SYNTHETIC APERTURE RADAR AUTOFOCUS VIA SEMIDEFINITE RELAXATION

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ABSTRACT

Synthetic Aperture Radar (SAR) imaging can suffer from image focus degradation due to unknown platform or target motion. Autofocus algorithms use signal processing techniques to remove the undesired phase errors. The recently proposed multichannel autofocus models formulate the problem as the solution to $\mathbf{A}e^{j\boldsymbol{\phi}} \approx \mathbf{0}$, where $\mathbf{A}$ is a given matrix and $\boldsymbol{\phi}$ are the unknown phases. Previous methods approximated $e^{j\boldsymbol{\phi}}$ using the null vector of $\mathbf{A}$. We propose to approximate $e^{j\boldsymbol{\phi}}$ using conic optimization and call this new autofocus algorithm Semidefinite Relaxation Autofocus (SDRA). Experimental results using a simulated SAR image shows that SDRA has promising performance advantages over existing autofocus methods.

***Index Terms*—** Synthetic aperture radar, Autofocus, Fourier-domain multichannel autofocus, Wide-angle SAR, Semidefinite relaxation.

1. INTRODUCTION

Synthetic aperture radar (SAR) is a means for producing high resolution images using an antenna of small size. In spotlight-mode SAR, the collected signals can be viewed as data lying on a polar annulus in the *Fourier domain* of the target reflectivity. A popular image reconstruction method is to first interpolate the collected Fourier data on the polar annulus to a Cartesian grid and then use a 2-D inverse Fourier transform to form the final image [1]. However, due to unknown signal delays resulting from inaccurate range measurements or signal propagation effects, there may be a demodulation timing error at the radar receiver. This will produce unknown phase errors in the collected Fourier data and can cause the reconstructed image to suffer severe distortion [1]. Autofocus algorithms apply sophisticated signal processing techniques to remove the undesired phase errors.

Most of the work on SAR autofocus algorithms is developed based on an 1-D phase error model that is accurate only when the radar's range of look angles is sufficiently small [1]. These approaches include Phase Gradient Autofocus (PGA) [2], a standard autofocus algorithm used in practice, and Multichannel Autofocus (MCA) [3]. Some other autofocus algorithms compensate the phase errors by maximizing the sharpness of the reconstructed image [4]. A popular yet heuristic sharpness measure is image entropy, which depends on the characteristics of the underlying scene [4]. A recently developed autofocus algorithm, Fourier-domain Multichannel Autofocus (FMCA) [5], does not depend on the 1-D phase error model and has been shown to provide superior image restoration capability as compared to PGA and MCA, especially when the range of look angles is large. The performance of MCA and FMCA does not depend on the characteristics of the image. Instead, they require prior knowledge of a region in the image that has pixels of zero or nearly zero value. Such a region can be identified within the low-return region of the antenna pattern. Mathematically, multichannel autofocus models formulate the autofocus problem as the solution to $\mathbf{A}e^{j\boldsymbol{\phi}} \approx \mathbf{0}$, where $\mathbf{A}$ is a given matrix and $\boldsymbol{\phi}$ are the unknown phases. Previous works approximated the solution to this problem using a simple heuristic [5].

In practice, the low-return region may deviate from zero due to bright reflectors in the antenna sidelobes. In this paper, we propose a new autofocus algorithm aimed at finding the unknown phases while minimizing the energy in the presumed low-return region. We call the new algorithm Semidefinite Relaxation Autofocus (SDRA). SDRA can be applied to MCA or FMCA and inherits all the advantages of MCA and FMCA. It uses the semidefinite relaxation (SDR) technique to estimate the phase errors. SDRA provides robust phase estimates at the cost of complicating the underlying optimization problem.

SDR has recently been applied to many problems in communications and signal processing [6][7][8][9]. SDR is an attempt to approximate a quadratic problem with quadratic constraints using a convex optimization problem by first lifting the problem to a higher dimension and then relaxing the nonconvex constraints. To the best of our knowledge, this is the first time that SDR has been applied to the problem of autofocus. Our SDRA problem formulation is similar to the discrete symbol detection problem in communication systems [7]. However, SDRA employs unit absolute value constraints on the variables, rather than constraining the variables to lie in a discrete set. Some theoretical results on the performance of SDR for the continuous symbol case are given in [10]. At the cost of solving a high-dimensional convex optimization problem, SDRA shows promising image restoration ability.

The organization of this paper is as follows. Section 2 presents the autofocus problem and discusses previous solutions. In Section 3 we presents the proposed autofocus algorithm SDRA. Section 4 compares SDRA with previous methods. Simulation results are given in Section 5. Finally, we summarize our work in Section 6.

In this paper, a superscript ^{H} denotes the hermitian transpose. The function $vec(\mathbf{A})$ stacks the columns of matrix $\mathbf{A}$ to produce a column vector. For a complex-valued vector $\mathbf{a}$, the function $\angle(\mathbf{a})$ retains the angular part of $\mathbf{a}$. We write $\mathbf{A} \succeq 0$ to indicate that $\mathbf{A}$ is a positive semidefinite matrix, and let $\mathrm{tr}(\mathbf{A})$ denote the trace of $\mathbf{A}$.

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2. PROBLEM FORMULATION

In this section we briefly review the tomographic formulation of SAR and refer the reader to [1] for more details. We formulate the multichannel autofocus model and state the problem. Due to lack of space, we only describe the FMCA model; however, the MCA model can be derived in a similar manner.

Suppose a target scene, represented by a complex-valued reflectivity function $g(x, y)$, is illuminated by SAR over a range of viewing angles, $\theta_m : \theta_{min} \leq \theta_m \leq \theta_{max}, m = 0, 1, \ldots, M-1$. The signal collected from look angle θ_m is a "slice" of the 2-D Fourier transform of $g(x, y)$, taken at angle θ_m. These signals are then collected into a matrix $G_p[n, m]$, which is the polar-format Fourier transform of $g(x, y)$. To reconstruct the image, it is standard to first interpolate $G_p[n, m]$ to a Cartesian grid $G_c[k, l]$ and then apply an inverse Fourier transform. We require a linear polar-to-Cartesian interpolation to preserve the linear structure for our problem. In practice, linear interpolation is almost always used, e.g. [1]. Using linear interpolation, the Cartesian grid data $G_c[k, l]$ can be expressed as

$$G_c[k, l] = \sum_{n,m} \alpha(k, l, n, m) G_p[n, m], \tag{1}$$

where $\alpha(k, l, n, m)$ are the interpolation coefficients. For example, a nearest-neighbor interpolation corresponds to $\alpha(k, l, n, m) = 1$ when $G_p[n, m]$ is closest to $G_c[k, l]$ and 0 otherwise. The image $g(x, y)$ is then reconstructed by using an inverse discrete Fourier transform,

$$g(x, y) = \frac{1}{KL} \sum_{k,l} G_c[k, l] e^{j2\pi\left(\frac{xk}{K} + \frac{yl}{L}\right)}. \tag{2}$$

In practice, due to unknown signal delays resulting from inaccurate range measurements or signal propagation effects, the collected Fourier data is corrupted by an unknown phase error function. The phase errors vary with look angle, but are modeled as constant for all data from a fixed angle. This results in

$$\tilde{G}_p[n, m] = G_p[n, m]\, e^{j\phi(m)}, \tag{3}$$

where $\phi(m) \in \mathbb{R}, m = 0, 1, \ldots, M-1$ are unknown [1]. The autofocus problem is to estimate $\phi(m)$ given the data $\tilde{G}_p[n, m]$ and assumptions about the imaging scenario.

In particular, the FMCA reconstruction method assumes there exists a known region in $g(x, y)$ with low valued pixel. This prior knowledge can be inferred by using the low-return region of the antenna pattern [3]. Let $g(x_r, y_r),\ r = 0, 1, \ldots, R-1$ be the set of pixels that are nearly zero. Together with (1), (2) and (3), we obtain

$$\frac{1}{KL} \sum_{k,l} \sum_{n,m} \alpha(k, l, n, m) \tilde{G}_p[n, m]\, e^{-j\phi(m)} e^{j2\pi\left(\frac{x_r k}{K} + \frac{y_r l}{L}\right)} \approx 0 \tag{4}$$

for all $r = 0, 1, \ldots, R-1$. In matrix notation, (4) can be written as

$$\mathbf{A}\, e^{-j\boldsymbol{\phi}} \approx \mathbf{0} \tag{5}$$

where $\boldsymbol{\phi} = [\phi(0)\ \phi(1) \ldots \phi(M-1)]^T$ and $\mathbf{A}$ is an R by M matrix with elements

$$\mathbf{A}_{rm} = \frac{1}{KL} \sum_{k,l} \sum_{n} \alpha(k, l, n, m) \tilde{G}_p[n, m] e^{j2\pi\left(\frac{x_r k}{K} + \frac{y_r l}{L}\right)}, \tag{6}$$

where $0 \leq r \leq R-1, 0 \leq m \leq M-1$. Note that MCA is derived under a similar model but without considering the polar-to-Cartesian interpolation.

The SAR autofocus problem can be expressed as finding the phases $\boldsymbol{\phi}$ that satisfy (5). By requiring that the low-return region has minimum energy, this leads to the following optimization problem

$$(P) \qquad \boldsymbol{\phi} = \arg \min_{\boldsymbol{\phi} \in \mathbb{R}^M} \|\mathbf{A}\, e^{-j\boldsymbol{\phi}}\|_2. \tag{7}$$

Problem (P) is very difficult to solve. Therefore, previous attempts replace (5) with strict equality ($\mathbf{A}\, e^{-j\boldsymbol{\phi}} = \mathbf{0}$), and approximate $e^{-j\boldsymbol{\phi}}$ using the null vector of $\mathbf{A}$. In practice, it is highly likely that $\mathbf{A}$ may not have a null vector. So, in MCA and FMCA, the right singular vector associated with the minimum singular value of $\mathbf{A}$ [5] is used instead. It has been found from experiment that this approach fails to produce a reasonable image as the pixel magnitudes in the low return region become larger. This motivates us to study a potentially better phase estimation scheme.

3. SDRA ALGORITHM

SDRA offers an alternative way to approximate the solution to (P). For this purpose, we first lift the solution space of (P) from vectors to positive semidefinite matrices to obtain

$$\begin{array}{lll} (P0) & \min_{\mathbf{X} \in \mathbb{C}^{M \times M}} & \mathrm{tr}(\mathbf{Q}\mathbf{X}) \\ & \quad s.t & \mathbf{X}_{ii} = 1\ \ , i = 0, \ldots, M-1 \\ & & \mathbf{X} \succeq 0 \\ & & rank(\mathbf{X}) = 1, \end{array} \tag{8}$$

where $\mathbf{Q} = \mathbf{A}^H \mathbf{A}$. Note that (P) and $(P0)$ are equivalent since the solution to $(P0)$, $\mathbf{X}$, can be expressed as $\mathbf{X} = \mathbf{x}\mathbf{x}^H$ with $|\mathbf{x}_i| = 1,\ i = 0, \ldots, M-1$, i.e., $\mathbf{x} = e^{-j\boldsymbol{\phi}}$. Problem $(P0)$ has a nonconvex feasible set due to the rank 1 constraint and cannot be solved efficiently. Instead, we relax the feasible set of $(P0)$ to obtain the revised problem (SDR):

$$\begin{array}{lll} (SDR) & \min_{\mathbf{X} \in \mathbb{C}^{M \times M}} & \mathrm{tr}(\mathbf{Q}\mathbf{X}) \\ & \quad s.t & \mathbf{X}_{ii} = 1\ \ , i = 0, \ldots, M-1 \\ & & \mathbf{X} \succeq 0. \end{array} \tag{9}$$

The above optimization problem is a relaxation of $(P0)$ and is a semidefinite programming (SDP) problem. SDP is a conic optimization problem with a linear objective function and linear constraints over a positive semidefinite cone. The advantage of the conic optimization formulation is that an exact solution can be found in polynomial time using off-the-shelf optimization packages such as SeDuMi [11].

Let $\hat{\mathbf{X}}$ be the solution to (SDR). If $rank(\hat{\mathbf{X}}) = 1$ then $\hat{\mathbf{X}} = \hat{\mathbf{x}}\hat{\mathbf{x}}^H$ is an optimal solution for $(P0)$, and SDRA has solved the autofocus problem exactly. Otherwise, we can use $\hat{\mathbf{X}}$ to obtain a feasible approximate solution to $(P0)$. There are several methods we might employ. Here we focus on the randomization method [8]. Let $\hat{\mathbf{X}} = \hat{\mathbf{V}}\hat{\mathbf{V}}^H$ where $\hat{\mathbf{V}} = [\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_n]$ is a square-root factor of $\hat{\mathbf{X}}$. Because we relax the rank 1 constraint for $\hat{\mathbf{X}}$, n may be greater than 1. The randomization method generates M_{rand} complex vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{M_{rand}}$ that are uniformly distributed on an n-dimensional unit sphere. It then computes $\mathbf{y}_i = \angle(\hat{\mathbf{V}}\mathbf{u}_i), i = 1, \ldots, M_{rand}$ and approximates a feasible solution, $\mathbf{x}^*$, to $(P0)$ as

$$\mathbf{x}^* = \arg \min_{\mathbf{y}_1, \ldots, \mathbf{y}_{M_{rand}}} \mathbf{y}_i^H \mathbf{Q} \mathbf{y}_i. \tag{10}$$

The SDR technique can be summarized as,

1. Lifting and then relaxing the feasible set of the original problem to obtain a semidefinite program.
2. Solving the relaxed problem in polynomial time.
3. Applying a randomization technique to convert the relaxation solution to a feasible approximation to the original problem.

The complexity of solving (SDR) using an interior-point method (IP) is on the order of $M^{3.5}$ which makes applying (SDR) to a real SAR system challenging, where M is usually in the thousands. However, there is a large body of research on large-scale SDP methods [12].

4. COMPARISON TO PREVIOUS SOLUTION

Now that we have described the SDRA, it is interesting to compare SDRA with previous approaches which approximate $e^{-j\boldsymbol{\phi}}$ as the minimum eigenvector problem (EV). Using the variational characterization of singular values, these approaches may be interpreted as solving the following optimization problem.

$$(EV) \quad \begin{array}{ll} \min_{\mathbf{x}\in\mathbb{C}^M} & \mathbf{x}^H\mathbf{Q}\mathbf{x} \\ s.t & \mathbf{x}^H\mathbf{x} = M, \end{array} \tag{11}$$

where $\mathbf{Q} = \mathbf{A}^H\mathbf{A}$. As before, (EV) can be reexpressed as

$$(EV) \quad \begin{array}{ll} \min_{\mathbf{X}\in\mathbb{C}^{M\times M}} & \mathrm{tr}(\mathbf{QX}) \\ s.t & \mathrm{tr}(\mathbf{X}) = M \\ & \mathbf{X} \succeq 0 \\ & rank(\mathbf{X}) = 1. \end{array} \tag{12}$$

Problems $(P0)$, (SDR) and (EV) share similar form. Therefore, it is informative to compare them in terms of their objective function value.

Theorem 1: Let v_p^*, v_{sdr}^* and v_{ev}^* represents the optimal objective function values found for problems (P), (SDR) and (EV), respectively. Then

$$v_p^* \geq v_{sdr}^* \geq v_{ev}^*. \tag{13}$$

Proof. First, we have shown that (SDR) is a relaxation of $(P0)$ and $(P0)$ is equivalent to (P). This immediately gives us

$$v_p^* \geq v_{sdr}^*. \tag{14}$$

To show the second inequality, we note that (EV) is equivalent to the following problem:

$$(EVR) \quad \begin{array}{ll} \min_{\mathbf{X}\in\mathbb{C}^{M\times M}} & \mathrm{tr}(\mathbf{QX}) \\ s.t & \mathrm{tr}(\mathbf{X}) = M \\ & \mathbf{X} \succeq 0. \end{array} \tag{15}$$

This is easily proved by showing that (EVR) is both a lower bound and an upper bound for (EV); thus it is tight. On the other hand, (EVR) can be viewed as a relaxation of (SDR) by relaxing the $\mathbf{X}_{ii} = 1, \forall i$ constraint to $\mathrm{tr}(\mathbf{X}) = M$, and therefore

$$v_{sdr}^* \geq v_{evr}^* = v_{ev}^*, \tag{16}$$

where v_{evr}^* is the optimal objective function value of (EVR). □

Theorem 1 shows that in terms of objective function value, SDRA, which uses (SDR), is a tighter relaxation than (EV) which is used by MCA and FMCA. However, it is important to emphasize that it does not provide any guarantees on the quality of phase estimates.

From a computational complexity perspective, while (SDR) provides a tighter approximation to (P) than (EV), solving (SDR) using the IP method is more expensive than finding the minimum right singular vector of $\mathbf{A}$.

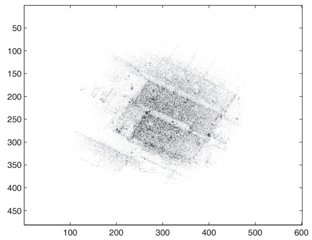
5. SIMULATION RESULTS

In this section, we give a few illustrative numerical examples showing the advantages of SDRA in comparison to existing autofocus algorithms. Figure 1 shows the performance of SDRA, FMCA, MCA and PGA, where the latter three are state-of-the-art techniques and PGA is widely used in practice. We adopted a scenario where the radar transmits and collects $M = 1307$ range lines across a 18 degree look angle. Due to the large problem size (problem dimension for this simulation is 1307) which makes IP methods infeasible, we used a bundle method proposed by Helmberg et al. to solve the problem [13][14]. In short, the bundle method transforms a large scale SDP into a series of simpler subproblems. Due to space limitations, we will not delve into the details of the implementation. After obtaining a solution for (SDR) using the bundle method, SDRA used randomization with $M_{rand} = 100$ to find the phase estimates and reconstruct the image. To simulate a SAR image, we applied a 2-D sinc antenna pattern and added complex gaussian noise to model system noise. Figure 1(a) shows the uncorrupted image with signal-to-noise-ratio (SNR) equal to 14.23 dB. Here SNR is calculated as $SNR = 20\log_{10}\{1/KL\sum_{kl}|G_p[k,l]|/\sigma_n\}$, where σ_n^2 is the variance of the added complex gaussian noise. The defocused image caused by unknown phase errors is shown in Fig. 1(b). The phase errors were independent and identically-distributed across the cross-range coordinate and uniformly distributed between $-\pi$ and π. The images restored by SDRA, FMCA, MCA and PGA are shown in Fig. 1(c), (d), (e) and (f), respectively. We note that although SDRA did not provide a perfect reconstruction, it did perform much better than all the other methods. In this wide-angle scenario, we did not expect MCA and PGA to work. SDRA also performs comparably well to MCA and PGA in the small angle case which they were designed for; however, those results are omitted here due to limited space.

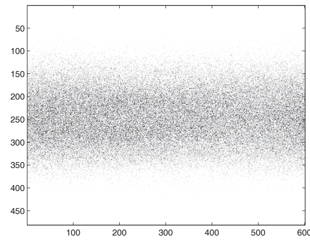
Next, we quantitatively characterize the performance of SDRA. We use the restoration quality metric SNR_{out} (i.e., output signal-to-noise ratio); this performance metric was also used in [3] . SNR_{out} is defined as

$$SNR_{out} = 20\log_{10}\frac{\|vec(g)\|_2}{\|(|vec(g)| - |vec(\hat{g})|)\|_2},$$

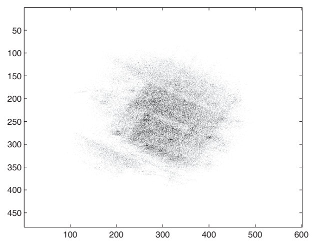
where $\hat{g}$ is the reconstructed image. A higher value of SNR_{out} corresponds to a better reconstruction quality. Here, the "noise" in SNR_{out} refers to the error in the magnitude of the reconstructed image $\hat{g}$ relative to the perfectly-focused image g. We used a simple toy image under the wide-angle SAR scenario for this part of the simulation. From the first part of simulation, we conclude that MCA and PGA will not work when the viewing angle is large, so here we only compare SDRA with FMCA. Figure 2 shows a Monte Carlo simulation of SNR_{out} verses different input SNR defined in the first part of simulation. It is expected that both methods will work when the noise is sufficiently small and will fail when the noise is large. From Fig. 2, we can see that SDRA outperforms FMCA over a wide range of SNR.



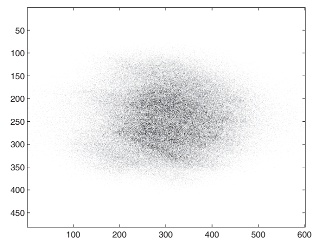
(a) Uncorrupted Image.



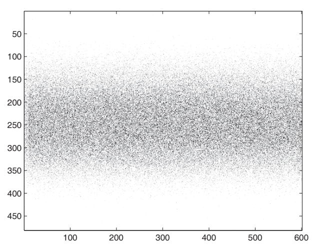
(b) Corrupted Image.



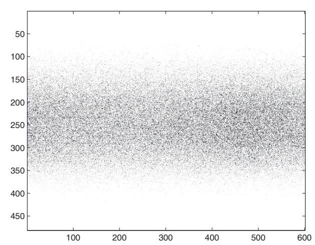
(c) Image restored by SDRA.



(d) Image restored by FMCA.



(e) Image restored by MCA.



(f) Image restored by PGA.

Fig. 1. Image restored by SDRA and contemporary autofocus algorithms.

6. CONCLUSION

We have proposed a new autofocus algorithm, SDRA, that is based on the semidefinite relaxation technique. SDRA is a robust algorithm that is suitable for solving both wide-angle monostatic SAR and bistatic SAR autofocus problems.

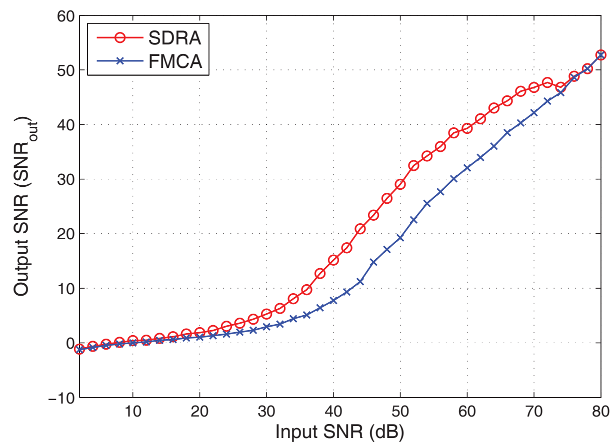


Fig. 2. Comparison of SDRA and FMCA.

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