9-2 Infinite Series

級數

師大工教一

Definition Given $\{a_n\}$, an expression of the form $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ is an **infinite series**. The number a_n is the n **th term** of the series. The sequence $\{s_n\}$ defined by

$$
\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
&\vdots \\
s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \\
&\vdots
\end{aligned}
$$

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is the **sequence of partial sums** of the series, the number s_n being the n**th partial sum**. If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

$S_n \to L$, $\overset{\Delta}{\Leftrightarrow}$ $\sum a_n$ converges, $\sum_{n=1}^{\infty} a_n = L$

L is called its sum

S_n diverges, $\overset{\Delta}{\Leftrightarrow}$ $\sum a_n$ diverges

Geometric series are series of the form $a+ar+ar^2+\cdots+ar^{n-1}+\cdots=\sum_{n=1}^{\infty} ar^{n-1}$. It

幾何級數(等比級數)

is also written as $\sum_{n=0}^{\infty} ar^n$.

$$
\begin{aligned}
s_n &= a+ar+ar^2+\cdots+ar^{n-1} \\
rs_n &= ar+ar^2+\cdots+ar^{n-1}+ar^n \\
s_n - rs_n &= a - ar^n \\
s_n(1-r) &= a\left(1-r^n\right) \\
s_n &= \frac{a\left(1-r^n\right)}{1-r}
\end{aligned}
$$

If $|r|<1$, $a+ar+ar^2+\cdots+ar^{n-1}+\cdots$ converges to $\frac{a}{1-r}$. If $|r|\geq 1$, the series diverges.

Ex2 Find the sum of the series $\sum_{n=0}^{\infty}\frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$.

$a=5$

$r=-\frac{1}{4}$ $\Rightarrow$ sum $= \frac{5}{1-(-\frac{1}{4})} = 4$

Ex4 Express the repeating decimal $5.232323\cdots$ as a fraction.

$= 5 + 0.23 + 0.0023 + \cdots$

$= 5 + \frac{0.23}{1-\frac{1}{100}} = 5 + \frac{23}{99}$

Ex5 Find the sum of the "telescoping" series $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$.

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$= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$

$S_n = (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{n} - \frac{1}{n+1})$

$= 1 - \frac{1}{n+1}$

$\therefore \lim_{n\to\infty} S_n = 1 = \sum_{n=1}^{\infty}\frac{1}{n(n+1)}$

The nth-Term Test for a Divergent Series

Theorem 7 If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

$$\sum_{n=1}^{\infty} a_n = L$$

$$S_n - S_{n-1} = a_n$$

$$\lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$L - L = 0$$

$\sum a_n$ conv. $\Rightarrow \lim_{n\to\infty} a_n = 0$

Theorem If $\lim_{n\to\infty} a_n$ fails to exist or is different from 0, $\sum_{n=1}^{\infty} a_n$ diverges.

$\lim_{n\to\infty} a_n$ DNE or $\lim_{n\to\infty} a_n \neq 0$

$\Rightarrow \sum a_n$ diverge

Warning: $\lim_{n\to\infty} a_n = 0 \not\Rightarrow \sum a_n$ conv.

Ex7 (a) $\sum_{n=1}^{\infty} n^2$ (b) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ (c) $\sum_{n=1}^{\infty} (-1)^{n+1}$ (d) $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$

Theorem 8 If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ are convergent series, then

1. $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = A + B$

2. $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n = A - B$

3. $\sum_{n=1}^{\infty} k a_n = k\sum_{n=1}^{\infty} a_n = kA$

Ex9 (a) $\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}}$ (b) $\sum_{n=0}^{\infty} \frac{4}{2^n}$

(a) $= \sum_{n=1}^{\infty}\left(\frac{3}{6}\right)^{n-1} - \sum_{n=1}^{\infty}\left(\frac{1}{6}\right)^{n-1}$

$= \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{6}}$

$= 2 - \frac{6}{5} = \frac{4}{5}$

(b) $\sum_{n=0}^{\infty} \frac{4}{2^n} = 4\sum_{n=0}^{\infty} \frac{1}{2^n} = 4 \cdot \frac{1}{1-\frac{1}{2}} = 8$

Adding or Deleting Terms

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_{k-1} + \sum_{n=k}^{\infty} a_n$$

The convergence or divergence of a series is not affected by its first few terms.

Reindexing

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots = \sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=5}^{\infty} \frac{1}{2^{n-5}} = \sum_{n=-4}^{\infty} \frac{1}{2^{n+4}}$$

HW9-2

- **HW:3,5,11,32,34,38,46,48,58,66,72**