13-5 Directional Derivatives and Gradient Vectors

師大工教一

Directional Derivatives in the Plane

Definition The **derivative of** f **at** $P_0(x_0, y_0)$ **in the direction of the unit vector** $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is the number

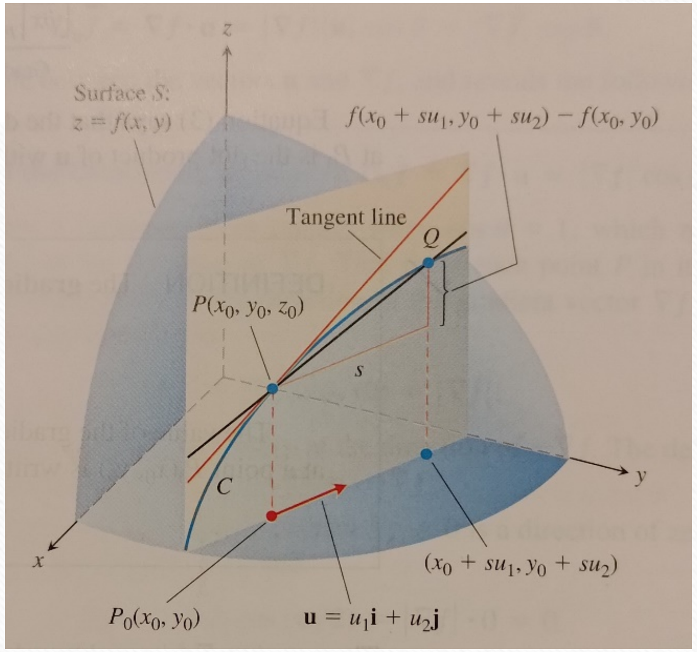
$D_{\vec{u}} f(P_0) \triangleq \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$

$$\left(\frac{df}{ds}\right)_{\vec{u}, P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

provided the limit exists.

Note 1: The **directional derivative** is also denoted by $D_{\vec{u}}f(P_0)$ or $D_{\vec{u}}f\big|_{P_0}$.

Note 2: The partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ are the directional derivatives of f at P_0 in the $\vec{i}$ and $\vec{j}$ directions.



Ex1(p754) Using the definition, find the derivative of $f(x,y) = x^2 + xy$ at $P_0(1,2)$ in the direction of the unit vector $\vec{u} = \left(\frac{1}{\sqrt{2}}\right)\vec{i} + \left(\frac{1}{\sqrt{2}}\right)\vec{j}$.

$$D_{\vec{u}} f(1,2) = \lim_{s\to 0} \frac{f(1+\frac{1}{\sqrt{2}}s,\ 2+\frac{1}{\sqrt{2}}s) - f(1,2)}{s}$$

$$= \lim_{s\to 0} \frac{[(1+\frac{s}{\sqrt{2}})^2 + (1+\frac{s}{\sqrt{2}})(2+\frac{s}{\sqrt{2}})] - [1+2]}{s}$$

$$= \lim_{s\to 0} \frac{(\cancel{1}+\sqrt{2}s + \frac{s^2}{2} + \cancel{2} + \frac{3}{\sqrt{2}}s + \frac{s^2}{2}) - (\cancel{1}+\cancel{2})}{s}$$

$$= \lim_{s\to 0} \frac{\frac{5}{\sqrt{2}}\cancel{s} + s^{\cancel{2}}}{\cancel{s}} = \frac{5}{\sqrt{2}}$$

$D_{\vec{u}} f(1,2)$

$= \nabla f(1,2) \cdot \vec{u}$

$= \langle 2x+y,\ x\rangle\big|_{(1,2)} \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$

$= \langle 4, 1\rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$

$= \frac{5}{\sqrt{2}}$

Definition The **gradient vector**(or **gradient**) of $f(x,y)$ is the vector

$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$. The value of the gradient vector obtained by evaluating the

partial derivatives at a point $P_0(x_0, y_0)$ is written $\nabla f|_{P_0}$ or $\nabla f(x_0, y_0)$.

Theorem 9—**The Directional Derivative Is a Dot Product**

If $f(x,y)$ is differentiable in an open region containing $P_0(x_0,y_0)$, then

$$\left(\frac{df}{ds}\right)_{\vec{u},P_0} = \nabla f\Big|_{P_0} \cdot \vec{u}$$, the dot product of the gradient ∇f at P_0 with the vector $\vec{u}$. In brief, $D_{\vec{u}} f = \nabla f \cdot \vec{u}$.

By Chain Rule,

$$\left(\frac{df}{ds}\right)_{\vec{u},P_0} = \frac{\partial f}{\partial x}\Big|_{P_0} \cdot \frac{dx}{ds} + \frac{\partial f}{\partial y}\Big|_{P_0} \cdot \frac{dy}{ds} = \frac{\partial f}{\partial x}\Big|_{P_0} \cdot u_1 + \frac{\partial f}{\partial y}\Big|_{P_0} \cdot u_2 = \left[\frac{\partial f}{\partial x}\Big|_{P_0} \vec{i} + \frac{\partial f}{\partial y}\Big|_{P_0} \vec{j}\right] \cdot \left[u_1\vec{i} + u_2\vec{j}\right]$$

Ex2(p757) Find the directional derivative of $f(x,y) = xe^y + \cos(xy)$ at the point $(2,0)$ in the direction of $\vec{v} = 3\vec{i} - 4\vec{j}$.

$\vec{u} = \frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}$

$D_{\vec{u}} f(2,0) = \nabla f(2,0) \cdot \vec{u}$

$= \langle e^y - y\sin xy, xe^y - x\sin xy \rangle \big|_{(2,0)} \cdot \langle \frac{3}{5}, -\frac{4}{5} \rangle$

$= \langle 1, 2 \rangle \cdot \langle \frac{3}{5}, -\frac{4}{5} \rangle$

$= \frac{3}{5} - \frac{8}{5} = -1$

$D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f||\vec{u}|\cos\theta = |\nabla f|\cos\theta$, where θ is the angle between the vector $\vec{u}$ and ∇f.

Properties of the Directional Derivative $D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f|\cos\theta$

1. At each point P in its domain, f increases most rapidly in the direction of gradient vector ∇f at P. The derivative is $D_{\vec{u}} f = |\nabla f|\cos(0) = |\nabla f|$.

2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\vec{u}} f = |\nabla f| \cos(\pi) = -|\nabla f|$.

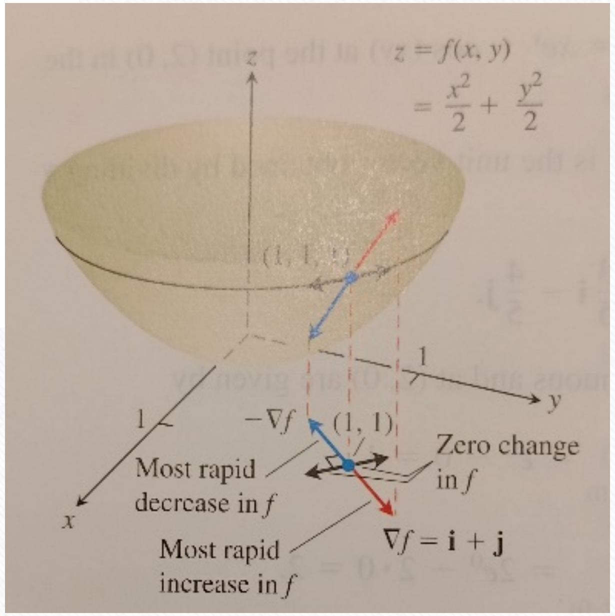
3. Any direction $\vec{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\frac{\pi}{2}$ and $D_{\vec{u}} f = |\nabla f| \cos\left(\frac{\pi}{2}\right) = |\nabla f| \cdot 0 = 0$.

Ex3(p757) Find the direction in which $f(x,y)=\left(\frac{x^2}{2}\right)+\left(\frac{y^2}{2}\right)$

(a) increases most rapidly at the point $(1,1)$.

(b) decreases most rapidly at $(1,1)$.

(c) What are the directions of zero change in f at $(1,1)$.



(a) $\nabla f\big|_{(1,1)} = (x\vec{i} + y\vec{j})\big|_{(1,1)} = \vec{i} + \vec{j}$, $\vec{u}_1 = \frac{\vec{i}+\vec{j}}{|\vec{i}+\vec{j}|} = \frac{\vec{i}}{\sqrt{2}} + \frac{\vec{j}}{\sqrt{2}}$

(b) $\vec{u}_2 = -\frac{\vec{i}}{\sqrt{2}} - \frac{\vec{j}}{\sqrt{2}}$

(c) $\begin{cases} \vec{n}_1 = -\frac{\vec{i}}{\sqrt{2}} + \frac{\vec{j}}{\sqrt{2}} \\ \vec{n}_2 = \frac{\vec{i}}{\sqrt{2}} - \frac{\vec{j}}{\sqrt{2}} \end{cases}$

Gradients and Tangents to Level Curves

Let $f(x,y)$ is a constant along a smooth curve $\vec{r} = g(t)\vec{i} + h(t)\vec{j}$, then

$f\left(g(t), h(t)\right) = c$. Thus,

$$\frac{d}{dt}\left(f\left(g(t), h(t)\right)\right) = \frac{d}{dt}(c) \Rightarrow \frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} = 0 \Rightarrow \underbrace{\left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt}\vec{i} + \frac{dh}{dt}\vec{j}\right)}_{\frac{d\vec{r}}{dt}} = 0$$

At every point (x_0, y_0) in the domain of a differentiable function $f(x, y)$ where the gradient of f is a nonzero vector, this vector is normal to the level curve through (x_0, y_0).

The line through a point $P_0(x_0, y_0)$ normal to a nonzero vector $\overrightarrow{N} = A\vec{i} + B\vec{j}$ has the equation $A(x - x_0) + B(y - y_0) = 0$. If the gradient is not zero vector, then equation for the tangent line to a level curve is

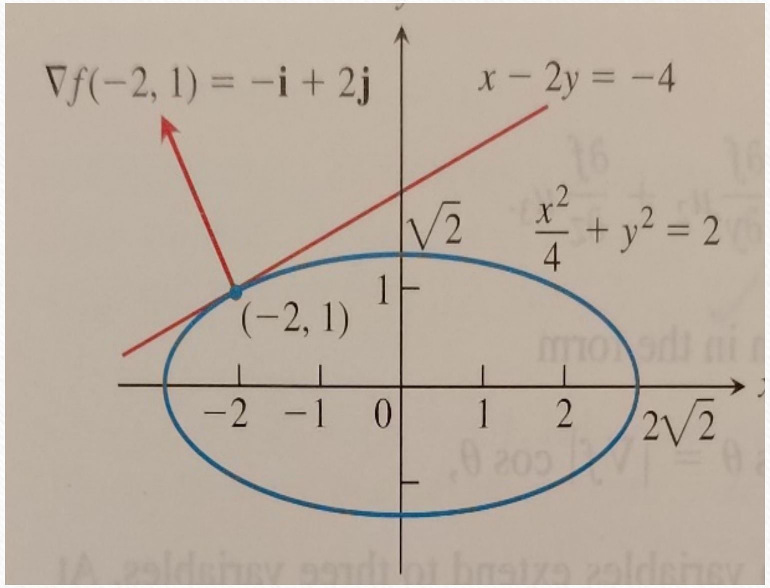
$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0$$

Ex4(p759) Find an equation for the tangent to the ellipse $\frac{x^2}{4}+y^2=2$ at the point $(-2,1)$.

$\nabla f|_{(-2,1)} = \langle \frac{x}{2}, 2y \rangle_{(-2,1)} = \langle -1, 2 \rangle$

$= -\vec{i} + 2\vec{j}$

$-(x+2)+2(y-1)=0$



Algebra Rules for Gradients

1. *Sum Rule* $$\nabla\left(f+g\right)=\nabla f+\nabla g$$

2. *Difference Rule* $$\nabla\left(f-g\right)=\nabla f-\nabla g$$

3. *Constant Multiple Rule* $$\nabla\left(kf\right)=k\nabla f$$

4. *Product Rule* $$\nabla\left(fg\right)=f\nabla g+g\nabla f$$

5. *Quotient Rule* $$\nabla\left(\frac{f}{g}\right)=\frac{g\nabla f-f\nabla g}{g^2}$$

Functions of Three Variables

For a differentiable function $f(x,y,z)$ and a unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ in space, we have $\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$ and $D_{\vec{u}} f = \nabla f \cdot \vec{u} = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 + \frac{\partial f}{\partial z}u_3$,

and $D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f||\vec{u}|\cos\theta = |\nabla f|\cos\theta$.

Ex6(p760)

(a) Find the derivative of $f(x,y,z) = x^3 - xy^2 - z$ at $P_0(1,1,0)$ in the direction of $\vec{v} = 2\vec{i} - 3\vec{j} + 6\vec{k}$.

(b) In what directions does f change most rapidly at P_0, and what are the rate of change in these direction?

(a) $\nabla f = (3x^2 - y^2)\vec{i} + (-2xy)\vec{j} + (-1)\vec{k}$

$\nabla f(1,1,0) = 2\vec{i} - 2\vec{j} - \vec{k}$

$\vec{u} = \frac{2}{7}\vec{i} - \frac{3}{7}\vec{j} + \frac{6}{7}\vec{k}$

$D_{\vec{u}} f(1,1,0) = \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}$

(b) Let $\vec{u}_1 = \frac{2}{3}\vec{i} - \frac{2}{3}\vec{j} - \frac{1}{3}\vec{k}$

$D_{\vec{u}_1} f(1,1,0) = 3$ change most rapidly

The Chain Rule for Path

If $\vec{r}(t)=x(t)\vec{i}+y(t)\vec{j}+z(t)\vec{k}$ is a smooth path C, and $w=f\left(\vec{r}(t)\right)$ is a scalar function evaluated along C, then by chain rule,

$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}$. Therefore, the derivative along a path

$$\frac{d}{dt}f\left(\vec{r}(t)\right)=\nabla f\left(\vec{r}(t)\right)\cdot\vec{r}'(t).$$

Ex(103 年，#4) Find the derivative of $f(x,y,z)=xyz$ of the velocity vector of the helix $\vec{r}(t)=(\cos 3t)\vec{i}+(\sin 3t)\vec{j}+3t\vec{k}$ at $t=\frac{\pi}{3}$.

$\nabla f(\vec{r}(t))=(yz, zx, xy)$

$\vec{r}'(t)=(-3\sin(3t), 3\cos(3t), 3)$

$\vec{r}(\frac{\pi}{3})=(-1, 0, \pi)$

$\nabla f(\vec{r}(\frac{\pi}{3}))=(0, -\pi, 0)$

$\vec{r}'(\frac{\pi}{3})=(0, -3, 3)$

$(0, -\pi, 0)\cdot(0, -3, 3)=3\pi$

HW13-5

- **HW:2,5,7,11,17,19,29**

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5. Let $f(x, y, z) = 3e^x \cos(yz)$ and $P_0(0, 0, 0)$ be a point in its domain.

(a) Find the gradient of f at the point P_0. (5 pts)

(b) Use (a) to find the directional derivative of f at the point P_0 in the direction of $\vec{v} = 2\vec{i} + \vec{j} - 2\vec{k}$. (5 pts)

(c) Find the direction in which f decreases most rapidly at the point P_0. (5 pts)

(a) $\nabla f = \langle 3e^x\cos(yz), -3ze^x\sin(yz), -3ye^x\sin(yz)\rangle$

$\nabla f(0,0,0) = \langle 3, 0, 0\rangle$

(b) $D_{\vec{u}} f(0,0,0) = \langle 3,0,0\rangle \cdot \langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\rangle$

$= \frac{6}{3} = 2$ #

(c) Let $\vec{u}_1 = \frac{-\nabla f}{\|\nabla f\|} =$

$\langle -1, 0, 0\rangle$

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5. 考慮函數 $f(x, y, z) = xe^y + z^2$ 及其定義域上一點 $P(1, \ln 2, \frac{1}{2})$。

(a) 求 f 在點 P 處的梯度 (gradient)。(5 分)

(b) 用 (a) 來求 f 在點 P 處沿方向 $\vec{\mathbf{v}} = 3\vec{\mathbf{i}} - 2\vec{\mathbf{j}} + \vec{\mathbf{k}}$ 的方向導數 (directional derivative)。(5 分)

(c) 求 f 在點 P 處遞增最遽烈的方向。(5 分)