12-2 Integrals of Vector Functions

師大工教一

Integrals of Vector Functions

A differentiable vector function $\vec{R}(t)$ is an **antiderivative** of a vector function $\vec{r}(t)$ on an interval I if $\dfrac{d\vec{R}}{dt}=\vec{r}$ at each point of I. If $\vec{R}$ is an antiderivative of $\vec{r}$ on I, then every antiderivative of $\vec{r}$ on I has the form $\vec{R}+\vec{C}$ for some constant vector $\vec{C}$. The set of all antiderivatives of $\vec{r}$ on I is the **indefinite integral** of $\vec{r}$ on I.

Definition The **indefinite integral** of $\vec{r}$ with respect to t is the set of all antiderivatives of $\vec{r}$, denoted by $\int \vec{r}(t)\,dt$. If $\vec{R}$ is an antiderivative of $\vec{r}$, then

$$\int \vec{r}(t)\,dt = \vec{R}(t) + \vec{C}.$$

Ex1(p689) Find $\int \left((\cos t)\vec{i} + \vec{j} - 2t\vec{k} \right) dt$.

$$= (\sin t + C_1)\vec{i} + (t + C_2)\vec{j} - (t^2 + C_3)\vec{k}$$

Definition If the components $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ are integrable over $[a,b]$, then so is $\vec{r}$, and the definite integral of $\vec{r}$ from a to b is

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\vec{i} + \left(\int_a^b g(t)\,dt\right)\vec{j} + \left(\int_a^b h(t)\,dt\right)\vec{k}$$

Ex2(p690) Find $\int_0^{\pi} \left((\cos t)\vec{i} + \vec{j} - 2t\vec{k}\right)dt$.

$$= \left(\sin t\,\vec{i} + t\vec{j} - t^2\vec{k}\right)\Big]_0^{\pi}$$

$$= \pi\vec{j} - \pi^2\vec{k}$$

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Fundamental Theorem of Calculus for continuous vector functions says that

$$\int_a^b \vec{r}(t)\,dt = \vec{R}(t)\Big|_a^b = \vec{R}(b) - \vec{R}(a),$$

where $\vec{R}$ is an antiderivative of $\vec{r}$, so that $\vec{R}'(t) = \vec{r}(t)$.

Ex3(p690) Solve the path of a hang glider $\vec{r}(t)$ with

$\vec{a}(t) = -(3\cos t)\vec{i} - (3\sin t)\vec{j} + 2\vec{k}$ and initially(at time 0) the glider departed

from the point $(4,0,0)$ with velocity $\vec{v}(0) = 3\vec{j}$.

$$\vec{V}(t) = (-3\sin t + C_1)\vec{i} + (3\cos t + C_2)\vec{j} + (2t + C_3)\vec{k}, \quad \vec{v}(0) = 3\vec{j} \to C_1 = 0, C_2 = 0, C_3 = 0$$

$$\vec{r}(t) = (3\cos t + C_1 t + b_1)\vec{i} + (3\sin t + C_2 t + b_2)\vec{j} + (t^2 + C_3 t + b_3)\vec{k}$$

$$= (3\cos t + b_1)\vec{i} + (3\sin t + b_2)\vec{j} + (t^2 + b_3)\vec{k} \quad (4,0,0)$$

$$= (3\cos t + 1)\vec{i} + (3\sin t)\vec{j} + (t^2)\vec{k}$$

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HW12-2

- **HW: 1,8,11,15,18,20**

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4. Suppose that a particle moves along the space curve given by

(a) $\vec{v} = (-5\sin t + 5\sin 5t)\vec{i} + (5\cos t - 5\cos 5t)\vec{j} + \frac{-1}{t^2}k$

$$\mathbf{r}(t) = (5\cos t - \cos 5t)\mathbf{i} + (5\sin t - \sin 5t)\mathbf{j} + \frac{1}{t}\mathbf{k}, \quad 0 < t \leq 2\pi,$$

$\vec{v}(\pi) = \pi^{-2}$ ※

where t denotes the time parameter. (時間參數)

(a) (10 pts) What is the speed (速率) of the particle at time $t = \pi$?

(b) (10 pts) Find the acceleration (加速度) vector of the particle at time $t = \pi/2$.

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6. (10 pts) Evaluate $\mathbf{r}(\frac{\pi}{4}) \times \mathbf{r}'(\frac{\pi}{4})$ if $\mathbf{r}(t)$ is a vector-valued function satisfying the following equation with initial conditions (初始條件)

$$\mathbf{r}''(t) = -4\cos t\mathbf{j} - 3\sin t\mathbf{k}, \quad \mathbf{r}'(0) = 3\mathbf{k}, \quad \mathbf{r}(0) = 4\mathbf{j}.$$

$\vec{r}'(t) = \int(-4\cos t\vec{j} - 3\sin t\vec{k})\,dt = (-4\sin t + C_1)\vec{j} + (3\cos t + C_2)\vec{k}$, $r'(0) = 3k$

$C_1 = 0$, $C_2 = 0$

t sint
1 $-\cos t$
$\int$

D I
t $+$ cost
1 $-$ sint
$\int$

$t\sin t - \int \sin t\,dt$

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6. Let

$$\vec{r}(t) = t^2\vec{i} + (\sin t - t\cos t)\vec{j} + (\cos t + t\sin t)\vec{k}, \quad 0 \le t \le \pi,$$

be a vector- valued function.

(a) (6 points) Evaluate the definite integral $\int_0^\pi \vec{r}(t)dt$.

(b) (6 points) Find the derivative of $\vec{r}(t)$ at $t = \frac{\pi}{4}$.

(c) (6 points) Find the arc length of the curve represented by $\vec{r}(t)$.

(a) $\int_0^\pi \vec{r}(t)dt$

$= \left(\frac{t^3}{2}\right)\vec{i} + (-\cos t - t\sin t - \cos t)\vec{j} + (\sin t - t\cos t + \sin t)\vec{k}$

(b) $\vec{r}\,'(t) = 2t\vec{i} + (\cancel{\cos t - \cos t} + t\sin t)\vec{j} + (\cancel{-\sin t + \sin t} + t\cos t)\vec{k}$

$\vec{r}\,'\left(\frac{\pi}{4}\right) = \frac{\pi}{2}\vec{i} + \frac{\pi\sqrt{2}}{8}\vec{j} + \frac{\pi\sqrt{2}}{8}\vec{k}$