Vehicle Dynamics and Simulation

Modelling and Simulation: Linearity and State Space

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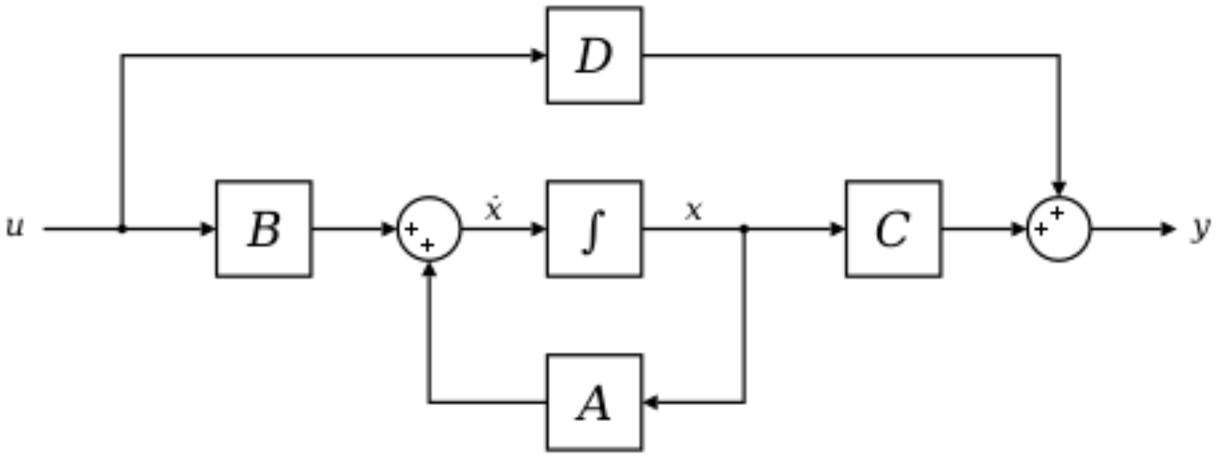
Lecture overview

- Linear systems
- State-space representation



Linear Systems

- In mathematics a system is said to be linear if;

$$y = ax$$

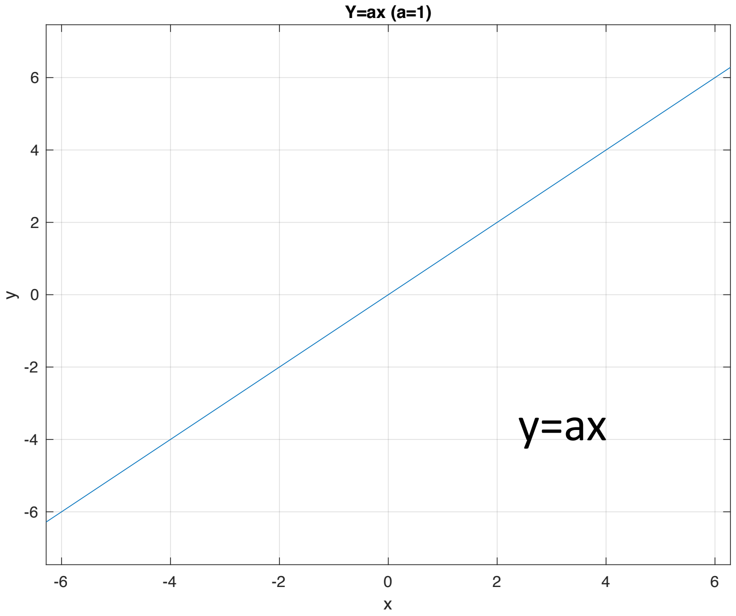
Or more generally

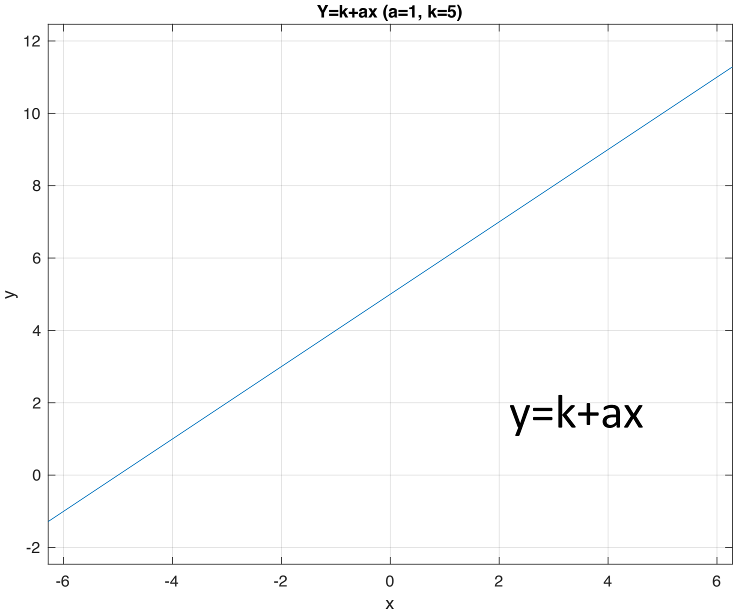
$$y = ax_1 + bx_2 + cx_3$$

- Technically, including a constant term makes the system non-linear;

$$y = k + ax_1 + bx_2 + cx_3$$

- This is known as an affine function (system) and can be dealt with in much the same way as a truly linear system.

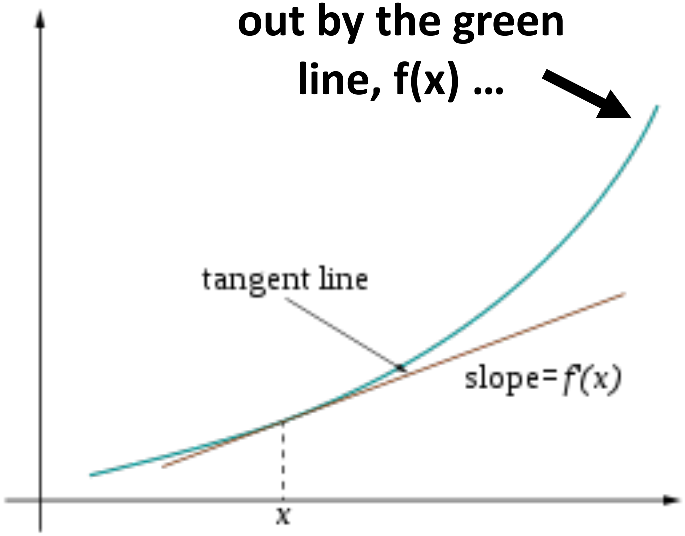




Linear Systems

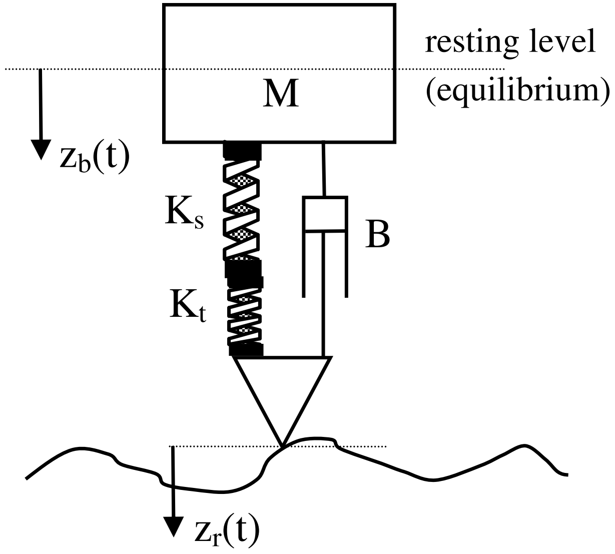
- Linear systems are important since the mathematics around them is more fully developed.
- There are many open questions around non-linear systems including finding of more general solutions.
- Most non-linear systems can be linearized (around a point) which provides a means of making some intractable nonlinear system problems tractable.

correction…traced out by the green line, f(x) …



Linearisation around x for f(x). Note that the approximation is good close to x but less good as one moves away from x to f(x+h) for example.

Linear or non-linear??



System Equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{K}{M}(x_3 - x_2) + \frac{B_s}{M}(u - x_2)$$

$$\dot{x}_3 = u$$

Linear Systems

- The mathematical description of the system states is written as a set of n coupled first order equations;

$$\begin{aligned}\dot{x}_1 &= f_1(\boldsymbol{x}, \boldsymbol{u}, t)\\ &\vdots\\ \dot{x}_n &= f_n(\boldsymbol{x}, \boldsymbol{u}, t)\end{aligned}$$

- If we restrict our type of system to those that are (very nearly) linear and time invariant, we can write the system state change i.e. *state equations* as a linear combination of inputs, u_i and states, x_i

$$\begin{aligned}\dot{x}_1(t) &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + \cdots + b_{1r}u_r\\ &\vdots\\ \dot{x}_n(t) &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}u_1 + \cdots + b_{nr}u_r\end{aligned}$$

Where n is the number of states and r the number of inputs.

State Space Representation

- Writing in matrix form;

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nr} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}$$

$$or$$

$$\boxed{\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}}$$

This tells us how the states, $\dot{\boldsymbol{x}}$ changes but not what the outputs are.

State Space Representation

- The states don’t necessarily include information necessary for engineering purposes.
- In addition states can be arbitrarily chosen and therefore may not represent anything physically meaningful.
- An output variable (arbitrary) can be written as linear combination of states x_i and inputs u_i;

$$y_1(t) = c_1x_1 + C_1x_2 + \cdots + c_nx_n + d_1u_1 + \cdots + d_ru_r$$

State Space Representation

- If we are interested in m output variables, we can write m equations as;

$$y_1(t) = c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n + d_{11}u_1 + \cdots + d_{1r}u_r$$
$$\vdots$$
$$y_m(t) = c_{m1}x_1 + c_{m2}x_2 + \cdots + c_{mn}x_n + d_{m1}u_1 + \cdots + d_{mr}u_r$$

- Or in matrix form;

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1r} \\ d_{21} & d_{22} & \cdots & d_{2r} \\ \vdots & \vdots & \cdots & \vdots \\ d_{m1} & d_{m2} & \cdots & d_{mr} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}$$

State space representation

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}$$

- Note how the output $\boldsymbol{y}$ (a vector) is a combination of inputs and current state after a transformation (multiplication by $\boldsymbol{C}$ and $\boldsymbol{D}$ respectively)
- For many real systems $\boldsymbol{D}\boldsymbol{u}$ is not necessary, so;

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$$

- The output is just the states multiplied by some vector $\boldsymbol{C}$ (a property of the system)

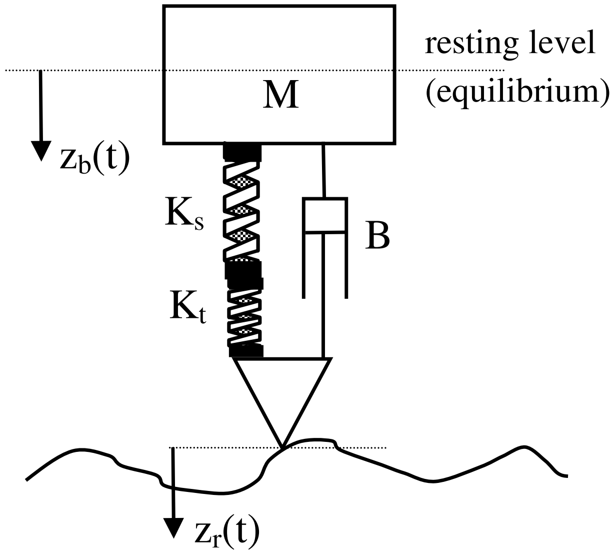
* Matrix transformation

Conclusion

- Linear systems
- State space representation

Tutorial

Tutorial - State Space Representation



Sign convention: +ve direction indicated by arrow heads

- Try for yourself;
  - Spring stiffness; $\frac{1}{K} = \frac{1}{K} + \frac{1}{K}$

$$F_s = \cdots$$