CS5800: Algorithms — Virgil Pavlu

Homework 12

Name:
Collaborators:

Instructions:

- Make sure to put your name on the first page. If you are using the LaTeX template we provided, then you can make sure it appears by filling in the `yourname` command.
- Please review the grading policy outlined in the course information page.
- You must also write down with whom you worked on the assignment. If this changes from problem to problem, then you should write down this information separately with each problem.
- Problem numbers (like Exercise 3.1-1) are corresponding to CLRS 4^{th} edition. While the 3^{rd} edition has similar problems with similar numbers, the actual exercises and their solutions are different, so make sure you are using the 4^{th} edition.

1. (20 points) (Exercise 24.1-3) Suppose that a flow network $G = (V, E)$ violates the assumption that the network contains a path $s \to v \to t$ for all vertices $v \in V$. Let u be a vertex for which there is no path $s \to u \to t$. Show that there must exist a maximum flow f in G such that $f(u, v) = f(v, u) = 0$ for all vertices $v \in V$.

Solution:
In essence this is a common sense problem. I'll analyze two examples though. The first example would be that vertice u is a single leaf with some incoming pipe and no outgoing pipes. This example leads to a very clear solution that in max flow there will never be any flow to u because it would break conservation. The next example we can compare, is similar to the first, but there is some subset of vertices connected to u, potentially in a cycle, such that water can flow into the section without breaking conservation. In this example, because we know there is no path from u -¿ t, there is also no path from any of the other vertices in the subset. That means that we can make a cut between the subset before u, and after u, solving for just the max flow from the subset before. We know this works because both s and t are guaranteed to be in the new subset. This max flow will result in no flow going to u at all or any flow through the subset that we cut as well.

2. (20 points) (Exercise 24.1-4) Let f be a flow in a network, and let α be a real number. The *scalar flow product*, denoted αf, is a function from $V \times V$ to $\mathbb{R}$ defined by

$$(\alpha f)(u, v) = \alpha \cdot f(u, v)$$

Prove that the flows in a network form a *convex set*. That is, show that if f_1 and f_2 are flows, then so is $\alpha f_1 + (1 - \alpha) f_2$ for all α in the range $0 \leq \alpha \leq 1$.

Solution:
By definition, a flow f is only a valid flow if it is $0 \leq f \leq f_{MAX}$ where f_{MAX} is the max flow. Therefore, we can frame flow as a function of f_1 and f_2:

$$f(f_1, f_2) = \alpha f_1 + (1 - \alpha) f_2$$

Now, we can see that $f(f_1, f_2)$ is a plane that rotates around the main diagonal of the plan $f_1 = f_2$ as α changes from 0 to 1. This means that the global maximum occurs at the corner of the plane where $f_1 = f_2 = f_{MAX}$ and the global minimum occurs at the opposite diagonal where $f_1 = f_2 = 0$. We can then calculate that $f(f_{MAX}, f_{MAX}) = f_{MAX}$ and $f(0, 0) = 0$. Therefore $f(f_1, f_2)$ always produces a valid flow because $0 \leq f(f_1, f_2) \leq f_{MAX}$ for all $0 \leq \alpha \leq 1$

3. (20 points) (Exercise 24.2-2) In Figure 24.1(b), what is the net flow across the cut $(\{s, v_2, v_4\}, \{v_1, v_3, t\})$? What is the capacity of this cut?

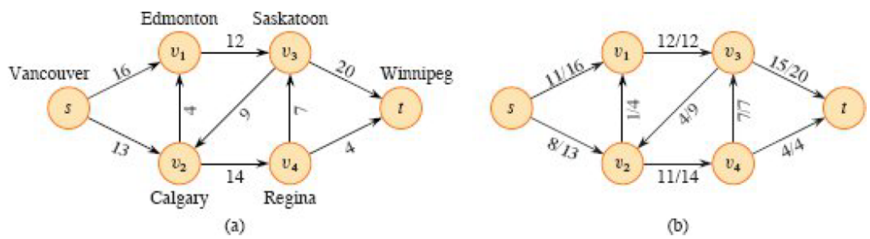

Figure 24.1 (a) A flow network $G = (V, E)$ for the Lucky Puck Company's trucking problem. The Vancouver factory is the source s, and the Winnipeg warehouse is the sink t. The company ships pucks through intermediate cities, but only $c(u, v)$ crates per day can go from city u to city v. Each edge is labeled with its capacity. **(b)** A flow f in G with value $|f| = 19$. Each edge (u, v) is labeled by $f(u, v)/c(u, v)$. The slash notation merely separates the flow and capacity and does not indicate division.

Solution:

$$\text{flow across cut } = f(s,v_1) + f(v_2,v_1) - f(v_3,v_2) + f(v_4,v_3) + f(v_4,t) = 11 + 1 - 4 + 7 + 4 = 19$$

$$\text{capacity across cut } = c(s,v_1) + c(v_2,v_1) + c(v_4,v_3) + c(v_4,t) = 16 + 4 + 7 + 4 = 31$$

4. (Extra Credit 20 points) Exercise 24.2-10

5. (30 points) Implement Push-Relabel for finding maximum flow.
Extra Credit: use relabel-to-front idea from Chapter 26.5 (3^{rd} edition) with the Discharge procedure.

Solution:

6. (15 points) Explain in a brief paragraph the following sentence from textbook page 737 (3^{rd} edition): "To make the preflow a legal flow, the algorithm then sends the excess collected in the reservoirs of overflowing vertices back to the source by continuing to relabel vertices to above the fixed height $|V|$ of the source".

Solution:
The preflow of push relabel works by initially flooding all of the immediately accessible junctions. This violates conservation and makes each of the targets overflow. Because there is an excess, they will be relabeled and pushed to the next neighbors. This will happen in succession until everything has been pushed to the sink. This next part may have occurred earlier as well depending on how it is implemented, but if the water has been pushed to the sink, and nodes are still in excess, then they are relabeled to push backwards. Eventually if those first targets are still in excess, they will be raised above the source and push flow backwards. At this point, the flow is at maximum because we've pushed all we can to the sink, and corrected previous pipes by pushing back to the source.

7. (Extra Credit 20 points) Exercise 26.4-4 (3^{rd} edition): Suppose that we have found a maximum flow in a flow network $G = (V, E)$ using a push-relabel algorithm. Give a fast algorithm to find a minimum cut in G.