Nonhomogeneous DE with Constant Coefficient(2.7) and Euler-Cauchy Equation (2.5)

Reference CENGAGE Learning

Form Rule

"guessing" the particular solution.

(1) (2)

Suitable for linear and constant coefficient DE.

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) = g(x)$$

(3) $g(x)$, $g'(x)$, $g''(x)$, $g'''(x)$, $g^{(4)}(x)$, $g^{(5)}(x)$, $\cdots\cdots\cdots$contain finite number of terms.

Form Rule

Hold on to one principle：

Particular solution always looks like g(x).

Please memorize the “rules” listed on the next page

(solve unknowns A, B, C, $\cdots$)

Trial Particular Solutions (from text page 143)

$g(x)$	Form of y_p
1 (any constant)	A
$5x + 7$	$Ax + B$
$3x^2 - 2$	$Ax^2 + Bx + C$
$x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
$\sin 4x$	$A\cos 4x + B\sin 4x$
$\cos 4x$	$A\cos 4x + B\sin 4x$
e^{5x}	Ae^{5x}
$(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
$e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
$5x^2\sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (Ex^2 + Fx + G)\sin 4x$
$xe^{3x}\cos 4x$	$(Ax + B)e^{3x}\cos 4x + (Cx + E)e^{3x}\sin 4x$

It comes from the "form rule."

Explanation of Form Rule

Form Rule: y_p should be a linear combination of $g(x)$, $g'(x)$, $g''(x)$, $g'''(x)$, $g^{(4)}(x)$, $g^{(5)}(x)$, ……………….

In this way, it will not generate **unnecessary** terms when comparing coefficients.

When $g(x) = x^n$

$$x^n \to x^{n-1} \to x^{n-2} \to x^{n-3} \to \cdots\cdots \to 1 \to 0$$

$$y_p = A_n x^n + A_{n-1} x^{n-1} + A_{n-2} x^{n-2} + \ldots\ldots + A_0$$

When $g(x) = \cos kx$

$$\cos kx \to \sin kx$$

$$y_p = A_1 \cos kx + A_2 \sin kx$$

When $g(x) = \exp(kx)$

$$e^{kx}$$

$$y_p = A \exp(kx)$$

When $g(x) = x^n \exp(kx)$

$$g'(x) = nx^{n-1}e^{kx} + kx^n e^{kx}$$

$$g''(x) = n(n-1)x^{n-2}e^{kx} + 2nkx^{n-1}e^{kx} + k^2x^n e^{kx}$$

$$g'''(x) = n(n-1)(n-2)x^{n-3}e^{kx} + 3kn(n-1)x^{n-2}e^{kx}$$
$$+3k^2nx^{n-1}e^{kx} + k^3x^n e^{kx}$$

$$\vdots$$

$$\vdots$$

We find that any-order differential of $g(x)$ only generate s the following terms:

$$x^n e^{kx}, x^{n-1}e^{kx}, x^{n-2}e^{kx}, x^{n-3}e^{kx}, \cdots\cdots, e^{kx}$$

$$\boxed{y_p = c_n x^n e^{kx} + c_{n-1}x^{n-1}e^{kx} + c_{n-2}x^{n-2}e^{kx} + \cdots\cdots + c_0 e^{kx}}$$

Example $y'' - y' + y = \underline{2\sin 3x}$

Step 1: Find the solution of the associated homogeneous equation

Guess

Step 2: Particular solution

$$\boxed{y_p = A\cos 3x + B\sin 3x}$$

$$y_p' = -3A\sin 3x + 3B\cos 3x$$

$$y_p'' = -9A\cos 3x - 9B\sin 3x$$

$$y_p'' - y_p' + y_p = (-8A - 3B)\cos 3x + (3A - 8B)\sin 3x = 2\sin 3x$$

$$\begin{cases} -8A - 3B = 0 \\ 3A - 8B = 2 \end{cases} \Longrightarrow A = 6/73,\ \ B = -16/73$$

$$\boxed{y_p = \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x}$$

Step 3: General solution:

$$\boxed{y = e^{x/2}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right) + \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x}$$

Example $y''-2y'-3y=\underline{4x-5}+\underline{6xe^{2x}}$

Step 1: Find the solution of

$$y''-2y'-3y=0.$$

$$y_c=c_1e^{3x}+c_2e^{-x}$$

Step 2: Particular solution

$$y''-2y'-3y=4x-5 \qquad y''-2y'-3y=6xe^{2x}$$

guess

$$\boxed{y_{p_1}=Ax+B}$$

$$y'_{p_1}=A$$

$$y''_{p_1}=0$$

$$-3Ax-2A-3B=4x-5$$

$$A=-\frac{4}{3},\quad B=\frac{23}{9}$$

$$\boxed{y_{p_1}=-\frac{4}{3}x+\frac{23}{9}}$$

guess

$$\boxed{y_{p_2}=Cxe^{2x}+Ee^{2x}}$$

$$y'_{p_2}=2Cxe^{2x}+Ce^{2x}+2Ee^{2x}$$

$$y''_{p_2}=4Cxe^{2x}+4Ce^{2x}+4Ee^{2x}$$

$$-3Cxe^{2x}+\left(2C-3E\right)e^{2x}=6xe^{2x}$$

$$C=-2,\quad E=-\frac{4}{3}$$

$$\boxed{y_{p_2}=-(2x+\frac{4}{3})e^{2x}}$$

Particular solution

$$y_p = y_{p_1} + y_{p_2} = -\frac{4}{3}x + \frac{23}{9} - (2x + \frac{4}{3})e^x$$

Step 3: General solution

$$y = y_c + y_p$$

$$\boxed{y = c_1 e^{3x} + c_2 e^{-x} - \frac{4}{3}x + \frac{23}{9} - (2x + \frac{4}{3})e^{2x}}$$

4-4-5 Glitch of the method:

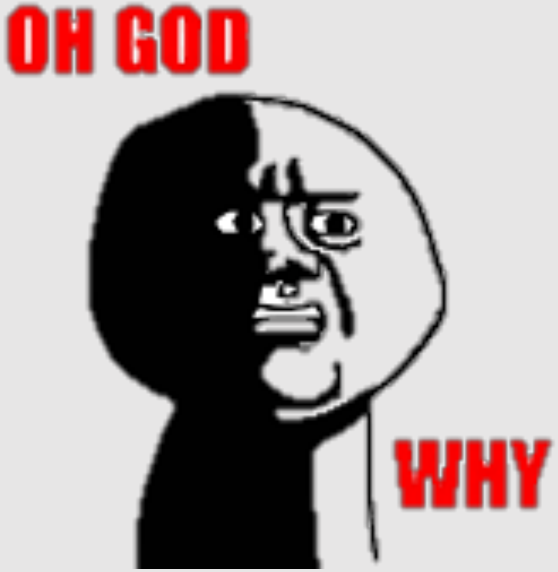
Example $y'' - 5y' + 4y = 8e^x$

Particular solution guessed by Form Rule:

$$y_p = Ae^x$$

$$y_p'' - 5y_p' + 4y_p = Ae^x - 5Ae^x + 4Ae^x = 8e^x$$

$$0 = 8e^x$$ (no solution)



Glitch condition 1:

The **particular solution** we guess belongs to the complementary function.

$$y'' - 5y' + 4y = 8e^x$$

Complementary function $\quad y_c = c_1 e^x + c_2 e^{4x} \qquad Ae^x \in y_c$

Solution：multiplied by x

$$\boxed{y_p = Axe^x} \qquad y'_p = Axe^x + Ae^x$$

$$y''_p = Axe^x + 2Ae^x$$

$$y''_p - 5y'_p + 4y_p = -3Ae^x = 8e^x \Longrightarrow A = -8/3$$

$$\boxed{y_p = -\frac{8}{3}xe^x}$$

$$\boxed{y = c_1 e^x + c_2 e^{4x} - \frac{8}{3}xe^x}$$

Example $y'' - 2y' + y = e^x$

$$y_c = c_1 e^x + c_2 x e^x$$

From Form Rule, the particular solution is Ae^x

$Ae^x \in y_c$

If multiplying by x is not enough, multiply by x^2

$Axe^x \in y_c$

$$\boxed{y_p = Ax^2 e^x} \qquad y_p' = (Ax^2 + 2Ax)e^x$$

$$y_p'' = (Ax^2 + 4Ax + 2A)e^x$$

$$y_p'' - 2y_p' + y_p = 2Ae^x = e^x \implies A = 1/2$$

$$\boxed{y_p = x^2 e^x / 2}$$

$$\boxed{y = c_1 e^x + c_2 x e^x + x^2 e^x / 2}$$

Example

$$y'' + y = 4x + 10\sin x \qquad y(\pi) = 0 \qquad y'(\pi) = 2$$

Step 1 $\quad y_c = c_1 \cos x + c_2 \sin x$

Step 2 $\quad y_p = \boxed{Ax + B} + \boxed{Cx\sin x + Ex\cos x}$

Note : both sin x and cos x should be multiplied by x

$$y_p = 4x - 5x\cos x$$

Step 3 $\quad \boxed{y = c_1 \cos x + c_2 \sin x + 4x - 5x\cos x}$ (solve IVP at the end)

Step 4 $\quad$ Solving c_1 and c_2 by initial conditions

$$y(\pi) = -c_1 + 4\pi + 5\pi = 0 \Longrightarrow c_1 = 9\pi$$

$$y' = -c_1 \sin x + c_2 \cos x + 4 - 5\cos x + 5x\sin x$$

$$y'(\pi) = -c_2 + 9 = 2 \Longrightarrow c_2 = 7$$

$$\boxed{y = 9\pi\cos x + 7\sin x + 4x - 5x\cos x}$$

$$y = e^{\alpha x}\left(c_1 \cos \beta x + c_2 \sin \beta x\right)$$

Glitch condition 2: $g(x)$, $g'(x)$, $g''(x)$, $g'''(x)$, $g^{(4)}(x)$, $g^{(5)}(x)$, ……………
contain infinite number of terms.

If $g(x) = \ln x$

$$\ln x \to \frac{1}{x} \to \frac{1}{x^2} \to \frac{1}{x^3} \to \cdots\cdots$$

If $g(x) = \exp(x^2)$

$$g'(x) \to 2xe^{x^2}$$
$$g''(x) \to (4x^2 + 2)e^{x^2}$$
$$g'''(x) \to (8x^3 + 12x)e^{x^2}$$

$$\vdots$$

$$\vdots$$

Euler-Cauchy Equation (2.5)

$x^2y'' + axy' + by = 0$

let $y = x^m$, $y' = mx^{(m-1)}\cdots$

$x^2m(m-1)x^{m-2} + amx^{m-1} + bx^m = 0$

Auxiliary function: $m^2 + (a-1)m + b = 0$

Three Cases

- Real different roots

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

- A real double root

$$y = (c_1 + c_2 lnx)x^{m_2}$$

- Complex roots
  - Same as real different roots

Example

$$x^2y''+xy'-y=0$$

$$m(m-1)+m-1=0$$

$$m^2-1=0$$

$$y=c_1x+c_2/x$$

Alternative Thinking

$x^2y'' + axy' + by = 0$

let $x = e^z$, $z = \ln x$

$dy/dx = (1/x) \cdot dy/dz$

$d^2y/dx^2 = (1/x^2) \cdot (d^2y/dz^2 - dy/dz)$

$y'' + (a-1)y' + by = 0$ differentiate w.r.t z

Exercise

1. $y'' - 3y' + 2y = xe^{2x}$

2. $y'' - 3y' + 2y = x^2(e^x + e^{-x})$,
   Write down the correct form of its particular solution, don't solve it.

Exercise

$3.\ y'' + 4y' + 4y = sin^2 t, find\ y_p(t)$ $\quad$ $4.\ y'' + 5y' + 6y = e^{-2t}, find\ y_p(t)$

Exercise

5. $Solve\ y'' - 6y' + 9y = 2e^{3x} + 9x + 3$

6. $x^2y'' - 2xy' + 2y = 0, find\ y_c$

Exercise

7. $x^2y'' - 3xy' + 3y = 0$