The Concept of Matrix

I-Wei Lai

Department of Electrical Engineering,
National Taiwan Normal University

September 19, 2023

Definition

- A **matrix** is a rectangular array of numbers or functions which we will enclose in brackets.
- The numbers (or functions) are called **entries** or, less commonly, elements of the matrix.

$$\begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

$$\begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix}, \quad \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ \frac{1}{2} \end{bmatrix}.$$

Definition

- m rows and n columns.
- $m \times n$ **matrix** (read m by n matrix).

$$\mathbf{A} = [a_{jk}] = \begin{array}{cc} \textcolor{blue}{\text{column}} & \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}. & \textcolor{red}{\text{row}} \end{array}$$

Vector

- A **vector** is a matrix with only one row or column.
- Its entries are called the **components** of the vector.

- **Row vector**

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}. \quad \text{Ex: } \mathbf{a} = \begin{bmatrix} 2 & 5 & 0.8 & 0 & 1 \end{bmatrix}.$$

- **Column vector**

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}. \quad \text{Ex: } \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}.$$

Vector

$$\mathbf{A}_{m\times n} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \cdots & \mathbf{A}_n \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1' \\ \mathbf{A}_2' \\ \vdots \\ \mathbf{A}_m' \end{bmatrix}.$$

For instance,

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 \end{bmatrix} \longrightarrow \mathbf{A}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \mathbf{A}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \ \mathbf{A}_3 = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{A}_1' \\ \mathbf{A}_2' \end{bmatrix} \longrightarrow \mathbf{A}_1' = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}, \ \mathbf{A}_2' = \begin{bmatrix} 2 & 4 & 6 \end{bmatrix}. \end{aligned}$$

Equality of Matrices

- Written $\mathbf{A} = \mathbf{B}$
- Two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ are **equal**
  - The same size
  - Corresponding entries are equal
    - $a_{11} = b_{11},\ a_{12} = b_{12},\ \cdots$

$$\mathbf{A}_{m\times n} = \mathbf{B}_{m\times n} \longleftrightarrow a_{ij} = b_{ij}, \quad \forall i,\ j$$

Matrix Algebraic Operation

- **Addition / Subtraction of Matrices**

$$\mathbf{A}_{m\times n} \pm \mathbf{B}_{m\times n} = \mathbf{C}_{m\times n} \;\longleftrightarrow\; a_{ij} \pm b_{ij} = c_{ij}, \quad \forall i,\ j$$

- **Scalar Multiplication (Multiplication by a Number)**

$$k\mathbf{A}_{m\times n} = \mathbf{A}_{m\times n}k = \mathbf{C}_{m\times n} \;\longleftrightarrow\; ka_{ij} = c_{ij}, \quad \forall i,\ j$$

- **Matrix Multiplication (Next Section)**

Rules for Matrix Addition and Scalar Multiplication

(a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

(b) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ (written $\mathbf{A} + \mathbf{B} + \mathbf{C}$)

(c) $\mathbf{A} + \mathbf{0} = \mathbf{A}$

(d) $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$.

$\mathbf{0}$: **zero matrix** (of size $m \times n$),
that is, the $m \times n$ matrix with all entries zero.

Rules for Matrix Addition and Scalar Multiplication(conti.)

$$\begin{array}{ll} \text{(a)} & c(\mathbf{A}+\mathbf{B}) = c\mathbf{A} + c\mathbf{B} \\ \text{(b)} & (c+k)\mathbf{A} = c\mathbf{A} + k\mathbf{A} \\ \text{(c)} & c(k\mathbf{A}) = (ck)\mathbf{A} \quad (\text{written } ck\mathbf{A}) \\ \text{(d)} & 1\mathbf{A} = \mathbf{A}. \end{array}$$

Matrix addition is *commutative* and *associative*.

Matrix Multiplication

Multiplication of a Matrix by a Matrix

- **Product $\mathbf{C} = \mathbf{AB}$**

$$\mathbf{A}_{m\times n}\ \mathbf{B}_{r\times p} = \mathbf{C}_{m\times p}, \quad r = n$$

$$\longrightarrow \quad \mathbf{A}_{m\times n}\ \mathbf{B}_{n\times p} = \mathbf{C}_{m\times p}.$$

$$\mathbf{AB} = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & 14 & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}.$$

Multiplication of a Matrix by a Matrix

- $n = 3$

$$m = 4 \left\{ \overbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}}^{n=3} \right. \overbrace{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}}^{p=2} = \left. \overbrace{\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \\ c_{41} & c_{42}. \end{bmatrix}}^{p=2} \right\} m = 4$$

Multiplication of a Matrix by a Matrix

- General form
- $\mathbf{A}_{m\times n}\,\mathbf{B}_{r\times p} = \mathbf{C}_{m\times p},\ \ r = n$

$$
\begin{bmatrix}
a_{11} & \cdots & \cdots & a_{1n} \\
\cdot & \cdots & \cdots & \cdot \\
a_{i1} & a_{i2} & \cdots & a_{in} \\
\cdot & \cdots & \cdots & \cdot \\
a_{m1} & \cdots & \cdots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\
\cdot & \cdots & b_{2j} & \cdots & b_{2p} \\
b_{i1} & \cdots & \cdots & \cdots & \cdot \\
\cdot & \cdots & \cdots & \cdots & \cdot \\
b_{r1} & \cdots & b_{rj} & \cdots & b_{rp}
\end{bmatrix}
=
\begin{bmatrix}
c_{11} & c_{12} & \cdots & \cdots & c_{1p} \\
c_{i1} & c_{i2} & \cdots & \cdots & c_{ip} \\
\cdot & \cdot & c_{ij} & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdots & \cdot \\
c_{m1} & c_{m2} & \cdots & \cdots & c_{mp}
\end{bmatrix}
$$

c_{ij} **: multiplication of** i **rows into** j **columns**

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Matrix Multiplication Is Not Commutative

- **AB $\neq$ BA in General**

$$\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{but} \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} = \begin{bmatrix} 99 & 99 \\ -99 & -99 \end{bmatrix}.$$

$<$ True or False $>$

$\mathbf{AB} = \mathbf{0} \longrightarrow \mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$ ( )

Matrix Multiplication Is Not Commutative

- **AB $\neq$ BA in General**

$$\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{but} \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} = \begin{bmatrix} 99 & 99 \\ -99 & -99 \end{bmatrix}.$$

$<$ True or False $>$

$\mathbf{AB} = \mathbf{0} \longrightarrow \mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$ (X)

Rules for Matrix Multiplication

(a) $(k\mathbf{A})\mathbf{B} = k(\mathbf{AB}) = \mathbf{A}(k\mathbf{B})$ (written $k\mathbf{AB}$ or $\mathbf{A}k\mathbf{B}$)

(b) $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ (written $\mathbf{ABC}$)

(c) $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$

(d) $\mathbf{C}(\mathbf{A} + \mathbf{B}) = \mathbf{CA} + \mathbf{CB}$.

k is any scalar

(b) is called the **associative law**

(c) and (d) are called the **distributive laws**

Example

- What is the solution of

$$\begin{bmatrix}1 & 2 & 3 & 1\\ 2 & 2 & 1 & 2\\ 3 & 1 & 1 & 3\\ 1 & 2 & 3 & 2\end{bmatrix}\begin{bmatrix}1 & 2 & 3 & 4\\ 5 & 6 & 7 & 8\\ 0 & 0 & 0 & 0\\ 9 & 10 & 11 & 12\end{bmatrix}+\begin{bmatrix}1 & -2 & -3 & -1\\ -2 & 2 & 1 & -2\\ -3 & 1 & 1 & 3\\ -1 & -2 & -3 & -2\end{bmatrix}\begin{bmatrix}1 & 2 & 3 & 4\\ 5 & 6 & 7 & 8\\ 0 & 0 & 0 & 0\\ 9 & 10 & 11 & 12\end{bmatrix}$$

Example

- What is the solution of

$$\begin{bmatrix}1 & 2 & 3 & 1\\ 2 & 2 & 1 & 2\\ 3 & 1 & 1 & 3\\ 1 & 2 & 3 & 2\end{bmatrix}\begin{bmatrix}1 & 2 & 3 & 4\\ 5 & 6 & 7 & 8\\ 0 & 0 & 0 & 0\\ 9 & 10 & 11 & 12\end{bmatrix}+\begin{bmatrix}1 & -2 & -3 & -1\\ -2 & 2 & 1 & -2\\ -3 & 1 & 1 & 3\\ -1 & -2 & -3 & -2\end{bmatrix}\begin{bmatrix}1 & 2 & 3 & 4\\ 5 & 6 & 7 & 8\\ 0 & 0 & 0 & 0\\ 9 & 10 & 11 & 12\end{bmatrix}$$

$< \text{Hint} >$ $\quad \mathbf{AC}+\mathbf{BC}=(\mathbf{A}+\mathbf{B})\mathbf{C}$

Solution

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 3 & 1 & 1 & 3 \\ 1 & 2 & 3 & 2 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 \\ 9 & 10 & 11 & 12 \end{bmatrix}}_{\mathbf{C}} + \underbrace{\begin{bmatrix} 1 & -2 & -3 & -1 \\ -2 & 2 & 1 & -2 \\ -3 & 1 & 1 & 3 \\ -1 & -2 & -3 & -2 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 \\ 9 & 10 & 11 & 12 \end{bmatrix}}_{\mathbf{C}}$$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 \\ 9 & 10 & 11 & 12 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 1 & 2 & 3 & 4 \\ 10 & 12 & 14 & 16 \\ 32 & 36 & 40 & 44 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 20 & 24 & 28 & 32 \\ 64 & 72 & 80 & 88 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example

If $\begin{bmatrix} a & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & b & 1 \end{bmatrix} \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$ and $a > 1$, then $(a,\ b) = ?$

Example

If $\begin{bmatrix} a & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & b & 1 \end{bmatrix} \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$ and $a > 1$, then $(a,\ b) = ?$

$$\begin{bmatrix} a & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} a & 1 & 0 \\ ab & a+b & 1 \\ 0 & ab & a+b \end{bmatrix} \quad \longrightarrow \quad ab = 1,\ \ a+b = 4$$

$$\longrightarrow \quad (4-b)b = -b^2 + 4b = 1$$

$$\longrightarrow \quad b = 2 \pm \sqrt{3} \ \ (\text{take } -) \quad \longrightarrow \quad b = 2 - \sqrt{3}$$

$$\therefore \quad a = \frac{1}{2-\sqrt{3}} = 2 + \sqrt{3} \quad \longrightarrow \quad (a,\ b) = (2+\sqrt{3},\ 2-\sqrt{3}).$$

True or False

- $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$ ( )

- $\mathbf{AB} = \mathbf{AC} \quad \longrightarrow \quad \mathbf{B} = \mathbf{C}$ ( )

- $\mathbf{B} = \mathbf{C} \quad \longrightarrow \quad \mathbf{AB} = \mathbf{AC}$ ( )
- $\mathbf{AB} = \mathbf{0} \quad \longrightarrow \quad \mathbf{A}$ or $\mathbf{B} = \mathbf{0}$ ( )

True or False

- $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$ (X)

  $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \mathbf{A}^2 - \mathbf{AB} + \mathbf{BA} - \mathbf{B}^2 \neq \mathbf{A}^2 - \mathbf{B}^2$

- $\mathbf{AB} = \mathbf{AC} \quad \longrightarrow \quad \mathbf{B} = \mathbf{C}$ ( )

- $\mathbf{B} = \mathbf{C} \quad \longrightarrow \quad \mathbf{AB} = \mathbf{AC}$ ( )
- $\mathbf{AB} = \mathbf{0} \quad \longrightarrow \quad \mathbf{A}$ or $\mathbf{B} = \mathbf{0}$ ( )

True or False

- $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$ (X)

  $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \mathbf{A}^2 - \mathbf{AB} + \mathbf{BA} - \mathbf{B}^2 \neq \mathbf{A}^2 - \mathbf{B}^2$

- $\mathbf{AB} = \mathbf{AC} \quad \longrightarrow \quad \mathbf{B} = \mathbf{C}$ (X) Only if **A** is invertible

  $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 3 & 4 \end{bmatrix}$

  $\mathbf{AB} = \mathbf{AC} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, but $\mathbf{B} \neq \mathbf{C}$.

- $\mathbf{B} = \mathbf{C} \quad \longrightarrow \quad \mathbf{AB} = \mathbf{AC}$ ( )
- $\mathbf{AB} = \mathbf{0} \quad \longrightarrow \quad \mathbf{A}$ or $\mathbf{B} = \mathbf{0}$ ( )

True or False

- $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$ (X)

  $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \mathbf{A}^2 - \mathbf{AB} + \mathbf{BA} - \mathbf{B}^2 \neq \mathbf{A}^2 - \mathbf{B}^2$

- $\mathbf{AB} = \mathbf{AC} \quad \longrightarrow \quad \mathbf{B} = \mathbf{C}$ (X) Only if **A** is invertible

  $$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 3 & 4 \end{bmatrix}$$

  $$\mathbf{AB} = \mathbf{AC} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{but } \mathbf{B} \neq \mathbf{C}.$$

- $\mathbf{B} = \mathbf{C} \quad \longrightarrow \quad \mathbf{AB} = \mathbf{AC}$ (O)
- $\mathbf{AB} = \mathbf{0} \quad \longrightarrow \quad \mathbf{A}$ or $\mathbf{B} = \mathbf{0}$ ( )

True or False

- $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$ (X)

  $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \mathbf{A}^2 - \mathbf{AB} + \mathbf{BA} - \mathbf{B}^2 \neq \mathbf{A}^2 - \mathbf{B}^2$

- $\mathbf{AB} = \mathbf{AC} \quad \longrightarrow \quad \mathbf{B} = \mathbf{C}$ (X) Only if $\mathbf{A}$ is invertible

  $$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 3 & 4 \end{bmatrix}$$

  $$\mathbf{AB} = \mathbf{AC} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{but } \mathbf{B} \neq \mathbf{C}.$$

- $\mathbf{B} = \mathbf{C} \quad \longrightarrow \quad \mathbf{AB} = \mathbf{AC}$ (O)
- $\mathbf{AB} = \mathbf{0} \quad \longrightarrow \quad \mathbf{A} \text{ or } \mathbf{B} = \mathbf{0}$ (X)

Transposition

- The **transpose of a matrix** by writing its rows as columns (or equivalently its columns as rows)
- $\mathbf{A}$ is the given matrix, then we denote its transpose by $\mathbf{A}^\intercal$

$$\textbf{Row} \xleftrightarrow[\text{Swap}]{\qquad\qquad} \textbf{Column}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \mathbf{A}^\intercal = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ . & . & \cdots & . \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Rules for transposition

$$
\begin{aligned}
&\text{(a)} \quad (\mathbf{A}^\intercal)^\intercal = \mathbf{A} \\
&\text{(b)} \quad (\mathbf{A}+\mathbf{B})^\intercal = \mathbf{A}^\intercal + \mathbf{B}^\intercal \\
&\text{(c)} \quad (c\mathbf{A})^\intercal = c\mathbf{A}^\intercal \\
&\text{(d)} \quad (\mathbf{A}\mathbf{B})^\intercal = \mathbf{B}^\intercal\mathbf{A}^\intercal.
\end{aligned}
$$

Special Matrices

- **Symmetric Matrix**
  - Square matrices
  - Transpose equals the matrix itself

$\mathbf{A}^\mathsf{T} = \mathbf{A}$

(thus $a_{kj} = a_{jk}$).

- **Skew-Symmetric Matrix**
  - Square matrices
  - Transpose equals **minus** the matrix

$\mathbf{A}^\mathsf{T} = -\mathbf{A}$

(thus $a_{kj} = -a_{jk},\ a_{jj} = 0$).

Example

- **Symmetric Matrix**
  - Square matrices
  - Transpose equals the matrix itself

$$\mathbf{A} = \begin{bmatrix} 20 & 120 & 200 \\ 120 & 10 & 150 \\ 200 & 150 & 30 \end{bmatrix}$$

- **Skew-Symmetric Matrix**
  - Square matrices
  - Transpose equals **minus** the matrix

$$\mathbf{B} = \begin{bmatrix} 0 & 120 & 200 \\ -120 & 0 & 150 \\ -200 & -150 & 0 \end{bmatrix}$$

Example

- Given $\mathbf{B} = \begin{bmatrix} 12 & 11 & -32 \\ -5 & 9 & 30 \\ 32 & -18 & 15 \end{bmatrix}$, write $\mathbf{B}$ as a sum of

  a symmetric and a skew-symmetric matrix.

$\mathbf{B} = \mathbf{C} + \mathbf{D}$

$$\mathbf{C} = \frac{\mathbf{B} + \mathbf{B}^\intercal}{2} = \begin{bmatrix} 12 & 3 & 0 \\ 3 & 9 & 6 \\ 0 & 6 & 15 \end{bmatrix}, \quad \mathbf{D} = \frac{\mathbf{B} - \mathbf{B}^\intercal}{2} = \begin{bmatrix} 0 & 8 & -32 \\ -8 & 0 & 24 \\ 32 & -24 & 0 \end{bmatrix}$$

Triangular Matrices

- **Upper triangular matrices**
  - Square matrices
  - Any entry below the diagonal must be zero

$$a_{ij} = 0; \;\; i > j$$

- **Lower triangular matrices**
  - Square matrices
  - Any entry on the diagonal must be zero

$$a_{ij} = 0; \;\; i < j$$

Example

- **Upper triangular matrices**
  - Square matrices
  - Any entry below the diagonal must be zero

$$\mathbf{A} = \begin{bmatrix} 20 & 120 & 200 \\ \color{red}{0} & 10 & 150 \\ \color{red}{0} & \color{red}{0} & 30 \end{bmatrix}$$

- **Lower triangular matrices**
  - Square matrices
  - Any entry on the diagonal must be zero

$$\mathbf{B} = \begin{bmatrix} 20 & \color{red}{0} & \color{red}{0} \\ 120 & 10 & \color{red}{0} \\ 200 & 150 & 30 \end{bmatrix}$$

Special Matrices

- **Diagonal Matrices**
  - Any entry above or below the main diagonal must be zero.

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & a_{22} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & \cdots & \cdots & a_{nn} \end{bmatrix}, \qquad a_{ij} = 0\ ;\ i \neq j.$$

Special Matrices

- **Unit matrix** (or **Identity matrix**)
  - Entries on the main diagonal are all 1.

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \qquad \mathbf{AI} = \mathbf{IA} = \mathbf{A}.$$

Gauss Elimination

Linear System

- A linear system: **m** equations in **n** unknowns $x_1, \cdots, x_n$

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m. \end{aligned} \tag{1}$$

- If b_j
  - are all zero $\longrightarrow$ (1) is a **homogeneous system**
    - always has at least **trivial solution:** $x_1 = 0, \cdots, x_n = 0$
  - at least one is not zero $\longrightarrow$ **nonhomogeneous system**

Matrix Form

$$\begin{aligned} a_{11}x_1 + \;\cdots\; + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \;\cdots\; + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + \;\cdots\; + a_{mn}x_n &= b_m. \end{aligned} \qquad \longrightarrow \qquad \mathbf{Ax} = \mathbf{b}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Example

- The complete solution to $\mathbf{Ax} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$ is $\mathbf{x} = \alpha \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \;\; \alpha \in \mathbb{R}$.
  Find $\mathbf{A}$.

Example

- The complete solution to $\mathbf{Ax} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$ is $\mathbf{x} = \alpha \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad \alpha \in \mathbb{R}$.

  Find $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \longrightarrow \quad \begin{aligned} 2b &= 0 \\ 2d &= 0 \end{aligned}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} \quad \longrightarrow \quad \begin{aligned} a &= -1 \\ c &= 1 \end{aligned}$$

$$\therefore \ \mathbf{A} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$$

Matrix Form (conti.)

- Augmented Matrix

$$\tilde{\mathbf{A}} = [\mathbf{A}|\mathbf{b}] = \begin{array}{c} \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] \\ \begin{array}{cccc} \uparrow & & & \uparrow \\ x_1 & & & x_n \end{array} \end{array}$$

$$\mathbf{Ax} = \mathbf{b} \quad \longrightarrow \quad \begin{array}{l} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m. \end{array}$$

Elementary Operation

- Interchange of two rows (r_{ij})
- Multiplication of a row by a **nonzero** constant k ($r_i(k)$)
- Addition of a constant multiple of one row to another row ($r_{ij}(k)$)

$$\begin{cases} 2x + y = 2 \\ x - 2y = 1 \end{cases}$$

Row equivalent system

: Have the same set of solutions

Equations $\begin{cases} 2x + y = 2 \\ x - 2y = 1 \end{cases}$ $\begin{cases} 4x + 2y = 4 \\ x - 2y = 1 \end{cases}$ $\begin{cases} 5x = 5 \\ x - 2y = 1 \end{cases}$

Matrices $\left[\begin{array}{cc|c} 2 & 1 & 2 \\ 1 & -2 & 1 \end{array}\right]$ $\left[\begin{array}{cc|c} 4 & 2 & 4 \\ 1 & -2 & 1 \end{array}\right]$ $\left[\begin{array}{cc|c} 5 & 0 & 5 \\ 1 & -2 & 1 \end{array}\right]$

Gauss Elimination

- Solve the linear system

$$\begin{array}{rrrcr} x_1 & -x_2 & +x_3 & = & 0 \\ -x_1 & +x_2 & -x_3 & = & 0 \\ & 10x_2 & +25x_3 & = & 90 \\ 20x_1 & +10x_2 & & = & 80 \end{array}$$

$$\text{Augmented Matrix } \tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right].$$

Gauss Elimination (conti.)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right] \qquad \begin{array}{rrrcr} x_1 & -x_2 & +x_3 & = & 0 \\ -x_1 & +x_2 & -x_3 & = & 0 \\ & 10x_2 & +25x_3 & = & 90 \\ 20x_1 & +10x_2 & & = & 80 \end{array}$$

$$\Big\downarrow \; r_{12}(1) \quad r_{14}(-20)$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right] \qquad \begin{array}{rrrcr} x_1 & -x_2 & +x_3 & = & 0 \\ & & 0 & = & 0 \\ & 10x_2 & +25x_3 & = & 90 \\ & 30x_2 & -20x_3 & = & 80 \end{array}$$

Gauss Elimination (conti.)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right] \qquad \begin{array}{rrrcr} x_1 & -x_2 & +x_3 & = & 0 \\ & & 0 & = & 0 \\ & 10x_2 & +25x_3 & = & 90 \\ & 30x_2 & -20x_3 & = & 80 \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{rrrcr} x_1 & -x_2 & +x_3 & = & 0 \\ & 10x_2 & +25x_3 & = & 90 \\ & 30x_2 & -20x_3 & = & 80 \\ & & 0 & = & 0 \end{array}$$

Gauss Elimination (conti.)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{rrrcr} x_1 & -x_2 & +x_3 & = & 0 \\ & 10x_2 & +25x_3 & = & 90 \\ & 30x_2 & -20x_3 & = & 80 \\ & & 0 & = & 0 \end{array}$$

$$\Big\downarrow\, r_{23}(-3)$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{rrrcr} x_1 & -x_2 & +x_3 & = & 0 \\ & 10x_2 & +25x_3 & = & 90 \\ & & -95x_3 & = & -190 \\ & & 0 & = & 0 \end{array} \;\Big\uparrow$$

Solve $x_3,\ x_2,\ x_1$

Example

- To solve the problem by Gauss Elimination Method

$$
\begin{array}{rrrrcl}
w & +x & +y & & = & 6 \\
-3w & -17x & +y & +2z & = & 2 \\
4w & -17x & +8y & -5z & = & 2 \\
 & -5x & -2y & +z & = & 2
\end{array}
$$

Solution

$$\tilde{\mathbf{A}} = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 6 \\ -3 & -17 & 1 & 2 & 2 \\ 4 & -17 & 8 & -5 & 2 \\ 0 & -5 & -2 & 1 & 2 \end{array}\right] \xrightarrow[r_{13}(-4)]{r_{12}(3)} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 6 \\ 0 & -14 & 4 & 2 & 20 \\ 0 & -21 & 4 & -5 & -22 \\ 0 & -5 & -2 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{r_2(-\frac{1}{14})} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 6 \\ 0 & 1 & -\frac{2}{7} & -\frac{1}{7} & -\frac{10}{7} \\ 0 & -21 & 4 & -5 & -22 \\ 0 & -5 & -2 & 1 & 2 \end{array}\right] \xrightarrow[r_{24}(5)]{r_{23}(21)} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 6 \\ 0 & 1 & -\frac{2}{7} & -\frac{1}{7} & -\frac{10}{7} \\ 0 & 0 & -2 & -8 & -52 \\ 0 & 0 & -\frac{24}{7} & \frac{2}{7} & -\frac{36}{7} \end{array}\right]$$

Solution (conti.)

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 6 \\ 0 & 1 & -\frac{2}{7} & -\frac{1}{7} & -\frac{10}{7} \\ 0 & 0 & -2 & -8 & -52 \\ 0 & 0 & -\frac{24}{7} & \frac{2}{7} & -\frac{36}{7} \end{array}\right] \xrightarrow{r_3(-\frac{1}{2})} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 6 \\ 0 & 1 & -\frac{2}{7} & -\frac{1}{7} & -\frac{10}{7} \\ 0 & 0 & 1 & 4 & 26 \\ 0 & 0 & -\frac{24}{7} & \frac{2}{7} & -\frac{36}{7} \end{array}\right]$$

$$\xrightarrow{r_{34}(\frac{24}{7})} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 6 \\ 0 & 1 & -\frac{2}{7} & -\frac{1}{7} & -\frac{10}{7} \\ 0 & 0 & 1 & 4 & 26 \\ 0 & 0 & 0 & 14 & 84 \end{array}\right] \xrightarrow{r_4(\frac{1}{14})} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 6 \\ 0 & 1 & -\frac{2}{7} & -\frac{1}{7} & -\frac{10}{7} \\ 0 & 0 & 1 & 4 & 26 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right]$$

Solution (conti.)

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 6 \\ 0 & 1 & -\frac{2}{7} & -\frac{1}{7} & -\frac{10}{7} \\ 0 & 0 & 1 & 4 & 26 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right] \longrightarrow \begin{array}{rrrrcr} w & +x & +y & & = & 6 \\ & x & -\frac{2}{7}y & -\frac{1}{7}z & = & -\frac{10}{7} \\ & & y & +4z & = & 26 \\ & & & z & = & 6 \end{array}$$

$$\longrightarrow \begin{cases} w = 4 \\ x = 0 \\ y = 2 \\ z = 6 \end{cases} \longrightarrow \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \\ 6 \end{bmatrix}$$

Exercise 1

Triangular matrix. If $\mathbf{U}_1$, $\mathbf{U}_2$ are upper triangular and $\mathbf{L}_1$, $\mathbf{L}_2$ are lower triangular, which of the following are triangular?

- $\mathbf{U}_1 + \mathbf{U}_2$
- $\mathbf{U}_1\mathbf{U}_2$
- $\mathbf{U}_1^2$
- $\mathbf{U}_1 + \mathbf{L}_1$
- $\mathbf{L}_1 + \mathbf{L}_2$

Exercise 2

Let $\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ -2 & 1 & 4 \\ 1 & 2 & -2 \end{bmatrix}$, $\quad \mathbf{B} = \begin{bmatrix} -1 & 3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Showing all intermediate results, calculate the following expressions or give reasons why they are undefined:

- $\mathbf{AB}$
- $\mathbf{AB}^\intercal$
- $\mathbf{BA}$
- $\mathbf{B}^\intercal\mathbf{A}$

Exercise 3

Let $\mathbf{B} = \begin{bmatrix} -1 & 3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \\ 2 & 0 \end{bmatrix}.$

Showing all intermediate results, calculate the following expressions or give reasons why they are undefined:

- $\mathbf{C}\mathbf{C}^\mathsf{T}$
- $\mathbf{B}\mathbf{C}$
- $\mathbf{C}\mathbf{B}$
- $\mathbf{C}^\mathsf{T}\mathbf{B}$

Exercise 4

Let $\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ -2 & 1 & 4 \\ 1 & 2 & -2 \end{bmatrix}$, $\mathbf{a} = \begin{bmatrix} -1 & -2 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$.

Showing all intermediate results, calculate the following expressions or give reasons why they are undefined:

- $\mathbf{Aa}$
- $\mathbf{Aa}^\intercal$
- $(\mathbf{Ab})^\intercal$
- $\mathbf{b}^\intercal \mathbf{A}^\intercal$

Exercise 5

Profit vector. Two factory outlets F_1 and F_2 in New York and Los Angeles sell sofas (S), chairs (C), and tables (T) with a profit of \$85, \$62, and \$30, respectively. Let the sales in a certain week be given by the matrix

$$\mathbf{A} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \begin{array}{ccc} S & C & T \end{array} & \\ \begin{bmatrix} 400 & 60 & 240 \\ 100 & 120 & 500 \end{bmatrix} & \begin{array}{c} F_1 \\ F_2 \end{array} \end{array}$$

Introduce a "profit vector" $\mathbf{p}$ such that the components of $\mathbf{v} = \mathbf{A}\mathbf{p}$ give the total profits of F_1 and F_2.

Exercise 6

Solve the linear system given explicitly or by its augmented matrix. Show details.

$$\begin{array}{rrcr} -3x & +8y & = & 5 \\ 8x & -12y & = & -11 \end{array}$$

Exercise 7

Solve the linear system given explicitly or by its augmented matrix. Show details.

$$\begin{array}{rrrcr} & 8y & +6z & = & -4 \\ -2x & +4y & -6z & = & 18 \\ x & +y & -z & = & 2 \end{array}$$

Exercise 8

Solve the linear system given explicitly or by its augmented matrix. Show details.

$$\left[\begin{array}{cc|c} 13 & 12 & 6 \\ -4 & 7 & 73 \\ 4 & 5 & 11 \end{array}\right]$$

Exercise 9

Solve the linear system given explicitly or by its augmented matrix. Show details.

$$\begin{array}{rrrcr} x & +y & -z & = & 9 \\ & 8y & +6z & = & -6 \\ -2x & +4y & -6z & = & 40 \end{array}$$

Exercise 10

Solve the linear system given explicitly or by its augmented matrix. Show details.

$$\begin{array}{rrrcl} & -2y & -2z & = & 8 \\ 3x & +4y & -5z & = & 8 \end{array}$$

Gauss-Jordan Elimination

Gauss-Jordan Elimination

- Let $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ -1 & -2 & 1 & 2 & 3 \\ 2 & 4 & -3 & 2 & 0 \\ -3 & -6 & 2 & 0 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 6 \\ 3 \\ 9 \end{bmatrix}.$$

(a) Obtain the reduced row echelon form of $[\mathbf{A}|\mathbf{b}]$

(b) Find the solution

Solution

$$\tilde{\mathbf{A}} = [\mathbf{A}|\mathbf{b}] = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}$$

$$\xrightarrow[\text{Elimination}]{\text{Gauss}} \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & -1 & 6 & 6 & 15 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & -1 & 6 & 6 & 15 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 8 & 8 & 16 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ Row-echelon matrix

Solution

$$\tilde{\mathbf{A}} = [\mathbf{A}|\mathbf{b}] = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}$$

$$\xrightarrow[\text{Elimination}]{\text{Gauss}} \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & -1 & 6 & 6 & 15 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & -1 & 6 & 6 & 15 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 8 & 8 & 16 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot element

Solution (conti.)

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Jordan}} \begin{bmatrix} 1 & 2 & -1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Reduced row-echelon matrix}$$

Solution (conti.)

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{cases} x_1 + 2x_2 - x_5 = -5 \\ x_3 = -3 \\ x_4 + x_5 = 2 \end{cases}$$

Let $x_2 = c_1, \; x_5 = c_2$

$$\mathbf{x} = \begin{bmatrix} -2c_1 + c_2 - 5 \\ c_1 \\ -3 \\ -c_2 + 2 \\ c_2 \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ 0 \\ -3 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{cases} x_1 + 2x_2 - x_5 = -5 \\ x_3 = -3 \\ x_4 + x_5 = 2 \end{cases}$$

Let $x_2 = c_1,\ x_5 = c_2$

$$\mathbf{x} = \begin{bmatrix} -2c_1 + c_2 - 5 \\ c_1 \\ -3 \\ -c_2 + 2 \\ c_2 \end{bmatrix} = \underbrace{\boxed{c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}}}_{\mathbf{x}_H} + \underbrace{\boxed{\begin{bmatrix} -5 \\ 0 \\ -3 \\ 2 \\ 0 \end{bmatrix}}}_{\mathbf{x}_P}$$

Exercise

$$\begin{cases} x_1 & +x_2 & 2x_3 & +x_4 & = & 2 \\ 2x_1 & +x_2 & +x_3 & -x_4 & = & 3 \\ x_1 & -2x_2 & -x_3 & -2x_4 & = & 5 \end{cases}$$

(a) Obtain the reduced row echelon matrix

(b) Find the solution

Solution

$$\begin{cases} x_1 & +x_2 & 2x_3 & +x_4 & = & 2 \\ 2x_1 & +x_2 & +x_3 & -x_4 & = & 3 \\ x_1 & -2x_2 & -x_3 & -2x_4 & = & 5 \end{cases} \longrightarrow \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & -1 \\ 1 & -2 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

(a) $$\begin{bmatrix} 1 & 1 & 2 & 1 & 2 \\ 2 & 1 & 1 & -1 & 3 \\ 1 & -2 & -1 & -2 & 5 \end{bmatrix} \xrightarrow[r_{13}(-1)]{r_{12}(-2)} \begin{bmatrix} 1 & 1 & 2 & 1 & 2 \\ 0 & -1 & -3 & -3 & -1 \\ 0 & -3 & -3 & -3 & 3 \end{bmatrix}$$

$$\xrightarrow[r_3(-\frac{1}{3})]{r_2(-1)} \begin{bmatrix} 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & 3 & 3 & 1 \\ 0 & 1 & 1 & 1 & -1 \end{bmatrix} \xrightarrow{r_{23}(-1)} \begin{bmatrix} 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & -2 & -2 & -2 \end{bmatrix}$$

Solution (conti.)

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & -2 & -2 & -2 \end{bmatrix} \xrightarrow{r_3(-\frac{1}{2})} \begin{bmatrix} 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[r_{32}(-3)]{r_{31}(-2)} \begin{bmatrix} 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{r_{21}(-1)} \begin{bmatrix} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

(b) $$\begin{cases} x_1 \qquad\qquad -x_4 = 2 \\ \quad x_2 \qquad\qquad\; = -2 \\ \qquad\quad x_3 + x_4 = 1 \end{cases} \xrightarrow{\text{Let } x_4 = c} \mathbf{x} = \begin{bmatrix} c+2 \\ -2 \\ 1-c \\ c \end{bmatrix} = c \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

Determinant

Definition

- A **determinant of order** n is a scalar associated with an $n \times n$ (hence **square!**) matrix $\mathbf{A} = [a_{jk}]$, and is denoted by

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Definition

- For $n \geq 2$, the determinant is defined by

$$\det \mathbf{A} = a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots + a_{jn}C_{jn} \qquad j = 1,\ 2, \cdots, \text{or } n$$

$$\det \mathbf{A} = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk} \qquad k = 1,\ 2, \cdots, \text{or } n.$$

- C_{jk}: the cofactor of a_{ij} in $\det \mathbf{A}$ $\longrightarrow$ $C_{jk} = (-1)^{j+k} M_{jk}$.
- M_{jk}: the determinant of the submatrix of **A** obtained from **A** by omitting the j^{th} row and the k^{th} column.

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned} \det \mathbf{A} &= 1 \times (-1)^{1+1} \begin{vmatrix} 6 & 4 \\ 0 & 2 \end{vmatrix} + 3 \times (-1)^{1+2} \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} + 0 \times (-1)^{1+3} \begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix} \\ &= 0 \times (-1)^{1+3} \begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix} + 4 \times (-1)^{2+3} \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} + 2 \times (-1)^{3+3} \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} \\ &= -12. \end{aligned}$$

Theorem

(a) *Interchange of two rows multiplies the value of the determinant by* -1.

(b) *Addition of a multiple of a row to another row does not alter the value of the determinant.*

(c) *Multiplication of a row by a nonzero constant* c *multiplies the value of the determinant by* c.

$$\det c\mathbf{A} = c^n \det \mathbf{A}$$

Application

- Evaluate determinants by reduction to triangular form, as in the Gauss elimination for a matrix

$$\det\begin{bmatrix} 2 & 0 & -4 & 6 \\ 4 & 5 & 1 & 0 \\ 0 & 2 & 6 & -1 \\ -3 & 8 & 9 & 1 \end{bmatrix} = \det\begin{bmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 2 & 6 & -1 \\ 0 & 8 & 3 & 10 \end{bmatrix}$$

$$= \det\begin{bmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & -11.4 & 29.2 \end{bmatrix} = \det\begin{bmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & 0 & 47.25 \end{bmatrix}$$

Discussion

If $\mathbf{A}$ is a

- Upper triangular matrix
- Lower triangular matrix
- Diagonal matrix,

then

$$\det \mathbf{A} = a_{11} \times a_{22} \times \cdots \times a_{nn}.$$

Theorem

(d) $\det \mathbf{A} = \det(\mathbf{A}^\top)$.

(e) *A zero row or column renders the value of a determinant zero.*

(f) *Proportional rows or columns render the value of a determinant zero. In particular, a determinant with two identical rows or columns has the value zero.*

(g) For any $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$

$$\det \mathbf{AB} = \det \mathbf{BA} = \det \mathbf{A} \times \det \mathbf{B}$$

Inverse of a Matrix

Definition

- The inverse of an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is denoted by $\mathbf{A}^{-1}$ and is an $n \times n$ matrix such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n.$$

- *If* $\mathbf{A}$ *has an inverse,* <u>*the inverse is unique*</u>

Assume both $\mathbf{B}$ and $\mathbf{C}$ are inverses of $\mathbf{A}$

$\longrightarrow$ $\mathbf{AB} = \mathbf{I}$ and $\mathbf{CA} = \mathbf{I}$

$\longrightarrow$ $\mathbf{B} = \mathbf{IB} = (\mathbf{CA})\mathbf{B} = \mathbf{C}(\mathbf{AB}) = \mathbf{CI} = \mathbf{C}.$

Definition

- $\mathbf{A}$ has an inverse $\iff$ Exist $\mathbf{A}^{-1}$

  $\iff \det \mathbf{A} \neq 0$

  $\iff$ $\mathbf{A}$ is **nonsingular**

- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

  proof.

  (1)

  $$
  \begin{aligned}
  &(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) \\
  &= \mathbf{B}^{-1}(\mathbf{A}^{-1}\mathbf{A})\mathbf{B} \\
  &= \mathbf{B}^{-1}\mathbf{I}_n\mathbf{B} \\
  &= \mathbf{I}_n
  \end{aligned}
  $$

  (2)

  $$
  \begin{aligned}
  &(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) \\
  &= \mathbf{A}(\mathbf{B}\mathbf{B}^{-1})\mathbf{A}^{-1} \\
  &= \mathbf{A}\mathbf{I}_n\mathbf{A}^{-1} \\
  &= \mathbf{I}_n
  \end{aligned}
  $$

Method of the Inverse

- Adjoint Matrix Method
- Gauss–Jordan Elimination

Adjoint Matrix Method

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}[C_{jk}]^{\mathsf{T}} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \cdot & \cdot & \cdots & \cdot \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

- C_{jk}: the cofactor of a_{ij} in $\det \mathbf{A}$ $\longrightarrow$ $C_{jk} = (-1)^{j+k} M_{jk}$.
- M_{jk}: the determinant of the submatrix of $\mathbf{A}$ obtained from $\mathbf{A}$ by omitting the j^{th} row and the k^{th} column.

Example 1

- $n = 2$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \longrightarrow \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \longrightarrow \mathbf{A}^{-1} = \frac{1}{10} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}$$

Example 2

- $n = 3$

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}, \qquad \det \mathbf{A} = 10,$$

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7, \qquad C_{21} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2, \qquad C_{31} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3,$$

$$C_{12} = -\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = -13, \qquad C_{22} = \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -2, \qquad C_{32} = -\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = 7,$$

Example 2 (conti.)

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}, \qquad \det \mathbf{A} = 10,$$

$$C_{13} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8, \qquad C_{23} = -\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} = 2, \qquad C_{33} = \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = -2,$$

$$\mathbf{A}^{-1} = \frac{1}{10} \begin{bmatrix} -7 & -13 & 8 \\ 2 & -2 & 2 \\ 3 & 7 & -2 \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}.$$

Gauss–Jordan Elimination

- Perform row operations on the augmented matrix $[\mathbf{A}|\mathbf{I}]$.

(1) Gauss Elimination
$$\left[\begin{array}{c|ccc} & 1 & 0 & 0 \\ \mathbf{A} & 0 & 1 & 0 \\ & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & * & * & \\ 0 & 1 & * & \mathbf{H} \\ 0 & 0 & 1 & \end{array}\right]$$

(2) Jordan Elimination
$$\left[\begin{array}{ccc|c} 1 & * & * & \\ 0 & 1 & * & \mathbf{H} \\ 0 & 0 & 1 & \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \mathbf{K} \\ 0 & 0 & 1 & \end{array}\right], \quad \mathbf{K} = \mathbf{A}^{-1}$$

Example

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix},$$

$$[\mathbf{A}|\mathbf{I}] \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 3.5 & 1.5 & 0.5 & 0 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right],$$

Example (conti.)

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 3.5 & 1.5 & 0.5 & 0 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0.6 & 0.4 & -0.4 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right],$$

$$\mathbf{A}^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}.$$

Exercise 1

- $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, then there exist $(a,\ b) \in \mathbb{R}$ such that $(\mathbf{I}_2 - \mathbf{A})^{-1} = a\mathbf{A} + b\mathbf{I}_2$,

  where $(a,\ b) =$?

Exercise 2

- Find the unique solution of the system, using the theorem that the unique solution is $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$ for a nonhomogeneous system $\mathbf{AX} = \mathbf{B}$ when $\mathbf{A}$ is nonsingular.

$$\begin{cases} 4x_1 & +6x_2 & -3x_3 & = & 0 \\ 2x_1 & +3x_2 & -4x_3 & = & 0 \\ x_1 & -x_2 & +3x_3 & = & -7 \end{cases}$$

Exercise 3

- Find the inverse by Gauss-Jordan.

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{bmatrix}$$

Exercise 4

- Find the inverse by Gauss-Jordan.

$$\begin{bmatrix} 0 & -0.2 & 0.75 \\ 0.4 & 1 & 2 \\ 0 & 0 & 8 \end{bmatrix}$$

Exercise 5

- Find the inverse by Gauss-Jordan.

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$$

Exercise 6

- Solve the following system of linear equations by Gauss-Jordan.

$$\begin{cases} -8x & -6y & = & -2 \\ 2x & +5y & = & -1 \end{cases}$$

Exercise 7

- Solve the following system of linear equations by Gauss-Jordan.

$$\begin{cases} 3x & +5y & = & 9 \\ 2x & +3y & = & 5 \end{cases}$$

Exercise 8

- Solve the following system of linear equations by Gauss-Jordan.

$$\left\{\begin{array}{rrrcl} 2x & -y & +3z & = & 5 \\ x & +4y & -2z & = & 1 \\ 3x & +y & +5z & = & 2 \end{array}\right.$$

Exercise 9

- Solve the following system of linear equations by Gauss-Jordan.

$$\left\{\begin{array}{rrrcl} x & +y & -z & = & 7 \\ x & -y & +2z & = & 3 \\ 2x & +y & +z & = & 9 \end{array}\right.$$

Exercise 10

- Solve the following system of linear equations by Gauss-Jordan.

$$\left\{\begin{array}{rrrrcr} 3w & -2x & +5y & -z & = & -8 \\ -w & +3x & -y & +4z & = & 9 \\ -2w & -x & +4y & +9z & = & -9 \\ w & & +3y & +2z & = & -2 \end{array}\right.$$

Vector Space

Definition

- In a **nonempty** set V, there are defined two algebraic operations
  - Vector addition: $\mathbf{a} + \mathbf{b}$
  - Scalar multiplication: $\alpha\mathbf{a}$

- For any two vector **a**, **b** in V, their linear combinations are also elements of V

$$\textbf{(1)} \quad \forall\ \mathbf{a},\ \mathbf{b} \in V \implies \mathbf{a} + \mathbf{b} \in V$$

$$\textbf{(2)} \quad \forall\ \mathbf{a} \in V \implies \alpha\mathbf{a} \in V.$$

Definition (conti.)

- A vector space V satisfies the following properties related to vector addition $\forall\ \mathbf{a},\ \mathbf{b},\ \mathbf{c} \in V$.

(1) Commutativity: $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

(2) Associativity: $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

(3) Additive Identity: Exist a zero vector $\mathbf{0}$ in V such that $\mathbf{a} + \mathbf{0} = \mathbf{a}$

(4) Additive Inverse: Exist a vector $-\mathbf{a} \in V$, such that $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$

Definition (conti.)

- A vector space V satisfies the following properties related to scalar multiplication $\forall$ $\mathbf{a}$, $\mathbf{b} \in V$ and scalars α, β:

(5) Distributivity: $\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}$

(6) Distributivity: $(\alpha + \beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}$

(7) Associativity: $\alpha(\beta\mathbf{a}) = (\alpha\beta)\mathbf{a}$

(8) Multiplicative Identity: $\forall$ $\mathbf{a} \in V$, $1\mathbf{a} = \mathbf{a}$.

Vector Space

- Euclidean Space

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \equiv \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^\mathsf{T};\ x_i \in F \right\}$$

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \equiv \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}^\mathsf{T}$$

Vector Space

- Matrix Space

$$\mathbb{R}^{m\times n} = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} ; \ a_{ij} \in F \right\}$$

$$\mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{m\times n}$$

Linear Combination

- S is given a set of vectors: $\mathbf{a}_1,\ \mathbf{a}_2,\ \cdots,\ \mathbf{a}_n$

$$S = \{\mathbf{a}_1,\ \mathbf{a}_2, \cdots,\ \mathbf{a}_n\}.$$

- A linear combination of these vectors is an expression of this form

$$\mathbf{u} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n$$

$c_1,\ c_2,\ \cdots,\ c_n$ are scalars.

Exercise

- Determine whether the matrix $\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$ is a linear combination of the matrices $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ in the vector space $\mathbb{R}^{2\times 2}$ of 2×2 matrices.

Solution

$$\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

$$\left\{ \begin{array}{rcrrr} -1 & = & c_1 & +2c_2 & \\ 7 & = & & -3c_2 & +c_3 \\ 8 & = & 2c_1 & & +2c_3 \\ -1 & = & c_1 & +2c_2 & \end{array} \right. \longrightarrow \begin{array}{rcr} c_1 & = & 3 \\ c_2 & = & -2 \\ c_3 & = & 1 \end{array}.$$

Linear Independence and Dependence of Vectors

- Consider the set of vectors: $S = \{\mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3\}$. Let the linear combination of these vectors be: $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 = 0$.

$$
\begin{aligned}
\textbf{(1)}\quad \text{If } c_1 \neq 0 \quad &\Longrightarrow \quad c_1\mathbf{a}_1 = -c_2\mathbf{a}_2 - c_3\mathbf{a}_3 \\
&\Longrightarrow \quad \mathbf{a}_1 = -\frac{c_2}{c_1}\mathbf{a}_2 - \frac{c_3}{c_1}\mathbf{a}_3 = \alpha_1\mathbf{a}_2 + \alpha_2\mathbf{a}_3 \\
&\Longleftrightarrow \quad \textbf{Linear dependent.}
\end{aligned}
$$

(2) If $c_1 = c_2 = c_3 = 0$

$$\Longrightarrow \quad \mathbf{a}_1 \neq \alpha_1 \mathbf{a}_2 + \alpha_2 \mathbf{a}_3$$

$$\Longrightarrow \quad \mathbf{a}_2 \neq \beta_1 \mathbf{a}_1 + \beta_2 \mathbf{a}_3$$

$$\Longrightarrow \quad \mathbf{a}_3 \neq \gamma_1 \mathbf{a}_1 + \gamma_2 \mathbf{a}_2$$

$\Longleftrightarrow$ **Linear independent**.

Linear Independence and Dependence of Vectors

- **Definition:**

Let $S = \{\mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3\}$ be a given set of vectors and consider the linear combination:

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 = 0.$$

- **(1)** If the scalars c_i are not all zero, then S is **linearly dependent**.
- **(2)** If $c_1 = c_2 = c_3 = 0$, then S is **linearly independent**.

Example

- Find all values of h so that the following vectors are linearly independent

$$x = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \quad y = \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \quad z = \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}.$$

Solution 1

$$\det \begin{bmatrix} 1 & -5 & 1 \\ -1 & 7 & 1 \\ -3 & 8 & h \end{bmatrix} = 2h + 20 \neq 0, \qquad \textbf{Ans: } h \neq -10.$$

(a) $\det \mathbf{A} \neq 0 \iff \operatorname{rank}(\mathbf{A}) = n$

$\iff$ The rows and columns of $\mathbf{A}$ are linear independent.

(b) $\det \mathbf{A} = 0 \iff \operatorname{rank}(\mathbf{A}) < n$

$\iff$ The rows and columns of $\mathbf{A}$ are linear dependent.

Solution 2

$$\begin{bmatrix} 1 & -1 & -3 \\ -5 & 7 & 8 \\ 1 & 1 & h \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -3 \\ 0 & 2 & -7 \\ 0 & 2 & h+3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -3 \\ 0 & 2 & -7 \\ 0 & 0 & h+10 \end{bmatrix}$$

$$\therefore \; h + 10 \neq 0.$$

Span

- The set of all **linear combinations** of given vectors $\mathbf{a}_1,\ \mathbf{a}_2,\ \cdots,\ \mathbf{a}_n$ with the same number of components.

$$S = \{\mathbf{a}_1,\ \mathbf{a}_2,\ \cdots,\ \mathbf{a}_n\},$$

$$\mathrm{span}(S) \equiv \{\mathbf{x} \mid \mathbf{x} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n\}.$$

- $\mathrm{span}(S)$ is a vector space.

Example

- $S_1 = \{[1,\ 0,\ 0]\}$

$$\text{span}(S_1) \equiv \{\mathbf{x} | \mathbf{x} = c[1,\ 0,\ 0]\} = \{\mathbf{x} \mid \mathbf{x} = [c,\ 0,\ 0]\} = x\text{-axis.}$$

- $S_2 = \{[1,\ 0,\ 0],\ [0,\ 1,\ 0]\}$

$$\begin{aligned}\text{span}(S_2) &\equiv \{\mathbf{x} \mid \mathbf{x} = c_1[1,\ 0,\ 0] + c_2[0,\ 1,\ 0]\} \\ &= \{\mathbf{x} \mid \mathbf{x} = [c_1,\ c_2,\ 0]\} = x\text{-}y \text{ plane.}\end{aligned}$$

Row Space and Column Space

$$\mathbf{A}_{m\times n} = \begin{bmatrix} \mathbf{A'}_1 \\ \mathbf{A'}_2 \\ \vdots \\ \mathbf{A'}_m \end{bmatrix} = [\mathbf{A}_1,\ \mathbf{A}_2,\ \cdots,\ \mathbf{A}_n]$$

$$\text{Row}(\mathbf{A}) = \text{row}(\mathbf{A}) \equiv \alpha_1\mathbf{A'}_1 + \alpha_2\mathbf{A'}_2 + \cdots + \alpha_m\mathbf{A'}_m$$

$$\text{Col}(\mathbf{A}) = \text{col}(\mathbf{A}) \equiv \beta_1\mathbf{A}_1 + \beta_2\mathbf{A}_2 + \cdots + \beta_n\mathbf{A}_n$$

Row Space and Column Space

- Example

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}_{2\times 3}$$

$$\textbf{(1)} \quad \mathrm{row}(\mathbf{A}) = \alpha_1 \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} = \mathrm{span}\{\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}\}$$

$$\textbf{(2)} \quad \mathrm{col}(\mathbf{A}) = \beta_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \beta_3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}.$$

Range Space and Null Space

$$\mathbf{A}_{m\times n} = \begin{bmatrix} \mathbf{A}'_1 \\ \mathbf{A}'_2 \\ \vdots \\ \mathbf{A}'_m \end{bmatrix} = [\mathbf{A}_1,\ \mathbf{A}_2,\ \cdots,\ \mathbf{A}_n],$$

$$\text{Range}(\mathbf{A}) = \text{col}(\mathbf{A}) \equiv \beta_1\mathbf{A}_1 + \beta_2\mathbf{A}_2 + \cdots + \beta_n\mathbf{A}_n,$$

$$\begin{aligned} \text{Null}(\mathbf{A}) = \ker(\mathbf{A}) &\equiv \{\mathbf{x} \mid \mathbf{A}\mathbf{x} = 0;\ \forall \mathbf{x} \in \mathbb{R}^{n\times 1}\} \\ &\equiv \text{the solution set of the homogeneous system } \mathbf{A}\mathbf{x} = 0. \end{aligned}$$

- Example

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}_{2\times 3}$$

$$\textbf{(3)} \quad \text{Range}(\mathbf{A}) = \text{col}(\mathbf{A}) = \beta_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \beta_3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}.$$

- Example

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}_{2\times 3}$$

(4) $$\mathrm{Null}(\mathbf{A}) \equiv \left\{ \mathbf{x} \mid \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \right\} = \mathrm{span} \left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\} \quad \because \ \mathbf{x} = c \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}.$$

Basis and Dimension

- V is a vector space and S is a subset in V, if:
  - The vectors in S are linearly independent
  - $\text{span}(S) = V$

  $\implies$ S is a **basis** for the vector space V

- $\dim(V)$: **Dimension** of V
  - The maximum number of linearly independent vectors in V
  - The number of vectors of a basis for V

Basis and Dimension

- $\mathbb{R}^n$: $S = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$ $\dim(\mathbb{R}^n) = n.$

- $\mathbb{R}^{m\times n}$: $S = \left\{ M_{11} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, M_{12} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \cdots \right.$

  $\left. , M_{mn} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \right\}$ $\dim(\mathbb{R}^{m\times n}) = m \times n.$

Rank

- **Definition:**

  The **rank** of a matrix $\mathbf{A}_{m\times n}$ is the maximum number of linearly independent row vectors of $\mathbf{A}$.

$$
\begin{aligned}
\text{rank}(\mathbf{A}) &= \dim(\text{Range}(\mathbf{A})) \\
&= \dim(\text{Col}(\mathbf{A})) \\
&= \dim(\text{Row}(\mathbf{A})) \\
&= \text{the number of the pivot elements} \\
&= \text{rank}(\mathbf{A}^\intercal).
\end{aligned}
$$

Nullity

- **Definition:**

  The dimension of the null space of $\mathbf{A}_{m\times n}$.

$$\begin{aligned}\text{nullity}(\mathbf{A}) &= \dim(\text{Null}(\mathbf{A}))\\ &= \text{The number of } c_i \text{ in the solution of the}\\ &\quad\ \text{homogeneous system } (\mathbf{Ax}=0)\\ &= \text{Number of columns of } \mathbf{A} - \text{rank}(\mathbf{A}).\end{aligned}$$

$$\textbf{rank(A)} + \textbf{nullity(A)} = \textbf{Number of columns of A}$$

Example

- If $\mathbf{B} = \begin{bmatrix} 2 & 1 & 3 & 1 & 0 & 4 \\ 1 & 2 & 3 & -1 & 1 & 5 \\ 0 & 1 & 1 & -1 & 2 & 2 \\ 1 & 1 & 2 & 0 & 2 & 3 \end{bmatrix}$ then $\text{rank}(\mathbf{B}) = ?$

Solution

$$\mathbf{B} \xrightarrow{r_{14}} \begin{bmatrix} 1 & 1 & 2 & 0 & 2 & 3 \\ 1 & 2 & 3 & -1 & 1 & 5 \\ 0 & 1 & 1 & -1 & 2 & 2 \\ 2 & 1 & 3 & 1 & 0 & 4 \end{bmatrix} \xrightarrow[r_{14}(-2)]{r_{12}(-1)} \begin{bmatrix} 1 & 1 & 2 & 0 & 2 & 3 \\ 0 & 1 & 1 & -1 & -1 & 2 \\ 0 & 1 & 1 & -1 & 2 & 2 \\ 0 & -1 & -1 & 1 & -4 & -2 \end{bmatrix}$$

$$\xrightarrow[r_{24}(1)]{r_{23}(-1)} \begin{bmatrix} 1 & 1 & 2 & 0 & 2 & 3 \\ 0 & 1 & 1 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & -5 & 0 \end{bmatrix} \xrightarrow[r_{34}(5)]{r_3(\frac{1}{3})} \begin{bmatrix} 1 & 1 & 2 & 0 & 2 & 3 \\ 0 & 1 & 1 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Ans: $\text{rank}(\mathbf{A}) = 3$

Exercise

- $\mathbf{A} = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix}$

(a) Row reduce $\mathbf{A}$ to its reduced echelon form

(b) Find a basis for $\mathrm{col}(\mathbf{A})$ and $\mathrm{row}(\mathbf{A})$

(c) Find a basis for $\mathrm{Null}(\mathbf{A})$

(d) Evaluate $\mathrm{rank}(\mathbf{A})$, $\dim(\mathrm{Null}(\mathbf{A}))$, $\mathrm{rank}(\mathbf{A}^\intercal)$, $\dim(\mathrm{Null}(\mathbf{A}^\intercal))$

Solution

(a) $\mathbf{A}_R = \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -\frac{7}{5} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix} \right\}$ is a basis for col($\mathbf{A}$)

$\left\{ \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \end{bmatrix}, \begin{bmatrix} 2 & 4 & -5 & 15 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \end{bmatrix} \right\}$
is a basis for row($\mathbf{A}$)

Solution (conti.)

(c) $\begin{cases} x_1 + 2x_2 + 4x_4 = 0 & \longrightarrow & x_2 = c_1 & \longrightarrow & x_1 = -2c_1 - 4c_2 \\ x_3 - \frac{7}{5}x_4 = 0 & \longrightarrow & x_4 = c_2 & \longrightarrow & x_3 = \frac{7}{5}c_2 \\ x_5 = 0 & \longrightarrow & x_5 = 0 & & \end{cases}$

$$\left\{ \mathbf{x} \mid \mathbf{x} = \begin{bmatrix} -2c_1 - 4c_2 \\ c_1 \\ \frac{7}{5}c_2 \\ c_2 \\ 0 \end{bmatrix} \right\} = \left\{ \mathbf{x} = c_1 \boxed{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} + c_2 \boxed{\begin{bmatrix} -4 \\ 0 \\ \frac{7}{5} \\ 1 \\ 0 \end{bmatrix}} \right\}$$

Solution (conti.)

(d) $\text{rank}(\mathbf{A}) = \dim(\text{col}(\mathbf{A})) = \dim(\text{row}(\mathbf{A})) = 3$

$\dim(\text{Null}(\mathbf{A})) = 5 - \text{rank}(\mathbf{A}) = 2$

$\text{rank}(\mathbf{A}^\intercal) = \text{rank}(\mathbf{A}) = 3$

$\dim(\text{Null}(\mathbf{A}^\intercal)) = 4 - \text{rank}(\mathbf{A}^\intercal) = 1.$

Exercise 1

- Prove vector addition is commutative.

  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \quad \forall\ \mathbf{u}, \mathbf{v} \in \mathbb{R}^n$

Exercise 2

- Prove vector addition is associative.

  $$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{v} + (\mathbf{u} + \mathbf{w}) \quad \forall\ \mathbf{u},\ \mathbf{v},\ \mathbf{w} \in \mathbb{R}^n$$

Exercise 3

- Prove scalar multiplication is distributive.

  $c(\mathbf{u} + \mathbf{v}) = c\mathbf{v} + c\mathbf{u} \quad \forall\ \mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and scalar c

Exercise 4

- Let $\mathbf{x}_1 = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$

  Is $\mathbf{b} = \begin{bmatrix} -7 \\ -15 \\ 6 \end{bmatrix}$ a linear combination of x_1, x_2, x_3 ?

Exercise 5

- $\left\{ \begin{bmatrix} 8 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} -13 \\ 6 \\ 3 \end{bmatrix} \right\} = V$. Is V linearly independent?

Exercise 6

- $\mathbf{A} = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$. Find the row space and column space of $\mathbf{A}$.

Exercise 7

- $\mathbf{B} = \begin{bmatrix} 3 & -6 & 9 & 0 \\ 2 & -4 & 7 & 2 \\ 3 & -6 & 6 & -6 \end{bmatrix}$. Find the null space of $\mathbf{B}$.

Exercise 8

- Find a basis and the dimension for $\operatorname{span}\left(\left\{\begin{bmatrix}1\\-4\\-3\end{bmatrix}, \begin{bmatrix}-3\\6\\7\end{bmatrix}, \begin{bmatrix}-4\\-2\\6\end{bmatrix}\right\}\right)$.

Exercise 9

- Find the ranks of the following matrices:

$$\mathbf{C} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & 4 \\ 4 & 1 & -3 \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \\ 2 & 4 & 2 \end{bmatrix},$$

$$\mathbf{E} = \begin{bmatrix} 3 & 2 & 1 \\ -6 & -4 & -2 \end{bmatrix}.$$

Exercise 10

- Find the null space and the nullity of $\mathbf{F} = \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$.

Orthogonal Matrix

Inner Product

- If $V = \mathbb{R}^n$; $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

$$\begin{aligned} \text{inner product } \langle \mathbf{x}, \mathbf{y} \rangle &\equiv x_1\overline{y_1} + x_2\overline{y_2} + \cdots + x_n\overline{y_n} \\ &\equiv \overline{\mathbf{y}}^\mathsf{T}\mathbf{x}. \end{aligned}$$

Inner Product

- If $V = C_{[a,b]}$; $f(x)$, $g(x)$; $x \in [a, b]$

$$\text{inner product } \langle f(x), g(x) \rangle \equiv \int_a^b w_k \cdot f(x)\overline{g(x)}\, dx$$

$$w_k : \text{weight.}$$

Norm

Definition

- $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$: length

$$\textbf{(1)} \quad \text{If}\|\mathbf{x}\| = 1 \ : \ \text{Unit Vecter}$$

$$\textbf{(2)} \quad \frac{\mathbf{x}}{\|\mathbf{x}\|} \ : \ \text{Normalize}$$

Example

- Consider the vector $\mathbf{u} = (2 + i,\ -1 - i),\ \mathbf{v} = (1 - i,\ 2 - i)$ in $\mathbb{C}^2$

  (a) Show $\mathbf{u}$ and $\mathbf{v}$ are orthogonal ($\langle \mathbf{u}, \mathbf{v} \rangle = 0$)

  (b) What is vector normal, $\|\mathbf{v}\|$?

Solution

(a) $\langle \mathbf{u}, \mathbf{v} \rangle = (1+i)(2+i) + (2+i)(-1-i) = (1+3i) + (-1-3i) = 0$

(b) $\|\mathbf{v}\| = \sqrt{(1+i)(1-i) + (2+i)(2-i)} = \sqrt{7}.$

Orthogonal Matrix

- $\mathbf{A}$ is an orthogonal, unitary matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \cdots & \mathbf{A}_n \end{bmatrix} \qquad \langle \mathbf{A}_i, \mathbf{A}_j \rangle \equiv \overline{\mathbf{A}}_j^{\mathsf{T}} \mathbf{A}_i = \begin{cases} 1 \,; & i = j \\ 0 \,; & i \neq j \end{cases}$$

$$\Longleftrightarrow \quad \overline{\mathbf{A}}^{\mathsf{T}} \mathbf{A} = \mathbf{I}_n$$

How about its determinant?

Orthogonal Matrix

proof.

$$\overline{\mathbf{A}}^\top \mathbf{A} = \begin{bmatrix} \overline{\mathbf{A}}_1^\top \\ \overline{\mathbf{A}}_2^\top \\ \vdots \\ \overline{\mathbf{A}}_n^\top \end{bmatrix} \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \cdots & \mathbf{A}_n \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{A}}_1^\top \mathbf{A}_1 & \overline{\mathbf{A}}_1^\top \mathbf{A}_2 & \cdots & \overline{\mathbf{A}}_1^\top \mathbf{A}_n \\ \overline{\mathbf{A}}_2^\top \mathbf{A}_1 & \overline{\mathbf{A}}_2^\top \mathbf{A}_2 & \cdots & \overline{\mathbf{A}}_2^\top \mathbf{A}_n \\ \cdot & \cdot & \cdots & \cdot \\ \overline{\mathbf{A}}_n^\top \mathbf{A}_1 & \overline{\mathbf{A}}_n^\top \mathbf{A}_2 & \cdots & \overline{\mathbf{A}}_n^\top \mathbf{A}_n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \mathbf{I}_n.$$

Gram-Schmidt Orthogonalization

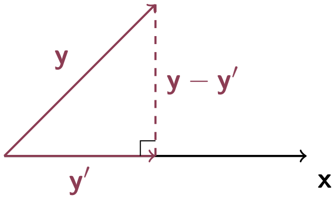
Orthogonal and Orthonormal Set

Orthogonal Set $\{\mathbf{x}_1, \ \mathbf{x}_2, \ \cdots \ , \ \mathbf{x}_n\}$ $\langle \mathbf{x}_i, \mathbf{x}_j \rangle = 0 \ ; \ i \neq j$

$$\Big\downarrow \ \frac{\mathbf{x}_k}{\|\mathbf{x}_k\|}$$

Orthonormal Set $\{\mathbf{e}_1, \ \mathbf{e}_2, \ \cdots \ , \ \mathbf{e}_n\}$ $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \begin{cases} 1 \ ; & i = j \\ 0 \ ; & i \neq j \end{cases}$

Orthogonal Projection



$\mathbf{y}'$: The projection of $\mathbf{y}$ in $\mathbf{x}$

$$\mathbf{y}' = \frac{\langle \mathbf{y}, \mathbf{x} \rangle}{\|\mathbf{x}\|^2}\mathbf{x}$$

proof.

$$\text{Let } \mathbf{y}' = C\mathbf{x}$$

$$\because (\mathbf{y}-\mathbf{y}') \perp \mathbf{x} \quad \Longrightarrow \quad \langle \mathbf{y} - C\mathbf{x}, \mathbf{x} \rangle = 0$$

$$\Longrightarrow \quad \langle \mathbf{y}, \mathbf{x} \rangle - C\langle \mathbf{x}, \mathbf{x} \rangle = 0 \quad \Longrightarrow \quad C = \frac{\langle \mathbf{y}, \mathbf{x} \rangle}{\|\mathbf{x}\|^2}$$

Gram-Schmidt Orthogonalization

- **Goal:** $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\} \longrightarrow \{\mathbf{e}_1,\ \mathbf{e}_2,\ \cdots,\ \mathbf{e}_n\}$

  A set of vectors → Orthonormal Set

- **Step 1:** $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n\} \xrightarrow{\text{GSO}} \{\mathbf{x}_1,\ \mathbf{x}_2,\ \cdots,\ \mathbf{x}_n\}$

  A set of vectors → Orthogonal Set

$$\mathbf{x}_1 = \mathbf{v}_1$$

$$\mathbf{x}_2 = \mathbf{v}_2 - \mathbf{v}_{2\parallel} = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{x}_1 \rangle}{\|\mathbf{x}_1\|^2}\mathbf{x}_1$$

$$\mathbf{x}_3 = \mathbf{v}_3 - \mathbf{v}_{3\parallel} = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{x}_1 \rangle}{\|\mathbf{x}_1\|^2}\mathbf{x}_1 - \frac{\langle \mathbf{v}_3, \mathbf{x}_2 \rangle}{\|\mathbf{x}_2\|^2}\mathbf{x}_2$$

$$\vdots$$

Gram-Schmidt Orthogonalization

- **Step 2:** $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \cdots\ ,\ \mathbf{x}_n\} \xrightarrow{\text{Normalize}} \{\mathbf{e}_1,\ \mathbf{e}_2,\ \cdots\ ,\ \mathbf{e}_n\}$

$$\mathbf{e}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|}$$

$$\mathbf{e}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|}$$

$$\mathbf{e}_3 = \frac{\mathbf{x}_3}{\|\mathbf{x}_3\|}$$

$$\vdots$$

Example

- In R^4, let $S = \{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3\}$,
  where $\mathbf{u}_1 = (1,\ 0,\ 1,\ 0)$, $\mathbf{u}_2 = (1,\ 1,\ 1,\ 1)$, and $\mathbf{u}_3 = (0,\ 1,\ 2,\ 1)$.
  Use the Gram-Schmidt process and compute an orthonormal basis $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$
  for the subspace $\text{span}(S)$.

Solution

$$\{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3\} \quad \longrightarrow \quad \{\mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_3\} : \text{Orthogonal Set}$$

(1) $\mathbf{x}_1 = \mathbf{u}_1 = (1,\ 0,\ 1,\ 0), \quad \|\mathbf{x}_1\|^2 = 2$

$$\mathbf{x}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{x}_1 \rangle}{\|\mathbf{x}_1\|^2}\mathbf{x}_1 = \mathbf{u}_2 - \mathbf{x}_1 = (0,\ 1,\ 0,\ 1), \quad \|\mathbf{x}_2\|^2 = 2$$

$$\mathbf{x}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{x}_1 \rangle}{\|\mathbf{x}_1\|^2}\mathbf{x}_1 - \frac{\langle \mathbf{u}_3, \mathbf{x}_2 \rangle}{\|\mathbf{x}_2\|^2}\mathbf{x}_2 = \mathbf{u}_3 - \mathbf{x}_1 - \mathbf{x}_2 = (-1,\ 0,\ 1,\ 0), \quad \|\mathbf{x}_3\|^2 = 2$$

Solution (conti.)

(2)

$$\mathbf{e}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \frac{1}{\sqrt{2}}(1,\ 0,\ 1,\ 0)$$

$$\mathbf{e}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \frac{1}{\sqrt{2}}(0,\ 1,\ 0,\ 2)$$

$$\mathbf{e}_3 = \frac{\mathbf{x}_3}{\|\mathbf{x}_3\|} = \frac{1}{\sqrt{2}}(-1,\ 0,\ 1,\ 0)$$

Exercise

- Let $\mathbf{A} = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$,

  Find an orthonormal basis for the column space of $\mathbf{A}$.

Do we have an orthonormal matrix?

Solution

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}$$

$$\mathbf{x}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{x}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{x}_1 \rangle}{\|\mathbf{x}_1\|^2}\mathbf{x}_1 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix} - \frac{6}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{5}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Solution (conti.)

$$\mathbf{x}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{x}_1 \rangle}{\|\mathbf{x}_1\|^2}\mathbf{x}_1 - \frac{\langle \mathbf{v}_3, \mathbf{x}_2 \rangle}{\|\mathbf{x}_2\|^2}\mathbf{x}_2 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix} - \frac{4}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\mathbf{e}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{e}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{e}_3 = \frac{\mathbf{x}_3}{\|\mathbf{x}_3\|} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

Exercise 1

- Consider the vectors $\mathbf{p} = (3 + 2i,\ 1 - i),\ \mathbf{q} = (4 - i,\ 2 + 3i)$ in $\mathbb{C}^2$

  (a) Find the inner product of $\mathbf{p}$ and $\mathbf{q}$, i.e., calculate $\langle \mathbf{p}, \mathbf{q} \rangle$.

  (b) Find the norm of vector $\mathbf{p}$, i.e., calculate $\|\mathbf{p}\|$.

Exercise 2

- Consider the vectors $\mathbf{r} = (1 + 2i,\ -3 + 4i)$, $\mathbf{s} = (2 - 3i,\ 5 + 2i)$ in $\mathbb{C}^2$

  (a) Determine whether vectors $\mathbf{r}$ and $\mathbf{s}$ are orthogonal.

  (b) Find the norm of vector $\mathbf{s}$, i.e., calculate $\|\mathbf{s}\|$.

Exercise 3

- $$\mathbf{B} = \begin{bmatrix} 2 & -2 & 18 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix},$$

  Find an orthonormal basis for the column space of $\mathbf{B}$.

Exercise 4

- Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ -2 \end{bmatrix}$,

  and let $S = \text{span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$.

  Find an orthonormal basis for S.

Matrix Decomposition

Why Do We Need Decomposition?

Row Elementary Matrix

- A matrix which performs a row elementary to identity matrix ($\mathbf{I}_n$)
- Example:

$$\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

(1)

$$\mathbf{R}_{12}\ \text{(Permutation matrix)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies \mathbf{R}_{12}\mathbf{A} = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$$

Row Elementary Matrix

(2)

$$\mathbf{R}_2(2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies \mathbf{R}_2(2)\mathbf{A} = \begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{bmatrix}$$

(3)

$$\mathbf{R}_{12}(-2) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies \mathbf{R}_{12}(-2)\mathbf{A} = \begin{bmatrix} a & b & c \\ d-2a & e-2b & f-2c \\ g & h & i \end{bmatrix}$$

Exercise

- Use Gauss-Jordan Elimination to find the inverse of $\mathbf{R}_{12}$, $\mathbf{R}_2(2)$, and $\mathbf{R}_{12}(-2)$

Answer

$$\mathbf{R}_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies \mathbf{R}_{12}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \mathbf{R}_{ij}^{-1} = \mathbf{R}_{ij}$$

$$\mathbf{R}_2(2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies \mathbf{R}_2^{-1}(2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \mathbf{R}_i^{-1}(k) = \mathbf{R}_i(\frac{1}{k})$$

$$\mathbf{R}_{12}(-2) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies \mathbf{R}_{12}^{-1}(-2) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \mathbf{R}_{ij}^{-1}(k) = \mathbf{R}_{ij}(-k)$$

LU Decomposition

- $\mathbf{A} \in \mathbb{R}^{n \times n}$
- Let $\mathbf{A} = \mathbf{LU}$

- $\mathbf{L}$: Lower triangular matrices
- $\mathbf{U}$: Upper triangular matrices

LU Decomposition

- Perform row elementary operation (Gauss Elimination) to $\mathbf{A}$

$$\mathbf{A} \xrightarrow{\text{Gauss Elimination}} \mathbf{U}$$

- Write elementary operation (with lower trianglur matrix) as row elementary matrix

$$\mathbf{L}_1 \, \mathbf{L}_2 \, \ldots \, \mathbf{L}_N$$

- $\mathbf{A} = \mathbf{L}\mathbf{U}$

$$\begin{aligned} \mathbf{A} &= \mathbf{L}^{-1}\mathbf{U} \\ &= \mathbf{L}\mathbf{U} \end{aligned}$$

Example

- Based on the LU-factorization, a matrix $\mathbf{A}$ can be expressed as $\mathbf{A} = \mathbf{LU}$, where $\mathbf{L}$ is a lower triangular matrix with all diagonal entries equal to 1 and $\mathbf{U}$ is an upper triangular matrix.

if $\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 2 \\ -2 & 4 & 3 \end{bmatrix}$, what are the matrices $\mathbf{L}$ and $\mathbf{U}$?

Solution

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 2 \\ -2 & 4 & 3 \end{bmatrix} \xrightarrow[r_{13}(-2)]{r_{12}(1)} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \end{bmatrix} \xrightarrow{r_{23}(-1)} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{U}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \mathbf{A} = \mathbf{U} \longrightarrow \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{U}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \mathbf{U} = \mathbf{LU}$$

Exercise

- $\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 1 \\ 3 & -1 & 2 \end{bmatrix}$.

(a) Find an LU decomposition of the matrix.

(b) Use LU decomposition to solve the system:

$$\begin{array}{rrrcr} x_1 & -x_2 & +x_3 & = & 4 \\ -x_1 & +2x_2 & +x_3 & = & -1 \\ 3x_1 & -x_2 & +2x_3 & = & 8 \end{array}$$

Solution

(a)

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 1 \\ 3 & -1 & 2 \end{bmatrix} \xrightarrow[r_{13}(-3)]{r_{12}(1)} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & -1 \end{bmatrix} \xrightarrow{r_{23}(-2)} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -5 \end{bmatrix} = \mathbf{U}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \mathbf{A} = \mathbf{U}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \mathbf{U} = \mathbf{LU}$$

Solution (conti.)

(b)

$$\mathbf{Ax} = \mathbf{b} \quad \longrightarrow \quad \mathbf{LUx} = \mathbf{b}$$

(1) Let $\mathbf{y} = \mathbf{Ux} \quad \longrightarrow \quad \mathbf{Ly} = \mathbf{b}$

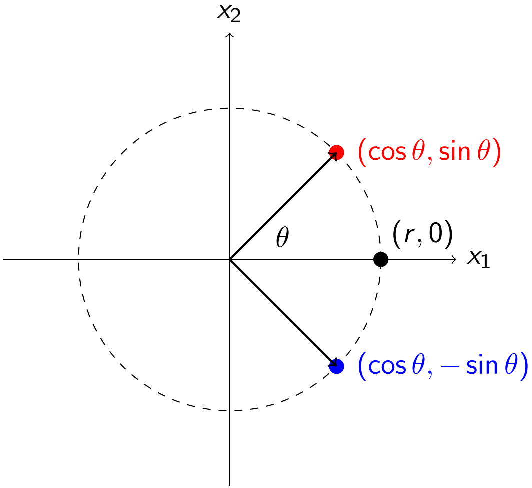
$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 8 \end{bmatrix} \quad \longrightarrow \quad \mathbf{y} = \begin{bmatrix} 4 \\ 3 \\ -10 \end{bmatrix}$$

(2) $\mathbf{Ux} = \mathbf{y}$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ -10 \end{bmatrix} \quad \longrightarrow \quad \mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

QR Decomposition with Givens Rotation

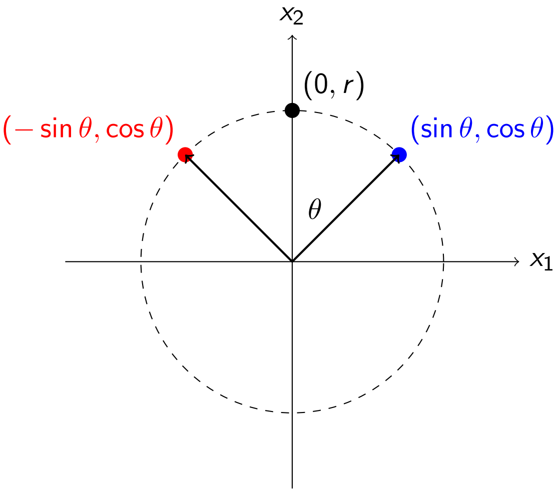
Rotation Matrix



$$\begin{bmatrix} r \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$ **Counter-clockwise**

$$\begin{bmatrix} r \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$$ **Clockwise**

Rotation Matrix



$$\begin{bmatrix} 0 \\ r \end{bmatrix} \longrightarrow \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$ **Counter-clockwise**

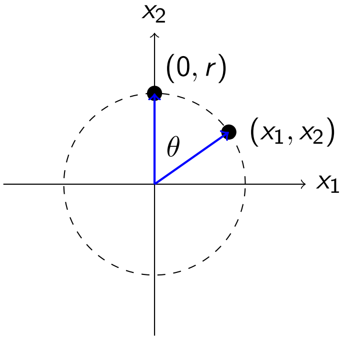
$$\begin{bmatrix} 0 \\ r \end{bmatrix} \longrightarrow \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}$$ **Clockwise**

Rotation Matrix

- $N = 2$, counter-clockwise

$$\mathbf{Q} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \mathbf{Q}_{12}$$

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$$



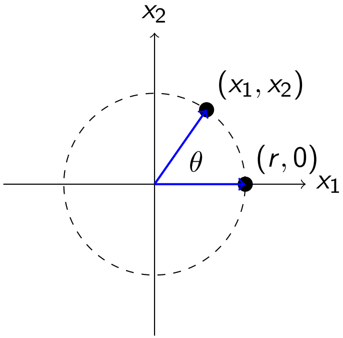
$$c = \cos\theta = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}, \qquad s = \sin\theta = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \qquad r = \sqrt{x_1^2 + x_2^2}$$

Rotation Matrix

- $N = 2$, clockwise

$$\mathbf{Q} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \mathbf{Q}_{21}$$

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$$



$$c = \cos\theta = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \qquad s = \sin\theta = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}, \qquad r = \sqrt{x_1^2 + x_2^2}$$

Rotation Matrix

- The inverse of the rotation matrix is its transpose: $\mathbf{R}^\intercal = \mathbf{R}^{-1}$

Counter-clockwise $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Clockwise $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

When $N > 2$

- How to write down the $\mathbf{Q}_{ij}$ matrix? (**Shift all the energy of axis-i to axis-j**)
  - $q_{ii} = c$
  - $q_{ij} = -s$
  - $q_{ji} = s$
  - $q_{jj} = c$

$$c = \cos\theta = \frac{x_j}{\sqrt{x_i^2 + x_j^2}}, \quad s = \sin\theta = \frac{x_i}{\sqrt{x_i^2 + x_j^2}}$$

Example:

$$\mathbf{Q}_{13} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \qquad \mathbf{Q}_{21} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

QR Decomposition

- $\mathbf{A} = \mathbf{QR}$, where $\mathbf{Q}^\intercal \mathbf{Q} = \mathbf{I}$, $\mathbf{R}$ is an upper triangular matrix
- $\mathbf{Q}^\intercal \mathbf{A} = \mathbf{R}$, easy to solve linear equation

Givens Rotation

- If we rotate all elements of $\mathbf{x} \in \mathbb{R}^n$ to x_1

$$\mathbf{Q}_{21}\mathbf{x} = \begin{bmatrix} \sqrt{x_1^2 + x_2^2} \\ 0 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{Q}_{31}\mathbf{Q}_{21}\mathbf{x} = \begin{bmatrix} \sqrt{x_1^2 + x_2^2 + x_3^2} \\ 0 \\ 0 \\ \vdots \\ x_n \end{bmatrix}$$

$$\mathbf{Q}_{n1} \cdots \mathbf{Q}_{31}\mathbf{Q}_{21}\mathbf{x} = \begin{bmatrix} \sqrt{x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \|\mathbf{x}\| \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Example

- A vector $\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

$$\mathbf{Q}_{21} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\cos\theta = \frac{3}{\sqrt{3^2+4^2}} = \frac{3}{5} \qquad \sin\theta = \frac{4}{\sqrt{3^2+4^2}} = \frac{4}{5}$$

Example (conti.)

$$\begin{bmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}$$

$$Q_{31} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\cos\theta = \frac{5}{\sqrt{5^2+5^2}} = \frac{1}{\sqrt{2}} \qquad \sin\theta = \frac{5}{\sqrt{5^2+5^2}} = \frac{1}{\sqrt{2}}$$

Example (conti.)

$$\mathbf{Q}_{31}\mathbf{Q}_{21}\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 5\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \xrightarrow{\text{Givens Rotation}} \|\mathbf{x}\| \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 5\sqrt{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Example

- Find the QR-decomposition of the following matrix

$$\mathbf{A} = \begin{bmatrix} 0 & -15 & 14 \\ 4 & 32 & 2 \\ 3 & -1 & 4 \end{bmatrix}$$

Solution

$$\mathbf{A} = \begin{bmatrix} 0 & -15 & 14 \\ 4 & 32 & 2 \\ 3 & -1 & 4 \end{bmatrix} \xrightarrow{\text{Givens Rotation}} \text{Upper triangular matrices}$$

$$\begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} : \qquad \mathbf{Q}_{21} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \cos\theta = \frac{0}{\sqrt{4^2+0^2}} = 0$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \sin\theta = \frac{4}{\sqrt{4^2+0^2}} = 1$$

Solution (conti.)

$$\mathbf{Q}_{21}\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -15 & 14 \\ 4 & 32 & 2 \\ 3 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 32 & 2 \\ 0 & 15 & -14 \\ 3 & -1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} : \qquad \mathbf{Q}_{31} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \qquad \cos\theta = \frac{4}{\sqrt{3^2+4^2}} = \frac{4}{5}$$

$$= \begin{bmatrix} \frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ -\frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix} \qquad \sin\theta = \frac{3}{\sqrt{3^2+4^2}} = \frac{3}{5}$$

Solution (conti.)

$$\mathbf{Q}_{31}\mathbf{Q}_{21}\mathbf{A} = \begin{bmatrix} \frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ -\frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix} \begin{bmatrix} 4 & 32 & 2 \\ 0 & 15 & -14 \\ 3 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 25 & 4 \\ 0 & 15 & -14 \\ 0 & -20 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 25 \\ 15 \\ -20 \end{bmatrix} : \qquad \mathbf{Q}_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \qquad \cos\theta = \frac{15}{\sqrt{(-20)^2 + 15^2}} = \frac{3}{5}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{bmatrix} \qquad \sin\theta = \frac{-20}{\sqrt{(-20)^2 + 15^2}} = -\frac{4}{5}$$

Solution (conti.)

$$\mathbf{Q}_{32}\mathbf{Q}_{31}\mathbf{Q}_{21}\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 5 & 25 & 4 \\ 0 & 15 & -14 \\ 0 & -20 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 25 & 4 \\ 0 & 25 & -10 \\ 0 & 0 & -10 \end{bmatrix}$$

$$\mathbf{Q}_{32}\mathbf{Q}_{31}\mathbf{Q}_{21}\mathbf{A} = \mathbf{R} \quad \longrightarrow \quad \begin{aligned} \mathbf{A} &= (\mathbf{Q}_{21}^{-1}\mathbf{Q}_{31}^{-1}\mathbf{Q}_{32}^{-1})\mathbf{R} \\ &= (\mathbf{Q}_{21}^{T}\mathbf{Q}_{31}^{T}\mathbf{Q}_{32}^{T})\mathbf{R} \\ &= \mathbf{QR} \end{aligned}$$

Exercise 1

- Find an LU-factorization of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix}.$$

Exercise 2

- Find an LU-factorization of the following matrix:

$$\mathbf{B} = \begin{bmatrix} 2 & 4 & 3 & 5 \\ -4 & -7 & -5 & -8 \\ 6 & 8 & 2 & 9 \\ 4 & 9 & -2 & 14 \end{bmatrix}.$$

Exercise 3

- Use LU decomposition to solve the system:

$$
\begin{array}{rrrcr}
x_1 & +x_2 & -x_3 & = & 4 \\
x_1 & -2x_2 & +3x_3 & = & -6 \\
2x_1 & +3x_2 & +x_3 & = & 7
\end{array}
$$

Exercise 4

- Consider the matrix:

$$\mathbf{C} = \begin{bmatrix} 0 & -1 & 1 \\ 4 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix}.$$

using Givens Rotation method, detertmine the QR decomposition.

Exercise 5

- Consider the matrix:

$$\mathbf{D} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 3 \end{bmatrix}.$$

using Givens Rotation method, detertmine the QR decomposition.