

Laplace Transform for DE

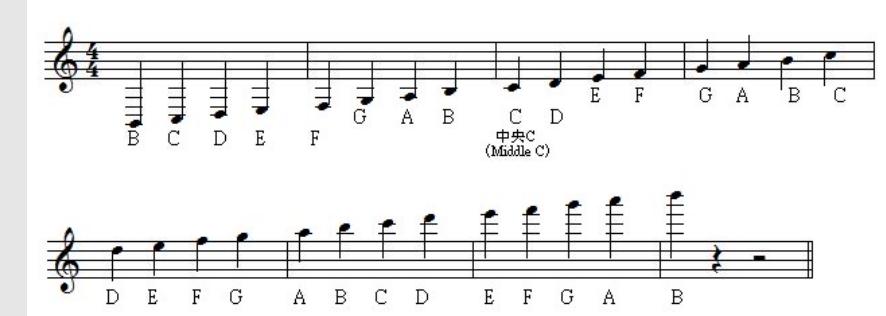
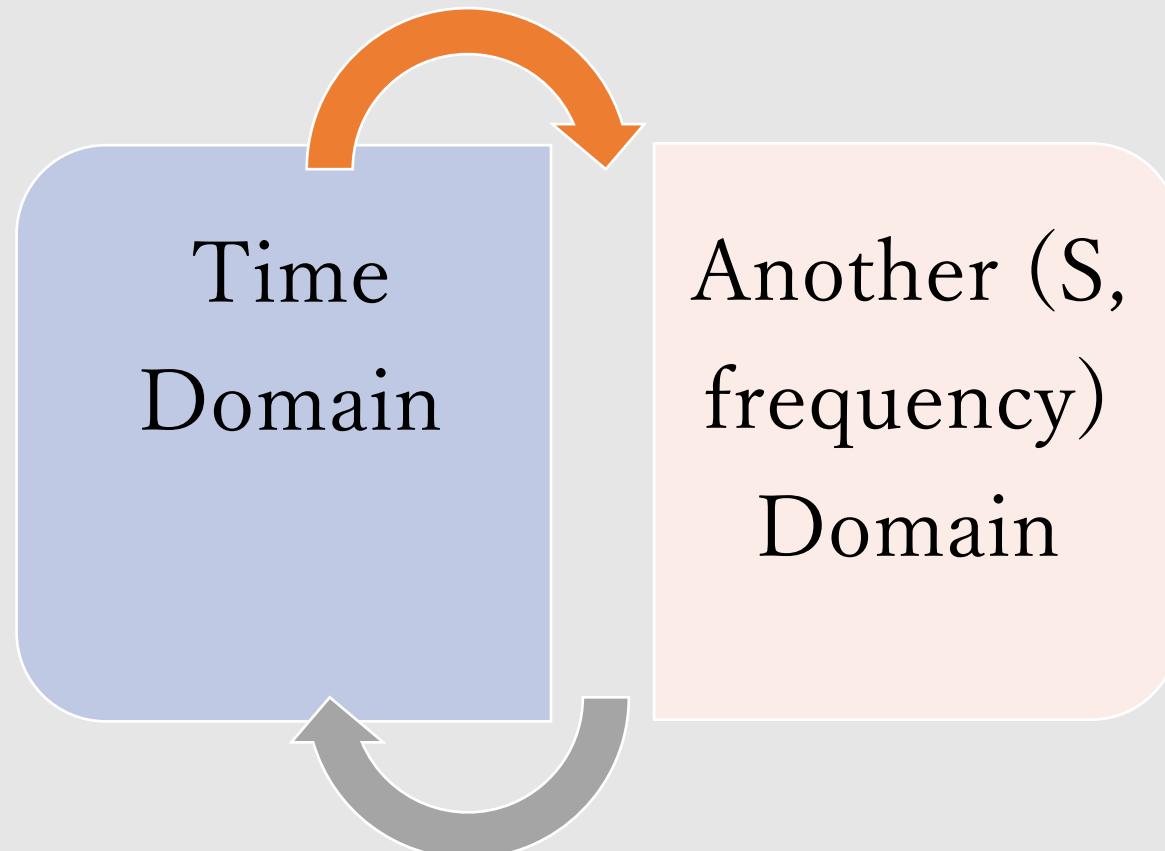
Why Laplace Transform

Laplace transform can change $\frac{d^k}{dt^k}y(t)$ to

$$s^k Y(s) - s^{k-1}y(0) - s^{k-2}y'(0) - \dots \dots - sy^{(k-2)}(0) - y^{(k-1)}(0)$$

Convert the DE problem into polynomial problem

What is Transform?



Definition of Laplace Transform

- Laplace Transform of $f(t)$

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

We usually use a capital letter to represent the result of Laplace transform.

$$F(s) = L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

The Laplace Transform is an **integral transform**

- **transform:** Change a function to another function
- **integral transform:** the operation that can be expressed as the integration form as follows:

$$F(s) = \int_a^b K(s, t)f(t)dt$$

• kernel

For the Laplace transform

$$K(s, t) = e^{-st}, \quad a = 0, \quad b \rightarrow \infty$$

Linear Property

$$\int_0^\infty e^{-st} [\alpha f(t) + \beta g(t)] dt = \alpha \int_0^\infty e^{-st} f(t) dt + \beta \int_0^\infty e^{-st} g(t) dt$$

$$L\{\alpha f(t) + \beta g(t)\} = \alpha L\{f(t)\} + \beta L\{g(t)\}$$

Every integral transform has the linear property.

Basic Functions of LT

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
$\exp(at)$	$\frac{1}{s - a}$
$\sin(kt)$	$\frac{k}{s^2 + k^2}$
$\cos(kt)$	$\frac{s}{s^2 + k^2}$
$\sinh(kt)$	$\frac{k}{s^2 - k^2}$
$\cosh(kt)$	$\frac{s}{s^2 - k^2}$

Example $L\{1\}$

$$L\{1\} = \int_0^{\infty} e^{-st} dt = -\frac{e^{-st}}{s} \Big|_0^{\infty} = -\frac{e^{-s \cdot \infty}}{s} - \left(-\frac{e^{-s \cdot 0}}{s}\right) = \frac{1}{s}$$

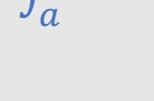
(1) $\lim_{b \rightarrow \infty} -\frac{e^{-s \cdot b}}{s}$ is a formal writing of $\frac{e^{-s \cdot \infty}}{s}$

(2) If $s > 0$ $-\frac{e^{-s \cdot \infty}}{s} = 0$

Example $L\{t\}$

$$\begin{aligned} L\{t\} &= \int_0^\infty te^{-st} dt \\ &= -\frac{te^{-st}}{s} \Big|_0^\infty + \int_0^\infty \frac{e^{-st}}{s} dt \\ &= -\frac{\infty \cdot e^{-s \cdot \infty}}{s} + \frac{0 \cdot e^{-s \cdot 0}}{s} \\ &= -\frac{e^{-s \cdot \infty}}{s^2} + \frac{e^{-s \cdot 0}}{s^2} \\ &= \frac{1}{s^2} \end{aligned}$$

$\int_a^b u(t)v'(t)dt = u(t)v(t) \Big|_a^b - \int_a^b u'(t)v(t)dt$



Example $L\{e^{-3t}\} = \frac{1}{s+3}$

Example $L\{\sin(2t)\}$

$$\sin(2t) = \frac{1}{2i} (e^{i2t} - e^{-i2t})$$

$$\begin{aligned} L\{\sin(2t)\} &= \frac{1}{2i} L\{e^{i2t}\} - \frac{1}{2i} L\{e^{-i2t}\} = \frac{1}{2i} \frac{1}{s-i2} - \frac{1}{2i} \frac{1}{s+i2} \\ &= \frac{1}{2i} \frac{s+i2 - (s-i2)}{(s-i2)(s+i2)} = \frac{1}{2i} \frac{i4}{s^2 + 4} = \frac{2}{s^2 + 4} \end{aligned}$$

Example

$$L\{1 + 5t\} = L\{1\} + 5L\{t\} = \frac{1}{s} + \frac{5}{s^2}$$

$$L\{4e^{-3t} - 10 \sin 2 t\} = 4L\{e^{-3t}\} - 10L\{\sin 2 t\} = \frac{4}{s+3} - \frac{20}{s^2+4}$$

Inverse Transforms and Transforms of Derivatives

When (1) $f_1(t)$ and $f_2(t)$ are piecewise continuous on $[0, \infty)$, and
(2) $f_1(t)$ and $f_2(t)$ are of exponential order, then

if $f_1(t) \neq f_2(t)$ \longrightarrow then $F_1(s) \neq F_2(s)$

On the other hand, the Laplace transform is one-to-one.

If the Laplace transform of $f_1(t)$ is $F_1(s)$,
then the inverse Laplace transform of $F_1(s)$ must be $f_1(t)$.

$F(s)$	$L^{-1}\{F(s)\}$
$\frac{1}{s}$	1
$\frac{n!}{s^{n+1}}$	t^n
$\frac{1}{s-a}$	$\exp(at)$
$\frac{k}{s^2+k^2}$	$\sin(kt)$
$\frac{s}{s^2+k^2}$	$\cos(kt)$
$\frac{k}{s^2-k^2}$	$\sinh(kt)$
$\frac{s}{s^2-k^2}$	$\cosh(kt)$

Example $L^{-1}\left\{\frac{1}{s^5}\right\}$

$$L^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{4!} L^{-1}\left\{\frac{4!}{s^5}\right\} = \frac{1}{4!} t^4$$

Example $L^{-1}\left\{\frac{-2s + 6}{s^2 + 4}\right\}$

$$\begin{aligned} L^{-1}\left\{\frac{-2s + 6}{s^2 + 4}\right\} &= L^{-1}\left\{\frac{-2s}{s^2 + 4} + \frac{6}{s^2 + 4}\right\} = -2L^{-1}\left\{\frac{s}{s^2 + 4}\right\} + 3L^{-1}\left\{\frac{2}{s^2 + 4}\right\} \\ &= -2 \cos(2t) + 3 \sin(2t) \end{aligned}$$

Decomposition of Fractions

Example $L^{-1} \left\{ \frac{s^2 + 6s + 9}{(s - 1)(s - 2)(s + 4)} \right\}$

$$\frac{s^2 + 6s + 9}{(s - 1)(s - 2)(s + 4)} = \frac{A}{s - 1} + \frac{B}{s - 2} + \frac{C}{s + 4}$$

$$L^{-1} \left\{ \frac{s^2 + 6s + 9}{(s - 1)(s - 2)(s + 4)} \right\} = Ae^t + Be^{2t} + Ce^{-4t}$$

problem : How to get A, B, and C ?

$$\frac{s^2 + 6s + 9}{(s - 1)(s - 2)(s + 4)} = \frac{A(s - 2)(s + 4) + B(s - 1)(s + 4) + C(s - 1)(s - 2)}{(s - 1)(s - 2)(s + 4)}$$

$$s^2 + 6s + 9 = (A + B + C)s^2 + (2A + 3B - 3C)s - 8A - 4B + 2C \quad \text{too complicated}$$

Fast Way of Decomposition

$$\frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4}$$

Multiply by $(s - 1)$ on both sides of equation

$$\frac{s^2 + 6s + 9}{(s-2)(s+4)} = A + (s-1)\frac{B}{s-2} + (s-1)\frac{C}{s+4}$$

$s = 1$ substitutes in $-\frac{16}{5} = A$ These two steps can combine

After multiplying by $(s - 2)$ on left equation , let $s = 2$

$$B = \left. \frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)} \right|_{s=2} = \frac{25}{6}$$

After multiplying by $(s + 4)$ on left equation , let $s = -4$

$$C = \left. \frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)} \right|_{s=-4} = \frac{1}{30}$$

rule : decomposition of a fraction (Cover up method)

$$\frac{K(s)}{(s-a_1)(s-a_2) \cdots (s-a_N)} = Q(s) + \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \cdots + \frac{A_N}{s-a_N}$$

a_1, a_2, \dots, a_N mutually different

(1) By division of polynomial to get $Q(s)$ quotient remainder

$$\frac{K(s)}{(s-a_1)(s-a_2) \cdots (s-a_N)} = Q(s) + \frac{K_1(s)^r}{(s-a_1)(s-a_2) \cdots (s-a_N)}$$

Let the order of $K_1(s) < N$

(2) Calculate A_n

$$A_n = \left. \frac{K_1(s)}{(s-a_1)(s-a_2) \cdots (s-a_{n-1})(s-a_n)(s-a_{n+1}) \cdots (s-a_N)} \right|_{s=a_n}$$

Example:

$$\frac{s^4 - 8s^3 + 31s^2 - 36s + 20}{(s-1)(s-2)(s-3)^2} = Q(s) + \frac{A_1}{s-1} + \frac{A_2}{s-2} + \frac{A_3 + A_4(s-3)}{(s-3)^2}$$

$$\frac{s^4 - 8s^3 + 31s^2 - 36s + 20}{(s-1)(s-2)(s-3)^2} = 1 + \frac{s^3 + 2s^2 + 3s + 2}{(s-1)(s-2)(s-3)^2}$$

$$A_1 = \left. \frac{s^3 + 2s^2 + 3s + 2}{(s-1)(s-2)(s-3)^2} \right|_{s=1} = -\frac{8}{4} = -2$$

$$A_2 = \left. \frac{s^3 + 2s^2 + 3s + 2}{(s-1)(s-2)(s-3)^2} \right|_{s=2} = 24$$

$$\frac{s^3 + 2s^2 + 3s + 2}{(s-1)(s-2)} = (s-3)^2 \frac{A_1}{s-1} + (s-3)^2 \frac{A_2}{s-2} + A_3 + A_4(s-3)$$

$$A_3 = \left. \frac{s^3 + 2s^2 + 3s + 2}{(s-1)(s-2)} \right|_{s=3} = \frac{56}{2} = 28$$

$$\frac{s^3 + 2s^2 + 3s + 2}{(s - 1)(s - 2)} = (s - 3)^2 \frac{A_1}{s - 1} + (s - 3)^2 \frac{A_2}{s - 2} + A_3 + A_4(s - 3)$$

$$A_4 = \left. \frac{d}{ds} \frac{s^3 + 2s^2 + 3s + 2}{(s - 1)(s - 2)} \right|_{s=3} = \left. \frac{(3s^2 + 4s + 3)(s - 1)(s - 2) - (s^3 + 2s^2 + 3s + 2)(2s - 3)}{(s - 1)^2 (s - 2)^2} \right|_{s=3}$$
$$= -21$$

$$\boxed{\frac{s^4 - 8s^3 + 31s^2 - 36s + 20}{(s - 1)(s - 2)(s - 3)^2} = 1 - \frac{2}{s - 1} + \frac{24}{s - 2} + \frac{28 - 21(s - 3)}{(s - 3)^2}}$$

- Tip: If there is only one unknown to be solved, we can substitute s by some value to get the solution quickly .

In the example on the last page, we substitute $s = 0$ to the equation

$$\frac{2}{18} = -A_1 - \frac{A_2}{2} + \frac{A_3 - 3A_4}{9} \quad A_4 = (-1 - 9A_1 - \frac{9A_2}{2} + A_3)/3 = -21$$

Example:

$$\frac{s^2 + 2s + 3}{(s - 1)(s^2 + 2s + 2)} = \frac{A_1}{s - 1} + \frac{A_2 s + A_3}{s^2 + 2s + 2}$$

$$A_1 = \left. \frac{s^2 + 2s + 3}{(s - 1)(s^2 + 2s + 2)} \right|_{s=1} = \frac{6}{5}$$

$$\frac{s^2 + 2s + 3}{(s - 1)(s^2 + 2s + 2)} - \frac{6/5}{s - 1} = \frac{1}{5} \frac{-s^2 - 2s + 3}{(s - 1)(s^2 + 2s + 2)} = \frac{1}{5} \frac{-s - 3}{s^2 + 2s + 2}$$

$$\frac{s^2 + 2s + 3}{(s - 1)(s^2 + 2s + 2)} = \frac{6/5}{s - 1} - \frac{1}{5} \frac{s + 3}{s^2 + 2s + 2}$$

Transforms of Derivatives

$$\begin{aligned} L\{f'(t)\} &= \int_0^\infty e^{-st} f'(t) dt = \left. e^{-st} f(t) \right|_0^\infty + s \int_0^\infty e^{-st} f(t) dt \\ &= 0 - f(0) + sL\{f(t)\} = sL\{f(t)\} - f(0) \end{aligned}$$

$$\int_a^b u(t)v'(t) dt = \left. u(t)v(t) \right|_a^b - \int_a^b u'(t)v(t) dt$$

$$\begin{aligned} L\{f''(t)\} &= sL\{f'(t)\} - f'(0) = s[sL\{f(t)\} - f(0)] - f'(0) \\ &= s^2L\{f(t)\} - sf(0) - f'(0) \end{aligned}$$

$$L\{f'''(t)\} = sL\{f''(t)\} - f''(0) = s^3L\{f(t)\} - s^2f(0) - sf'(0) - f''(0)$$

Derivative Property of the Laplace Transform

$$L\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Example $y'(t) + 3y(t) = 13 \sin 2t$ $y(0) = 6$

$$sY(s) - y(0) + 3Y(s) = 13 \frac{2}{s^2 + 4}$$

$$(s+3)Y(s) = 6 + \frac{26}{s^2 + 4}$$

$$Y(s) = \frac{6}{s+3} + \frac{26}{(s+3)(s^2+4)}$$

$$Y(s) = \frac{8}{s+3} + \frac{-2s+6}{s^2+4}$$

$$Y(s) = \frac{8}{s+3} + -2\frac{s}{s^2+4} + 3\frac{2}{s^2+4}$$

$$y(t) = 8e^{-3t} - 2\cos 2t + 3\sin 2t$$

$$\left. \frac{26}{(s+3)(s^2+4)} \right|_{s=-3} = \frac{26}{13} = 2$$

$$\begin{aligned} & \frac{26}{(s+3)(s^2+4)} - \frac{2}{s+3} \\ &= \frac{-2s^2 + 18}{(s+3)(s^2+4)} = \frac{-2s + 6}{s^2 + 4} \end{aligned}$$

Solving the Constant Coefficient Linear DE by Laplace Transforms

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \dots + a_1 y'(x) + a_0 y = g(x)$$

↓
Laplace transform

$$\begin{aligned} & a_n [s^n Y(s) - s^{n-1} y(0) - s^{n-2} y'(0) - \dots - s y^{(n-2)}(0) - y^{(n-1)}(0)] \\ & + a_{n-1} [s^{n-1} Y(s) - s^{n-2} y(0) - s^{n-3} y'(0) - \dots - s y^{(n-3)}(0) - y^{(n-2)}(0)] \\ & + \dots \\ & + a_1 [s Y(s) - y(0)] + a_0 Y(s) = G(s) \end{aligned}$$

$$\begin{aligned}
 & a_n [s^n Y(s) - s^{n-1} y(0) - s^{n-2} y'(0) - \dots - s y^{(n-2)}(0) - y^{(n-1)}(0)] \\
 & + a_{n-1} [s^{n-1} Y(s) - s^{n-2} y(0) - s^{n-3} y'(0) - \dots - s y^{(n-3)}(0) - y^{(n-2)}(0)] \\
 & + \dots \\
 & + a_1 [s Y(s) - y(0)] + a_0 Y(s) = G(s)
 \end{aligned}$$

$$P(s)Y(s) - Q(s) = G(s)$$

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$$

$$\begin{aligned}
 Q(s) &= a_n [s^{n-1} y(0) + s^{n-2} y'(0) + \dots + s y^{(n-2)}(0) + y^{(n-1)}(0)] \\
 & + a_{n-1} [s^{n-2} y(0) + \dots + s y^{(n-3)}(0) + y^{(n-2)}(0)] \\
 & + \dots \\
 & + a_2 [s y(0) + y'(0)] \\
 & + a_1 [y(0)]
 \end{aligned}$$

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}$$

$$Y(s) = W(s)Q(s) + W(s)G(s)$$

$G(s)$: Laplace transform of the input

$$W(s) = \frac{1}{P(s)}$$

$Q(s)$: caused by initial conditions

$Y(s)$: Laplace transform of the response

$W(s)$: transform function

$L^{-1}[W(s)Q(s)]$: zero-input response or state response

$L^{-1}[W(s)G(s)]$: zero-state response or input response

Example

$$y''(t) - 3y'(t) + 2y(t) = e^{-4t} \quad y(0) = 1 \quad y'(0) = 5$$

↓
Quickly Solving

$$\begin{aligned} & (s^2 - 3s + 2)Y(s) = s + 2 + \frac{1}{s+4} \\ Y(s) &= \frac{s+2}{s^2 - 3s + 2} + \frac{1}{(s^2 - 3s + 2)(s+4)} \\ &= \frac{s+2}{(s-1)(s-2)} + \frac{1}{(s-1)(s-2)(s+4)} \\ &= -\frac{3}{s-1} + \frac{4}{s-2} - \frac{1}{5} \frac{1}{s-1} + \frac{1}{6} \frac{1}{s-2} + \frac{1}{30} \frac{1}{s+4} \\ &= -\frac{16}{5} \frac{1}{s-1} + \frac{25}{6} \frac{1}{s-2} + \frac{1}{30} \frac{1}{s+4} \end{aligned}$$

$$y(t) = -\frac{16}{5}e^t + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$$

What type of lover are you

$$\frac{dR}{dt} = aR + bJ$$

$$\frac{dJ}{dt} = cR + dJ$$



- Fanatic type ($a > 0, b > 0$) :
The love from yourself or your lover will make your emotions warm up
- Prudent type ($a < 0, b > 0$) :
Restrain your love, you will love more because of the your lover's love
- Narcissistic type ($a > 0, b < 0$)
- Hermit type ($a < 0, b < 0$)

4 types, 16 combinations of lovers

1. $f(t) = 3 + 12t + 42t^3 - 3e^{2t}$, find $F(s)$

2. Let $f(t) = 4e^{3t} + 3 \sin(2t)$, find the Laplace transform of function f(t)

3. $L^{-1} \left[\frac{5}{s+1} - \frac{6}{s^2+4} + \frac{1}{s^4} \right], \text{ find } f(t)$

4. Find the inverse Laplace transform of $\frac{s+4}{(s+1)(s+2)}$

5. Use Laplace Transform to find the solution of $y' - 4y = 1; y(0) = 1$

$$6. D^3x - D^2x = 0 ; x(0) = x'(0) = x''(0) = 3$$

7. Use Laplace Transform to find the solution of $y'' + 4y' + 3y = e^t$

$$y(0) = 0, y'(0) = 2$$

8. Solve $y'' + 4y = t, y(0) = 1, y'(0) = 2$ with Laplace transform.