13-3 Partial Derivatives

師大工教一

(107) 1. Find the following limits (if it exists). If the limit does not exist, explain why.

(a) (5 pts) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$ (b) (5 pts) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5xy^2}{x^2+2y^4}$

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$

$0 \le \frac{y^2}{x^2+y^2} \le 1$

$0 \le \frac{x^2y^2}{x^2+y^2} \le x^2$

$\displaystyle\lim_{(x,y)\to(0,0)} 0 = 0$

$\displaystyle\lim_{(x,y)\to(0,0)} x^2 = 0$

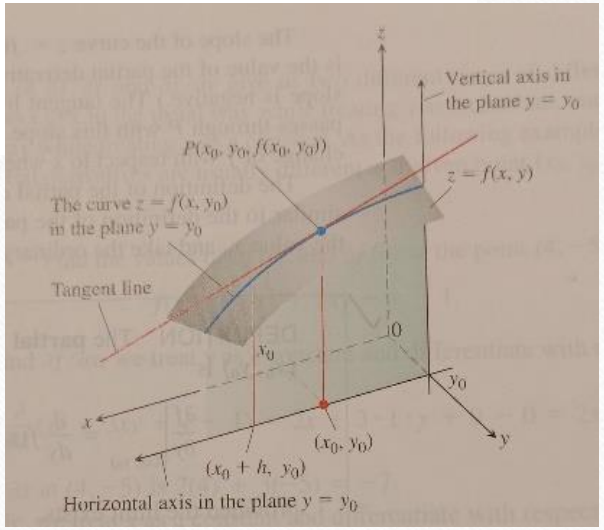
by Sandwich Thm. 所求$=0$

Partial Derivatives of a Function of Two Variables

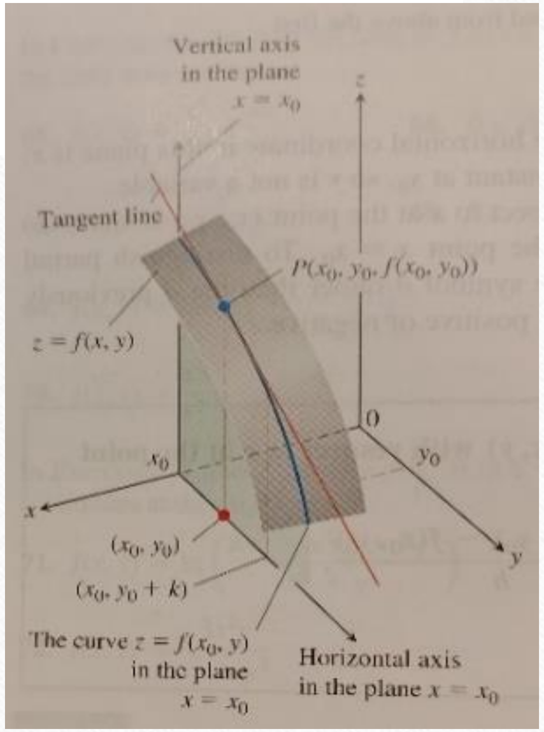
Definition The **partial derivatives of** $f(x,y)$ **with respect to** x **at the point** (x_0, y_0) is $\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim\limits_{h \to 0} \dfrac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$ provided the limit exists.

Note 1: $\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \dfrac{d}{dx} f(x, y_0)\Big|_{x=x_0}$.

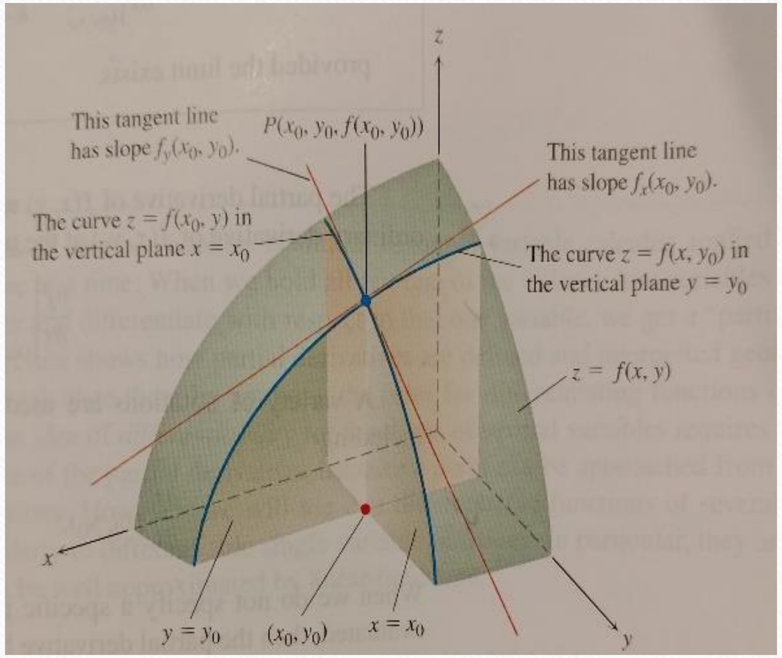
Note 2: $\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,y_0)} = f_x(x_0, y_0) = \left.\dfrac{\partial z}{\partial x}\right|_{(x_0,y_0)}$.



Definition The **partial derivatives of** $f(x,y)$ **with respect to** y **at the point** (x_0,y_0) is $\left.\dfrac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim\limits_{k\to 0}\dfrac{f(x_0,y_0+k)-f(x_0,y_0)}{k}$ provided the limit exists.



Note: $\left.\dfrac{\partial f}{\partial y}\right|_{(x_0,y_0)}$, $f_y(x_0,y_0)$, $\dfrac{\partial f}{\partial y}$, f_y .



z
This tangent line has slope $f_y(x_0, y_0)$.
$P(x_0, y_0, f(x_0, y_0))$
This tangent line has slope $f_x(x_0, y_0)$.
The curve $z = f(x_0, y)$ in the vertical plane $x = x_0$
The curve $z = f(x, y_0)$ in the vertical plane $y = y_0$
$z = f(x, y)$
x
y
$y = y_0$
(x_0, y_0)
$x = x_0$

Ex1(p735) Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4,-5)$ if

$f(x,y) = x^2 + 3xy + y - 1$.

$$\frac{\partial f}{\partial x} = 2x + 3y \Big|_{(4,-5)} = 8 - 15 = -7$$

$$\frac{\partial f}{\partial y} = 3x + 1 \Big|_{(4,-5)} = 13$$

Ex2(p735) Find $\frac{\partial f}{\partial y}$ as a function $f(x,y) = y \sin xy$.

$$\frac{\partial f}{\partial y} = \sin xy + xy \cos xy$$

Ex3(p735) Find f_x and f_y as functions if $f(x,y)=\dfrac{2y}{y+\cos x}$.

$f_x = 2y\left(\dfrac{\sin x}{(y+\cos x)^2}\right)$

$f_y = \dfrac{2(y+\cos x)-2y}{(y+\cos x)^2}$

Ex4(p736) Find $\dfrac{\partial z}{\partial x}$ assume that the function $yz-\ln z = x+y$ defines z as a function of the two independent variables x and y and the partial derivative exists.

$z=f(x,y)$

$y\dfrac{\partial z}{\partial x} - \dfrac{\partial(\ln z)}{\partial x} = 1$

Let $u=\ln z$

$\partial u = \frac{1}{z}\partial z$

$y\dfrac{\partial z}{\partial x} - \dfrac{1}{z}\dfrac{\partial z}{\partial x} = 1$

$\dfrac{\partial z}{\partial x} = \dfrac{z}{zy-1}$

Ex5(p736) The plane $x=1$ intersects the paraboloid $z=x^2+y^2$ in a parabola. Find the slope of the tangent line to the parabola at $(1,2,5)$.

$$\frac{\partial z}{\partial y}=2y$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2,5)}=4$$

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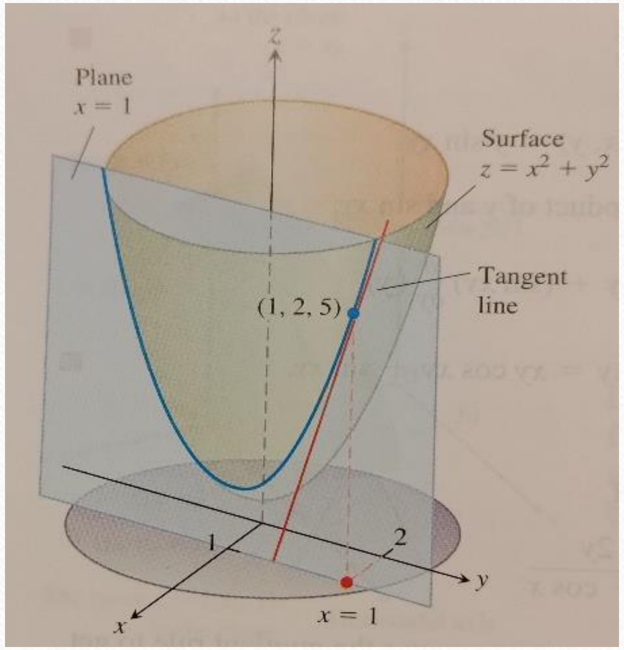


Functions of More Than Two Variables

Ex6(p736) If $x, y,$ and z are independent variables and

$f(x, y, z) = x\sin(y+3z)$, find f_z.

$$\frac{\partial f}{\partial z} = 3x\cos(y+3z)$$

Partial Derivatives and Continuity

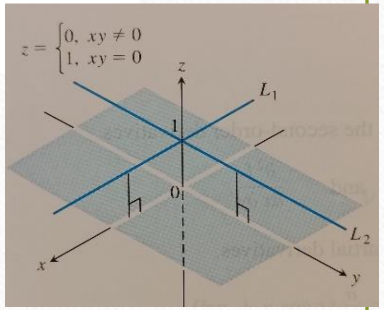
Ex8(p737) Let $f(x,y)=\begin{cases}0, & xy\neq 0\\ 1, & xy=0\end{cases}$.

(a) Find the limit of f as (x,y) approaches $(0,0)$ along the line $y=x$.

(b) Find the limit of f as (x,y) approaches $(0,0)$ along the line $y=0$.

(c) Prove that f is not continuous at the origin.

(d) Show that both partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at the origin.



Second-Order Partial Derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \text{ or } f_{xx} \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \text{ or } f_{xy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \text{ or } f_{yx} \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \text{ or } f_{yy}$$

Ex9(p739) If $f(x, y) = x\cos y + ye^x$, find the second-order derivatives.

$$\frac{\partial f}{\partial x} = \cos y + ye^x \qquad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x \quad \text{※} \qquad \frac{\partial^2 f}{\partial y \partial x} = -\sin y + e^x \quad \text{※}$$

$$\frac{\partial f}{\partial y} = -x\sin y + e^x \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x\cos y \quad \text{※}$$

Theorem 2- The Mixed Derivative Theorem

If $f(x,y)$ and its partial derivatives f_x, f_y, f_{xy}, f_{yx} are defined throughout an open region containing a point (a,b) and are all continuous at (a,b), then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Partial Derivatives of Still Higher Order

Ex11(p740) Find f_{yxyz} if $f(x,y,z) = 1 - 2xy^2z + x^2y$.

$f_y = -4xyz + x^2$ $\quad f_{yx} = -4yz + 2x$ $\quad f_{yxy} = -4z$ $\quad f_{yxyz} = -4$ #

Differentiability

Definition A function $z = f(x, y)$ is **differentiable at** (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$ satisfies an equation of the form $\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$, in which each of $\varepsilon_1, \varepsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$. We call f **differentiable** if it is differentiable at every point in the domain, and say that its graph is a **smooth surface**.

Theorem 3-The Increment Theorem for Functions of Two Variables

Suppose that the first partial derivatives of $f(x,y)$ are defined throughout an
1

open region R containing the point (x_0,y_0) and that f_x and f_y are
2

continuous at (x_0,y_0). Then the change $\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$ in the value of f that results from moving from (x_0,y_0) to another point $(x_0+\Delta x, y_0+\Delta y)$ in R satisfies an equation of the form

$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$ in which each of $\varepsilon_1,\varepsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$.

Corollery of Theorem 3

If the partial derivatives f_x and f_y of a function $f(x,y)$ are continuous throughout an open region R, then f is differentiable at every point of R.

Theorem 4—Differentiability implies Continuity

If a function $f(x,y)$ is differentiable at (x_0,y_0), then f is continuous at (x_0,y_0).

HW13-3

- **HW:2,3,16,21,23,41,86**

105年(分)#4

4. Find both first partial derivatives of following function. (8 points)

$$f(x,y)=\int_x^y (2t+1)\,dt+\int_y^x (2t-1)\,dt\,.$$

$$\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)\,dt\right]=f(h(x))\cdot h'(x)-f(g(x))\cdot g'(x)$$

$$f_x=-(2x+1)+(2x-1)=-2$$

$$f_y=(2y+1)-(2y-1)=2$$