using the $QSP(R, \langle 0| \cdot |0\rangle)$ convention.

(b) Sign function approximation using the Gauss error function erf with $d = 19$, $k = 10$. Every point is simulated numerically using the $QSP(R, \langle 0| \cdot |0\rangle)$ convention.

Figure 4: QSP response functions of application specific polynomials

An intuitive way of thinking about these subspaces is by way of visualizing the two concentric Bloch spheres as in Figure 5, borrowed from [9]. Each pair of states spans a qubit space. The rotation operators A_ϕ and B_ϕ just become rotations around the Z axis for these Bloch spheres.

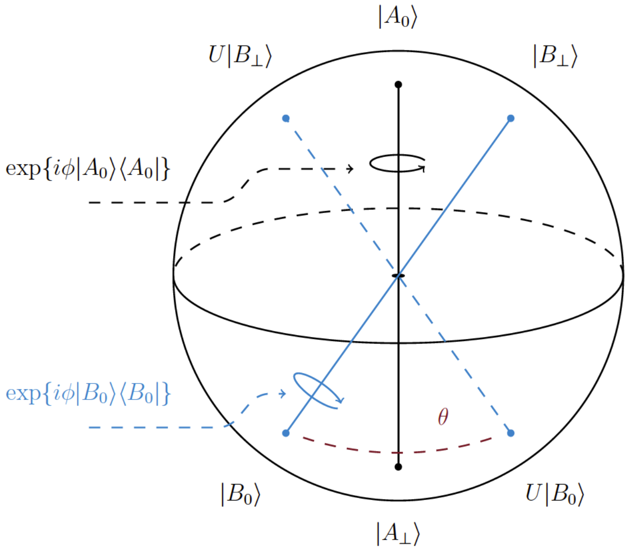


Figure 5: An illustration of amplitude amplification. As the figure is from [9], the rotation operators are in a different form, but only differ in a global phase. The key intuition is that the two bloch spheres defined by the orthogonal (thus opposite on the Bloch sphere) states $|A_0\rangle$ $|A_\perp\rangle$ and $|B_0\rangle$ $|B_\perp\rangle$ are in black and blue. The U operator switches from A to B and $U^\dagger$ switches it back, making the rotation operators Z rotations.

As U has the format above, we can then use the QSP reflection convention! We will state this by way of a formal theorem:

Theorem III.1. *Amplitude amplification. Given unitary operators U and its inverse $U^\dagger$, and operators $A_\phi = e^{i\phi(2|A_0\rangle\langle A_0|-I)}$, $B_\phi = e^{i\phi(2|B_0\rangle\langle B_0|-I)}$, for a polynomial Poly of degree at most d, which satisfies the properties on P in Theorem II.1, there exist $\vec{\phi} \in \mathbb{R}^d$ that satisfies the following for even d:*

$$\langle A_0| \left[\prod_{k=1}^{d/2} B_{\phi_{2k-1}} U^\dagger A_{\phi_{2k}} U \right] |B_0\rangle = Poly(a) \tag{17}$$

, and for odd d

$$\langle B_0| A_{\phi_1} \left[\prod_{k=1}^{(d-1)/2} U B_{\phi_{2k}} U^{\dagger} A_{\phi_{2k+1}} \right] U |B_0\rangle = Poly(a) \tag{18}$$

One might notice that this is stated differently from Theorem 2 in [9], given our experience with the simulations, there is lack of clarity around even and odd cases, especially regarding the right use of the projectors. We used, from [3] Definition 15 and Theorem 17 to find the correct form.

III.2 Numerical methods

We are interested in reproducing the following results:

- We will ensure that the Chebyshev polynomials T_1, T_2, T_3, T_4, T_5 are produced correctly in this setup as well, similarly to the single qubit QSP
- We will run the amplitude amplification polynomial

These experiments are captured in the `simple_qubitization.py` file in the repository.

Our choice of states were $|A_0\rangle = |2\rangle, |B_0\rangle = |3\rangle$, for which we constructed the rotation operators. One of the first challenges were to figure out a multi-qubit unitary where we can create a real amplitudes that we can control. Our choice for this was the following construction, given two basis vectors $|a_0\rangle, |b_0\rangle$:

$$U_\theta := e^{i\theta(\mathrm{CX})} \tag{19}$$

$$U(s, a, b) := s * \frac{(\langle a_0| U_\theta |b_0\rangle)^*}{\|\langle a_0| U_\theta |b_0\rangle\|} U_\theta \tag{20}$$

, where $s = \pm 1$, so that we can have positive and negative values as well, CX is the unitary for a CNOT gate: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

This allows us to map $[0, \pi]$ to $[-1, 1]$ and we can do the scans to evaluate $Poly(a)$.

III.3 Results

We successfully reproduced the Chebyshev polynomials, simulated points matching the target functions, and the FPSearch function as shown by Figure 6 and 7. However, there were significant challenges at this phase, it was notoriously hard to get these results from the papers directly, given the following small details, mainly because how Theorem 2 from [9] is stated:

- There is no clear definition of even versus odd cases, and difference between projectors in the two cases. And while though this is understandable in hindsight, as amplitude amplification should relate to the same amplitude, while in the even case we are getting a different amplitude for the $Poly(a)$
- The number of Us (and $U^\dagger$s) in is $d + 1$ but it seems that in all other theorems and based on our experiments that for d degree polynomials typically d U's are required.
- Differences between conventions are explained well, but the conversion between R and W_x phases seem wrong in the [9] appendix.

The sensitivity of QSP is very high regarding to one or two of these things missing, and thus days were spent experimenting to see what can possibly be so wrong to get chaotic plots instead of the expected polynomials. This does raise an interesting question about the sensitivity of QSP to the angles and might be a worthwhile study to execute on in the future.

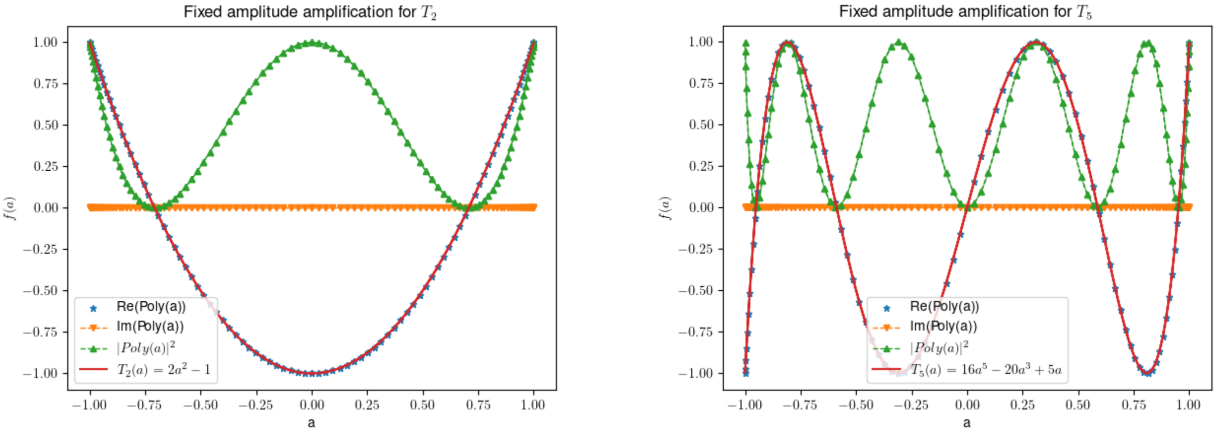
IV Discussion

We managed to reproduce some of the results of the QSVT framework via numerical simulations in Cirq. Due to the complications there was no time to finish all the planned stages, namely the Hamiltonian simulation and the QAOA experiment, which will be tackled separately, outside of this project.

QSVT is surprisingly powerful and provides a completely different design space of polynomials that allows for a different way of thinking about quantum algorithms. This new, promising, flexible framework for quantum algorithms hopefully will generate new ideas in quantum algorithm research and move the field of quantum computing forward.

V Acknowledgements

The author is thankful for the teachings of professor Ken Brown and the TAs, Travis Hurant and Evan Anderson for their hard work throughout this semester, which enabled this project. We acknowledge the helpful hint from András Gilyén to use the Chebyshev polynomials as test cases to debug the issues with the amplitude amplification example.



(a) Simulation by fixed amplitude amplification of $T_2 = 2x^2 - 1$ Chebyshev polynomial, for the W_x signal $[0, 0, 0]$, transformed to $QSP(R, \langle 0| \cdot |0\rangle)$ convention. The simulated points agree completely with the target polynomial.

(b) Simulation by fixed amplitude amplification of $T_5 = 16x^5 - 20x^3 + 5x$ Chebyshev polynomial, for the W_x signal $[0, 0, 0, 0, 0, 0]$, transformed to $QSP(R, \langle 0| \cdot |0\rangle)$ convention. The simulated points agree completely with the target polynomial.

Figure 6: Amplitude amplification with Chebyshev polynomials

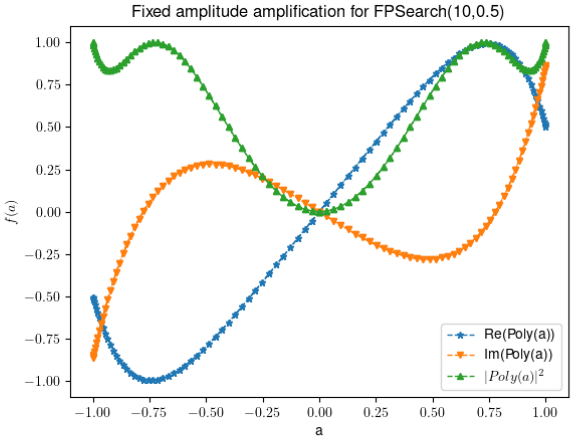


Figure 7: Fixed amplitude amplification simulation of the FPSearch function

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