Advanced Engineering Mathematics 賴以威

Laplace Transform for DE

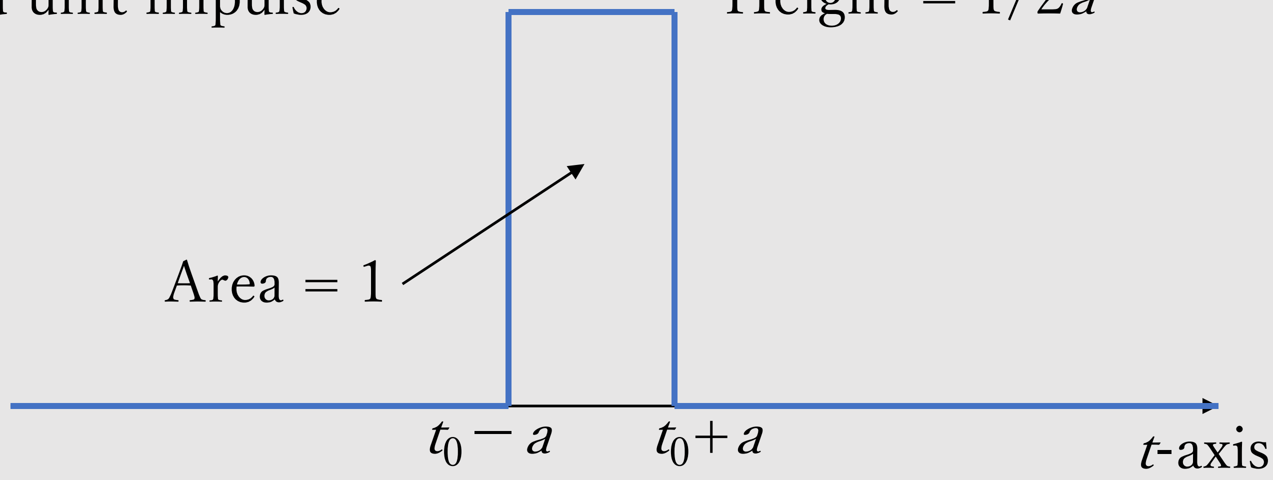
Reference CENGAGE Learning

The Dirac Delta Function

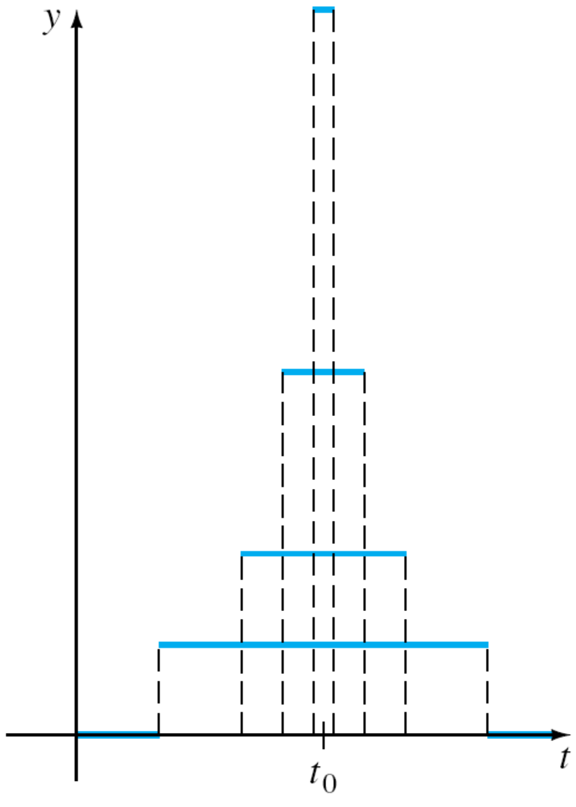
$$\delta_a(t-t_0)=\begin{cases} 0 & \text{for } t< t_0-a \quad \text{or } t> t_0+a \\ \dfrac{1}{2a} & \text{for } t_0-a \le t \le t_0+a \end{cases}$$

called unit impulse

Height $= 1/2a$



The Dirac Delta Function



(b) behavior of δ_a as $a \to 0$

$$\delta(t - t_0) = \lim_{a\to 0} \delta_a (t - t_0)$$

$$\delta_a(t - t_0) = \begin{cases} \infty & \text{for } t = t_0 \\ 0 & \text{for } t \neq t_0 \end{cases}$$

Properties of the Dirac Delta Function

(1) Integration $\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$

(2) Sifting $\int_p^q f(t)\delta(t-t_0)\,dt = f(t_0)$ when $t_0 \in [p, q]$

Proof: $$\int_p^q f(t)\delta(t-t_0)\,dt = \lim_{a\to 0}\int_p^q f(t)\delta_a(t-t_0)\,dt$$

$$\cong f(t_0)\lim_{a\to 0}\int_p^q \delta_a(t-t_0)\,dt = f(t_0)$$

When a is very small, $f(t) = f(t_0)$ for $t_0 - a \le t \le t_0 + a$

(3) Laplace transform of $\delta(t - t_0)$

$$L\{\delta(t-t_0)\} = e^{-t_0 s} \quad \text{when } t_0 > 0$$

Proof: $\int_0^\infty e^{-st}\delta(t-t_0)dt$

Note: Theorem 7.1.3 can’t be applied on the Dirac delta function.

(4) Relation with the unit step function

$$\int_{-\infty}^{t} \delta(\tau - t_0)d\tau = u(t-t_0)$$

$$\frac{d}{dt}u(t-t_0) = \delta(\tau - t_0)$$

Example

$$y'' + y = 4\delta(t - 2\pi) \qquad y(0) = 1 \qquad y'(0) = 0$$

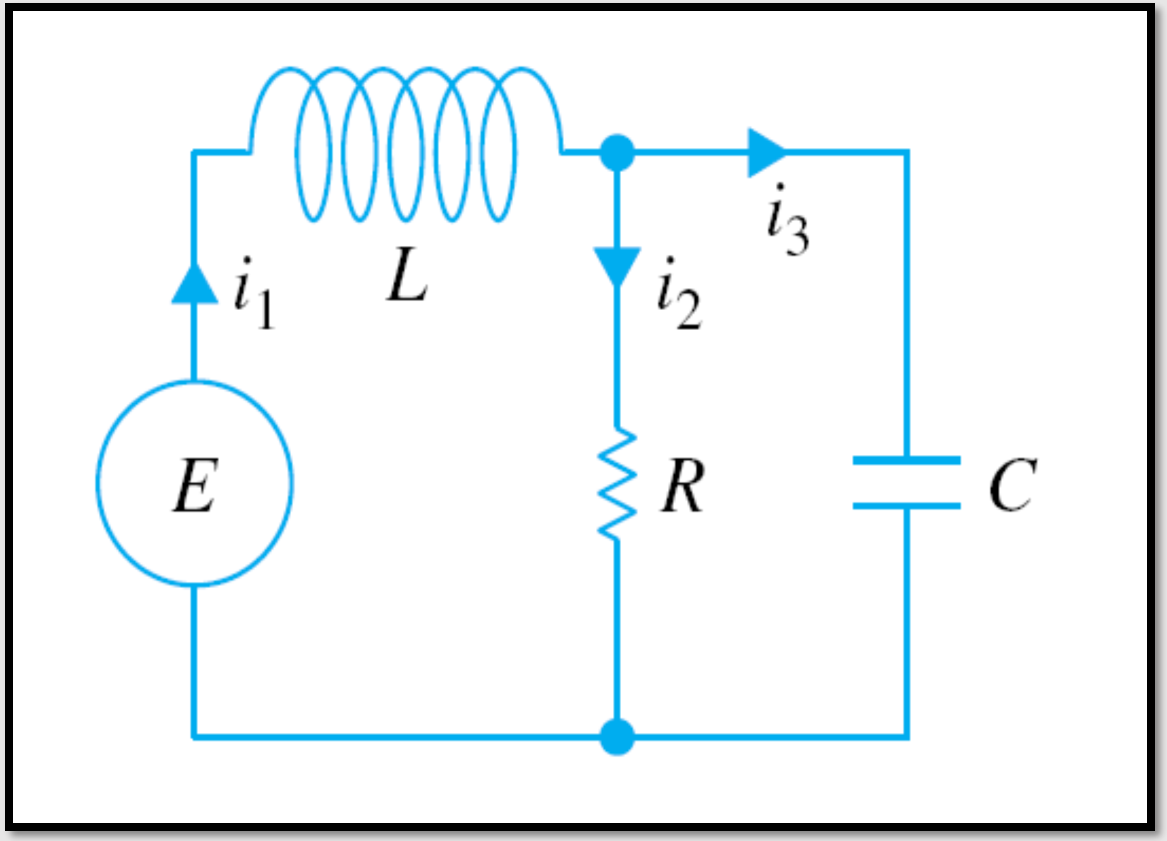
$$s^2 Y(s) - s + Y(s) = 4e^{-2\pi s}$$

$$Y(s) = \frac{s}{s^2+1} + 4\frac{e^{-2\pi s}}{s^2+1} \qquad L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

$$\begin{aligned} y(t) &= \cos t + 4\sin(t - 2\pi)u(t - 2\pi) \\ &= \cos t + 4\sin t \cdot u(t - 2\pi) \end{aligned}$$

$$y(t) = \begin{cases} \cos t & 0 \le t < 2\pi \\ \cos t + 4\sin t & t \ge 2\pi \end{cases}$$

Systems of Linear Differential Equations



$$\begin{cases} L\dfrac{di_1(t)}{dt} + i_2R_2 = E(t) \quad (1) \\ \dfrac{q_3}{C} = i_2R \quad (2) \\ i_1 = i_2 + i_3 \quad (3) \end{cases}$$

by (2) and (3)

$$i_1 = i_2 + \frac{d}{dt}q_3 = i_2 + \frac{d}{dt}RCi_2$$

$$\begin{cases} L\dfrac{di_1(t)}{dt} + i_2R_2 = E(t) \\ RC\dfrac{d}{dt}i_2 + i_2 - i_1 = 0 \end{cases}$$

$$\begin{cases} L\dfrac{di_1(t)}{dt} + i_2R_2 = E(t) \\ RC\dfrac{d}{dt}i_2 + i_2 - i_1 = 0 \end{cases}$$

Example

$E(\mathrm{t}) = 60\ \mathrm{V},\ \ L = 1\ \mathrm{h},\ \ R = 50\ \Omega,\ \ C = 10^{-4}\ \mathrm{f},\ \ i_1(t) = i_2(t) = 0$

$$\begin{cases} \dfrac{di_1(t)}{dt} + 50i_2 = 60 \\ 0.005\dfrac{d}{dt}i_2 + i_2 - i_1 = 0 \end{cases} \longrightarrow \begin{cases} sI_1(s) + 50I_2(s) = \dfrac{60}{s} \quad \cdots\cdots\cdots (1) \\ -I_1(s) + (0.005s + 1)I_2(s) = 0 \cdots (2) \end{cases}$$

$(1) \times 1 + (2) \times s$

$$(0.005s^2 + s + 50)I_2(s) = \frac{60}{s} \qquad (s^2 + 200s + 10000)I_2(s) = \frac{12000}{s}$$

$$I_2(s) = \frac{12000}{s(s+100)^2} = \frac{6/5}{s} - \frac{120}{(s+100)^2} - \frac{6/5}{s+100} \quad (1)$$

Review: how to compute the numerator ?

$$\boxed{i_2(t) = \frac{6}{5} - 120te^{-100t} - \frac{6}{5}e^{-100t}}$$

substitute $I_2(s) = \frac{6/5}{s} - \frac{120}{(s+100)^2} - \frac{6/5}{s+100}$ into (1)

$$sI_1(s) = \frac{60}{s} - \frac{60}{s} + \frac{6000}{(s+100)^2} + \frac{60}{s+100}$$

$$I_1(s) = \frac{6000}{s(s+100)^2} + \frac{60}{s(s+100)}$$

$$I_1(s) = \frac{6000}{s(s+100)^2} + \frac{60}{s(s+100)} = \frac{a}{s} + \frac{b + c(s+100)}{(s+100)^2}$$

$$a = \left.\frac{6000}{(s+100)^2} + \frac{60}{(s+100)}\right|_{s=0} = \frac{6}{5}$$

$$b = \left.\frac{6000}{s} + \frac{60(s+100)}{s}\right|_{s=-100} = -60$$

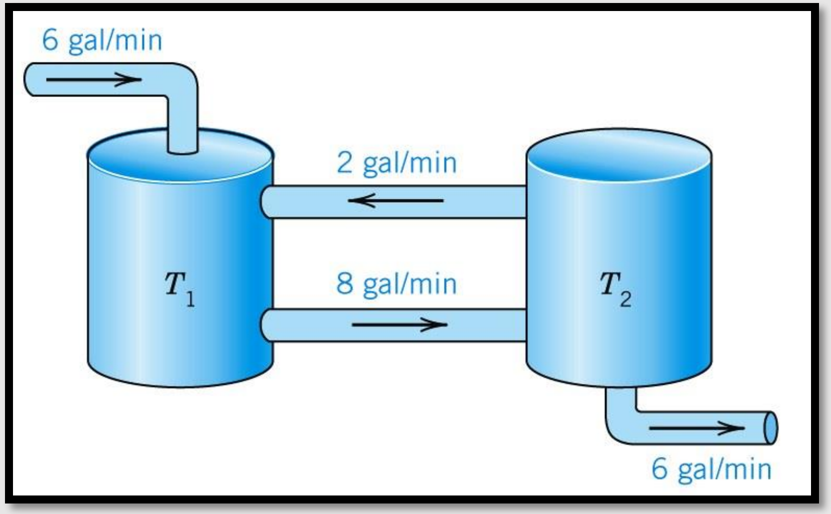
$$c = \left.\frac{d}{ds}\left(\frac{6000}{s} + \frac{60(s+100)}{s}\right)\right|_{s=-100} = \left.-\frac{6000}{s^2} - \frac{6000}{s^2}\right|_{s=-100} = -\frac{6}{5}$$

$$I_1(s) = \frac{6/5}{s} - \frac{60}{(s+100)^2} - \frac{6/5}{s+100}$$

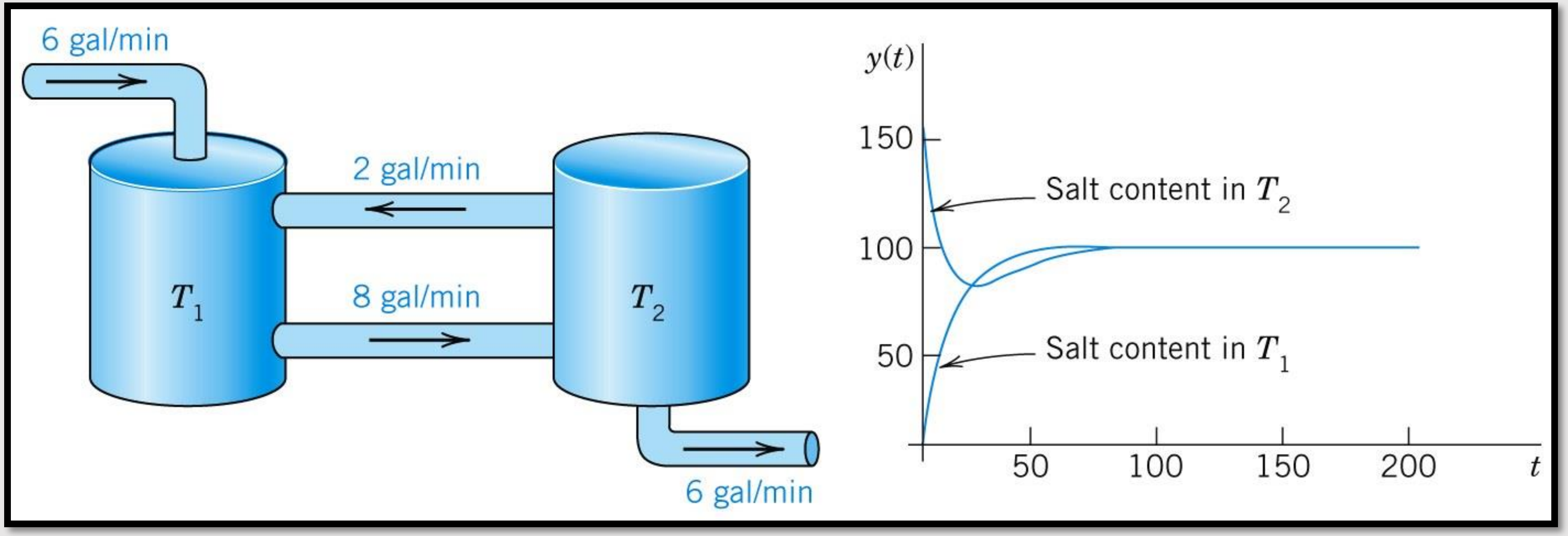
$$\boxed{i_1(t) = \frac{6}{5} - 60te^{-100t} - \frac{6}{5}e^{-100t}}$$

Mixing Two Tanks

- Tank T_1 initially contains 100 gal of pure water. Tank T_2 initially contains 100 gal of water in which 150 lb of salt are dissolved. The inflow into T_1 is 2 gal/min from T_2 and 6 gal/min containing 6 lb of salt from the outside. The inflow into T_2 is 8 gal/min from T_1.
- The outflow from T_2 is $2 + 6 = 8$ gal/min, as shown in the figure. The mixtures are kept uniform by stirring. Find and plot the salt contents $y_1(t)$ and $y_2(t)$ in T_1 and T_2, respectively.



Mixing Two Tanks



Solution

The model is obtained in the form of two equations

Time rate of change = Inflow/min − Outflow/min

for the two tanks,

$$y_1' = -\frac{8}{100}y_1 + \frac{2}{100}y_2 + 6.$$

$$y_2' = \frac{8}{100}y_1 - \frac{8}{100}y_2.$$

Solution

- The initial conditions are $y_1(0) = 0$, $y_2(0) = 150$. From this

$$y_1' = -\frac{8}{100}y_1 + \frac{2}{100}y_2 + 6.$$
$$y_2' = \frac{8}{100}y_1 - \frac{8}{100}y_2.$$

$$(-0.08 - s)Y_1 + 0.02Y_2 = -\frac{6}{s}$$
$$0.08Y_1 + (-0.08 - s)Y_2 = -150$$

Solution

$$Y_1 = \frac{9s+0.48}{s(s+0.12)(s+0.04)} = \frac{100}{s} - \frac{62.5}{s+0.12} - \frac{37.5}{s+0.04}$$

$$Y_2 = \frac{150s^2+12s+0.48}{s(s+0.12)(s+0.04)} = \frac{100}{s} + \frac{125}{s+0.12} - \frac{75}{s+0.04}$$

$$y_1 = 100 - 62.5e^{-0.12t} - 37.5e^{-0.04t}$$

$$y_2 = 100 + 125e^{-0.12t} - 75e^{-0.04t}.$$

Exercise

1. $y'' + 2y' + 2y = \delta(t-3), y(0) = y'(0) = 0$

2. $y'' + 5y' + 4y = 3 + \delta(t)$

Exercise

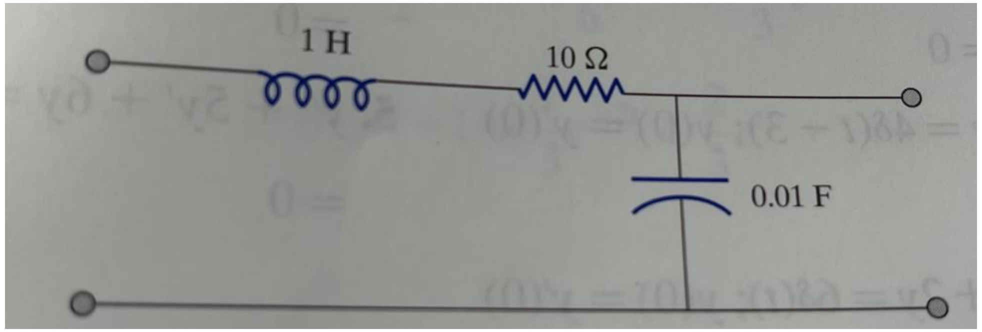
3. *Use the Laplace transform to solve the following initial value problem:*

$$y''(t) - 4y'(t) + 4y(t) = \delta(t-1), \; y(0) = 0, y'(0) = 1$$

4. $y'' + 4y = \delta(t-\pi) - \delta(t-2\pi), y(0) = 0, y'(0) = 0$

Exercise

5. $The\ figure\ below\ depicts\ an\ RLC\ circuit, assuming\ that\ at\ time\ t\ =\ 0,$
$the\ current\ and\ charge\ on\ the\ capacitor\ are\ both\ zero. Furthermore,$
$we\ consider\ introducing\ an\ electromotive\ force\ E(t) =\ \delta(t).$
$According\ to\ Kirchhoff's\ voltage\ law, for\ the\ current\ i(t):$
$i' + 10i + 100q = \delta(t), with\ the\ charge\ q(t) = i'(t), we\ obtain\ a\ second$
$-order\ DE. q'' + 10q' + 100q = \delta(t). Given\ the\ initial\ conditions,$
$\mathrm{q}(0) = \mathrm{q}'(0) = 0, please\ solve\ q(t)$



Exercise

6. *Complex circuits and mechanical systems with multiple components can be modeled using systems of ordinary differential equations. Please employ the Laplace transform to solve the following system of equations:*

$$\begin{cases} x'' - 2x' + 3y' + 2y = 4 \\ 2y' - x' + 3y = 0 \\ x(0) = x'(0) = y(0) = 0 \end{cases}$$

Exercise

7. $Assuming\ that\ the\ switch\ in\ the\ circuit\ below\ closes\ at\ time\ t\ =\ 0,$
$with\ the\ initial\ conditions\ of\ zero\ current\ and\ charge.$
$Let\ E(t) = 2u(t-4) - u(t-5). By\ applying\ Kirchhoff's\ current\ law,$
$we\ can\ write\ the\ equations\ for\ the\ loop\ currents\ as\ follows:$

$$2i_1 + 5(i_1 - i_2)' + 3i_1 = E(t) = 2u(t-4) - u(t-5)$$

$$i_2 + 4i_2 + 5(i_2 - i_1)' = 0$$