106 (分部)

1. (a) $\lim\limits_{x\to\infty}\dfrac{2x^3+1}{6x^3-5x^2-1}=\dfrac{1}{3}$

(b) $\lim\limits_{x\to 0} x^{\frac{1}{3}}e^{\cos\frac{1}{x}}=0 \quad \left(\because\ x^{\frac{1}{3}}e^{-1}\le x^{\frac{1}{3}}e^{\cos\frac{1}{x}}\le x^{\frac{1}{3}}e^{1},\ \lim\limits_{x\to 0}x^{\frac{1}{3}}e^{-1}=0=\lim\limits_{x\to 0}x^{\frac{1}{3}}e^{1}\right)$

(c) $\lim\limits_{x\to 4}\dfrac{\sqrt{1+2x}-3}{\sqrt{x}-2}=\lim\limits_{x\to 4}\dfrac{(\sqrt{1+2x}-3)(\sqrt{x}+2)(\sqrt{1+2x}+3)}{(\sqrt{x}-2)(\sqrt{x}+2)(\sqrt{1+2x}+3)}=\lim\limits_{x\to 4}\dfrac{(2x-8)(\sqrt{x}+2)}{(x-4)(\sqrt{1+2x}+3)}=\dfrac{8}{6}=\dfrac{4}{3}$

(d) $\lim\limits_{x\to 0^+}\left(\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}-\sqrt{\frac{1}{x}-\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}\right)=\lim\limits_{x\to 0^+}\dfrac{2\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}{\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}+\sqrt{\frac{1}{x}-\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}}=\lim\limits_{x\to 0^+}\dfrac{2\sqrt{1+\sqrt{x}}}{\sqrt{1+\sqrt{x+\sqrt{x^3}}}+\sqrt{1-\sqrt{x+\sqrt{x^3}}}}=1$

(e) $\lim\limits_{x\to -\infty}\left(x+\sqrt{x^2+1}\right)=\lim\limits_{x\to -\infty}\dfrac{(x+\sqrt{x^2+1})(\sqrt{x^2+1}-x)}{\sqrt{x^2+1}-x}=\lim\limits_{x\to -\infty}\dfrac{1}{\sqrt{x^2+1}-x}=0$

105 (分部)

1. (a) $\lim\limits_{x\to\infty}\dfrac{x^6+2x^4+3x^2+1}{5x^6+6x^5+6x^3+x}=\dfrac{1}{5}$

(b) $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+x+4}-2}{3x}=\lim\limits_{x\to 0}\dfrac{(\sqrt{x^2+x+4}-2)(\sqrt{x^2+x+4}+2)}{3x\cdot(\sqrt{x^2+x+4}+2)}=\lim\limits_{x\to 0}\dfrac{(x^2+x)}{3x\cdot(\sqrt{x^2+x+4}+2)}$

$=\lim\limits_{x\to 0}\dfrac{x+1}{3(\sqrt{x^2+x+4}+2)}=\dfrac{1}{12}$

(c) $\lim\limits_{x\to 0}\dfrac{\tan(3x^2)}{\sin(2x^3)}=\lim\limits_{x\to 0}\dfrac{\sin 3x^2}{\cos 3x^2}\cdot\dfrac{1}{\sin 2x^3}=\lim\limits_{x\to 0}\dfrac{\sin 3x^2}{3x^2}\cdot\dfrac{2x^3}{\sin 2x^3}\cdot\dfrac{3}{2x}\cdot\dfrac{1}{\cos 3x^2}=\lim\limits_{x\to 0}\dfrac{3}{2x}$. 不存在

(d) $\lim\limits_{x\to 0}(1+3x)^{\frac{1}{x}}=\lim\limits_{y\to 0}\left[(1+y)^{\frac{1}{y}}\right]^3=e^3$

104 (分部)

1. (a) $\lim\limits_{x\to 9}\dfrac{9x-x^2}{3-\sqrt{x}}=\lim\limits_{x\to 9}\dfrac{(9x-x^2)(3+\sqrt{x})}{(3-\sqrt{x})(3+\sqrt{x})}=\lim\limits_{x\to 9}x(3+\sqrt{x})=54$

(b) $\lim\limits_{x\to\infty}\dfrac{5x^4+2x-7}{2x^4-3x^3-x+6}=\dfrac{5}{2}$

(c) $\lim\limits_{x\to\infty}e^{\frac{1}{x}}\cos\left(\frac{1}{x}\right)=1 \quad \left(=\lim\limits_{y\to 0}e^{y}\cos y=1\right)$

(d) $\lim\limits_{x\to 0}\dfrac{\tan^2(3x)}{x\cdot\sin(5x)}=\lim\limits_{x\to 0}\dfrac{\sin 3x}{3x}\cdot\dfrac{\sin 3x}{3x}\cdot\dfrac{5x}{\sin 5x}\cdot\dfrac{9}{5}\cdot\dfrac{1}{\cos 3x}\cdot\dfrac{1}{\cos 3x}=\dfrac{9}{5}$

(e) $\lim\limits_{x\to 0^-}\left(x^2-\dfrac{1}{x}\right)=\infty$

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I (a) $\lim\limits_{x\to\infty}\left(\sqrt{x+12}-\sqrt{x+5}\right)=\lim\limits_{x\to\infty}\dfrac{7}{\sqrt{x+12}+\sqrt{x+5}}=0$

(b) $\lim\limits_{x\to 0}\dfrac{\tan 5x}{\sin 2x\cdot\cos 3x}=\lim\limits_{x\to 0}\dfrac{\sin 5x}{5x}\cdot\dfrac{1}{\cos 5x}\cdot\dfrac{2x}{\sin 2x}\cdot\dfrac{1}{\cos 3x}\cdot\dfrac{5}{2}=\dfrac{5}{2}$

✱ (c) $\lim\limits_{x\to 0}\dfrac{4(x-\sin x)}{x^3}=\dfrac{2}{3}$

(d) If $\lim\limits_{x\to 1}\dfrac{f(x)-1}{x-1}=3$. $\Rightarrow f(1)=1\Rightarrow\lim\limits_{x\to 1}\dfrac{f(x)}{x-2}=-1$

✱ $\lim\limits_{x\to 0}\dfrac{4(x-\sin x)}{x^3}$

① L'Hôpital's Rule. $\lim\limits_{x\to 0}\dfrac{4(x-\sin x)}{x^3}=\lim\limits_{x\to 0}\dfrac{4(1-\cos x)}{3x^2}=\lim\limits_{x\to 0}\dfrac{4\sin x}{6x}=\dfrac{2}{3}$

② 令 $A=\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}=\lim\limits_{t\to 0}\dfrac{3t-\sin 3t}{(3t)^3}=\lim\limits_{t\to 0}\dfrac{3t-3\sin t+4\sin^3 t}{27t^3}=\dfrac{1}{9}A+\dfrac{4}{27}$

$\Rightarrow A=\dfrac{1}{6}$. $\therefore\lim\limits_{x\to 0}\dfrac{4(x-\sin x)}{x^3}=\dfrac{2}{3}$

③ $\sin x=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\cdots$

$\dfrac{x-\sin x}{x^3}=\dfrac{1}{3!}-\dfrac{x^2}{5!}+\cdots$ $\therefore\lim\limits_{x\to 0}\dfrac{4(x-\sin x)}{x^3}=\dfrac{2}{3}$

102

1. (a) $\lim\limits_{x\to 3}\dfrac{x^3-9x}{\sqrt{x^2+7}-4}=\lim\limits_{x\to 3}\dfrac{(x^3-9x)(\sqrt{x^2+7}+4)}{x^2-9}=\lim\limits_{x\to 3}x(\sqrt{x^2+7}+4)=24$

(b) $\lim\limits_{x\to\infty}\dfrac{x^2}{e^{2x}}=0$. $\left(\because\dfrac{1}{e^{2x}}\le\dfrac{x^2}{e^{2x}}\le\dfrac{2^x}{e^{2x}}\right.$ 當 $x\to\infty$. 又 $\left.\lim\limits_{x\to\infty}\dfrac{1}{e^{2x}}=0=\lim\limits_{x\to\infty}\dfrac{2^x}{e^{2x}}\right)$

(c) $\lim\limits_{x\to 0}\dfrac{x^2+\sin x}{e^{2x}}=0$

(d) $\lim\limits_{x\to\infty}\dfrac{2x^3+7}{x^3+3x^2-7}=2$

(e) $\lim\limits_{x\to\infty}\dfrac{\log_2 x}{\log_3(x+3)}=\lim\limits_{x\to\infty}\dfrac{\ln x}{\ln 2}\cdot\dfrac{\ln 3}{\ln(x+3)}=\lim\limits_{x\to\infty}\dfrac{\ln 3}{\ln 2}=\log_2 3$

(f) $\lim\limits_{x\to 0}\dfrac{\sin 5x}{\sin 3x\cdot\cos 3x}=\lim\limits_{x\to 0}\dfrac{\sin 5x}{5x}\cdot\dfrac{3x}{\sin 3x}\cdot\dfrac{1}{\cos 3x}\cdot\dfrac{5}{3}=\dfrac{5}{3}$

$\left[\lim\limits_{x\to\infty}\dfrac{\ln(x+3)-\ln x}{\ln x}=\lim\limits_{x\to\infty}\dfrac{\ln(1+\frac{3}{x})}{\ln x}=0\right.$. 又 $\lim\limits_{x\to\infty}\dfrac{\ln x}{\ln x}=1$. 故 $\left.\lim\limits_{x\to\infty}\dfrac{\ln(x+3)}{\ln x}=1\right]$