MDSC - 301

Sri Sathya Sai Institute of Higher Learning

(Deemed to be University)

Department of Mathematics and Computer Science

Muddenahalli Campus

Course: **M.Sc. Data Science and Computing** Date: **July 16, 2022**

Subject: **Machine Learning** Module : **Mathematics for ML**

Answer the following:

1. Compute the column rank of $\begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$

2. Compute the following in $\mathbb{R}^4$

   (a) $U_1 = \langle \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \rangle$

   (b) $U_2 = \langle \begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ -2 \\ -1 \end{bmatrix} \rangle$

   (c) $U_1 \cap U_2$

3. Define ℓ^p-norm of an element $v \in \mathbb{R}^n$.

4. Prove that the matrix $\begin{bmatrix} 9 & 6 \\ 6 & 3 \end{bmatrix}$ is positive definite.

5. Find the angle between the vectors $x = [1, 1]^T$ and $y = [1, 2]^T$ in $\mathbb{R}^2$.

6. What is the geometric object of all the elements x such that $\|x\|_1 = 1$?

7. Let a, b be two scalar vector in $\mathbb{R}^n, A$ be a square matrix of order n and $x = [x_1, \ldots, x_n]^T$ variable vector in $\mathbb{R}^n$. Consider the differentiable function $f : \mathbb{R}^n \to \mathbb{R}$. Define

$$\frac{\partial f}{\partial x} = \Big[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\Big]^T$$

   Then prove the following:

   (a) $\dfrac{\partial}{\partial x}(a^T x) = a$

   (b) $\dfrac{\partial}{\partial x}(b^T A x) = A^T b$

(c) $\frac{\partial}{\partial x}(x^T A x) = (A + A^T)x$

8. Consider the differentiable function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$. Define

$$\frac{\partial f}{\partial \mathbf{X}} = \begin{bmatrix} \frac{\partial f}{\partial x_{11}} & \cdots & \frac{\partial f}{\partial x_{1n}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial x_{m1}} & \cdots & \frac{\partial f}{\partial x_{mn}} \end{bmatrix}$$

Prove the following:

(a) $\frac{\partial}{\partial \mathbf{X}}(a^T \mathbf{X} b) = ab^T$

(b) $\frac{\partial}{\partial \mathbf{X}}(a^T \mathbf{X}^T b) = ba^T$

(c) $\frac{\partial}{\partial \mathbf{X}} tr(A\mathbf{X}^T B) = A^T B^T$

(d) $\frac{\partial}{\partial \mathbf{X}} tr(\mathbf{X}^T A) = A$

(e) $\frac{\partial}{\partial \mathbf{X}} tr(\mathbf{X}^T A \mathbf{X}) = (A + A^T)\mathbf{X}$

9. Find the projection of $x = [6, 0, 0]^T$ on the subspace $U = \langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \rangle$. Also find the projection matrix.

10. Compute the distance between $x = [1, 2, 3]^T,\ y = [-1, -1, 0]^T$ using

(a) $d(x, y) = \|x - y\|_2$

(b) $d(x, y) = \sqrt{x^T A y}$, where $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

11. Find the eigen-values and eigen-vectors of $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$

12. Define algebraic multiplicity and geometric multiplicity.

13. Explore the connection between eigen-vectors of a matrix and Googl'e PageRank.

14. Consider the univariate function

$$f(x) = x^3 + 6x^2 - 3x - 5$$

Find its stationary points and indicate whether they are maximum, minimum, or saddle points.