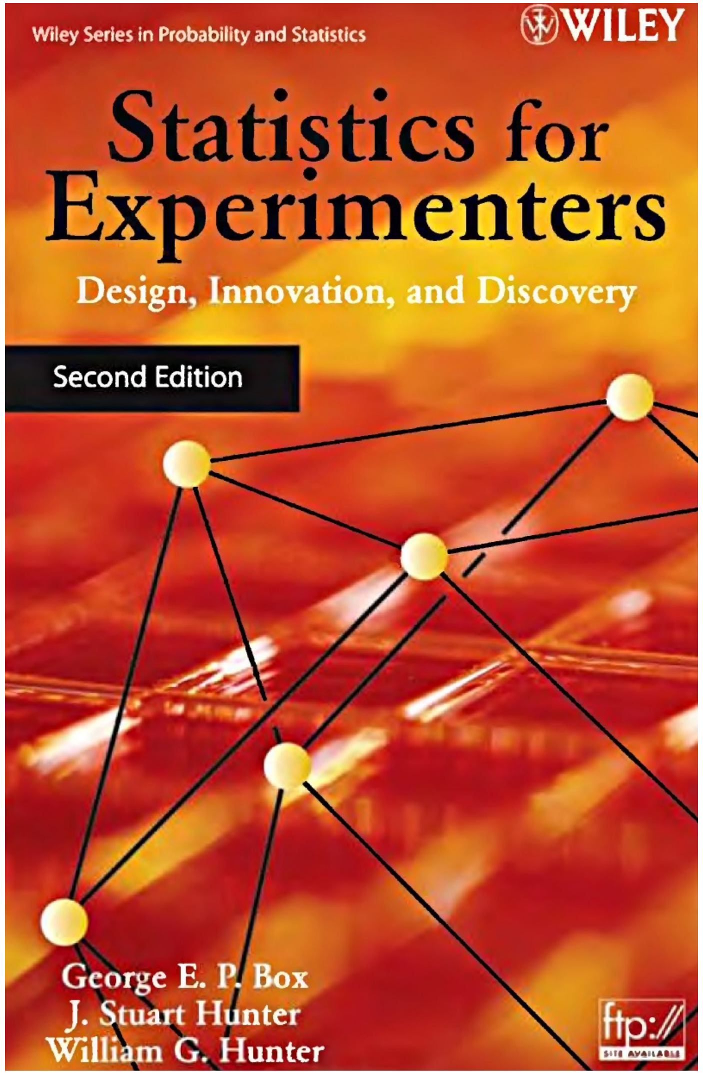
# Powertrain Calibration Optimisation

## Introduction to Statistics

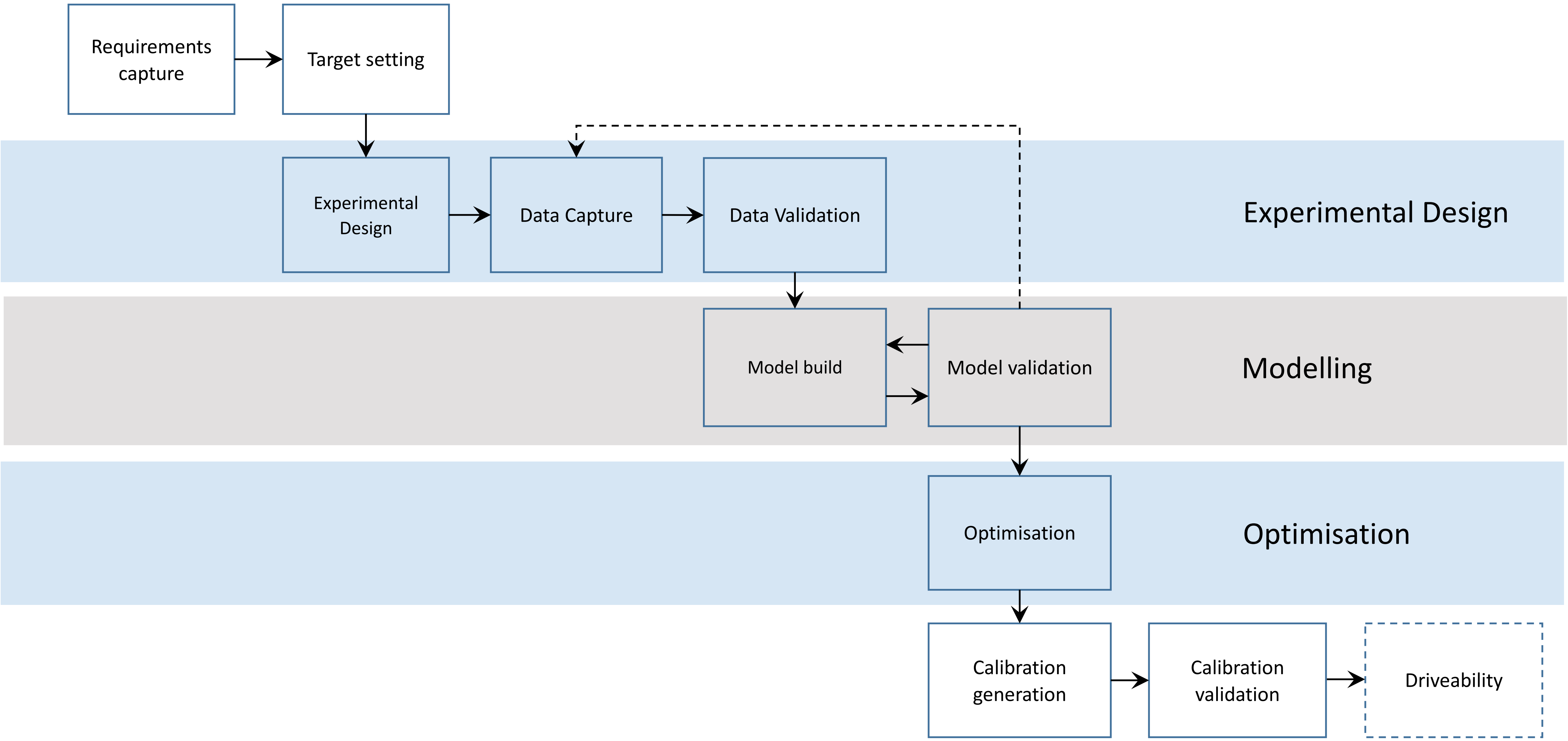
# Overview

- Process overview
- Basic concepts
- Continuous distributions
- Estimation
- Significance tests
- Regression
- ANOVA

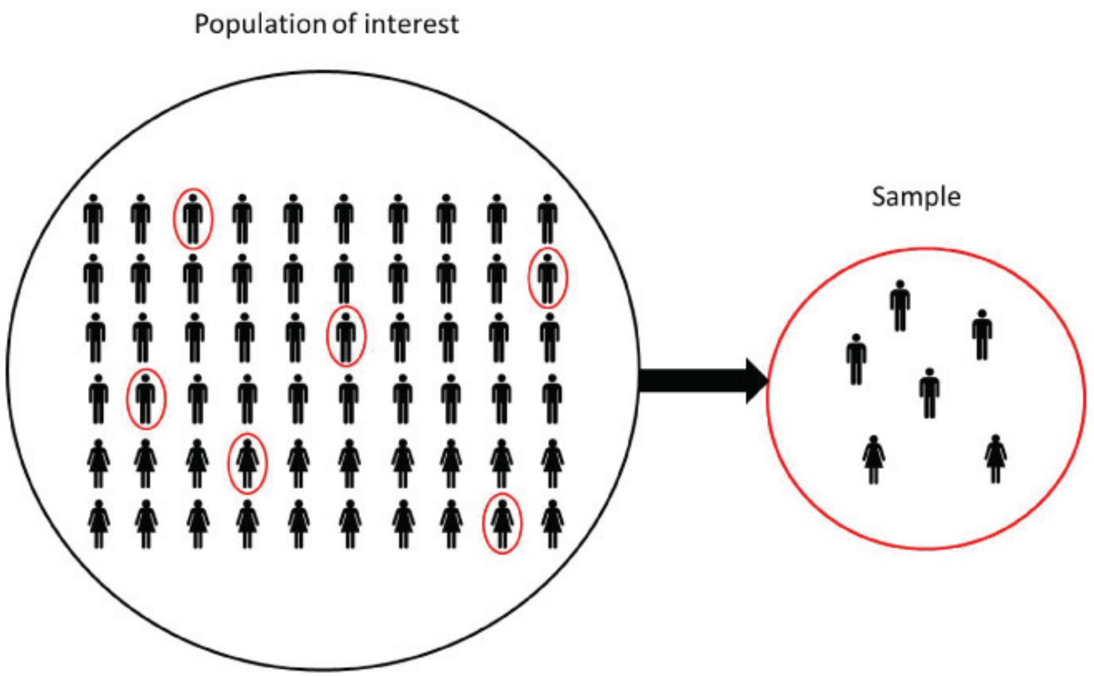

Wiley Series in Probability and Statistics
WILEY
Statistics for Experimenters
Design, Innovation, and Discovery
Second Edition
George E. P. Box
J. Stuart Hunter
William G. Hunter
ftp://

Powertrain Calibration Optimisation

# High Level Overview



# Population vs sample




As the sample size, n increases, the sample becomes more representative of the population from which it is drawn.

# Definition - Degrees of Freedom

- How many choices?
- Degrees of freedom* relate to the number of 'observations' that are free to vary when estimating statistical parameters

$$mean = \frac{x_1 + x_2 + \cdots x_n}{n}$$

In calculating the mean only $n-1$ observations are 'free to vary'

* we also talk about control degrees of freedom which is the control inputs that we can change to modify the behaviour of the system.

# Making measurements – location and spread

Mean, $\bar{x} = \frac{x_1+x_2+\cdots+x_N}{N} = \sum_N \frac{x_i}{N}$

Also known as the expectation of $x$ i.e. $E(x)$.

Variance, $s^2$

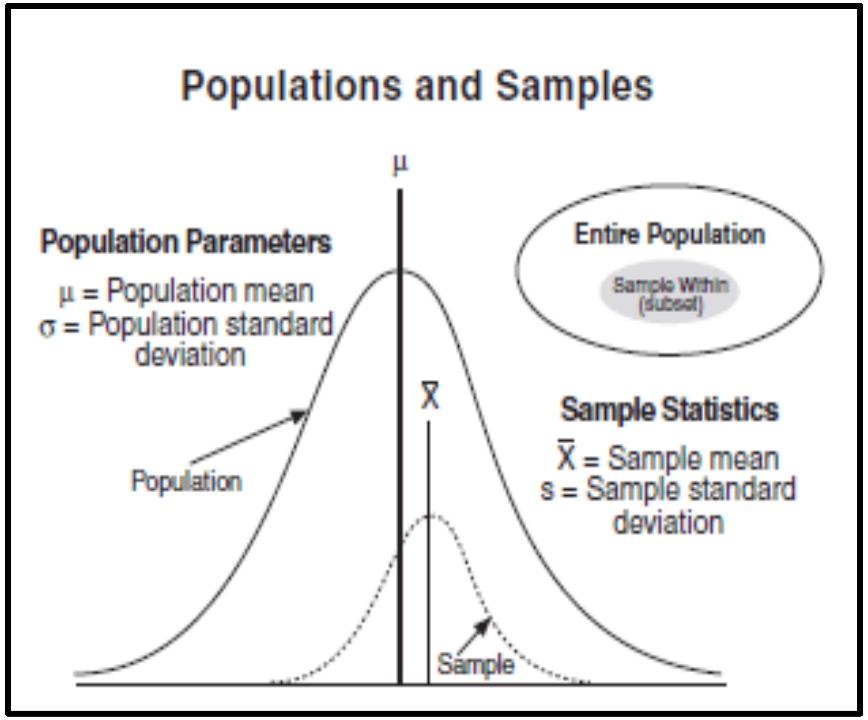
$$s^2 = \frac{(x_1-\bar{x})^2 + (x_2-\bar{x})^2 + \cdots + (x_N-\bar{x})^2}{N-1} = \sum_N \frac{(x_i-\bar{x})^2}{N-1}$$

Why $N-1$?

Standard deviation, $s$,

$$s = \sqrt{s^2}$$

What are the units of $s, s^2, \sigma, \sigma^2$?

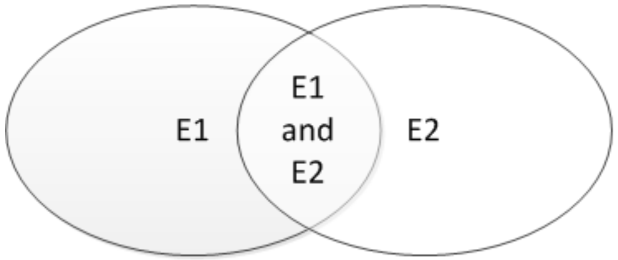


# Probability

Probability: *the extent to which an event is likely to occur, measured by the ratio of the cases of interest to the whole number of cases possible*

Sample space: *the set of all possible outcomes of an experiment.*

Events:



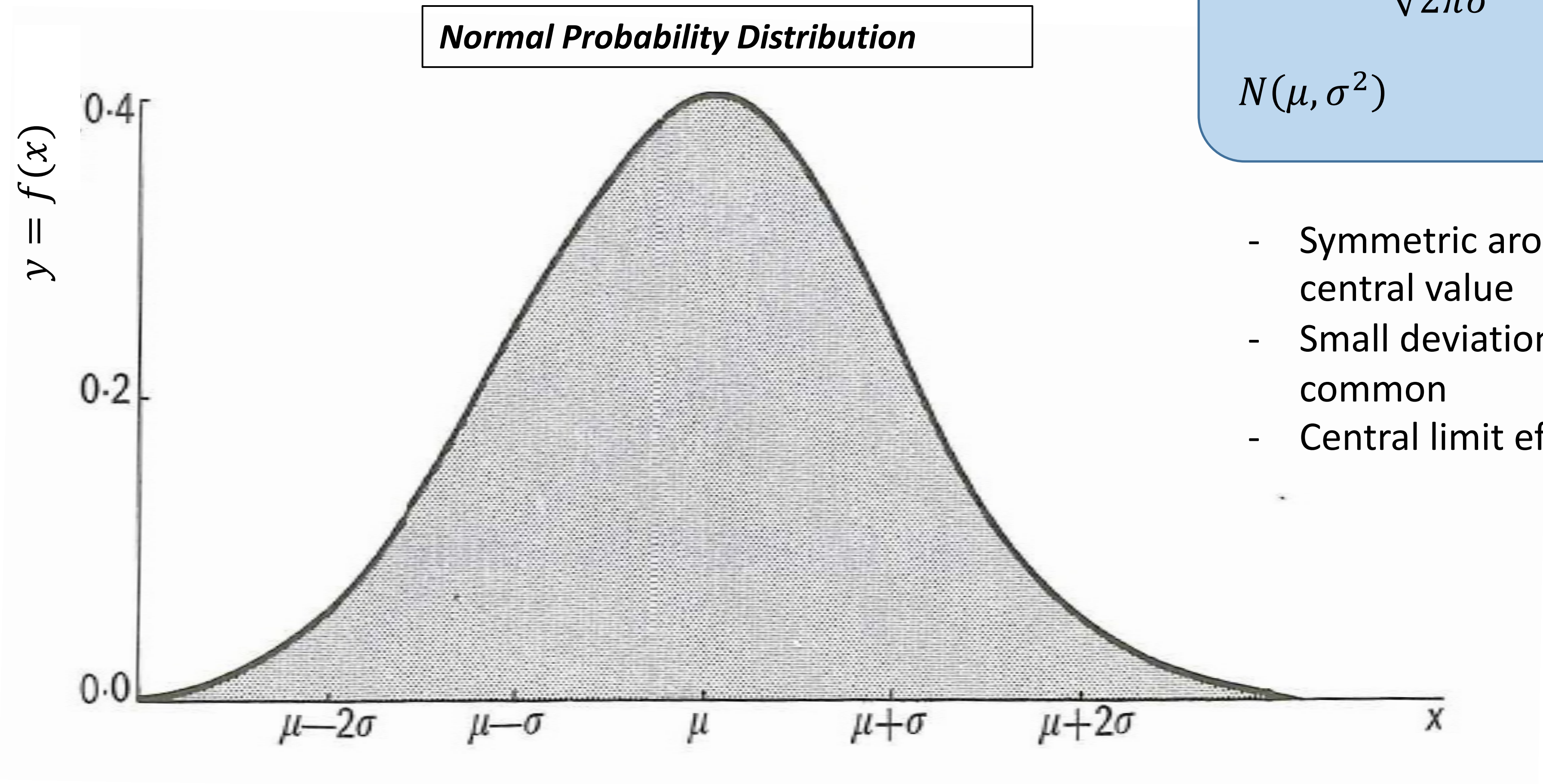


Probabilities: $p(E_1), p(E_2), p(E_1E_2)$

$$P(E_1|E_2) = \frac{\sum_{E_1E_2} P(Sample\ points\ common\ to\ E_1 and\ E_2)}{\sum_{E_2} P(Sample\ points\ in\ E_2)} = \frac{P(E_1E_2)}{P(E_2)}$$

*“$E_2$ has already happened.” What is the probability of E1?*

# Probability distributions

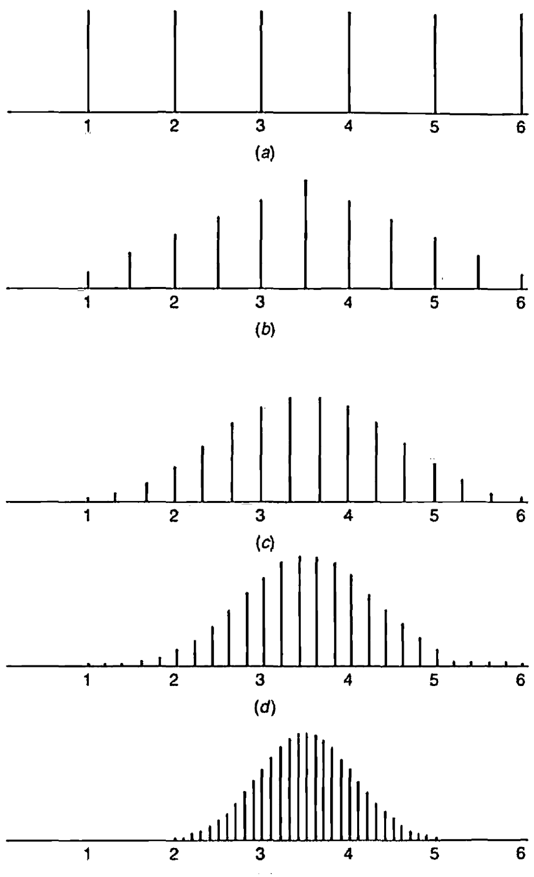



$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)^2 / 2\sigma^2}$$

$N(\mu, \sigma^2)$

- Symmetric around some central value
- Small deviations more common
- Central limit effect

# Central limit effect




<u>Average scores of (100 rolls)</u>

One die

Two die

Three die

Five die

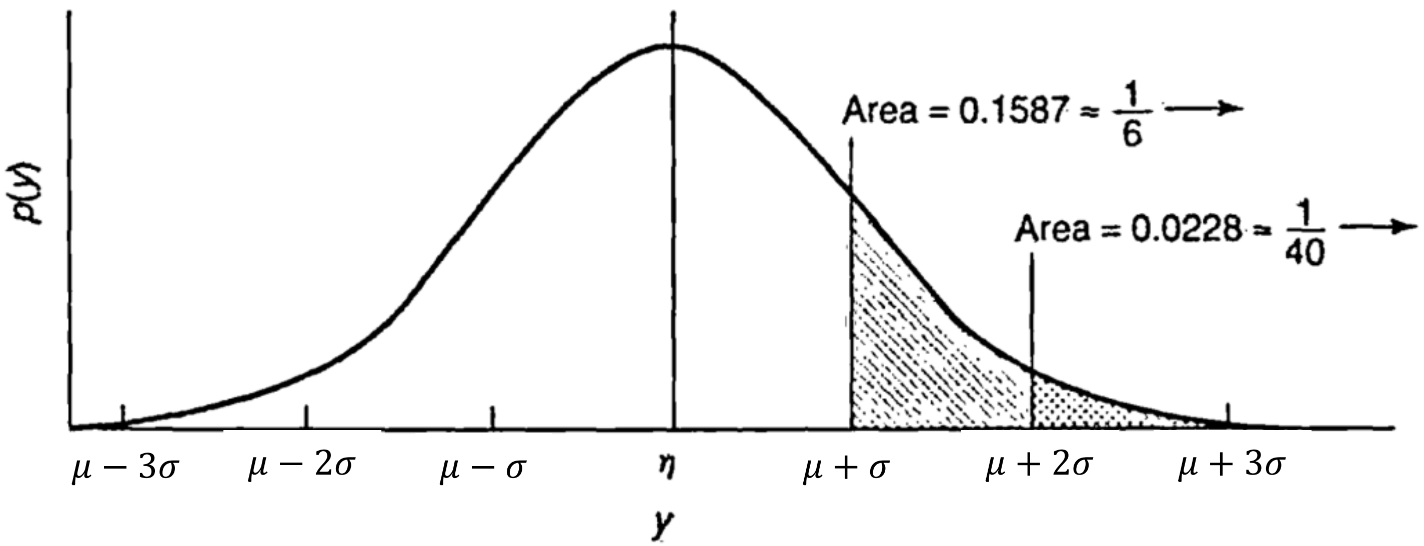
Ten die

In many experiments the error is an aggregate of a number of component errors and the distribution will tend to be "normal".

This is important since it is true for 'many' experiments.

# Probability

$\mu$ and $\sigma^2$ fully characterise a normal distribution, $N(\mu, \sigma^2)$




**Probability density** is given by a point on the line p(y)

$p(y > \mu + \sigma) = \frac{1}{6}$ i.e. the area under the curve.

Often it is easier to express probability in terms of the standard deviate;

$$z = \frac{y - \mu}{\sigma}$$

$$z(0, 1)$$

i.e. z has a mean of 0 and variance, $s^2 = 1$. So that;

$$= p(y > \mu + \sigma)$$

$$= p(y - \mu > \sigma)$$

$$= p\left(\frac{y-\mu}{\sigma} > 1\right)$$

$$= p(z > 1)$$

(which can be easily found from tables)

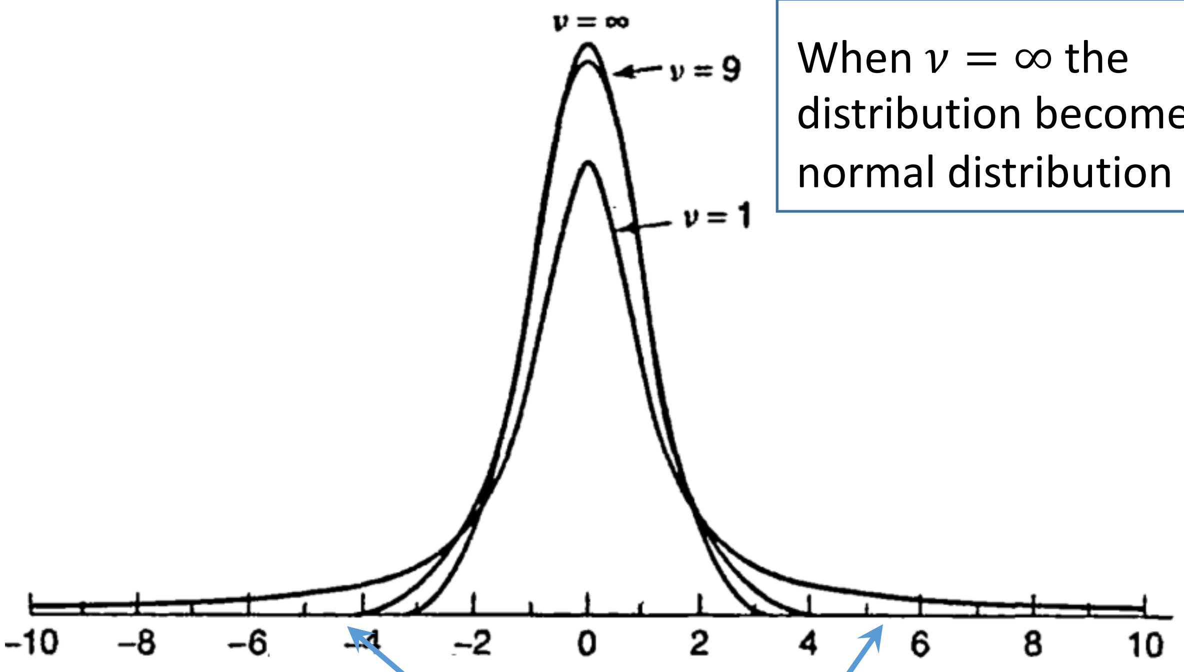
# If $\sigma$ is unknown (which is normally the case)

A substitution can be made for $\sigma$ *using* $s$, the sample standard deviation;

$$z = \frac{y - \mu}{\sigma}$$

i.e.

$$t = \frac{y - \mu}{s}$$

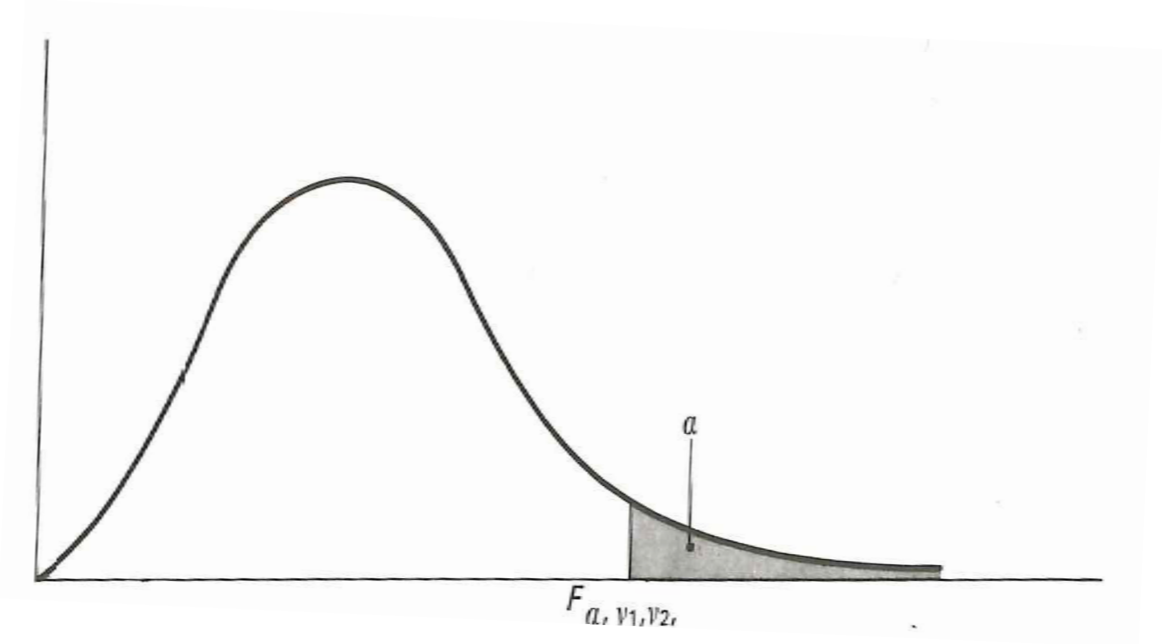



When $\nu = \infty$ the distribution becomes the normal distribution

Fatter tail compared with normal distribution

the 'student' or 't' distribution depends on degrees of freedom available for estimation of $s$.

# F-distribution




<u>**Can obtain ratio of two sample variances;**</u>

$F$ statistic is ${s_1}^2/{s_2}^2$

$F$ depends on the estimates and the DOF of the variance estimates

Degrees of freedom of population variances;

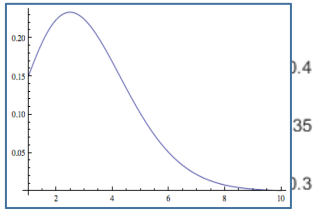
$\nu_1 = n_1 - 1$
$\nu_2 = n_2 - 1$

So F test statistic is designated

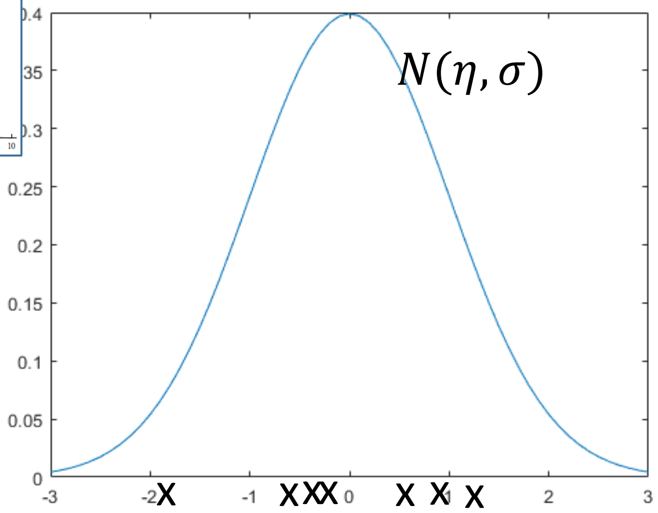
$F_{\nu_1,\nu_2}$

# Standard Error of the Mean

- Take $n$ random samples from a normal distribution with mean, $\mu$ and standard deviation, $\sigma$. Calculate the sample $\bar{x}$ and $s$. Repeat.
- The sample means will form a distribution with the same mean, $\mu$ but a <u>**smaller standard deviation**</u> $\sigma/\sqrt{n}$ (the **standard error of the sample mean**).



‘Normal like’




For a sample of size $n$ the sample mean is $\bar{x}$

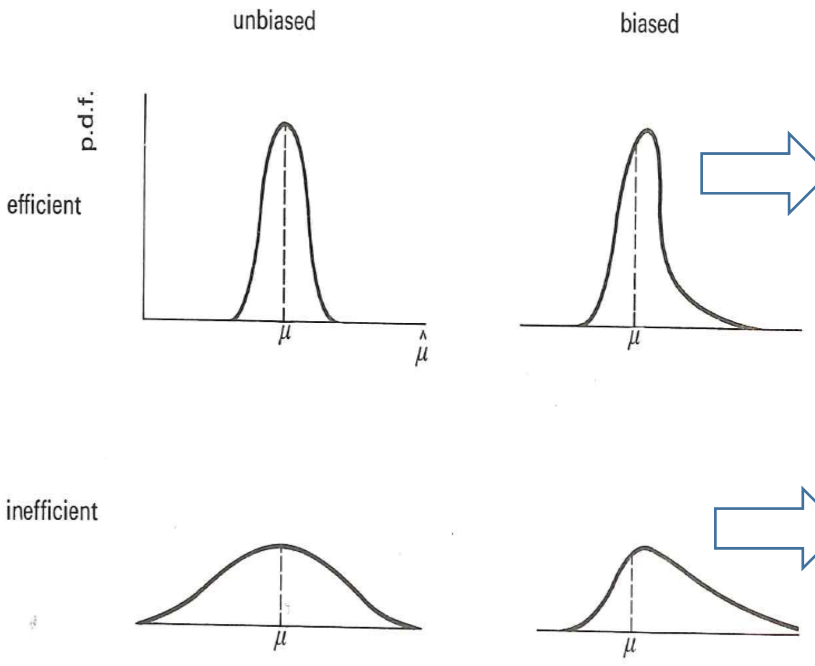
The standard error is an estimate of the standard deviation of the sample means for sample size $n$

$$SE_m = \frac{\sigma}{\sqrt{n}}$$

**Intuitively it is a measure of how sample size affects the dispersion of sample means relative to the population mean.**

# Bias and efficiency

- **Bias** – an estimator is said to be biased if the mean of its sampling distribution is not equal to the value it is estimating.
- **Efficiency** – an efficient unbiased estimator is the minimum variance unbiased estimator (MVUE).




$bias(\hat{\mu}) = \mathrm{E}(\mu) - \mu$

$var(\hat{\mu}_1) < var(\hat{\mu}_2)$

more efficient estimator $\hat{\mu}_1$

# Significance testing

## Testing a theory about the population

| | |
|---|---|
| Null hypothesis | $H_0$ |
| Alternative hypothesis | $H_1$ |

What we are testing ….

An alternative ….

- Test statistic
- Level of significance
- One tailed and two tailed tests

# How to know if a treatment is significant?

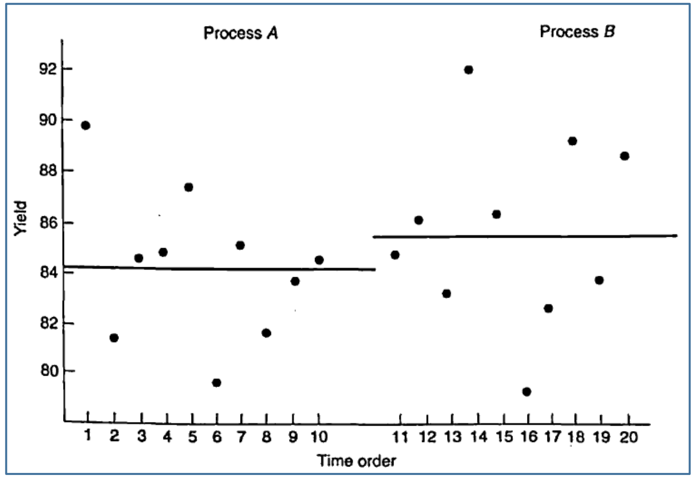
Batch Process A

Batch Process B

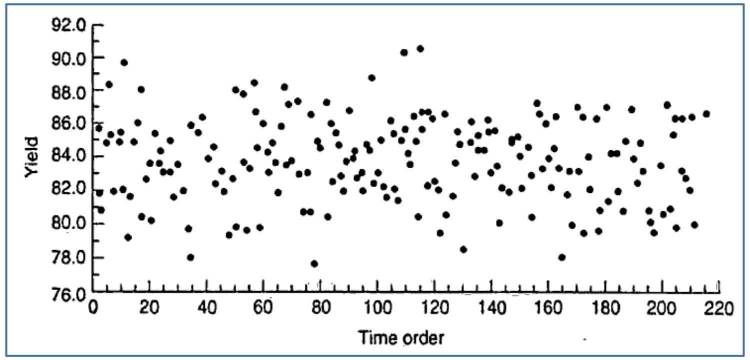
$\bar{y}_a = 84.24$

Difference 1.3

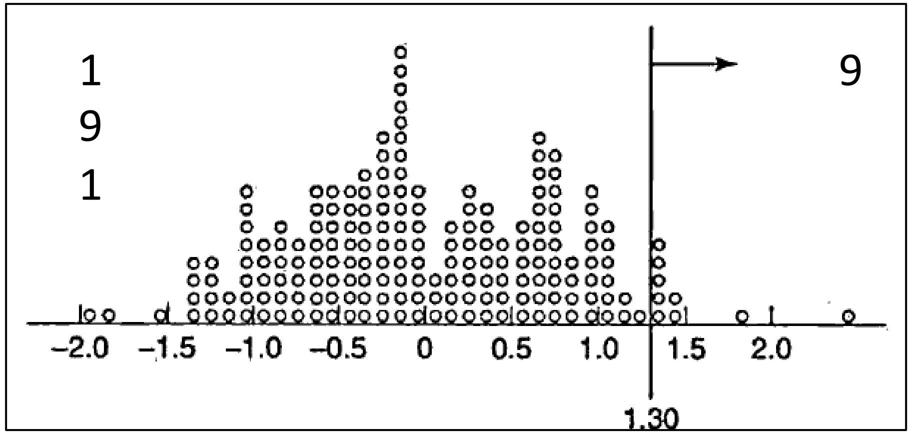
$\bar{y}_b = 85.54$




Previous yield data




$$\frac{9}{191} = 0.047$$



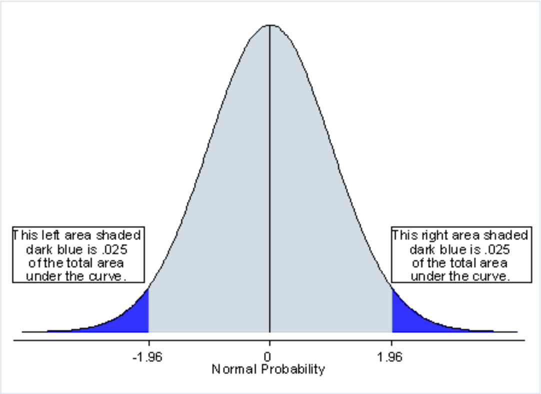

Difference in means of sequential batches

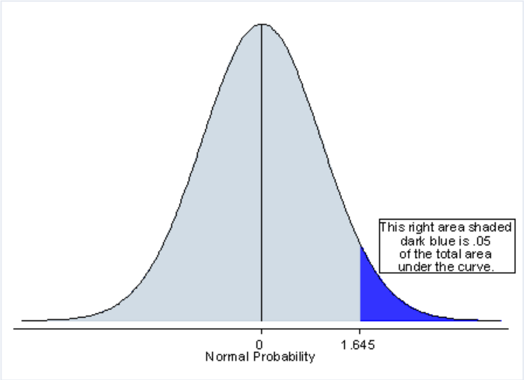
## An example - composition of a chemical compound

- The iron content of a compound should be 12.1%. Tests on nine different samples are being used to examine this assumption.

- Null hypothesis i.e. there is no difference in the sample $(n = 9)$ mean
  - $H_0: \mu = 12.1\%$

- Alternative hypothesis
  - $H_1: \mu \neq 12.1\%$



# Example (continued)

The analysis of nine samples gave the following values for % content of iron.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 11.7 | 12.2 | 10.9 | 11.4 | 11.3 | 12.0 | 11.1 | 10.7 | 11.6 |

$$\bar{y} = 11.43$$
$$s^2 = 0.24$$
$$s = 0.49$$

$$t = \frac{(\bar{y} - \eta)}{s/\sqrt{n}} = \frac{(11.43 - 12.1)}{0.49/\sqrt{9}} = -4.1$$

Standard deviation of the mean (estimate)

Degrees of freedom: eight because nine samples and one DoF used for population mean, $\bar{x}$

## Example (continued)

1. Two tailed test
2. 5% level of significance
3. Eight degrees of freedom (from tables)

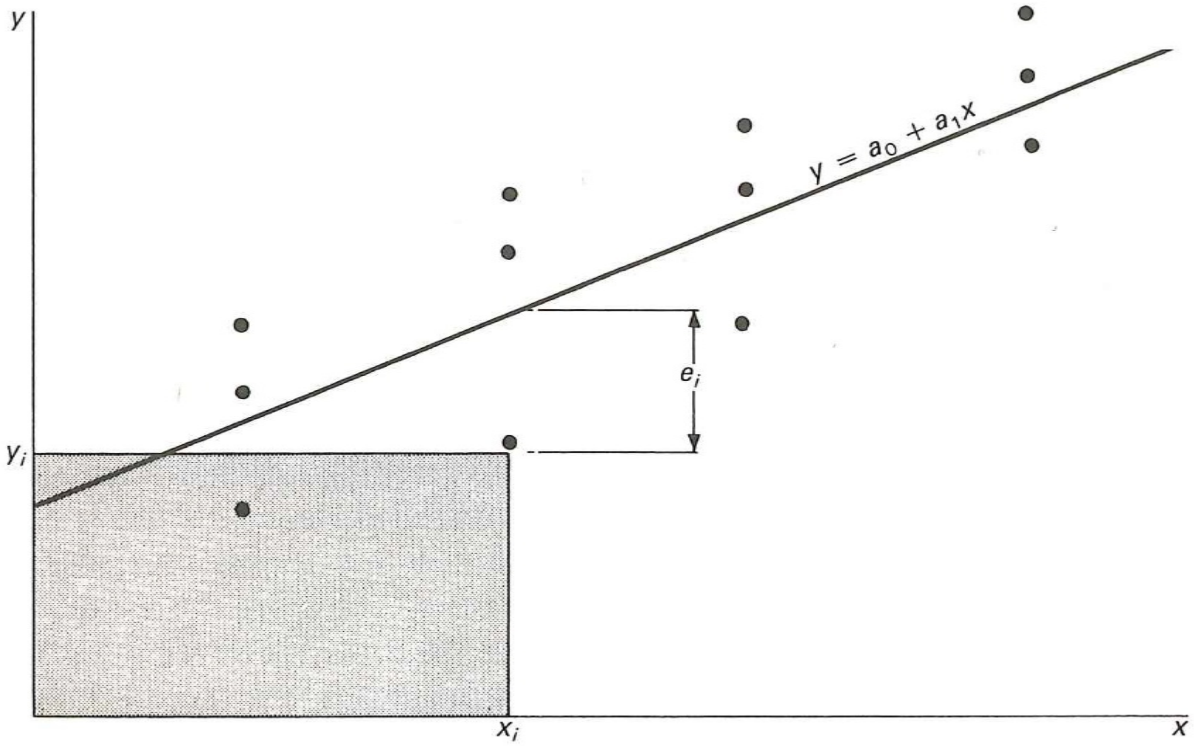
Test statistic is designated: $t_{0.025,8} = 2.31$ (from tables)

In fact, $t_{0.005,8} = 3.36$ (from tables)

So even at the 1% level, the result is significant.

# Regression

Fitting a line or curve to the data in order to predict the mean value of the dependent variable for a given value of the controlled variable

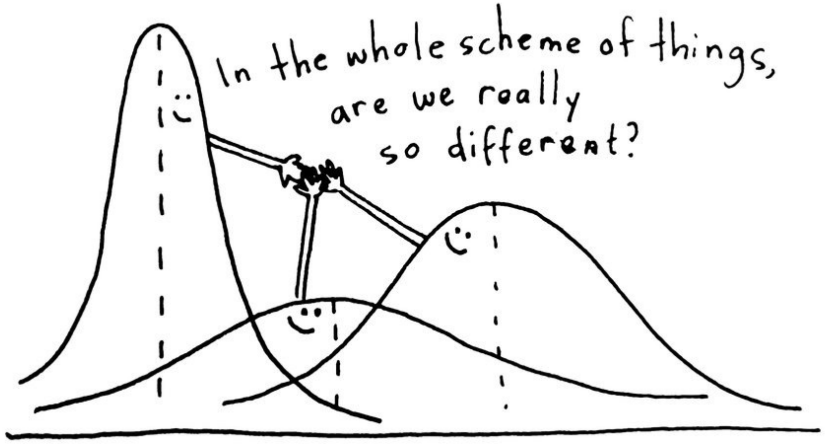



Model = $f(x, \beta)$

$$\min\left(\sum y_i - f(x_i, \beta)\right)$$

# Analysing variance (ANOVA)
## For comparing more than two entities

| | A | B | C | D |
|---|---|---|---|---|
| | 62 | 63 | 68 | 56 |
| | 60 | 67 | 66 | 62 |
| | 63 | 71 | 71 | 60 |
| | 59 | 64 | 67 | 61 |
| | 63 | 65 | 68 | 63 |
| | 59 | 66 | 68 | 64 |
| **Group avg** | **61** | **66** | **68** | **61** |
| **Overall avg** | **64** | **64** | **64** | **64** |



## ANOVA

- Review the topic and evaluate the data on the previous slide.

Deviation from overall average

Deviations **between** treatments

deviations **within** treatments

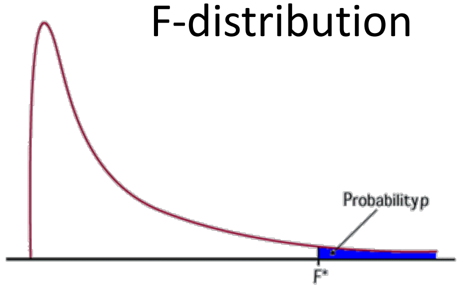
| | A | B | C | D | $y_{ti} - \bar{y}$ | $\bar{y}_t - \bar{y}$ | $y_{ti} - \bar{y}_t$ |
|---|---|---|---|---|---|---|---|
| $y_{ti}$ | 62 | 63 | 68 | 56 | -2 -1 4 -8 | -3 2 4 -3 | 1 -3 0 -5 |
| | 60 | 67 | 66 | 62 | -4 3 2 -2 | -3 2 4 -3 | -1 1 -2 1 |
| | 63 | 71 | 71 | 60 | -1 7 7 -4 | -3 2 4 -3 | 2 5 3 -1 |
| | 59 | 64 | 67 | 61 | -5 0 3 -3 | -3 2 4 -3 | -2 -2 -1 0 |
| | 63 | 65 | 68 | 63 | -1 1 4 -1 | -3 2 4 -3 | 2 -1 0 2 |
| | 59 | 66 | 68 | 64 | 5 2 4 0 | -3 2 4 -3 | -2 0 0 3 |
| **Sum of squares** | | | | | **340** | **228** | **112** |
| **Degrees of freedom** | | | | | **23** | **3** | **20** |

$y_{ti}$ individual results
$\bar{y}_t$ treatment average
$\bar{y}$ overall average

## ANOVA Table

| Source of variation | Sum of squares | d.f. | $\frac{\chi^2}{\nu}$ |
|---|---|---|---|
| Between treatments | $\sum(\bar{y}_t - \bar{y})^2 = 228$ | $n - 1 = 3$ | $\frac{\sum(\bar{y}_t - \bar{y})^2}{n-1} = 76$ |
| Within treatments | $\sum(y_{ti} - \bar{y}_t)^2 = 112$ | $n - 1 = 20$ | $\frac{\sum(y_{ti} - \bar{y}_t)^2}{n-1} = 5.6$ |
| **Total about the overall average** | **340** | **23** | |

F-distribution




$$F_{\nu_1,\nu_2} = \left.\frac{\sum(\bar{y}_t - \bar{y})^2}{n-1}\middle/\frac{\sum(y_{ti} - \bar{y}_t)^2}{n-1}\right.$$

$$F_{3,20} = 13.6$$

Significant at 0.001 i.e. we can be confident that treatments do result in different means, we can reject $H_0$