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EXACT OPTIMAL SOLUTION FOR FACILITY LAYOUT: DECIDING WHICH PAIRS OF LOCATIONS SHOULD BE ADJACENT

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Abstract—This paper describes the development of a mathematical model for determining optimum block layout systems, exploiting 0–1 integer programming as the optimization component. The objective function is to maximize a weighted sum of adjacent departments. The selected weight is a measure of the flow of material between departments. Developing a set of constraints, and changing the objective function to minimization of adjacency of departments with no interaction, it is shown that the procedure is efficient in determining the optimal location in small size problems. To apply it to larger size problems, some variables are eliminated and some constraints are combined, both of which help reduce the number of iterations for the integer program to converge to an optimal solution. Even though some improvements can be envisioned, the model's performance is greatly a function of the size of the problem, and reduces with an increase in the problem size. Few examples and some experiments are reported. The special case of departments of the same size is modeled and an example of its application is provided.

INTRODUCTION

In most manufacturing systems different amounts of material flow from any given department to other departments, so there is no need for the adjacency of departments to have the same priority. Two departments with maximum in-between flow have the highest priority for being adjacent, and any two departments that have minimal interaction need not be adjacent. This paper presents a mathematical model that can be used to determine an optimal layout for the manufacturing systems that have less than 10 interrelated departments. In this model a rectangular building is being used to site N different rectangular departments. The objective is to lay out the departments in the building such that the sum of the weighted adjacency values for all departments is maximized—two departments are adjacent if they share at least a portion of a wall.

A review of the literature shows that due to its significance, many researchers have discussed the above-mentioned optimization problem in the layout of facilities in any manufacturing and/or service industry. But, the authors are not aware of any published model offering a procedure to determine the optimal solution that maximizes the weighted adjacency between facilities, for facilities of different areas. Therefore, even though no comparisons are made for the general case, the performance of the model in finding optimal solution to the special case of facilities with the same area can be compared with that of others.

The facilities layout problem involves allocation of the available space to a variety of activities that have different interrelationships. One application of such a model is in minimization of total handling cost in a manufacturing system in which material handling between different departments is costly and needs to be minimized.

LITERATURE REVIEW

Since 1960, many attempts have been made to solve the facility layout problems. Most of these techniques have already been reviewed by El-Rayah and Hollier [1], Moore [2], and Foulds [3]. Foulds in his review of the literature on the techniques for facility layout has listed over 80 sources, some of which have responded to the question of which pairs of activities should be adjacent [3]. In his survey Foulds reviewed two facilities design formulations, and concluded that it is unlikely that either can be used to design an efficient algorithm as they represent NP-complete problems. In addition, Foulds introduced a new approach based on graph theory that appeared to be a most promising line for research into the development of efficient facility design problem heuristics. For more information, the readers are referred to Foulds [3].

Drezner [4] developed a simple and efficient algorithm yielding guidelines for the layout of facilities. These guidelines are a scatter diagram showing the relative positioning of the facilities on the plane. The designer can plan the required layout with the aid of this scatter diagram. Drezner's approach differs from other methods in that the final output of the algorithm is not the final solution, but can be used as an aid to build the final configuration.

Montreuil *et al.* [5] introduced an interactive block layout system which utilized b-matching as the optimization component. Their model is a relaxation of the layout problem, which means that for every feasible block there is a corresponding feasible solution to the b-matching model. However, some feasible solutions to the matching model may not have corresponding feasible block layouts. Hence in the Montrueil *et al.* model, human interaction is a critical element needed to overcome the factors not captured by the model. Montreuil and Ratliff [6] proposed a cut tree approach as the design skeleton for a family of facility layouts. Their approach has some very attractive properties when used as part of an interactive system to support the designer in the analysis, construction, and improvement of facility layouts.

Due to existence of survey articles (see Foulds [3] for a comprehensive list of references), rather than repeating the literature it suffices to say that to date there is no single exact solution to the layout problem that can solve large problems with different departmental shapes in economically feasible computational time.

MATHEMATICAL MODEL

The main site available for laying out the departments is rectangular in shape and has an area of size $\mathbf{A} = \mathbf{a} \times \mathbf{b}$. Each department $\mathbf{i}$ is also rectangular in shape and has an area of size $\mathbf{A_i} = \mathbf{a_i} \times \mathbf{b_i}$. To represent the location of each department, the configuration shown in Fig. 1 is used for a site with $\mathbf{a}$ cells along the x-axis and $\mathbf{b}$ cells along the y-axis. To account for every possible layout, departments are allowed to be oriented with their a_i or b_i side along the x-axis, thus allowing the designer to possibly create a better layout.

The following notation is used to describe the mathematical model for the maximization of the sum of the weighted adjacency between all the departments:

N = number of departments;
i = the index used to denote the department number; $i = 1, \ldots, N$;
j = the index used to denote the cell number on the x-axis, $j = 1, \ldots, \mathbf{a}$;
k = the index used to denote the cell number on the y-axis, $k = 1, \ldots, \mathbf{b}$;
X_{ijk} = the decision variable representing the location of department i with regard to jth cell on the x-axis and kth cell on the y-axis, i.e. $X_{ijk} = 1$ if location (j, k) is occupied by department $\mathbf{i}$, 0 otherwise;
Y_{mn} = The decision variable representing whether departments $\mathbf{m}$ and $\mathbf{n}$ are adjacent or not, i.e. $Y_{mn} = 1$ if department $\mathbf{i}$ is adjacent to department $\mathbf{j}$, 0 otherwise;
C_{mn} = the magnitude of flow between departments $\mathbf{m}$ and $\mathbf{n}$.

Objective function: one objective of interest in determining Y_{mn} so as to:

$$\text{Maximize} \sum_{m=1}^{N} \sum_{n=\mathbf{m}+1}^{N} C_{mn} \cdot Y_{mn}. \tag{1}$$

There are N departments with $N \cdot (N-1)/2$ interrelationships. The contribution (to the objective function) of two departments m and n being adjacent is $C_{mn} \cdot Y_{mn}$, therefore the objective function shows total weighted adjacency benefits for all possible interactions.

To represent the layout problem correctly, the following sets of constraints are introduced and their roles in modeling the problem is described.

First group of constraints

One of the unique features of the current model is that it allows each department to remain intact in its entirety, i.e. all the blocks that belong to a department remain together and as a solid

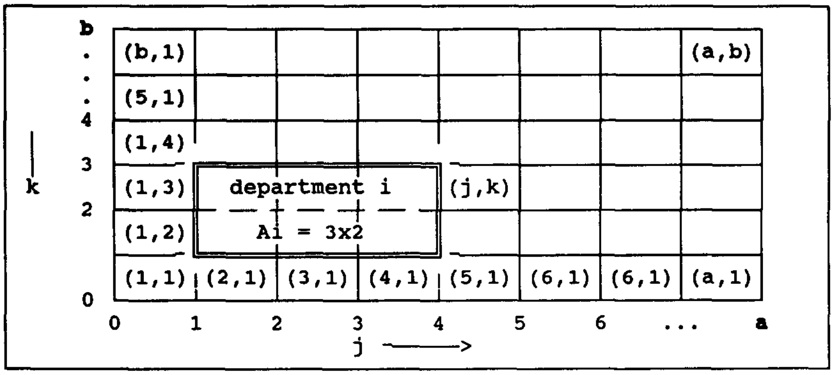

Fig. 1. The schematic representation of the main site. Department i is of dimension $a_i = 3$, $b_i = 2$, and occupies cells (2, 2), (3, 2), (4, 2), (2, 3), (3, 3), and (4, 3).

rectangle. Modification to any other geometric shape follows the same rules. This requires that for some **J** and some **K**:

either

$$\sum_{j=\mathbf{J}}^{\mathbf{J}+\mathbf{a}_i-1} \sum_{k=\mathbf{K}}^{\mathbf{k}+\mathbf{b}_i-1} X_{ijk} = \mathbf{A}_i, \quad \text{for some } \mathbf{J} = 1, \mathbf{a} - \mathbf{a}_i, \quad \mathbf{K} = 1, \mathbf{b} - \mathbf{b}_i \tag{2a}$$

or

$$\sum_{j=\mathbf{J}}^{\mathbf{J}+\mathbf{b}_i-1} \sum_{k=\mathbf{K}}^{\mathbf{K}+\mathbf{a}_i-1} X_{ijk} = \mathbf{A}_i, \quad \text{for some } \mathbf{J} = 1, \mathbf{a} - \mathbf{b}_i, \quad \mathbf{K} = 1, \mathbf{b} - \mathbf{a}_i. \tag{2b}$$

The difference between the two sets of constraints is in the way a rectangle can be laid on the x–y axis. The question is how can this restriction be expressed in a simple mathematical form? To answer the question, all possible ways of fitting department i into the site should be determined. For instance if $\mathbf{A} = \mathbf{4} \times \mathbf{3}$ and $\mathbf{A}_i = \mathbf{2} \times \mathbf{1}$, there are $(3 \times 3 + 2 \times 4 = 17)$ ways of fitting department i into the main site. The first 9 ways are associated with having the length of **i** along the x-axis, and the last 8 ways are associated with having the width if **i** along the x-axis.

Examples are: [cells (1, 1)&(1, 2)], [cells (1, 2)&(1, 3)], [cells (1, 3)&(1, 4)], ... Each one of the sites, say [(1, 1), (1, 2)], is called a block. Let:

$$n_i = \text{total possible number of blocks for department } \mathbf{i}.$$

Then it can be shown that for a non-square shaped department:

$$n_i = [a - a_i + 1) * (b - b_i + 1)] + [(a - b_i + 1) * (b - a_i + 1)],$$

and for a squared shape department:

$$n_i = [(a - a_i + 1) * (b - b_i + 1)].$$

Define:

Z_{il} = The decision variable representing whether department **i** occupies the **l**th block ($\mathbf{l} = 1, \ldots, n_i$), i.e. $Z_{il} = 1$ if department **i** occupies lth block, 0 otherwise.

The restriction on contiguity (rigidity) of each department can be expressed as:

$$Z_{il} \leqslant \left\{ \sum_{j=\mathbf{J}}^{\mathbf{J}+\mathbf{a}_i-1} \sum_{k=\mathbf{K}}^{\mathbf{K}+\mathbf{b}_i-1} X_{ijk} \right\} \Big/ A_i \quad \text{for all } 1 \leqslant J \leqslant a - a_i, \text{ and } 1 \leqslant K \leqslant b - b_i \tag{3a}$$

$$Z_{il} \leqslant \left\{ \sum_{j=\mathbf{J}}^{\mathbf{J}+\mathbf{b}_i-1} \sum_{k=\mathbf{K}}^{\mathbf{K}+\mathbf{a}_i-1} X_{ijk} \right\} \Big/ A_i \quad \text{for all } 1 \leqslant J \leqslant a - b_i, \text{ and } 1 \leqslant K \leqslant b - a_i \tag{3b}$$

$$\sum_{l=1}^{n_i} Z_{il} = 1 \quad \text{for all departments } i = 1, \ldots, N. \tag{3c}$$

The first two sets of constraints [inequalities (3a) and (3b)] guarantee that $Z_{il} \leqslant 1$ and Z_{il} can accept the value **1** only if $X_{ijk} = 1$ for all cells in the **l**th block. The last group of constraints [equations (3c)] guarantees that one (and only one) block will be allocated to each department. Therefore, the three constraints together guarantee that $Z_{il} = 1$ iff $X_{ijk} = 1$ for all cells in the **l**th block.

Second group of constraints

This group of constraints is to make sure that every cell is occupied by one of the departments.

$$\sum_{i=1}^{N} X_{ijk} = 1 \quad \text{for all cells } (j, k). \tag{4a}$$

In addition to make sure that every department occupies A_i cells, the following sets of constraints are required:

$$\sum_{j=1}^{a} \sum_{k=1}^{b} X_{ijk} = \mathbf{A_i} \quad \text{for all departments } i = 1, \ldots, N. \tag{4b}$$

Third group of constraints

This group of constraints is used to identify the departments that are adjacent. Two departments **m** and **n** are adjacent iff at least one of the cells of **m** is adjacent to one of the cells of **n**, i.e. they are in contact in at least one cell. The question is how can this statement be quantified? One way of expressing this requirement is to use equation (5) to relate X_{ijk} and Y_{mn}:

$$Y_{mn} = \text{Maximum } \{[X_{mjk} + X_{n,j+1,k} - 1], [X_{mjk} + X_{n,j,k+1} - 1], [X_{mjk} + X_{n,j-1,k} - 1],$$
$$\times [X_{mjk} + X_{n,j,k-1} - 1]\} \quad \text{for all } j \text{ and } k. \tag{5}$$

The first expression in the maximization term examines every cell (j, k) that belongs to **m** to see if the cell to its right, i.e. $(j + 1, k)$, belongs to **n**. The next three expressions check every cell (j, k) that belongs to **m** to see if the cell above it, i.e. $(j, k + 1)$, to its left, i.e. $(j - 1, k)$, or below it, i.e. $(j, k - 1)$, belongs to **n**, respectively.

Each expression in equation (5) can assume only values of **0**, **1**, or **−1** depending on whether each of the **X** values if **0** or **1**. Therefore, Y_{mn}, which is the maximum value for all j and k, can assume only the values **0** or **1**. Furthermore, it can only assume the value of 1 iff m and n are neighbors. Equation (5) is not in the linear form. To transform it to a form acceptable by any LP software, we use the argument detailed in the Appendix.

Introducing the **0–1** variables $\mathbf{S}_{mnjkp}$, $p = 1, \ldots, 4$; equation (5) can be rewritten as:

$$Y_{mn} \geqslant X_{mjk} + X_{n,j+1,k} - 1, \quad Y_{mn} \leqslant X_{mjk} + X_{n,j+1,k} - 1 + M \cdot S_{mnjk1} \quad \text{for all } j \text{ and } k \tag{6a}$$

$$Y_{mn} \geqslant X_{mjk} + X_{n,j,k-1} - 1, \quad Y_{mn} \leqslant X_{mjk} + X_{n,j,k-1} - 1 + M \cdot S_{mnjk2} \quad \text{for all } j \text{ and } k \tag{6b}$$

$$Y_{mn} \geqslant X_{mjk} + X_{n,j+1,k} - 1, \quad Y_{mn} \leqslant X_{mjk} + X_{n,j+1,k} - 1 + M \cdot S_{mnjk3} \quad \text{for all } j \text{ and } k \tag{6c}$$

$$Y_{mn} \geqslant X_{mjk} + X_{n,j,k-1} - 1, \quad Y_{mn} \leqslant X_{mjk} + X_{n,j,k-1} - 1 + M \cdot S_{mnjk4} \quad \text{for all } j \text{ and } k \tag{6d}$$

$$\sum_{j=1}^{\mathbf{a}} \sum_{k=1}^{\mathbf{b}} \sum_{p=1}^{4} S_{mnjkp} \leqslant 4 \cdot A - 1 \quad \text{for } 1 \leqslant m < n \leqslant N. \tag{6e}$$

SOLUTION PROCEDURE

Putting things together, it is seen that the facility layout problem will reduce to the following **0–1** linear programming problem with four sets of binary decision variables (X_{ijk}, Y_{mn}, Z_{il}, and S_{mnjkp}):

Objective function: equation (1)

Subject to constraints: equations (3, 4, 6).

That is:

Objective function

$$\text{Maximize} \sum_{m=1}^{N} \sum_{n=m+1}^{N} C_{mn} \cdot Y_{mn}.$$

Subject to constraints:

$$Z_{il} \leqslant \left\{ \sum_{j=J}^{J+a_i-1} \sum_{k=K}^{K+b_i-1} X_{ijk} \right\} \Big/ A_i \quad \text{for all } 1 \leqslant J \leqslant a - a_i, \quad \text{and } 1 \leqslant K \leqslant b - b_i$$

$$Z_{il} \leqslant \left\{ \sum_{j=J}^{J+b_i-1} \sum_{k=K}^{K+a_i-1} X_{ijk} \right\} \Big/ A_i \quad \text{for all } 1 \leqslant J < a - b_i, \quad \text{and } 1 \leqslant K \leqslant b - a_i$$

$$\sum_{l=1}^{n_i} Z_{il} = 1 \quad \text{for all departments } i = 1, \ldots, N$$

$$\sum_{i=1}^{N} X_{ijk} = 1 \quad \text{for all cells } (j, k)$$

$$\sum_{j=1}^{a} \sum_{k=1}^{b} X_{ijk} = A_i \quad \text{for all departments } i = 1, \ldots, N$$

$$Y_{mn} \geqslant X_{mjk} + X_{n,j+1,k} - 1, \quad Y_{mn} \leqslant X_{mjk} + X_{n,j+1,k} - 1 + M \cdot S_{mnjk1} \quad \text{for all } j \text{ and } k$$

$$Y_{mn} \leqslant X_{mjk} + X_{n,j,k-1} - 1, \quad Y_{mn} \leqslant X_{mjk} + X_{n,j,k-1} - 1 + M \cdot S_{mnjk2} \quad \text{for all } j \text{ and } k$$

$$Y_{mn} \geqslant X_{mjk} + X_{n,j+1,k} - 1, \quad Y_{mn} \leqslant X_{mjk} + X_{n,j+1,k} - 1 + M \cdot S_{mnjk3} \quad \text{for all } j \text{ and } k$$

$$Y_{mn} \geqslant X_{mjk} + X_{n,j,k-1} - 1, \quad Y_{mn} \leqslant X_{mjk} + X_{n,j,k-1} - 1 + M \cdot S_{mnjk4} \quad \text{for all } j \text{ and } k$$

$$\sum_{j=1}^{a} \sum_{k=1}^{b} \sum_{p=1}^{4} S_{mnjkp} \leqslant 4 \cdot A - 1 \quad \begin{array}{l} \text{for } 1 \leqslant m < n \leqslant N \\ \text{for all } j \text{ and } k \end{array}$$

This LP problem has a maximum of $\{NA + N(N-1)/2 + (n_l + \cdots + n_N) + 2N(N-1)A\}$ variables, and a maximum of $\{A + 2N + 4.5N(N-1)A + (n_l + \cdots + n_N)\}$ constraints. It is seen that the number of variables and constraints is of the order of $O(N^2A)$. It is known that the required computational time for solving 0–1 integer programming problems is more related to the number of integer variables than the number of constraints. In this model, the number of variables as well as constraints are so large that even for a simple layout problem with $N = 10$ departments, the computational time is prohibitive. As an example for a problem with $N = 4$ and $A = 16$, which will be discussed in the next section (case 3), there are *507 variables and 941 constraints*. Using LINDO [7] on a VAX 11/785 to solve the problem as a linear problem, and forcing the variables to be 0–1 integers, it was observed that 8–10 min of the computer time was required for finding the optimal solution.

A difficulty in solving the above LP problem lies in the S_{mnjkp} variables. They drastically increase the number of integer variables as well as constraints. Eliminating them (and their corresponding constraints) introduces another difficulty in handling the objective function which is in the form of maximization. The argument employed in the Appendix proves that minimization of Y_{mn} would have helped to reduce the set of constraints (6) by 50% and eliminate the need for the S_{mnjkp} variables. Therefore, to reduce the number of variables and constraints (and improve the modeling time as well as computation time), the optimization problem will be transformed into a minimization problem, but with the following modification on the cost coefficients C_{mn} in the objective function: Substitute C_{mn} with $D_{mn} = 1/C_{mn}$, i.e.

$$\text{Minimize} \sum_{m=1}^{N} \sum_{n=m+1}^{N} [1/C_{mn}] \cdot Y_{mn}. \tag{7}$$

Subject to constraints:

$$Z_{il} \leqslant \left\{ \sum_{j=J}^{J+a_i-1} \sum_{k=K}^{K+b_i-1} X_{ijk} \right\} \Big/ A_i \quad \text{for all } 1 \leqslant J \leqslant a - a_i, \text{ and } 1 \leqslant K \leqslant b - b_i$$

$$Z_{il} \leqslant \left\{ \sum_{j=J}^{J+b_i-1} \sum_{k=K}^{K+a_i-1} X_{ijk} \right\} \Big/ A_i \quad \text{for all } 1 \leqslant J \leqslant a - b_i, \text{ and } 1 \leqslant K \leqslant b - a_i$$

$$\sum_{l=1}^{n_i} Z_{il} = 1 \quad \text{for all departments } i = 1, \ldots, N$$

$$\sum_{i=1}^{N} X_{ijk} = 1 \quad \text{for all cells } (j, k)$$

$$\sum_{j=1}^{a} \sum_{k=1}^{b} X_{ijk} = \mathbf{A_i} \quad \text{for all departments } i = 1, \ldots, N$$

$$Y_{mn} \geqslant X_{mjk} + X_{n,j+1,k} - 1,$$

$$Y_{mn} \geqslant X_{mjk} + X_{n,j,k-1} - 1,$$

$$Y_{mn} \geqslant X_{mjk} + X_{n,j+1,k} - 1,$$

$$Y_{mn} \geqslant X_{mjk} + X_{n,j,k-1} - 1,$$

This LP problem has a maximum of $\{NA + N(N-1)/2 + (n_1 + \cdots + n_N)\}$ variables, and a maximum of $\{A + 2N + 2N(N-1)A + (n_1 + \cdots + n_N)\}$ constraints. Even with such a modification, the number of variables and constraints are so large that the computational time is restraining. As an example, for the same problem (mentioned before) with $N = 4$ and $A = 16$, which will be discussed in the next section, there are *123 variables and 461 constraints.*

To improve the quality of the model, several attempts were made to reduce the number of constraints. Note that equation (5) can be replaced by:

$$Y_{mn} = \text{Maximum } \{[X_{mjk} + 1/4(X_{n,j+1,k} + X_{n,j,k+1} + X_{n,j-1,k} + X_{n,j,k-1}) - 1]\} \quad \text{for } 1 \leqslant m < n \leqslant N \text{ for all } j \text{ and } k. \tag{8}$$

This modification reduces the number of inequality constraints (6) by 75%. The new set of constraints is:

$$Y_{mn} \geqslant X_{mjk} + 1/4(X_{n,j+1,k} + X_{n,j,k+1} + X_{n,j-1,k} + X_{n,j,k-1}) - 1 \quad \text{for } 1 \leqslant m < n \leqslant N \text{ for all } j \text{ and } k. \tag{9}$$

This reduces the number of constraints to a maximum of $\{A + 2N + 0.5N(N-1)A + (n_1 + \cdots + n_N)\}$. As an example, for the same problem with $N = 4$ and $A = 16$, there are *123 variables and 173 constraints.* Other attempts to improve the model were to reduce the number of integer variables, and/or add more informative constraints that result in faster convergence to the optimal solution. Details of these modifications are demonstrated later. Next, four sample problems are presented.

SAMPLE PROBLEMS

Using the solution procedure developed in the previous sections, the following example problems were solved. Figure 2 represents the geometrical shape of the layouts for all the cases:

Case 1

This case consists of $N = 3$ departments with $C_{12} = 4$, $C_{13} = 3$, and $C_{23} = 2$. Total area available is $A = 3 \times 3$ and $A_i = 1 \times 3$ for all departments. The objective function of the modified problem is:

$$\text{Min. } Z = 0.25 * Y_{12} + 0.33 * Y_{13} + 0.50 * Y_{23}.$$

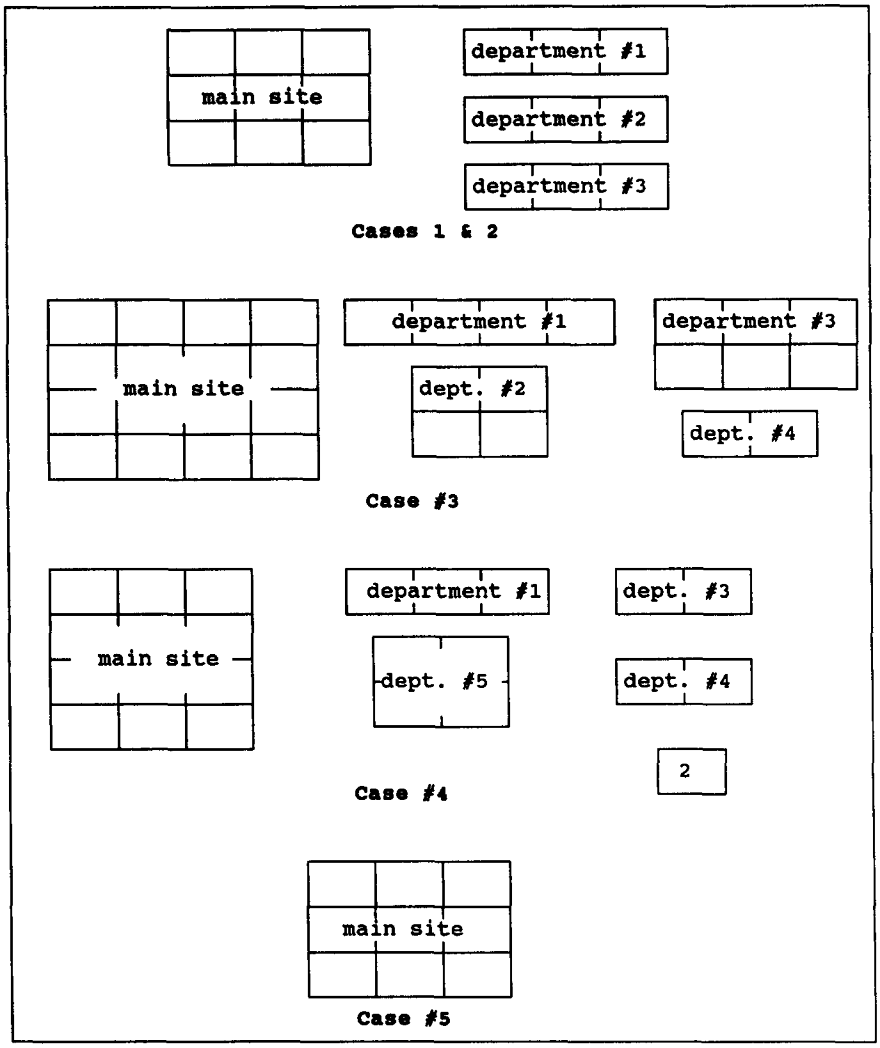

Fig. 2. Geometrical shape of the layouts for the case studies.

There are *48 variables and 60 constraints*. LINDO used 1902 iterations to generate the following optimal integer solution: $Z^* = 0.58$, with $Y_{12} = 1$, $Y_{13} = 1$, and $Y_{23} = 0$, which states that department 1 should be in the center and the other two departments on either sides.

Case 2

The same case was run once more, but with new $C_{12} = 5$, $C_{13} = 2$, and $C_{23} = 4$ values. The objective function of the modified problem is:

$$\text{Min. } Z = 0.2\ Y_{12} + 0.5\ Y_{13} + 0.25\ Y_{23}.$$

LINDO used 2188 iterations to generate the following optimal integer solution: $Z^* = 0.45$, with $Y_{12} = 1$, $Y_{13} = 0$, and $Y_{23} = 1$; i.e. department 2 with the highest inbetween flow should be in the center of the building with the other two departments located on either side.

Case 3

There are $N = 4$ departments with $C_{1j} = 8.3$, 1.66, and 5, $C_{2j} = 6.6$, and 5, and $C_{34} = 3.3$. Total area available is $A = 4 \times 4$, and $A_1 = 1 \times 4$, $A_2 = 2 \times 2$, $A_3 = 2 \times 3$, and $A_4 = 1 \times 2$. The objective function of the modified problem is:

$$\text{Min. } Z = 0.12 * Y_{12} + 0.60 * Y_{13} + 0.20 * Y_{14} + 0.15 * Y_{23} + 0.20 * Y_{24} + 0.30 * Y_{34}.$$

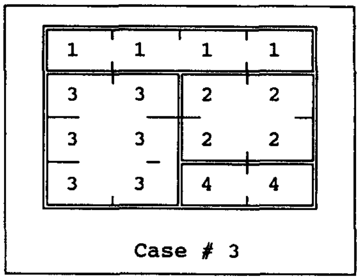

Fig. 3. Optimal solution for case 3.

There are *123 variables and 173 constraints.* LINDO used 6084 iterations to generate the following optimal integer solution: $Z^* = 1.37$, with $Y_{12} = Y_{13} = Y_{23} = Y_{24} = Y_{34} = 1$. Figure 3 depicts the optimal layout for this case study.

Case 4

There are $N = 5$ departments with the corresponding cost values, C_{ij}, as listed in Fig. 4. Total area available is $A = 3 \times 4$, and $A_1 = 1 \times 3$, $A_2 = 1 \times 1$, $A_3 = 1 \times 2$, $A_4 = 1 \times 2$, and $A_5 = 2 \times 2$. The objective function of the modified problem is:

$$\text{Min. } Z = (0.1*Y_{12} + 0.2*Y_{13} + 0.5*Y_{14} + 0.3*Y_{15}) + (0.1*Y_{23} + 0.1*Y_{24} + 1*Y_{25}) + (0.2*Y_{34} + 0.1*Y_{35}) + (0.1*Y_{45}).$$

There are *126 variables and 203 constraints.* LINDO used 2432 iterations to generate the following optimal integer solution: $Z^* = 0.7$, with $Y_{12} = Y_{13} = Y_{24} = Y_{35} = Y_{45} = 1$. Figure 4 depicts the optimal layout for this case study.

MODIFICATIONS

The fact that solving case 3 took up to 10 min of computer time persuaded us to apply the following two modifications to the procedure. The result was a significant reduction in the computational time.

The first modification was to reduce the number of integer 0–1 variables as much as possible. To do so, the problem was solved with the integer conditions on X_{ijk} being relaxed. The relaxed LP (having $N \cdot A$ less integer variables that the original model) ended up generating the original optimal 0–1 solution in less computational time. This is due to the structure of constraints (3) and (4).

A second modification was to add more constraints to the problem (for a detailed description of use of strong inequalities and or equalities as cutting planes see Van Roy and Wolsey [8] and Johnson and Padberg [9]). To do so, it was noticed that convergence is more sensitive to the constraints on Y_{mn}, therefore attempts were made to add as many powerful constraints on Y_{mn} as possible.

Dij	2	3	4	5
1	0.1	0.2	0.5	0.3
2		0.1	0.1	1.0
3			0.2	0.1
4				0.1

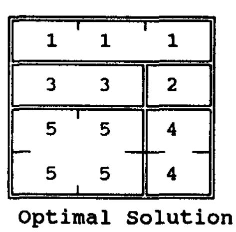

Fig. 4. The cost function and the optimal solution for case 4.

Example 1

For the first two cases, irrespective of the layout the following constraints hold true:

$$\sum_{n=1}^{N} Y_{mn} = 2 \quad \text{for all } m = 1, \ldots, N.$$

Such constraints help us find the optimal solution for cases 1 and 2 in 126 and 383 iterations, respectively.

Example 2

For the third case, irrespective of the layout the following constraints hold true:

$$\sum_{n=1}^{N} Y_{mn} \geqslant 2 \quad \text{for all } m = 1, \ldots, N$$

$$Y_{12} + Y_{13} + Y_{14} + Y_{23} + Y_{24} + Y_{34} \leqslant 5.$$

Such constraints help us find the optimal solution for the third case in 1066 iterations.

SPECIAL CASE

The special case of facility layout problem in which all the departments are of the same size deserves special attention. Under the assumption of equal size departments, considering the blocks to be of size 1, i.e. A = N, there is not need for set of constraints (3) on contiguity (rigidity). Therefore, the model reduces to:

$$\text{Minimize} \quad \sum_{m=1}^{N} \sum_{n=m+1}^{N} [1/C_{mn}] \cdot Y_{mn} \tag{10}$$

$$\text{Subject to:} \quad \sum_{i=1}^{N} X_{ijk} = 1 \quad \text{for all cells } (j, k) \tag{11}$$

$$\sum_{j=1}^{a} \sum_{k=1}^{b} X_{ijk} = 1 \quad \text{for all departments } i \tag{12}$$

$$Y_{nm} \geqslant X_{mjk} + 1/4(X_{n,j+1,k} + X_{n,j,k+1} + X_{n,j-1,k} + X_{n,j,k-1}) - 1 \quad \text{for } 1 \leqslant m < n \leqslant N \text{ for all } j \text{ and } k \tag{13}$$

$$\sum_{m=1}^{N} \sum_{n=m+1}^{N} Y_{mn} = 2(N - N^{0.5}). \tag{14}$$

The last constraint [equation (14)] is to guarantee that irrespective of the layout, total number of adjacencies is fixed. The new problem has $\{0.5N(3N-1)\}$ variables and $\{2N + 0.5N^2(N-1) + 1\}$ constraints. For $N = 6$ departments there are *51 variables and 103 constraints*, and for $N = 9$ departments there are *117 variables and 343 constraints* (compare the values for $N = 9$ with *198 variables and 432 constraints* for the general model).

Case 5

There are $N = 9$ departments with the corresponding cost values, C_{ij}, as listed in Fig. 5. Total area available is $A = 3 \times 3$, and $A_i = 1 \times 1$. There are *117 variables and 343 constraints*. Figure 5 depicts the optimal layout for this case study.

Comparison with the general case shows the advantage of using the special case to approximate the optimal solution for a system in which the departments do not require exactly the same area. The resulting layout can be used as an initial solution for further detailed analysis.

RESULTS AND CONCLUSIONS

In this paper a mathematical model for block layout systems was developed that utilizes 0–1 integer programming as the optimization component. In this model a rectangular building is being

Dij	2	3	4	5	6	7	8	9
1	1.0	1.0	1.0	0.02	0.2	1.0	0.01	1.0
2		0.2	1.0	1.0	0.04	0.2	1.0	1.0
3			0.01	0.04	0.2	0.05	1.0	1.0
4				.005	1.0	0.02	1.0	0.01
5					1.0	1.0	1.0	1.0
6						0.025	0.2	0.02
7							0.1	1.0
8								0.05

9	4	5
8	7	3
1	6	2

Optimal Solution

Fig. 5. The cost function and the optimal solution for case 5.

used to site N different departments of possible different sizes, each of which is rectangular in shape. The objective of interest is to lay out the departments in the building such that the sum of the weighted adjacency values for all departments is maximized.

The procedure solves small size problems ($N < 10$). To apply it to larger size problems, some modifications are made that reduce the number of constraints drastically. For the new improved model additional tight constraints on the number of adjacencies are developed. The modifications help reduce the number of iterations for the integer program to converge to optimal solution. But the fact remains unchanged: the number of variables and constraints which are functions of the size of the problem, grow in such a magnitude that unless the process of constraints-writing for the computer is done by a text generator, the time spent in constructing the model becomes restrictive for any large problem.

The special case of departments of the same size is modeled and comparison is made with the general case.

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APPENDIX

Suppose Z is the maximum of X_1, X_2, and X_3, i.e.:

$$Z = \mathrm{Max}\{X_1, X_2, X_3\}. \tag{A1}$$

How can equation (A1) be expressed in linear form?

(1) Z has to be larger than X_1, X_2, and X_3, i.e.:

$$Z \geqslant X_1 \tag{A2}$$

$$Z \geqslant X_1 \tag{A3}$$

$$Z \geqslant X_3. \tag{A4}$$

(2) In addition, Z has to be equal to the largest of the quantities X_1, X_2, and X_3. To express this constraint quantitatively, we require one or more of the following inequalities to hold:

$$Z \leqslant X_1 \tag{A5}$$

$$Z \leqslant X_2 \tag{A6}$$

$$Z \leqslant X_3. \tag{A7}$$

The rationale is to say $Z \leqslant X_2$, then because simultaneously Z has to be greater than or equal to X_2, we conclude that $Z = X_2$.

To make sure that at least one of the inequalities (A5)–(A7) holds, we introduce three 0–1 integer variables S_1, S_2, and S_3, and rewrite inequalities (A5)–(A7) as:

$$Z \leqslant X_1 + M \cdot S_1 \tag{A8}$$

$$Z \leqslant X_2 + M \cdot S_2 \tag{A9}$$

$$Z \leqslant X_3 + M \cdot S_3 \tag{A10}$$

$$S_1 + S_2 + S_3 \leqslant 2 \tag{A11}$$

where M is a sufficiently large number. Therefore, equation (A1) is equivalent to (A2)–(A4) plus (A8)–(A11).

Note that, if we were to minimize Z, then there would be no need for inequalities (A8)–(A11). This is due to the fact that the LP optimization attempts to find the smallest Z that fulfills the requirements of (A2)–(A5), i.e. it will be equal to the maximum of the three variables.