9-3 The Integral Test

積分審斂法

師大工教一

Nondecreasing Partial Sums

正項級數 $a_n \geq 0$, $\Rightarrow S_1 \leq S_2 \leq S_3 \leq \cdots$

$\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \geq 0$ for all $n \Rightarrow s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n \leq s_{n+1} \leq \cdots$

Corollery of Theorem 6

A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

Ex1(p536) **Harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$ diverges.

調和級數

$S_1 = 1$

$S_2 = 1 + \frac{1}{2}$

$S_{2^2} = S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + 2 \times \frac{1}{2}$

$S_{2^3} = S_8 = 1 + \frac{1}{2} + \cdots + \frac{1}{8} \geq 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 1 + 3 \times \frac{1}{2}$

$\vdots$

$S_{2^n} \geq 1 + \frac{n}{2}$ $\lim_{n \to \infty} S_{2^n} = \infty$

Theorem 9—The Integral Test

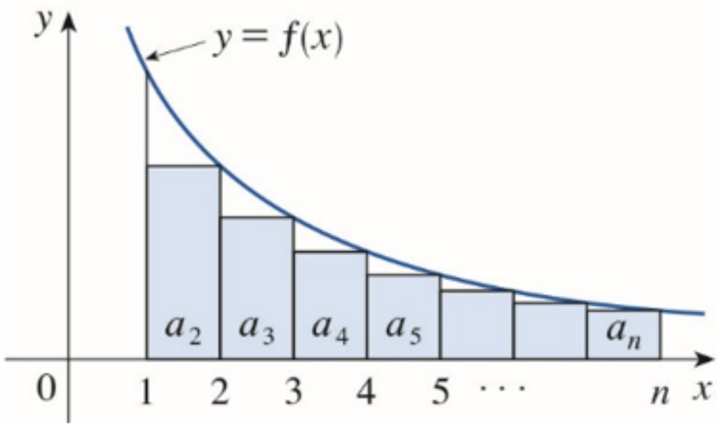
f must be continuous positive decreasing.

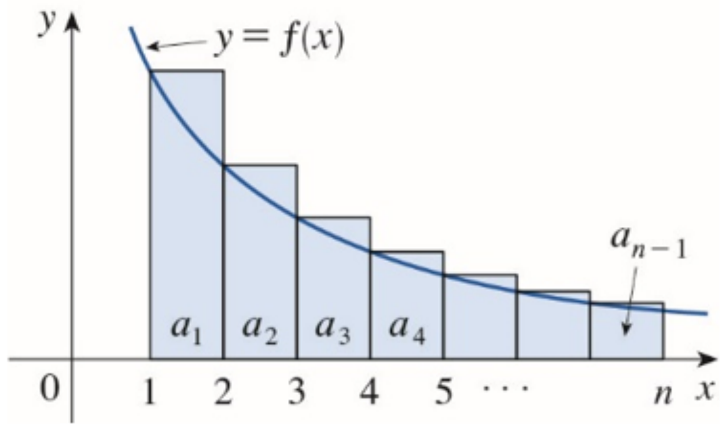
$\sum_{n=1}^{\infty} a_n$, $\int_1^{\infty} f(x)dx$ 有相同敛散性

P-series P级数

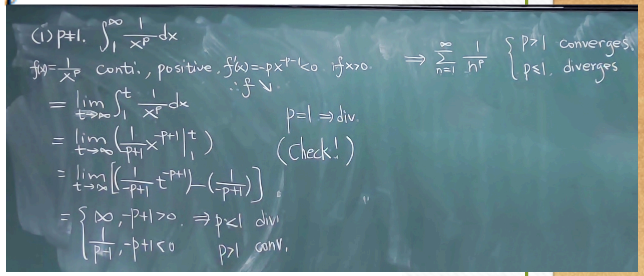
$\sum_{n=1}^{\infty} \frac{1}{n^p}$ $p>1$ 收敛 $p\le 1$ 发散

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \ge N$ (N is a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.





Ex3(p538) Show that **the** $p-$**series** $\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}+\cdots$ (p is a real constant) converges if $p>1$ and diverges if $p\le 1$.



Ex4(p538) Determine the convergence or divergence of $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$.

$\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ $\quad \int_1^\infty \frac{1}{x^2+1}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2+1}dx = \lim_{t\to\infty}\left(\tan^{-1}x\Big|_1^t\right) = \lim_{t\to\infty}\left(\tan^{-1}t-\frac{\pi}{4}\right)=\frac{\pi}{4}$ conv.

$\frac{1}{x^2+1}$ conti., positive, decreasing (Check!)

By integral test.

$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2+1}$ conv.

Ex5(p538) Determine the convergence or divergence of the series.

(a) $\sum_{n=1}^{\infty} ne^{-n^2}$ $\qquad$ (b) $\sum_{n=1}^{\infty}\frac{1}{2^{\ln n}}$

Cf. (105 本部) $\sum_{n=2}^{\infty}\frac{1}{(\ln n)^{\ln n}}$

(a) $\sum_{n=1}^{\infty} ne^{-n^2}$

$\int_1^\infty xe^{-x^2}dx$

$=\lim_{t\to\infty}\int_1^t xe^{-x^2}dx = \lim_{t\to\infty}\left(-\frac{1}{2}e^{-t^2}+\frac{1}{2}e^{-1}\right)=\frac{1}{2}e^{-1}$ conv.

$\int_1^t xe^{-x^2}dx = \int_1^{t^2} e^{-u}\left(\frac{1}{2}du\right)$

Let $u=x^2$, $du=2x\,dx$

$=-\frac{1}{2}e^{-u}\Big|_1^{t^2}$

$=-\frac{1}{2}e^{-t^2}+\frac{1}{2}e^{-1}$

By integral test

$\sum_{n=1}^{\infty} ne^{-n^2}$ conv.

(b) $\sum_{n=1}^{\infty}\frac{1}{2^{\ln n}}$ $\quad f(x)=\frac{1}{2^{\ln x}}=2^{-\ln x}$ conti. positiv decreasing.

$\int_1^\infty \frac{1}{2^{\ln x}}dx$

$=\lim_{t\to\infty}\int_1^t \frac{1}{2^{\ln x}}dx$

$=\lim_{t\to\infty}\left(-\frac{1}{2}e^{-t^2}+\frac{1}{2}e^{-1}\right)=\frac{1}{2}e^{-1}$ conv. $\Rightarrow$ By integral test $\sum_{n=1}^{\infty} ne^{-n^2}$ conv.

$\int_1^t \frac{1}{2^{\ln x}}dx = \int_0^{\ln t}\frac{e^u}{2^u}du$

Let $u=\ln x$, $x=e^u$, $dx=e^u du$

$=\frac{1}{\ln\left(\frac{e}{2}\right)}\left(\frac{e}{2}\right)^u\Big|_0^{\ln t} = \frac{1}{\ln\left(\frac{e}{2}\right)}\left[\left(\frac{e}{2}\right)^{\ln t}-1\right]$

HW9-3

- **HW: 6,7,43**