9-9 Convergence of Taylor Series

師大工教一

Theorem 23—**Taylor's Theorem**

Let f and its first n derivatives $f', f'', \cdots, f^{(n)}$ are continuous on the closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a constant c between a and b such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

Taylor's Formula

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+R_n(x) \qquad (1)$$

where $R_n(x)=\dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a and x. (2)

Note: Rewrite Equation (1) by $f(x)=P_n(x)+R_n(x)$ and Eq. (1) is called

Taylor's formula. The function $R_n(x)$ is called the **remainder of order** n

餘式

or the **error term** for the approximation of f by $P_n(x)$ over I.

誤差

If $R_n(x) \to 0$ as $n \to \infty$ for all $x \in I$, we say that the Taylor series generated by f at $x = a$ **converges** to f on I and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k .$$

Ex1(p578) Show that the Taylor series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every value of x.

$f(x) = e^x = 1 + x + \frac{x^2}{2!} \cdots + \frac{x^n}{n!} + R_n(x)$

$R_n(x) = \frac{\overbrace{f^{(n+1)}(c)}^{=e^c}}{(n+1)!} x^{n+1}$, c 介於 $0, x$ 之間

$$\begin{cases} x<0, & e^c<1 \\ x=0, & e^c=1 \\ x>0, & 1<e^c<e^x \end{cases} \Rightarrow \left\{ \begin{aligned} & x \le 0, \ R_n(x) \le \frac{x^{n+1}}{(n+1)!} \\ & x>0, \ R_n(x) \le \frac{e^x x^{n+1}}{(n+1)!} \end{aligned} \right\} \Rightarrow \lim_{n\to\infty} R_n(x) = 0, \ \forall x \in \mathbb{R}$$

$\sum_{n=0}^{\infty} \frac{x^n}{n!} \to f(x), \ \forall x \in \mathbb{R}$

Estimating the Remainder

$|f^{(n+1)}(t)| \le M$, $\forall t$ lies between a & x
$\Rightarrow R_n(x) \to 0$
$\Rightarrow$ Taylor series converges to generating function

Theorem 24—**The Remainder Estimation Theorem**

估計

If there is a positive constant M such that $\left|f^{(n+1)}(t)\right| \le M$ for all t between a and x, inclusive, then the remaider term $R_n(x)$ in Taylor's Theorem satisfies the inequality $\left|R_n(x)\right| \le M\dfrac{|x-a|^{n+1}}{(n+1)!}$. If the inequality holds for every n, and the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

Ex2(p579) Show that the Taylor series for $\sin x$ at $x = 0$ converges for all x.

$f^{(2k+2)}(c)$

$$f(x) = \sin x \qquad f'(x) = \cos x$$

$$f''(x) = -\sin x \qquad f'''(x) = -\cos x$$

$$\vdots \qquad \vdots$$

$$f^{(2k)}(x) = (-1)^k \sin x \qquad f^{(2k+1)}(x) = (-1)^k \cos x$$

$$f^{(2k)}(0) = (-1)^k \qquad f^{(2k+1)}(0) = (-1)^k$$

$\because f(x) = \sin x$

$f^{(2k+2)}(x) = \pm\sin x, \pm\cos x$

$|f^{(2k+2)}(c)| \le 1$

By Remainder Estimation theor.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + R_{2k+1}(x)$$

$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \to \sin x$

$R_{2k+1}(x) = \frac{f^{(2k+2)}(c)}{(2k+2)!} x^{2k+2}$

$\forall x \in \mathbb{R}$

All derivatives of $\sin x$ are one of $\pm\sin x, \pm\cos x$. Hence, $\left|f^{(2k+2)}(t)\right| \le 1$ for all t lies between 0 and x. By the Remainder Estimation Theorem,

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Ex3(p580) Show that the Taylor series for $\cos x$ at $x = 0$ converges to $\cos x$ for every value of x.

Ex4(p580) Using known series, find the first few terms of the Taylor series for the given function by using power series operations.

(a) $\frac{1}{3}\left(2x + x\cos x\right)$ (b) $e^x \cos x$

By Theorem 20, if the Taylor series generated by $f(x)$ at $x=0$,

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots$$

Converges absolutely for $|x|<R$, and if u is a continuous function, then the

series $\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}(u(x))^k$ converges absolutely on the set of points x where

$|u(x)|<R$.

E.g.,

$$\cos 2x = \sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!} = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots$$

$$= 1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} - \frac{2^6 x^6}{6!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} x^{2k}}{(2k)!}$$

Table 9.1(p589) Frequently Used Taylor Series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad |x| < \infty$$

$$\frac{1}{1-x}$$

$$f(x) = \frac{1}{1-x}$$

$$f'(x) = \frac{-(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}$$

$$f''(x) = \frac{-[(1-x)^2]'}{(1-x)^4} = \frac{-2\cancel{(1-x)}(-1)}{(1-x)^{\cancel{4}\,3}} = \frac{2}{(1-x)^3}$$

$$f'''(x) = \frac{-2((1-x)^3)'}{(1-x)^6}$$

$$= \frac{-6(1-x)(-1)}{(1-x)^6} = \frac{6}{(1-x)^5}$$

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$$

$$f(x) = \sum_{k=0}^{\infty} \frac{\frac{\cancel{k!}}{(1-a)^{k+1}}}{\cancel{k!}} (x-a)^k = \sum_{k=0}^{\infty} x^k$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots + \frac{(-1)^{n-1} x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x \le 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^{n-1} x^{2n+1}}{2n+1} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n+1}}{2n+1}, \quad |x| \le 1$$

HW9-9

- **HW:1,3,7**