

Homework3

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1 Problem1: LU algorithm

1.1 problem description

Prove that the time complexity of the LU decomposition algorithm is $O(n^3)$, here L means lower triangular matrix and U upper triangular matrix.

1.2 solution

Proof:

For a nonsingular matrix $A \in \mathbb{R}^{n \times n}$ (with partial pivoting if needed), there exist a permutation matrix P , a unit lower triangular matrix L , and an upper triangular matrix U such that

$$PA = LU \quad (1)$$

The upper and lower triangular matrix satisfies:

$$a_{ij} = \sum_{s=1}^n l_{is}u_{sj}, \quad l_{ii} = 1, \quad l_{ij} = 0 \quad (j > i), \quad u_{ij} = 0 \quad (i > j) \quad (2)$$

(1) Time complexity of LU decomposition: the decomposition can be divided into 3 steps:
step1: get the first column of L and the first row of U

$$\begin{cases} l_{i1} = a_{i1} & i = 1, 2, \dots, n \\ u_{1j} = \frac{a_{1j}}{l_{11}} & j = 1, 2, \dots, n \end{cases} \quad (3)$$

step2: get the second and following elements of L

$$l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik}u_{kj} \quad j = 2, 3, \dots, n \quad i = j, j+1, \dots, n \quad (4)$$

step3: use results from step2 to get elements of U

$$u_{ji} = \frac{1}{l_{jj}}(a_{ji} - \sum_{k=1}^{j-1} l_{jk}u_{ki}) \quad j = 2, 3, \dots, n-1 \quad i = j+1, j+2, \dots, n \quad (5)$$

In the whole loop ,step2 and step3 is coupled with each other, each will use the output elements of the other one.The first step has complexity of $O(n)$. And in step2, the total number of loop is

$$\sum_{j=2}^n (n-j+1)(j-1) = \frac{1}{2}n^2(n-1) - \frac{1}{6}n(n-1)(2n-1) = \frac{1}{6}n^3 - \frac{1}{6}n \quad (6)$$

The same applies to step3, total number of loop in step3:

$$\sum_{j=2}^{n-1} (n-j)(j-1) = \frac{1}{6}n^3 - \frac{1}{2}n^2 + \frac{1}{3}n \quad (7)$$

(2)Time complexity of forward and backward substitution:

In solving $LUx = b$, forward substitution means solving $Ly = b$, backward substitution means solving $Ux = y$, the two process has the same time complexity $O(n^2)$, for a lower or upper triangular matrix, the total number of loop is:

$$\sum_{j=1}^n (j-1) = \frac{n(n-1)}{2} \quad (8)$$

Sum over equation(6)(7)(8), the leading term being $\frac{1}{3}n^3$, hence the LU decomposition algorithm has time complexity $O(n^3)$.

2 Problem2:

2.1 problem description

Solve systems of equations using the Gaussian elimination algorithm and partial-pivoting scheme (Note: Write a general program applicable to solving the following equations; select the pivot element with the largest coefficient among all columns for each elimination step; Optional: Compute the inverse matrix of the coefficient matrix).

$$\begin{cases} 2x_1 + 3x_2 + 5x_3 &= 5, \\ 3x_1 + 4x_2 + 8x_3 &= 6, \\ x_1 + 3x_2 + 3x_3 &= 5. \end{cases} \quad (9)$$

2.2 algorithm description

We solve $Ax = B$ by Gaussian elimination with partial pivoting on the augmented matrix $[A \mid B]$, here B can be generalized as $B \in \mathbb{R}^{n \times m}$.

(1) Forward elimination (with partial pivoting):

First for $k = 1, \dots, n$, Select the pivot row $p = \max_{i \geq k} |a_{ik}|$ in column k and swap rows $k \leftrightarrow p$ in the whole augmented matrix.

Second for each $i = k+1, \dots, n$, form the multiplier $l_{ik} = a_{ik}/a_{kk}$, then apply the row operation to each row below k

$$\text{row}_i \leftarrow \text{row}_i - l_{ik} \text{row}_k, \quad (10)$$

updating both the left block (turning A into an upper triangular U) and the right block B simultaneously. After this phase, the left block is U , and the right block is $B' = L^{-1}B$ (forward substitution has been implicitly applied by the same row operations).

(2) Back substitution (simultaneous for all right-hand sides): Solve $Ux = B'$ from bottom to top. For $i = n, \dots, 1$ and for each RHS column r ,

$$x_{ir} = \frac{1}{u_{ii}} \left(b'_{ir} - \sum_{j=i+1}^n u_{ij} x_{jr} \right). \quad (11)$$

(3) Matrix inverse (optional):

To get inversion matrix of A , just set $B = I_n$ as identity matrix, and run the same procedure. The result x equals A^{-1} .

(4) Time complexity Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$.

Pivot search: $\sum_{k=1}^n (n - k + 1) = O(n^2)$ comparisons.

Updating the left block (turning A into U): $2 \sum_{k=1}^{n-1} (n - k)^2 = \frac{2}{3}n^3 + O(n^2)$ flops.

Updating the right block during elimination (equivalent to forward substitution for $n \times m$ RHS): about $mn^2 + O(mn)$ flops.

Back substitution for $n \times m$ RHS: about $mn^2 + O(mn)$ flops.

Total: flops = $\frac{2}{3}n^3 + 2mn^2 + O(n^2 + mn) = O(n^3)$.

2.3 output

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Solution x:
['2.00000000', '2.00000000', '-1.00000000']

A^{-1}:
 6.00000000 -3.00000000 -2.00000000
 0.50000000 -0.50000000  0.50000000
-2.50000000  1.50000000  0.50000000

Check A x:
['5.00000000', '6.00000000', '5.00000000']

Check A * A^{-1}:
 1.00000000  0.00000000 -0.00000000
 0.00000000  1.00000000  0.00000000
 0.00000000  0.00000000  1.00000000

```

Figure 1: output of problem2.py

3 Problem3:

3.1 problem description

Solve the 1D Schrodinger equation with the potential (i) $V(x) = x^2$; (ii) $V(x) = x^4 - x^2$ with the variational approach using a Gaussian basis (either fixed widths or fixed centers). Consider the three lowest energy eigenstates.

The Gaussian basis functions are defined as: $\phi_i(x) = \left(\frac{v_i}{\pi}\right)^{1/2} \exp(-v_i(x - s_i)^2)$. This function has two variational parameters: v_i the width of the Gaussian, and s_i the center

of the Gaussian. For simplicity, we only vary one of these parameters at a time and do calculations with either fixed widths or fixed centers.

3.2 algorithm description

To solve the one-dimensional Schrödinger equation in Hartree units

$$\hat{H}\psi(x) = E\psi(x), \quad \hat{H} = -\frac{1}{2}\frac{d^2}{dx^2} + V(x), \quad (12)$$

with the potentials $V(x) = x^2$ and $V(x) = x^4 - x^2$

The trial wave function is expanded as

$$\psi(x) = \sum_i c_i \phi_i(x), \quad \phi_i(x) = \sqrt{\frac{v_i}{\pi}} e^{-v_i(x-s_i)^2}, \quad (13)$$

where v_i is the Gaussian width and s_i its center.

For each pair of basis functions, analytical integrals are used to build:

$$S_{ij} = \langle \phi_i | \phi_j \rangle, \quad T_{ij} = \frac{1}{2} \int \phi'_i(x) \phi'_j(x) dx, \quad V_{ij} = \langle \phi_i | V(x) | \phi_j \rangle. \quad (14)$$

The Hamiltonian matrix is $H = T + V$. Because the basis is non-orthogonal ($S \neq I$), the eigenvalue equation can be written in matrix form. $c = [c_1, c_2, \dots, c_n]^T$

$$Hc = \varepsilon Sc \quad (15)$$

S can be decomposed using Cholesky method as $S = R^T R$, R being the upper triangular matrix. Multiply R^{-T} on the left side:

$$R^{-T} H R^{-1} R c = \varepsilon R^{-T} R^T R c \rightarrow Ay = \varepsilon y, \quad A = R^{-T} H R^{-1}, \quad y = R c \quad (16)$$

So we just need to find eigenvalues of A . In the code I use fixed-center basis, I take 9 Gaussian functions and set their centers in $[-3, 3]$ with the same interval, and they share the same width v for simplicity.

Then a grid of width $v \in [0.2, 3.0]$ is scanned. The value minimizing the ground-state energy $E_0(v)$ is chosen as the optimal width v_{best} to get E_0 . Then the second and third excitation energies are computed in the same way.

3.3 output

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• [Fixed centers] potential = x2    basis size = 9
  best width = 0.7220  energy 1  =0.70710683
  best width = 1.1492  energy 2  =2.12228693
  best width = 0.8644  energy 3  =3.53553854
[Fixed centers] potential = x4-x2    basis size = 9
  best width = 1.6237  energy 1  =0.33801422
  best width = 1.4339  energy 2  =1.62150652
  best width = 0.9593  energy 3  =3.68652237

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Figure 2: output of problem3.py

the output includes the best width for each energy.