交錯級數

9-6 Alternating Series and Conditional Convergence

條件收斂

師大工教一

A series in which the terms are alternately positive and negative is the **alternating series**.

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{(-1)^{n}}{n}+\cdots$$

$$-2+1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots+\frac{(-1)^{n}4}{2^{n}}+\cdots$$

$$1-2+3-4+\cdots+(-1)^{n+1}n+\cdots$$

Theorem 15—**The Alternating Series Test**

The series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$ converges if the following conditions are satisfied:

1. The u_n's are all positive.
2. The u_n's are eventually nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$ for some integer N.
3. $u_n \to 0$

Ex1(p555) The alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ is convergent.

1. $\frac{1}{n}$: positive
2. $\frac{1}{n}$: decreasing
3. $\lim_{n\to\infty}\frac{1}{n}=0$

By Alternating series test $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ conv.

Ex2(p555) Check the 2nd condition of alternating series test for

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{10n}{n^2+16}.$$

$U_n=\frac{10n}{n^2+16}$

consider $f(x)=\frac{10x}{x^2+16}$

$f'(x)=\frac{10(x^2+16)-10x\cdot 2x}{(x^2+16)^2}$

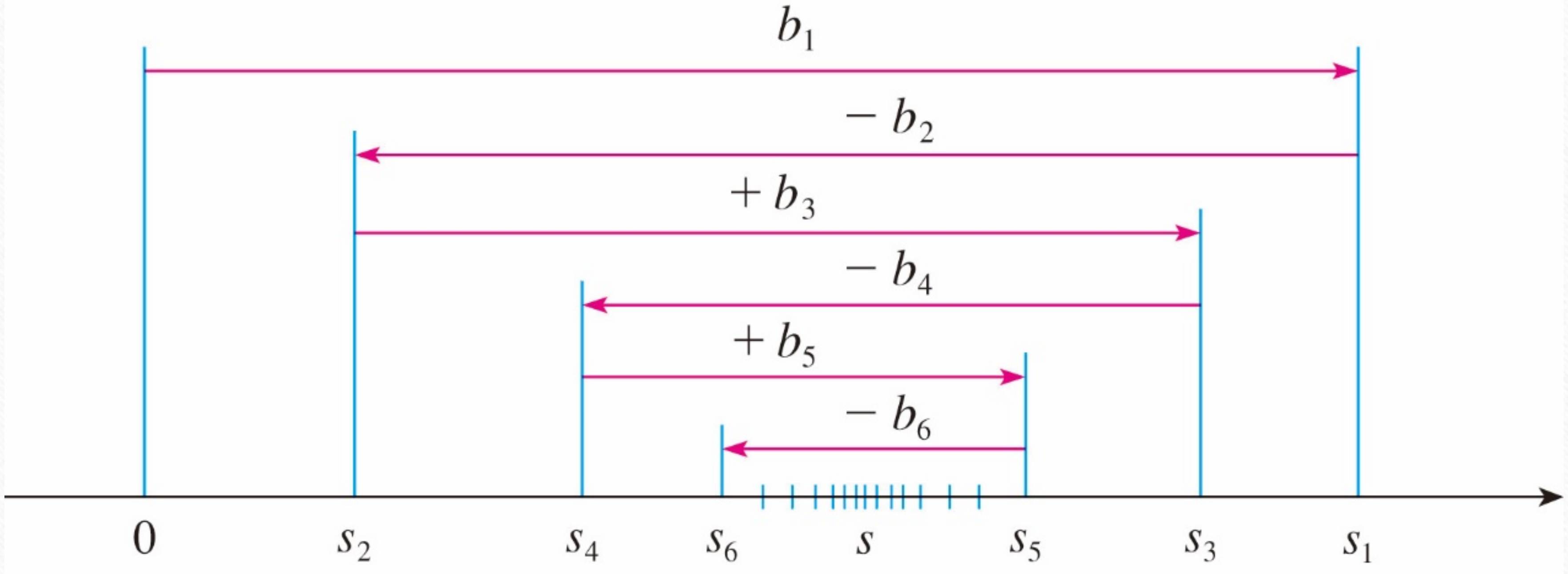
$=\frac{160-10x^2}{(x^2+16)^2}$

$f'(x)<0$ if $x>4$

$\therefore U_n \searrow$ as $n>4$

Facts: 1. $|L - s_n| < u_{n+1}$ for $n \geq N$.

2. L lies between s_n and s_{n+1} for $n \geq N$.



Conditional Convergence

Definition A series that is convergent but not absolutely convergent is called **conditional convergent**.

Ex4(p557) Discuss the convergence of the alternating p-series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^p}=1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\cdots$$

(1) $p>1$: $\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{n^p}\right| = \sum_{n=1}^{\infty}\frac{1}{n^p}$ converges.

$\therefore \sum (-1)^{n+1}\frac{1}{n^p}$ converges absolutely

(2) $0<p\le 1$: $\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{n^p}\right| = \sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges.

$u_n=\frac{1}{n^p}$

(a) u_n : positive

(b) u_n decreasing

(c) $u_n \to 0$

$\therefore \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^p}$ converges conditionally.

Rearranging Series

Theorem 17—The Rearrangement Theorem for Absolutely Convergent Series

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \cdots, b_n, \cdots$ is any arrangement of the sequence $\{a_n\}$, then $\sum_{n=1}^{\infty} b_n$ converges absolutely and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$.

$\sum a_n$ converges absolutely

$\sum b_n$ rearrangement of $\sum a_n$

$\Rightarrow \sum b_n = \sum a_n$

$\sum a_n$ converges conditionally

$\forall r \in \mathbb{R}$ $\exists \sum b_n$ rearrangement of a_n

such that $\sum b_n = r$.

Ex5(p557) Consider the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

$$2L = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 2\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \cdots\right)$$

$$= 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} - \cdots$$

$$= (2-1) - \frac{1}{2} + \left(\frac{2}{3} - \frac{1}{3}\right) - \frac{1}{4} + \left(\frac{2}{5} - \frac{1}{5}\right) - \frac{1}{6} + \left(\frac{2}{7} - \frac{1}{7}\right) - \frac{1}{8} + \cdots$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = L$$

Summary of Tests to Determine Convergence or Divergence

1. **The nth-Term Test for Divergence**: Unless $a_n \to 0$, the series diverges.

2. **Geometric series**: $\sum ar^n$ converges if $|r| < 1$; otherwise, it diverges.

3. p **-series**: $\sum \frac{1}{n^p}$ converges if $p > 1$; otherwise, it diverges.

4.**Series with nonnegative terms**: Try the Integral Test or try comparing to a known series with the Direct Comparison Test or the Limit Comparison Test. Try the Ratio Test or Root Test.

5.**Series with some negative terms**: Does $\sum |a_n|$ converge by the Ratio Test or Root Test, or by another of the tests listed above?

6.**Alternating Series**: $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.

HW9-6

- **HW:1,5,10,13,17,19,41.**

Determine the convergence or divergence of the following series.

(109, #6(e)) $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n}}{(2n+1)!}$

$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} 2^{4n+4}}{(2n+3)!}}{\frac{(-1)^n 2^{4n}}{(2n+1)!}} \right| = \lim_{n\to\infty} \frac{16}{(2n+2)(2n+3)} = 0 < 1$ $\therefore \sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n}}{(2n+1)!}$ converges absolutely

(105 分, #1) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{\frac{5}{7}}}$

consider $u_n = \frac{1}{n\sin(\frac{1}{n})}$

$\lim_{n\to\infty} u_n = \lim_{x\to\infty} \frac{1}{x\sin(\frac{1}{x})}$

$= \lim_{x\to\infty} \frac{\frac{1}{x}}{\sin\frac{1}{x}} \left(\frac{0}{0}\right)$

$= \lim_{x\to\infty} \frac{\cancel{-\frac{1}{x^2}}}{\cos(\frac{1}{x})\cancel{(-\frac{1}{x^2})}} = 1$

$\therefore \lim_{n\to\infty} (-1)^n u_n$ DNE

$\Rightarrow \sum \frac{(-1)^n}{n\sin(\frac{1}{n})}$ diverges

(106 本, #3(a)) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sin\left(\frac{1}{n}\right)}$