1. Fourier Transform

延續複數型的結論，我們把指數中的 π/l 寫成常數 ω_0，那麼 $f(t)$ 可以寫為：

$$f(t)=\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}\ ,\ c_n=\frac{1}{2l}\int_{-l}^{l} f(t)e^{-in\omega_0 t}\,\mathrm{d}t$$

把 c_n 代入，再將 $n\omega_0$ 寫為變數 ω，則 $\Delta\omega=(n+1)\omega_0-n\omega_0=\omega_0=\pi/l$：

$$\begin{aligned} f(t)&=\sum_{n=-\infty}^{\infty}\left[\frac{1}{2l}\int_{-l}^{l} f(u)e^{-i\omega u}\,\mathrm{d}u\right]e^{i\omega t}\\ &=\sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi}\frac{\pi}{l}\int_{-l}^{l} f(u)e^{-i\omega u}\,\mathrm{d}u\right]e^{i\omega t}\\ &=\sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-l}^{l} f(u)e^{-i\omega u}\,\mathrm{d}u\right]e^{i\omega t}\Delta\omega \end{aligned}$$

在第二個等號，為了湊出 $\Delta\omega$，分式上下各多了一個 π. 現在前面有 $\sum$、後面有 $\Delta\omega$，看起來非常像一個積分的形式. 因此我們很自然的，把 $l\to\infty$，得到 $\Delta\omega\to\ \mathrm{d}\omega$：

$$\begin{aligned} f(t)&=\sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-l}^{l} f(u)e^{-i\omega u}\,\mathrm{d}u\right]e^{i\omega t}\Delta\omega\\ &=\frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\left[\int_{-\infty}^{\infty} f(u)e^{-i\omega u}\,\mathrm{d}u\right]}_{\text{記為}F(\omega)}e^{i\omega t}\,\mathrm{d}\omega \end{aligned}$$

現在裡面的 $f(t)$，經過裡面那層積分，變成 $F(\omega)$；而 $F(\omega)$ 再經過外面那層積分，又變回了 $f(t)$：

$$f(t)\ \underset{\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{i\omega u}\,\mathrm{d}\omega}{\overset{\int_{-\infty}^{\infty}f(t)e^{-i\omega t}\,\mathrm{d}t}{\rightleftarrows}}\ F(\omega)$$

這就是大名鼎鼎的傅立葉變換.

Definition 函數 $f(t)$ 之**傅立葉變換**（**Fourier Transform, F.T.**）定義為：

$$\mathcal{F}\{f(t)\}=F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,\mathrm{d}t$$

我們通常會用手寫體的 $\mathcal{F}\{\cdots\}$ 來表示傅立葉變換；帶有上標的 $\mathcal{F}^{-1}\{\cdots\}$ 來表示逆變換；用 $F(\omega)$ 表示變換後的函數. 我們稱變換前的 $f(t)$ 是在**時域**（Time-Domain、t-domain）、變換後的 $F(\omega)$ 是在**頻域**（Frequency-Domain、ω-domain）.

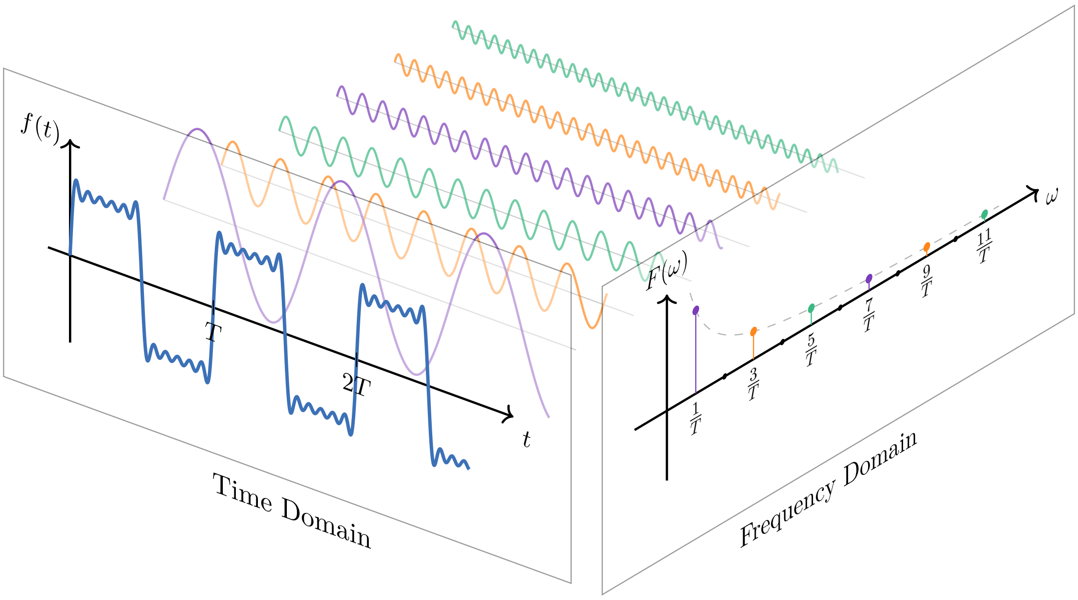

Figure 1: 傅立葉變換示意圖：將時域中的信號分解，並在頻域中標出各頻率的分佈.

Theorem. **(Dirichlet Conditions)** 若函數 $f(t)$ 同時滿足以下條件，則 $f(t)$ 的傅立葉變換收斂.

1. $f(t)$ 在一個週期內絕對可積 (absolutely integrable).
2. $f(t)$ 在一個週期內只有有限個極值點 (maxima and minima).
3. $f(t)$ 在一個週期內只有有限個不連續點 (finite discontinuities).

2. Laplace Transform

Definition 函數 $f(t)$ 之**拉普拉斯變換**（**Laplace Transform, L.T.**）定義為：

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t$$

其中 $s = \alpha + i\omega$.

我們用手寫體的 $\mathcal{L}\{\cdots\}$ 來表示拉普拉斯變換；$\mathcal{L}^{-1}\{\cdots\}$ 表示逆變換；大寫的 $F(s)$ 表示變換後的函數. 稱變換前的 $f(t)$ 是在**時域**（Time-Domain, t-domain）；稱變換後的 $F(s)$ 是在**複頻域** (Complex Frequency-Domain, s-domain).

Theorem. **(Exponential Order Conditions)** 若分段連續函數 $f(t)$ 滿足：存在常數 $M > 0$ 和 $c \in \mathbb{R}$，使得對於所有充分大的 t，我們有：

$$|f(t)| \leq Me^{ct}$$

則 $f(t)$ 的拉普拉斯變換變換收斂.

2.1. Laplace Transform of Common Functions

Question Find the Laplace transform of following functions.

▷ $\mathcal{L}\{\delta(t)\}$.

$$\begin{aligned}\mathcal{L}\{\delta(t)\} &= \int_0^{+\infty} \delta(t)e^{-st}\,\mathrm{d}t \\ &= e^{-st}\Big|_{t=0} = 1\end{aligned}$$

▷ $\mathcal{L}\{u(t)\} = \mathcal{L}\{1\}$.

$$\begin{aligned}\mathcal{L}\{u(t)\} &= \int_0^{+\infty} u(t)e^{-st}\,\mathrm{d}t \\ &= \int_0^{+\infty} e^{-st}\,\mathrm{d}t \\ &= \frac{-1}{s}e^{-st}\Big|_{t=0}^{t=\infty} \\ &= 0 - \left(\frac{-1}{s}\right) = \frac{1}{s}\end{aligned}$$

▶ $\mathcal{L}\{e^{at}\}$.

$$\begin{aligned}\mathcal{L}\left\{e^{at}\right\} &= \int_0^{+\infty} e^{at}e^{-st}\,\mathrm{d}t \\ &= \int_0^{+\infty} e^{-(s-a)t}\,\mathrm{d}t \\ &= \frac{-1}{s-a}e^{(s-a)t}\Big|_{t=0}^{t=\infty} \\ &= 0 - \left(\frac{-1}{s-a}\right) = \frac{1}{s-a}\end{aligned}$$

▷ $\mathcal{L}\{\sin(at)\}$, $\mathcal{L}\{\cos(at)\}$

$$\begin{aligned}\mathcal{L}\left\{e^{iat}\right\} &= \frac{1}{s-ia} = \frac{s+ia}{s^2+a^2} \\ &= \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2} \\ &= \mathcal{L}\{\cos(at) + i\sin(at)\} \\ &= \mathcal{L}\{\cos(at)\} + i\mathcal{L}\{\sin(at)\}\end{aligned}$$

$$\Rightarrow \mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2},\ \mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$$

$f(t)$	$\delta(t)$	$u(t)$	t^n	e^{at}	$\sin(at)$	$\cos(at)$	$\sinh(at)$	$\cosh(at)$
$F(s)$	1	$\frac{1}{s}$	$\frac{n!}{s^{n+1}}$	$\frac{1}{s-a}$	$\frac{a}{s^2+a^2}$	$\frac{s}{s^2+a^2}$	$\frac{a}{s^2-a^2}$	$\frac{s}{s^2-a^2}$

Table 1: 基本函數拉普拉斯變換

Remark 歐拉公式 $e^{ix} = \cos(x) + i\sin(x)$.

2.2. Laplace Transform Properties

Question Show the following Laplace transform properties holds.

▷ $\mathcal{L}\left\{(f*g)(t)\right\} = F(s)\cdot G(s)$.

$$\begin{aligned}
\mathcal{L}\left\{f(t)*g(t)\right\} &= \int_0^\infty e^{-st}\int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \int_0^\infty \int_0^t e^{-st} f(\tau)g(t-\tau)\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \int_0^\infty \int_0^\infty e^{-st} f(\tau)g(t-\tau)\,\mathrm{d}t\,\mathrm{d}\tau \\
&= \int_0^\infty \int_0^\infty e^{-s(u+\tau)} f(\tau)g(u)\,\mathrm{d}u\,\mathrm{d}\tau \\
&= \int_0^\infty e^{-s\tau} f(\tau)\int_0^\infty e^{-su} g(u)\,\mathrm{d}u\,\mathrm{d}\tau \\
&= \left[\int_0^\infty e^{-s\tau} f(\tau)\,\mathrm{d}\tau\right]\left[\int_0^\infty e^{-su} g(u)\,\mathrm{d}u\right] \\
&= \mathcal{L}\left\{f(t)\right\}\cdot\mathcal{L}\left\{g(t)\right\} \\
&= F(s)\cdot G(s)
\end{aligned}$$

▶ $\mathcal{L}\left\{f'(t)\right\} = sF(s) - f(0)$.

$$\begin{aligned}
\mathcal{L}\left\{f'(t)\right\} &= \int_0^\infty e^{-st} f'(t)\,\mathrm{d}t \\
&= \left[e^{-st}f(t)\right]_0^\infty - \int_0^\infty (-se^{-st})f(t)\,\mathrm{d}t \\
&= \left(e^{-s(\infty)}f(\infty) - e^0 f(0)\right) + s\int_0^\infty e^{-st}f(t)\,\mathrm{d}t \\
&= (0 - f(0)) + s\mathcal{L}\left\{f(t)\right\} \\
&= sF(s) - f(0)
\end{aligned}$$

▷ $\mathcal{L}\left\{\int_0^t f(\tau)\,\mathrm{d}\tau\right\} = \frac{1}{s}F(s)$.

$$\begin{aligned}
\mathcal{L}\left\{\int_0^t f(\tau)\,\mathrm{d}\tau\right\} &= \mathcal{L}\left\{\int_0^t 1\cdot f(\tau)\,\mathrm{d}\tau\right\} \\
&= \mathcal{L}\left\{f(t)*1\right\} \\
&= \mathcal{L}\left\{f(t)\right\}\cdot\mathcal{L}\left\{1\right\} \\
&= F(s)\cdot\frac{1}{s}
\end{aligned}$$

2.3. Laplace Transform of General Functions

$f(t)$	$F(s)$	$f(t)$	$F(s)$
$\delta(t)$	1	$c_1 f(t) + c_2 g(t)$	$c_1 F(s) + c_2 G(s)$
$u(t)$	$\frac{1}{s}$	$f(t) * g(t)$	$F(s) \cdot G(s)$
t^n	$\frac{n!}{s^{n+1}}$	$e^{at} f(t)$	$F(s-a)$
e^{at}	$\frac{1}{s-a}$	$f(t+a)u(t+a)$	$e^{as} F(s)$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$\frac{d^n}{dt^n} f(t)$	$s^n F(s) - s^{n-1} f(0) \cdots - f^{(n-1)}(0)$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n} F(s)$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$\int_0^t f(\tau)\, \mathrm{d}\tau$	$\frac{1}{s} F(s)$
$\cosh(at)$	$\frac{s}{s^2-a^2}$	$\frac{1}{t} f(t)$	$\int_s^\infty F(\xi)\, \mathrm{d}\xi$

Table 2: 基本函數拉普拉斯變換與拉普拉斯變換之性質

Question Find the Laplace transform of following functions.

▷ $\mathcal{L}\left\{te^{-2t}\sin(t)\right\}$.

$$\begin{aligned}\mathcal{L}\left\{te^{-2t}\sin(t)\right\} &= -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\left\{e^{-2t}\sin(t)\right\}\\ &= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\left.\frac{1}{s^2+1}\right|_{s\to s+2}\right)\\ &= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{(s+2)^2+1}\right) = \frac{2(s+2)}{((s+2)^2+1)^2}\end{aligned}$$

▶ $\mathcal{L}\left\{te^{t}\cos(2t)\right\}$.

$$\begin{aligned}\mathcal{L}\left\{te^{t}\cos(2t)\right\} &= -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\left\{e^{t}\cos(2t)\right\}\\ &= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\left.\frac{s}{s^2+4}\right|_{s\to s-1}\right)\\ &= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s-1}{(s-1)^2+4}\right) = \frac{(s-1)^2-4}{((s-1)^2+4)^2}\end{aligned}$$

▷ $\mathcal{L}\left\{e^{-t}\int_0^t \sin(2\tau)\,\mathrm{d}\tau\right\}$.

$$\begin{aligned}\mathcal{L}\left\{e^{-t}\int_0^t \sin(2\tau)\,\mathrm{d}\tau\right\} &= \mathcal{L}\left\{\int_0^t \sin(2\tau)\,\mathrm{d}\tau\right\}\Bigg|_{s\to s+1}\\ &= \left(\frac{1}{s}\cdot\frac{2}{s^2+4}\right)\Bigg|_{s\to s+1}\\ &= \frac{2}{(s+1)((s+1)^2+4)}\end{aligned}$$

▶ $\mathcal{L}\left\{e^{2t}\int_0^t \tau e^{-3\tau}\,\mathrm{d}\tau\right\}$.

$$\begin{aligned}\mathcal{L}\left\{e^{2t}\int_0^t \tau e^{-3\tau}\,\mathrm{d}\tau\right\} &= \mathcal{L}\left\{\int_0^t \tau e^{-3\tau}\,\mathrm{d}\tau\right\}\Bigg|_{s\to s-2}\\ &= \left(\frac{1}{s}\mathcal{L}\left\{\tau e^{-3\tau}\right\}\right)\Bigg|_{s\to s-2}\\ &= \left(\frac{1}{s}\cdot\frac{1}{(s+3)^2}\right)\Bigg|_{s\to s-2}\\ &= \frac{1}{(s-2)(s+1)^2}\end{aligned}$$

▷ $\mathcal{L}\left\{t\int_0^t e^{-2\tau}\,\mathrm{d}\tau\right\}$.

$$\begin{aligned}\mathcal{L}\left\{t\int_0^t e^{-2\tau}\,\mathrm{d}\tau\right\} &= -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\left\{\int_0^t e^{-2\tau}\,\mathrm{d}\tau\right\}\\ &= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s}\mathcal{L}\left\{e^{-2t}\right\}\right)\\ &= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s(s+2)}\right)\\ &= \frac{2s+2}{s^2(s+2)^2}\end{aligned}$$

▶ $\mathcal{L}\left\{t\int_0^t \cos(\tau)\,\mathrm{d}\tau\right\}$.

$$\begin{aligned}\mathcal{L}\left\{t\int_0^t \cos(\tau)\,\mathrm{d}\tau\right\} &= -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\left\{\int_0^t \cos(\tau)\,\mathrm{d}\tau\right\}\\ &= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s}\cdot\frac{s}{s^2+1}\right)\\ &= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2+1}\right)\\ &= \frac{2s}{(s^2+1)^2}\end{aligned}$$

$\triangleright$ $\mathcal{L}\left\{\int_0^t (t-\tau)e^{2\tau}\cos(\tau)\,\mathrm{d}\tau\right\}$.

$$\begin{aligned}\mathcal{L}\left\{t*(e^{2t}\cos t)\right\} &= \mathcal{L}\left\{t\right\}\cdot\mathcal{L}\left\{e^{2t}\cos t\right\}\\ &= \frac{1}{s^2}\cdot\left(\left.\frac{s}{s^2+1}\right|_{s\to s-2}\right)\\ &= \frac{1}{s^2}\cdot\frac{s-2}{(s-2)^2+1}\\ &= \frac{s-2}{s^2(s^2-4s+5)}\end{aligned}$$

$\blacktriangleright$ $\mathcal{L}\left\{\int_0^t e^{-(t-\tau)}\tau\sin(\tau)\,\mathrm{d}\tau\right\}$.

$$\begin{aligned}\mathcal{L}\left\{e^{-t}*(t\sin t)\right\} &= \mathcal{L}\left\{e^{-t}\right\}\cdot\mathcal{L}\left\{t\sin t\right\}\\ &= \frac{1}{s+1}\cdot\left(-\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s^2+1}\right)\\ &= \frac{1}{s+1}\cdot\frac{2s}{(s^2+1)^2}\\ &= \frac{2s}{(s+1)(s^2+1)^2}\end{aligned}$$

$\triangleright$ $\mathcal{L}\left\{ty''(t)+y'(t)+ty(t)\right\}$ with $y(0)=1,\ y'(0)=0$.

$$\begin{aligned}\mathcal{L}\left\{ty''+y'+ty\right\} &= -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\left\{y''\right\}+\mathcal{L}\left\{y'\right\}-\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\left\{y\right\}\\ &= -\frac{\mathrm{d}}{\mathrm{d}s}\left(s^2Y(s)-s\right)+(sY(s)-1)-\frac{\mathrm{d}}{\mathrm{d}s}Y(s)\\ &= -(2sY(s)+s^2Y'(s)-1)+sY(s)-1-Y'(s)\\ &= -(s^2+1)Y'(s)-sY(s)\end{aligned}$$

$\blacktriangleright$ $\mathcal{L}\left\{ty''(t)+(1-t)y'(t)\right\}$ with $y(0)=1,\ y'(0)=0$.

$$\begin{aligned}\mathcal{L}\left\{ty''+y'-ty'\right\} &= -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\left\{y''\right\}+\mathcal{L}\left\{y'\right\}+\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\left\{y'\right\}\\ &= -\frac{\mathrm{d}}{\mathrm{d}s}\left(s^2Y(s)-s\right)+(sY(s)-1)+\frac{\mathrm{d}}{\mathrm{d}s}(sY(s)-1)\\ &= -(s^2Y'(s)+2sY(s)-1)+sY(s)-1+(sY'(s)+Y(s))\\ &= (s-s^2)Y'(s)+(1-s)Y(s)\end{aligned}$$

3. Pole-Zero Intuition

在對於一個線性系統，輸入 $X(s)$ 與輸出 $Y(s)$ 的關係為：

$$Y(s) = H(s) \cdot X(s) = A\frac{(s-z_1)(s-z_2)\cdots(s-z_n)}{(s-p_1)(s-p_2)\cdots(s-p_m)} \cdot X(s)$$

其中，使分子為零的跟 z_i 稱為系統的**零點 (Zeros)**；使分母為零 p_i 稱為**極點 (Poles)**. 當我們輸入單位脈衝 $\delta(t)$，其拉普拉斯變換為 $X(s) = 1$. 此時系統的輸出直接等於轉移函數：

$$Y(s) = H(s) \cdot 1 = \sum \frac{A_i}{s - p_i} \quad \xrightarrow{\mathcal{L}^{-1}} \quad y(t) = \sum A_i e^{p_i t}$$

這告訴我們一個重要結論：「極點 p_i 決定了系統自然響應的行為」. 無論外力如何，系統總是傾向於以 $e^{p_i t}$ 的形式運動：

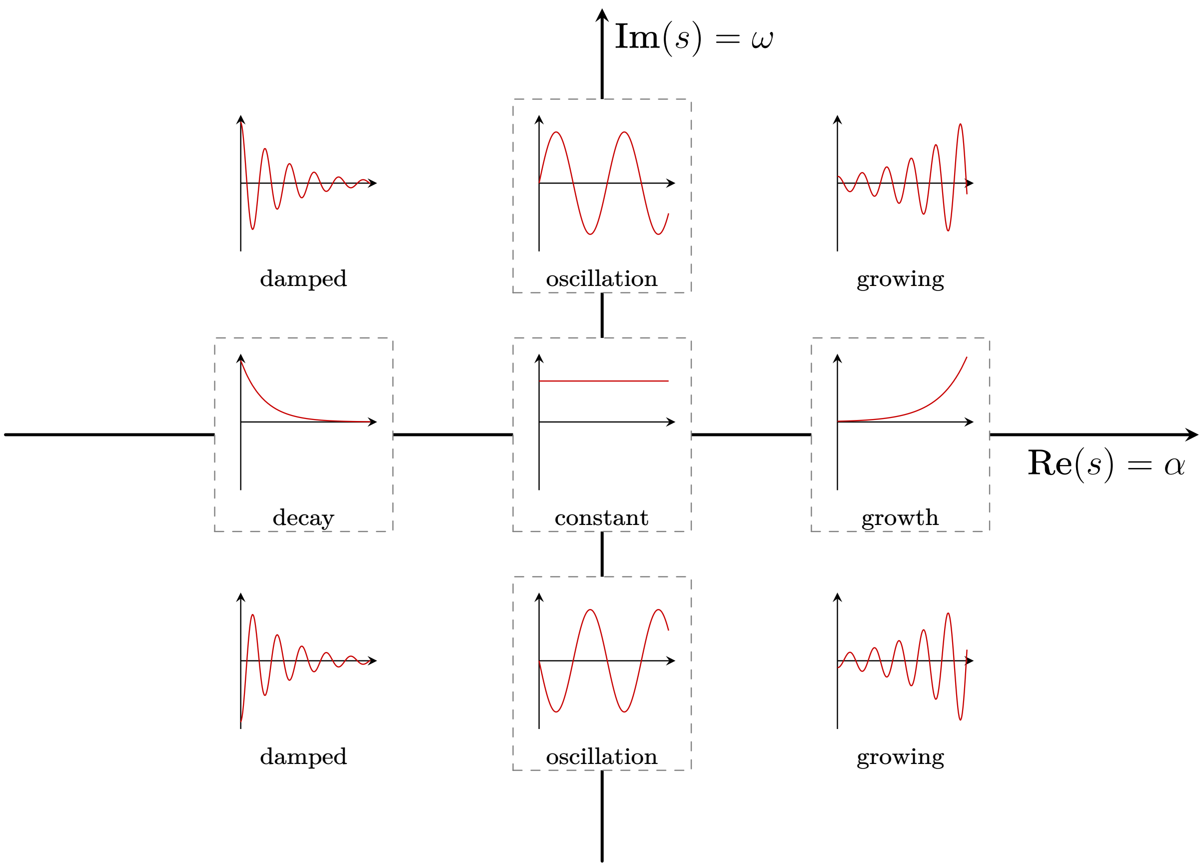

Figure 2: The s-Plane

Next Week: Inverse Laplace Transform, Laplace Transform for solving DEs.