Vehicle Dynamics and Simulation

Differential Equations and Numerical Integration

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Overview

- Differential equations

- Model generation examples

- Numerical integration
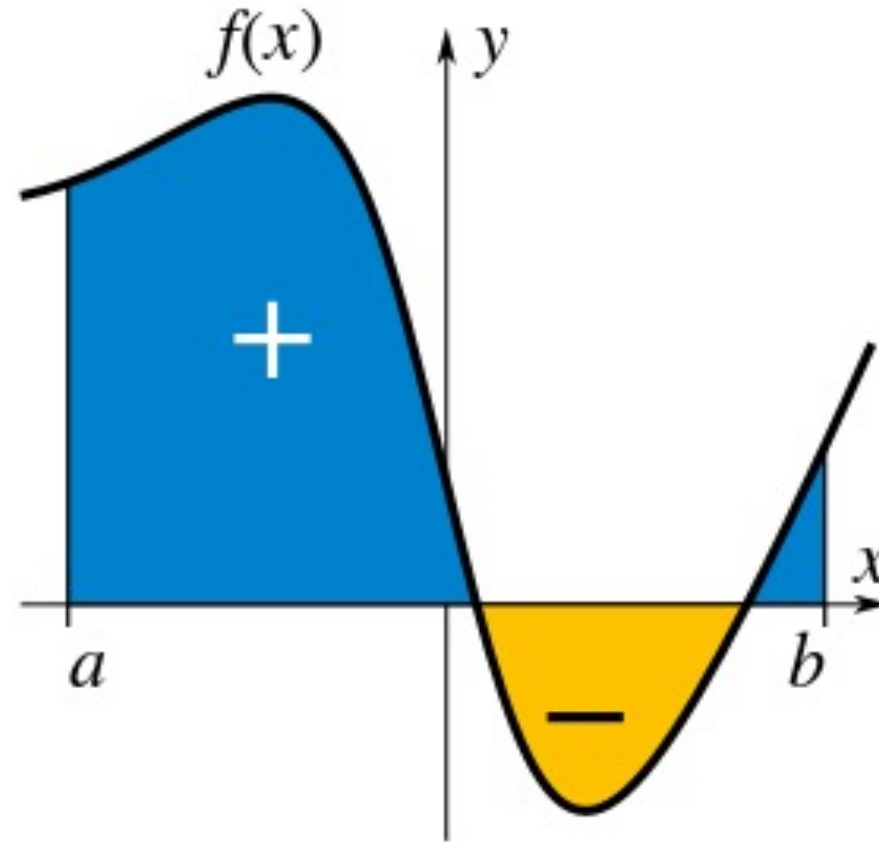
Dynamic Systems Modelling: Differential Equations

$= x$ $y =$

- Dynamic numerical models often make use of differential equations, for example;

y

$$y = 3x^2 + 2$$

$$\frac{dy}{dx} = 6x$$

$$\int y = x^3 + 2x + c$$

y

M

- A simple model may look something like this;

y

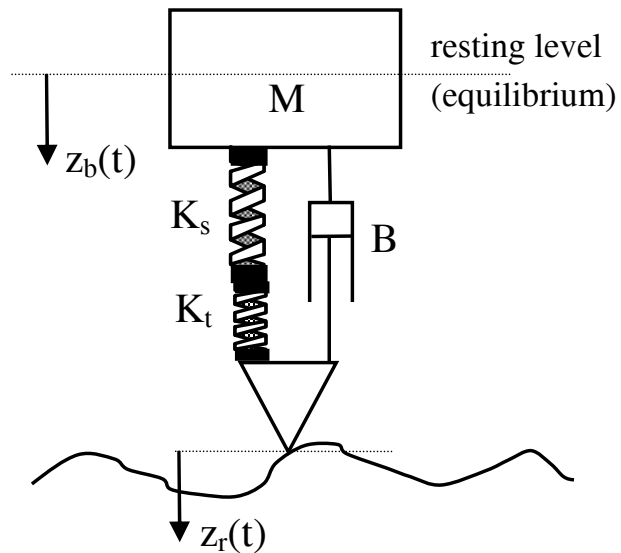$$M\frac{dv}{dt} = F - \frac{1}{2}\rho A C_d v^2$$

M

How does velocity change over time, if you apply constant engine torque (and hence F *in the model above) to a car ? What are the initial and final values of $\dfrac{dv}{dt}$ and* v *?*

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Modelling Example 1

- Try for yourself;

  - Spring stiffness; $\frac{1}{K} = \frac{1}{K} + \frac{1}{K}$

  $$F_s = \cdots$$

Sign convention: +ve direction indicated by arrow heads

Modelling Example 1

- To find a solution we need to express the Equation (1) as a system of first order equations by choosing the system 'states' correctly. Note that this is an arbitrary definition and many other choices are possible.

$$x_1 = z_b$$
$$x_2 = \dot{z}_b$$
$$x_3 = z_r$$
$$u = \dot{z}_r$$

- Making the substitutions into Equation (1)

$$M\dot{x}_2 = K(x_3 - x_1) + B_s(u - x_2)$$

Modelling Example 1

- Rearranging in terms of the states, x_1, x_2 and x_3;

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{K}{M}(x_3 - x_2) + \frac{B_s}{M}(u - x_2)$$
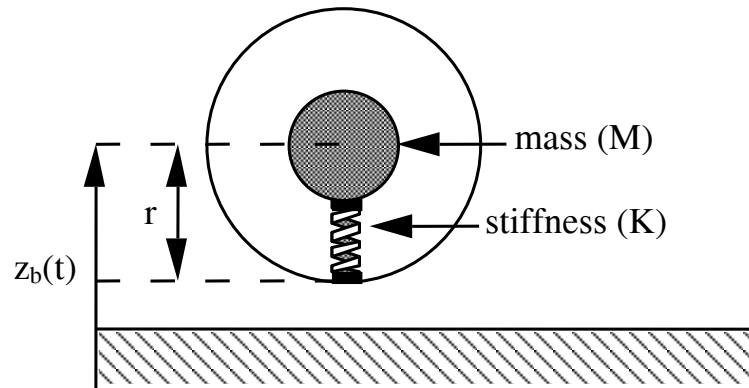
$$\dot{x}_3 = u$$

- An alternative representation using $x_1 = z_r - z_b$ i.e. new definition of x_1.

$$\dot{x}_1 = u - x_2$$

$$\dot{x}_2 = \frac{K}{M}x_1 + \frac{B_s}{M}(u - x_2)$$

Modelling Example 2

Looking back at the previous example and notes make sure you can obtain the following system equations;
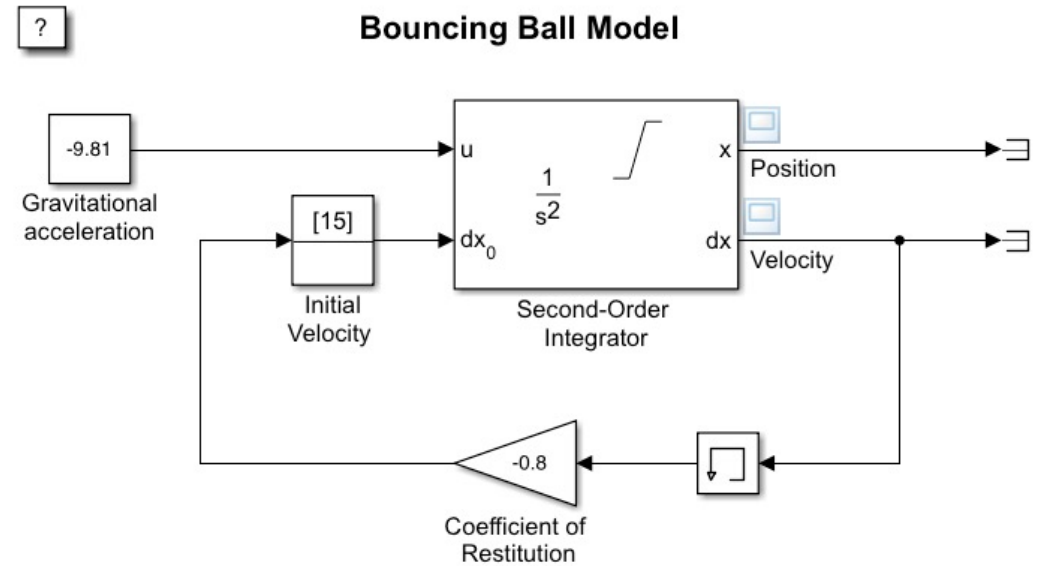
$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{K}{M(r - x_1)} - \frac{B}{M}x_2$$

with states defined as follows;

$$\dot{x}_1 = z_b \text{ and } x_2 = \dot{z}_b$$

mass (M)

stiffness (K)

r

$z_b(t)$

Modelling Example 2

- From Simulink help open the bouncing ball model

- Familiarize yourself with the model implementation.

- Try changing the initial conditions and see how the model behavior changes.
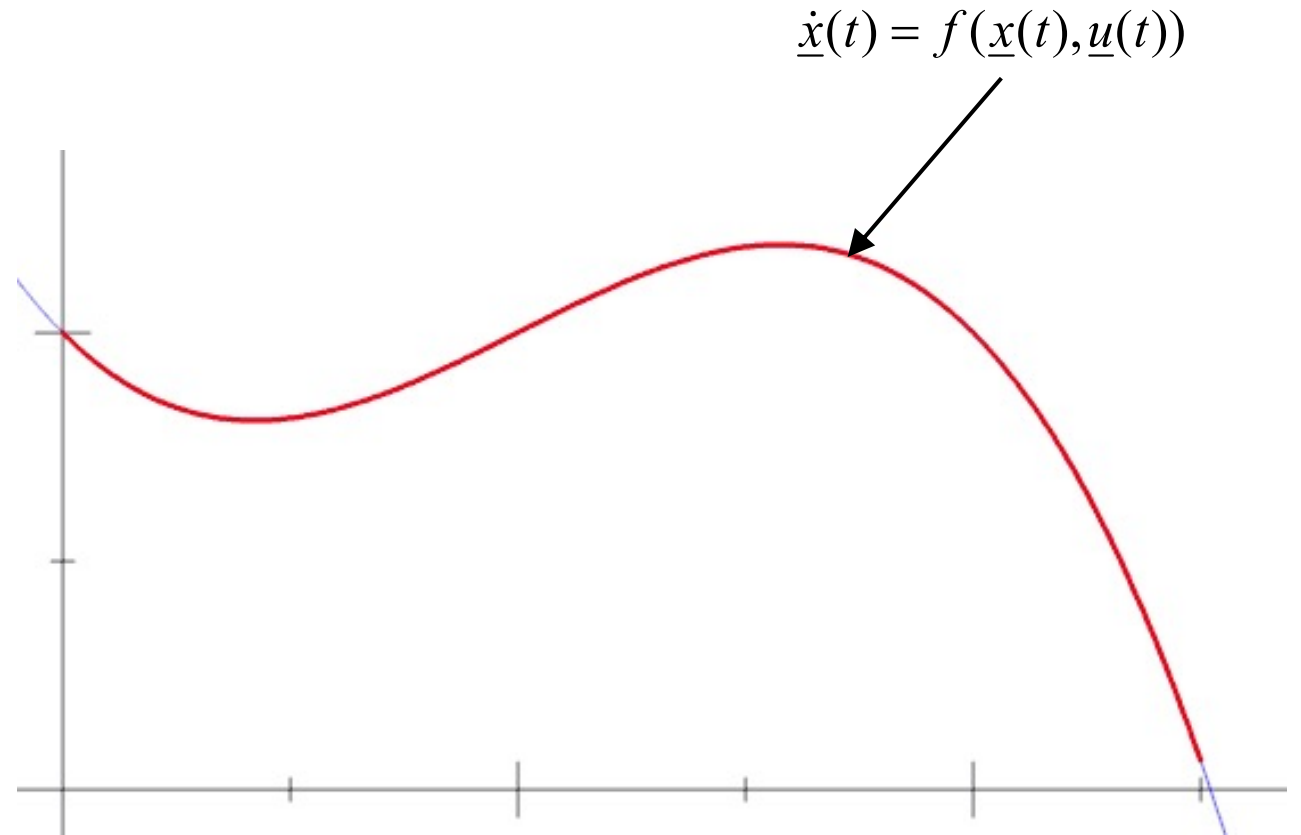

Bouncing Ball Model

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Numerical Integration of ODE's

$$\dot{\underline{x}}(t) = f(\underline{x}(t), \underline{u}(t))$$
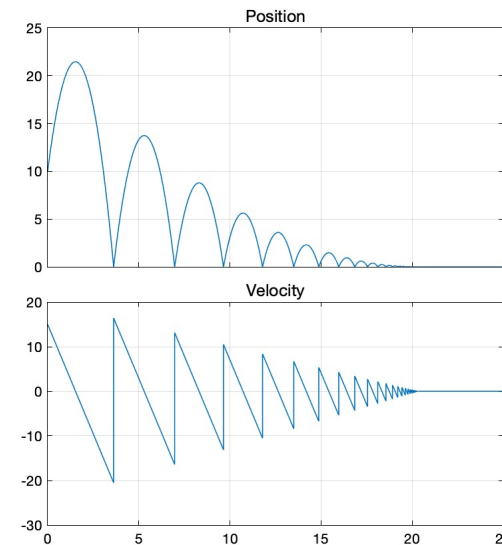
- We often need to find the definite integral of some function (solution).
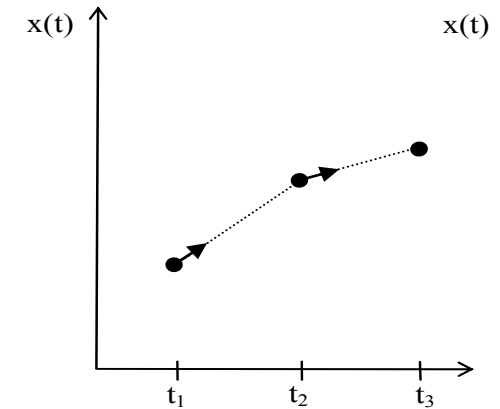
$$x(t) = \int_a^b \dot{x}\,(t)$$

Numerical Integration of ODE's

Euler's method

- Euler's forward method is a numerical integration technique that enables us to do this.

- $x(t + h)$ is evaluated using the gradient at t.

- h is the step size (small).

- Limitations;
  - Accuracy improved by reducing h i.e. the step size.
  - Large h can result in low accuracy and numerical instability.
  - Errors $\mathcal{O}(h^2)$

x(t)   x(t)

t₁   t₂   t₃

$$x \quad =$$

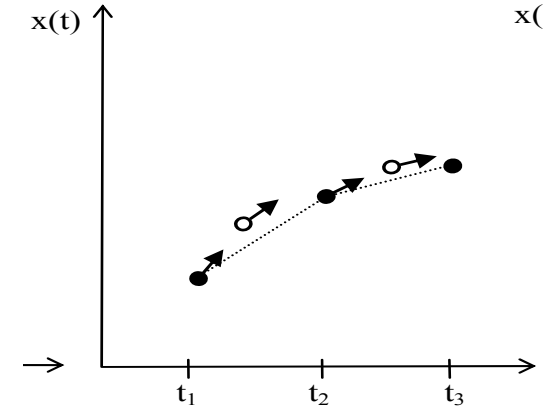$$\underline{\dot{x}}(t) = f(\underline{x}(t), \underline{u}(t))$$

$$\underline{x}(t+h) = \underline{x}(t) + h\underline{\dot{x}}(t)$$

x $\underline{x}$ +

What is the consequence of reducing the size of h on the calculation?

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x

x

Numerical Integration of ODE's

Midpoint method

- The midpoint method evaluates $x(t+h)$ using the gradient at $t+h/2$

- Errors $\mathcal{O}(h^3)$

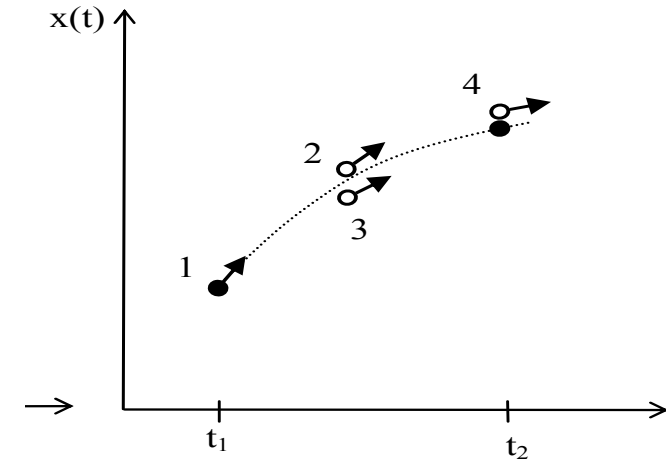$$\underline{k}_1 = hf(\underline{x}(t),\ \underline{u}(t))$$

$$\underline{k}_2 = hf(\underline{x}(t) + \underline{k}_1/2,\ \underline{u}(t+h/2))$$

$$\underline{x}(t+h) = \underline{x}(t) + \underline{k}_2 + O(h^3)$$

Numerical Integration of ODE's

Runge-Kutta 4th order

- RK4 evaluates $x(t+h)$ using the gradient at $t+h$ and $t+h/2$
- $\mathcal{O}(h^5)$
- RK4 is the most used fixed step solver

x

$x(t)$

1   2   3   4

t_1 t_2

f

$$\underline{k}_1 = hf(\underline{x}(t), \ \underline{u}(t))$$

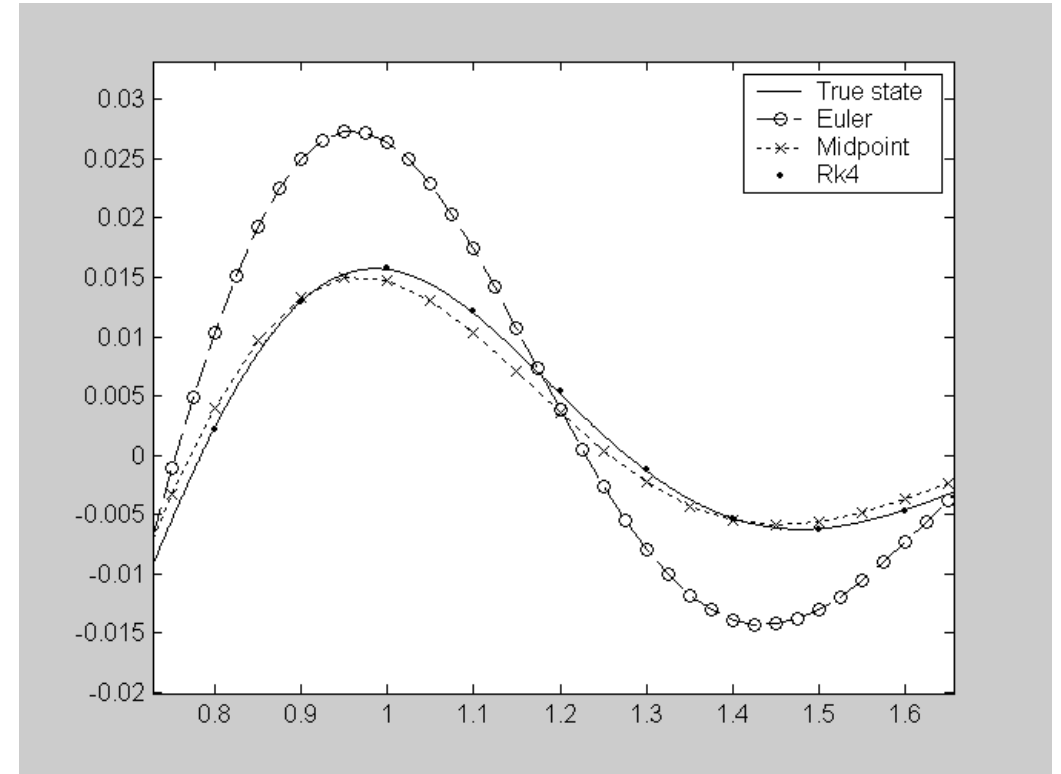$$\underline{k}_2 = hf(\underline{x}(t) + \underline{k}_1/2, \ \underline{u}(t+h/2))$$

$$\underline{k}_3 = hf(\underline{x}(t) + \underline{k}_2/2, \ \underline{u}(t+h/2))$$

$$\underline{k}_4 = hf(\underline{x}(t) + \underline{k}_3, \ \underline{u}(t+h))$$

$$\underline{x}(t+h) = \underline{x}(t) + \frac{\underline{k}_1}{6} + \frac{\underline{k}_2}{3} + \frac{\underline{k}_3}{3} + \frac{\underline{k}_4}{6} + O(h^5)$$

Numerical Integration of ODE's

- Compare the results on the right for the three different integration algorithms.

In what circumstance would one opt for a lower accuracy method?

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Numerical Integration of ODE's

- Fixed and variable step 'solvers' are the two main categories.

- Variable step solvers change the step size during the solution.

- Example: bouncing ball. It is not always obvious what the solution is going to look like.

- Fixed step solvers are required for real time simulation.
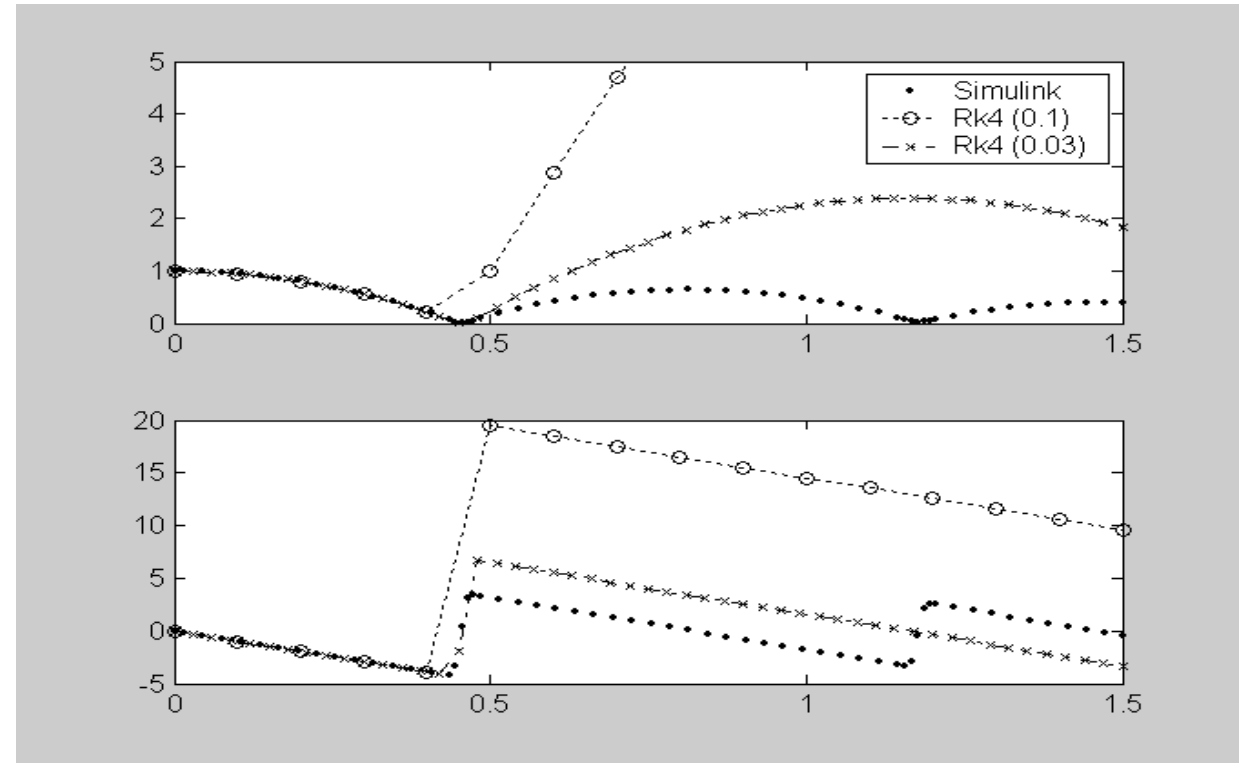
Tutorial

- See the exercise at the end of the notes in Section 2: Ordinary Differential Equations