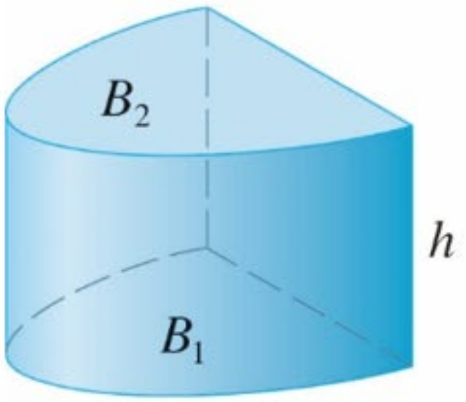
用截面積求体積

6-1 Volumes Using Cross-Sections

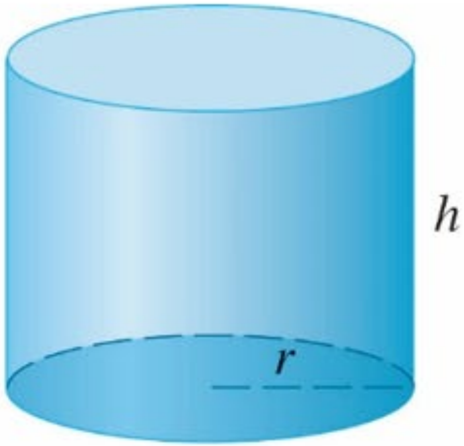
師大工教一

Volumes of Cylinders

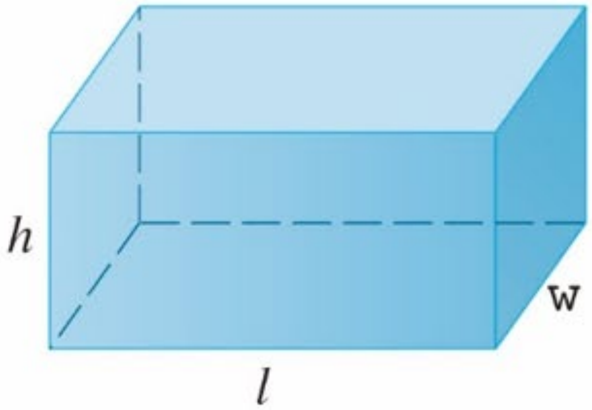
柱体



(a) Cylinder $V = Ah$

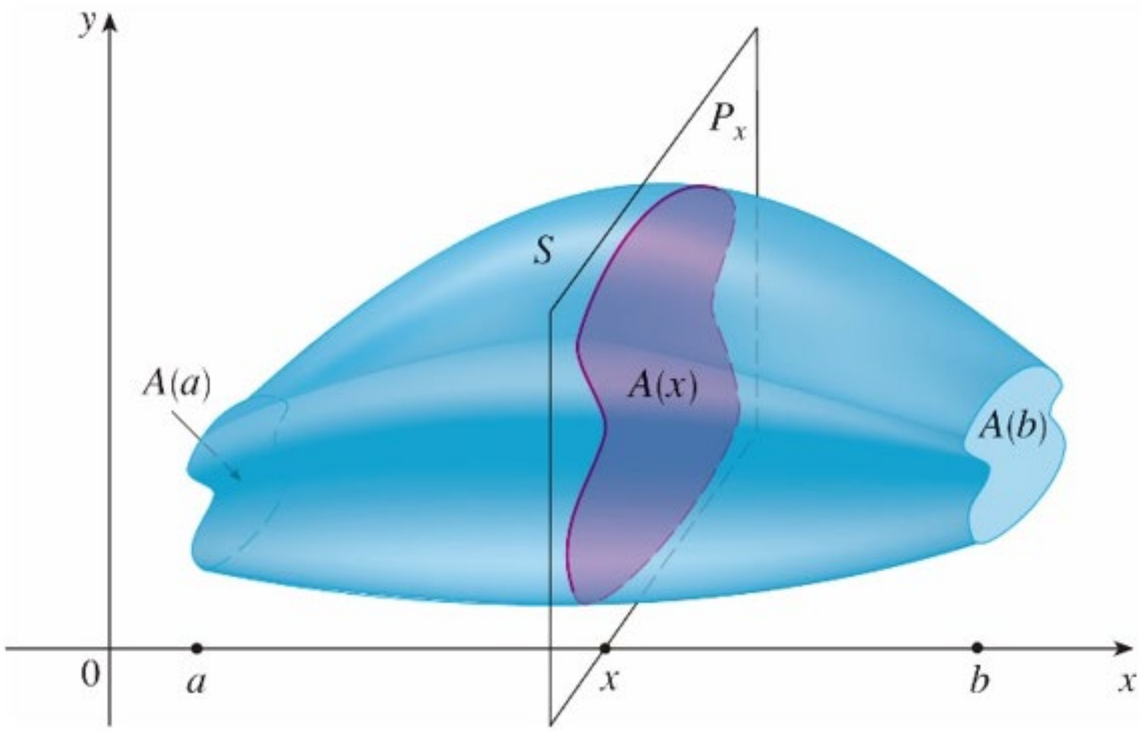


(b) Circular cylinder $V = \pi r^2 h$

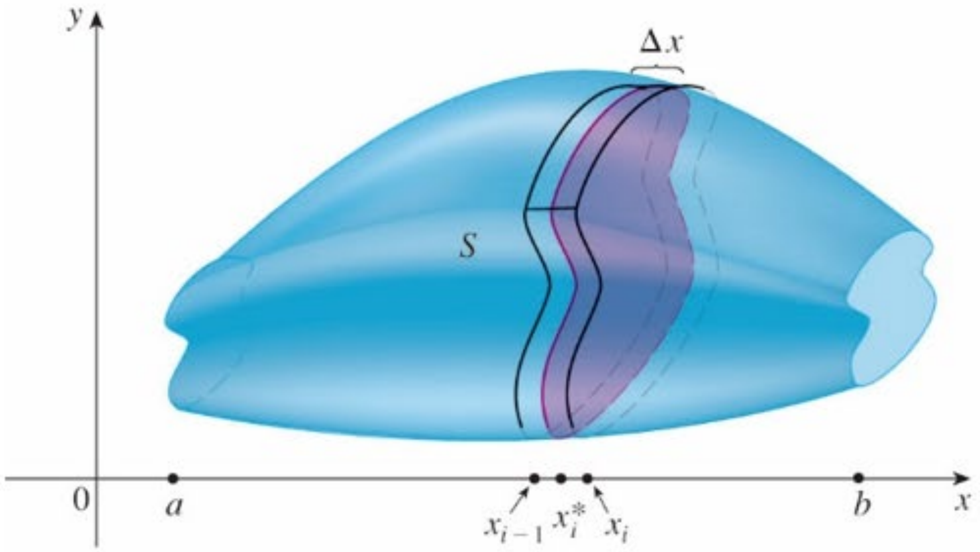


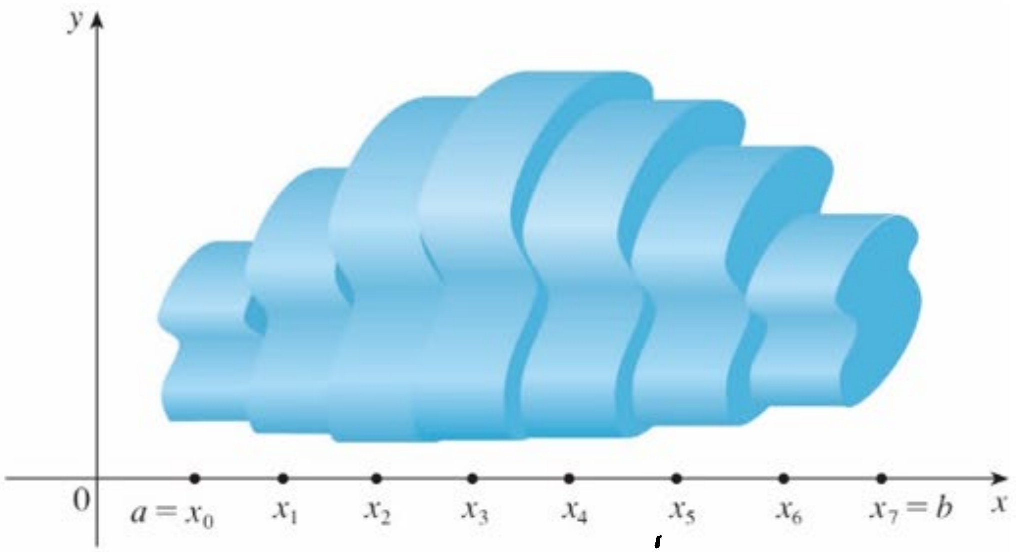
(c) Rectangular box $V = lwh$

Cross-sections ：(橫)截面



Method of Slicing





$$V \approx \sum_{k=1}^{n} V_k = \sum_{k=1}^{n} A(x_k)\,\Delta x_k$$

$$V = \int_a^b A(x)\,dx$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} A(x_k)\,\Delta x_k = \int_a^b A(x)\,dx$$

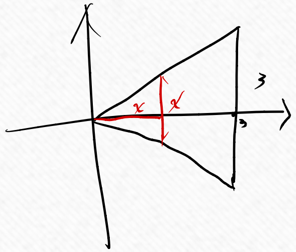
Definition The **volume** of a solid of integrable cross-sectional area $A(x)$ from $x=a$ to $x=b$ is the integral of A from a to b, $V=\int_a^b A(x)\,dx$.

Calculating the Volume of a Solid

1. Sketch the solid and a typical cross-section.
2. Find a formula for $A(x)$, the area of a typical cross-section.
3. Find the limits of integration.
4. Integrate $A(x)$ to find the volume.

金字塔

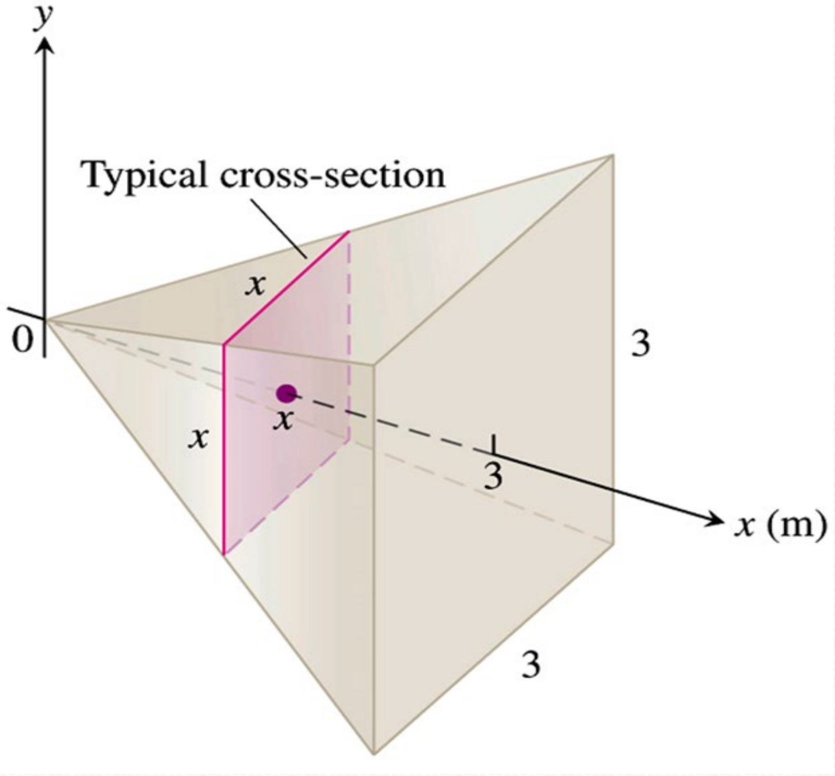
Ex1(p375) A pyramid 3 m high has a square base that is 3 m on a side. Find the volume of the pyramid.



$A = x^2$

$$V = \int_0^3 x^2\,dx = \frac{x^3}{3}\Big|_0^3 = 9$$

錐体体積 $= \frac{1}{3}$ 柱体体積

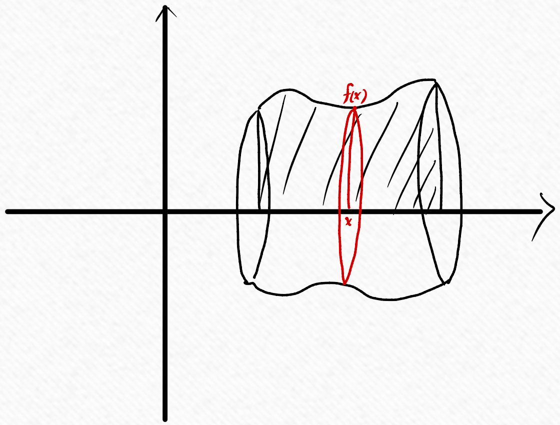


Solid of Revolution: The Disk Method

Volume by Disks for Rotation About the $x-$Axis

$$V=\int_a^b A(x)\,dx=\int_a^b \pi\left[R(x)\right]^2 dx$$

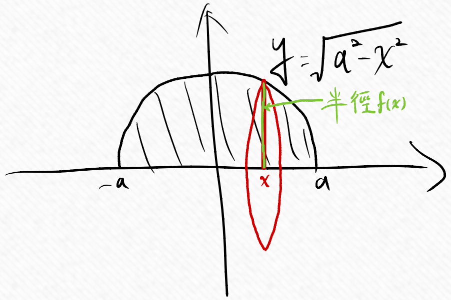
旋転体：圓盤法



$$V=\int_a^b A(x)\,dx=\int_a^b \pi(f(x))^2\,dx$$

Ex5(p377) The circle $x^2 + y^2 = a^2$ is rotated about the $x-$axis to generate a sphere. Find its volume.

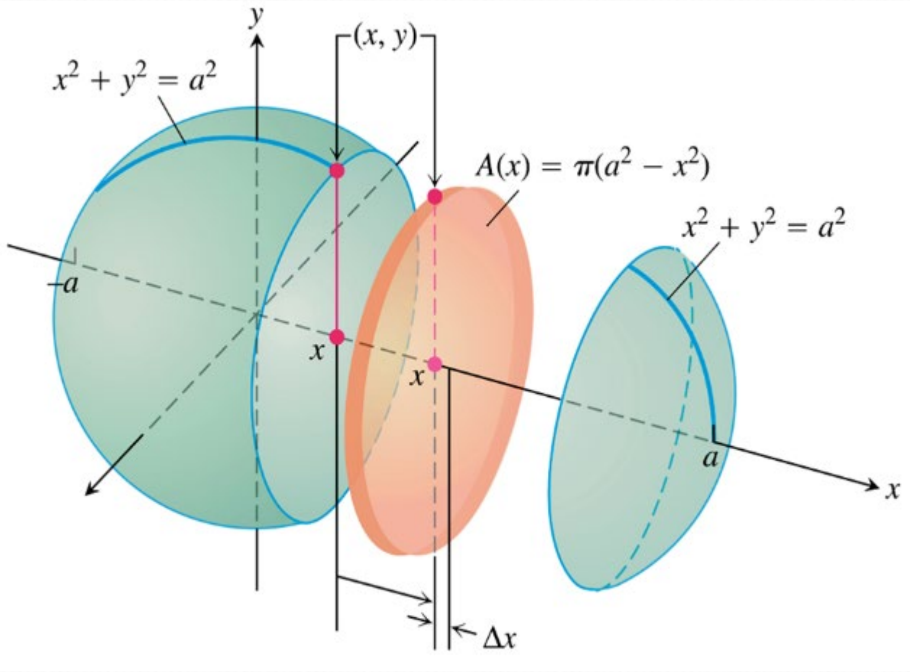
球面



ball 球 半径 a, $V=\frac{4}{3}\pi a^3$

$$V=\int_{-a}^{a}\pi\left(\sqrt{a^2-x^2}\right)^2dx$$
$$=\int_{-a}^{a}\pi(a^2-x^2)\,dx$$
$$=2\pi\int_0^a(a^2-x^2)\,dx$$
$$=2\pi\left(a^2x-\frac{1}{3}x^3\Big|_0^a\right)$$
$$=2\pi\left(a^3-\frac{1}{3}a^3\right)$$
$$=\frac{4}{3}\pi a^3$$

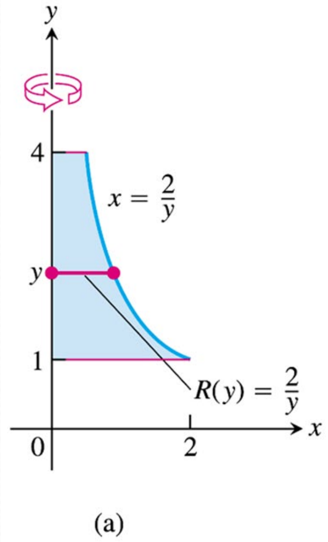
$$x^2+y^2=a^2$$
$$y^2=a^2-x^2$$
$$y=\pm\sqrt{a^2-x^2}$$

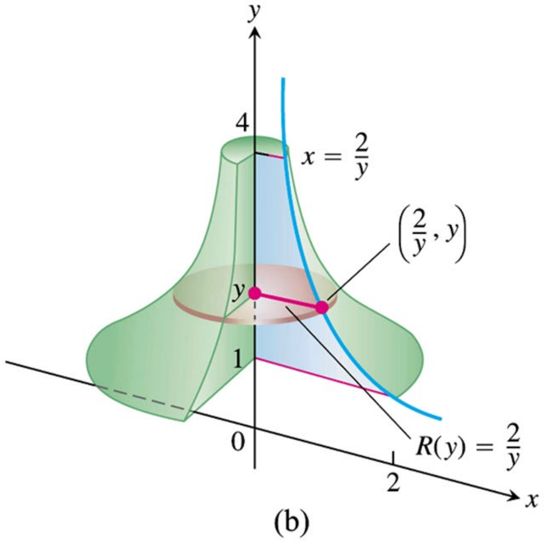


Volumes by Disks for Rotation About the $y-$Axis

$$V=\int_c^d A(y)\,dy=\int_a^b \pi\left[R(y)\right]^2 dy$$

Ex7(p379) Find the volume of the solid generated by revolving the region between the $y-$axis and the curve $x=\dfrac{2}{y}, 1\le y\le 4$, about the $y-$axis.





$$V=\int_1^4 \pi\left(\frac{2}{y}\right)^2 dy$$

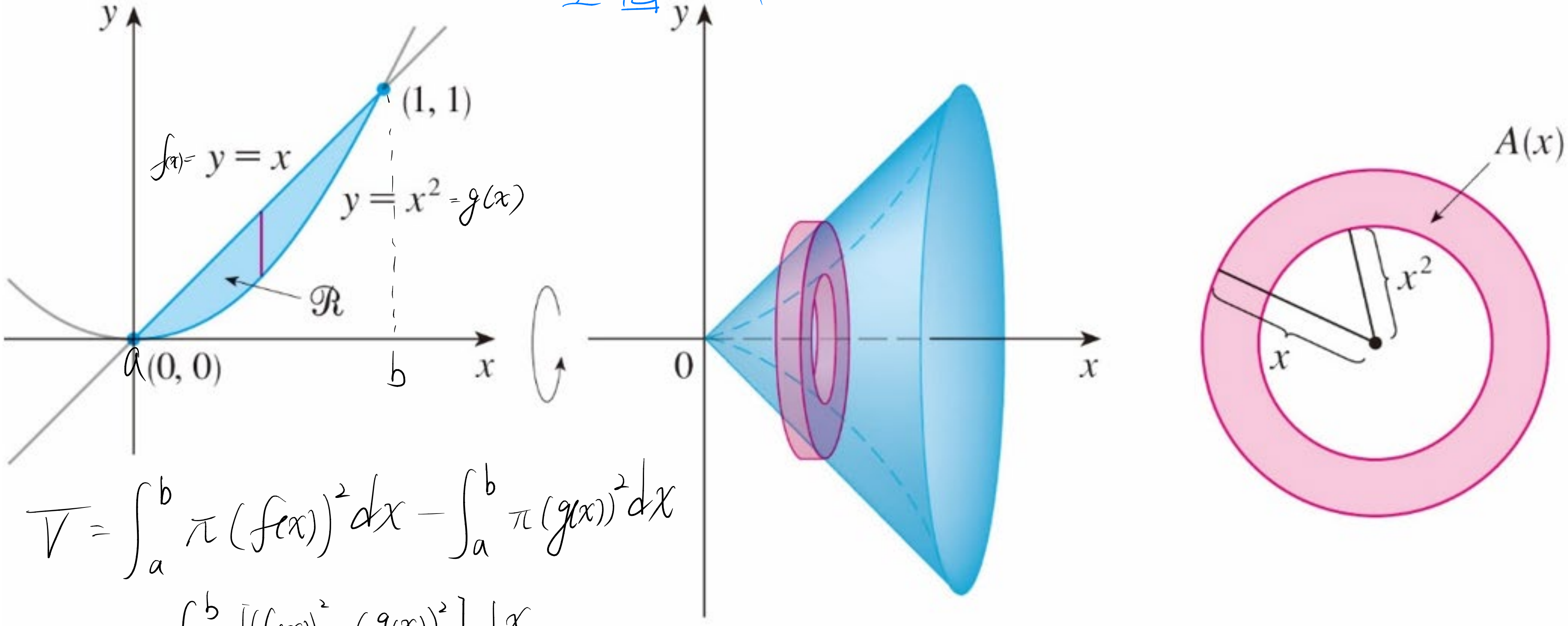
$$=\pi\int_1^4 \frac{4}{y^2}\,dy$$

$$=\pi\left(\frac{-4}{y}\Big|_1^4\right)$$

$$=\pi(-1+4)=3\pi$$

Solid of Revolution: The Washer Method

垫圈法



$$V = \int_a^b \pi (f(x))^2 dx - \int_a^b \pi (g(x))^2 dx$$

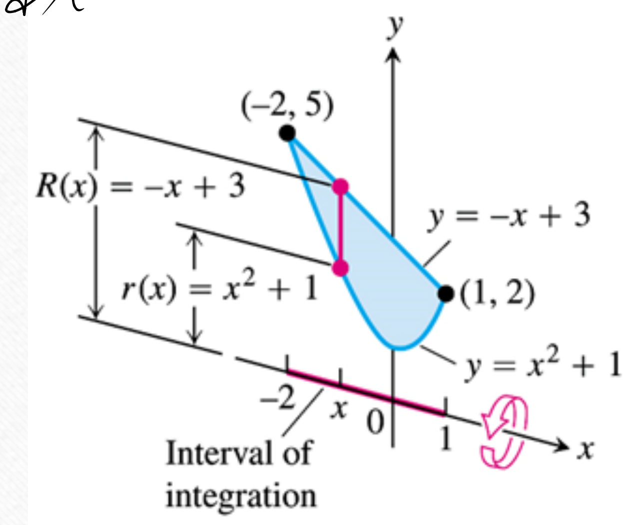
$$= \pi \int_a^b [(f(x))^2 - (g(x))^2] dx$$

(a) (b) (c)

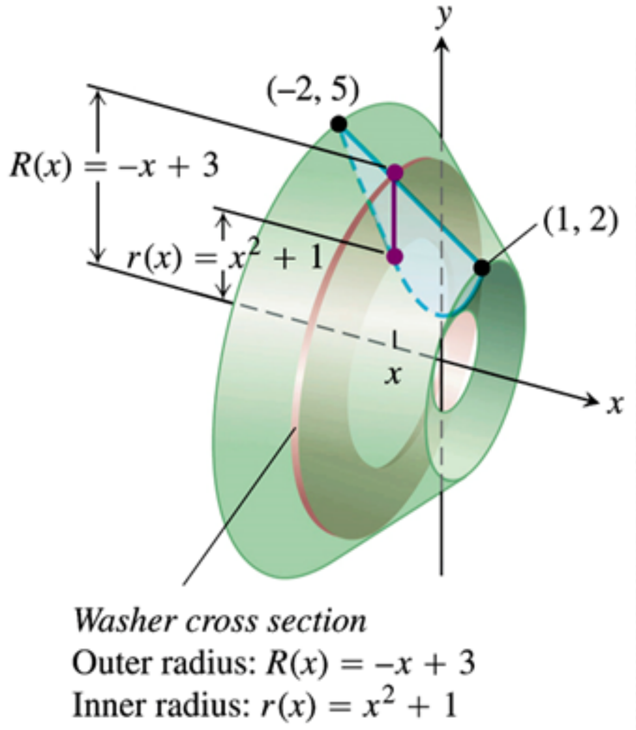
Ex9(p380) The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the $x-$axis to generate a solid. Find the volume of the solid.

$$x^2 + 1 = -x + 3$$
$$x^2 + x - 2 = 0$$
$$x = -2, 1$$

$$V = \int_{-2}^{1} \pi \left[(3-x)^2 - (x^2+1)^2 \right] dx$$
$$= \pi \int_{-2}^{1} \left[9 - 6x + x^2 - (x^4 + 2x^2 + 1) \right] dx$$
$$= \pi \int_{-2}^{1} \left[9 - 6x + x^2 - x^4 - 2x^2 - 1 \right] dx$$
$$= \pi \int_{-2}^{1} (-x^4 - x^2 - 6x + 8)\, dx$$
$$= \pi \left(-\frac{x^5}{5} - \frac{x^3}{3} - 3x^2 + 8x \Big|_{-2}^{1} \right)$$
$$= \pi \left[-\frac{1}{5} - \frac{1}{3} - 3 + 8 - \left(\frac{32}{5} + \frac{8}{3} - 12 - 16 \right) \right]$$
$$= \pi \left[-\frac{33}{5} - 3 + 33 \right] = \pi \left(\frac{117}{5} \right)$$



(a)



(b)

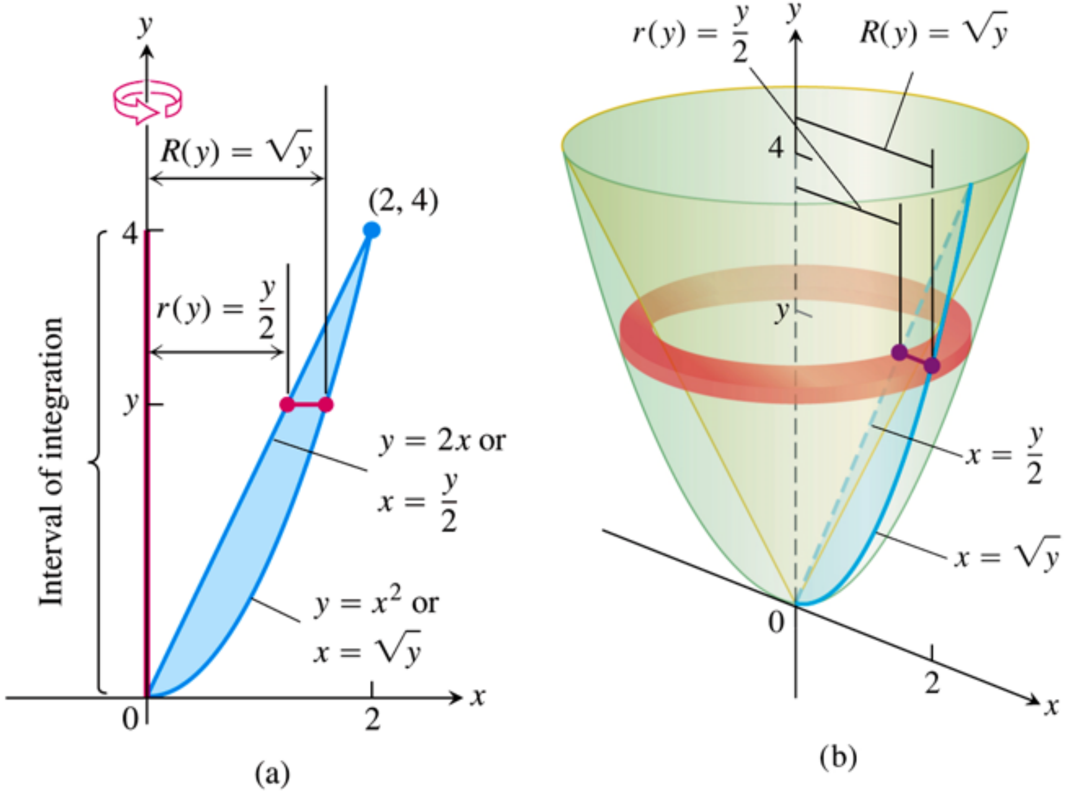
Ex10(p381) The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the $y-$axis to generate a solid. Find the volume of the solid.

$$V = \pi\int_0^4 \left[(\sqrt{y})^2 - \left(\frac{y}{2}\right)^2\right]dy$$

$$= \pi\int_0^4 \left(y - \frac{y^2}{4}\right)dy$$

$$= \pi\left(\frac{y^2}{2} - \frac{y^3}{12}\Big|_0^4\right)$$

$$= \pi\left(\frac{4^2}{2} - \frac{4^3}{12}\right) = \frac{8}{3}\pi$$



HW6-1

- **HW: 2,5,10,11,18,26,28,29,37,38,50.**