2-2 Limit of a Function and Limit Laws

師大工教一

※Intuitive Definition of a Limit

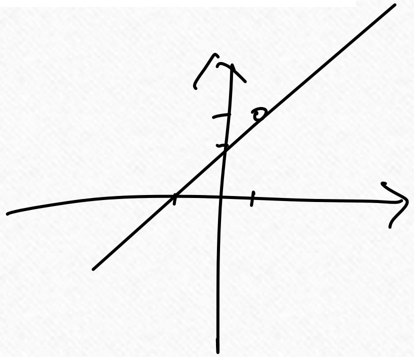
We write $\lim_{x \to a} f(x) = L$ and say

"the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

Ex1(p76): How does the function $f(x)=\dfrac{x^2-1}{x-1}$ behave near $x=1$?

(1) $x \neq 1$ $\quad f(x) = x+1$



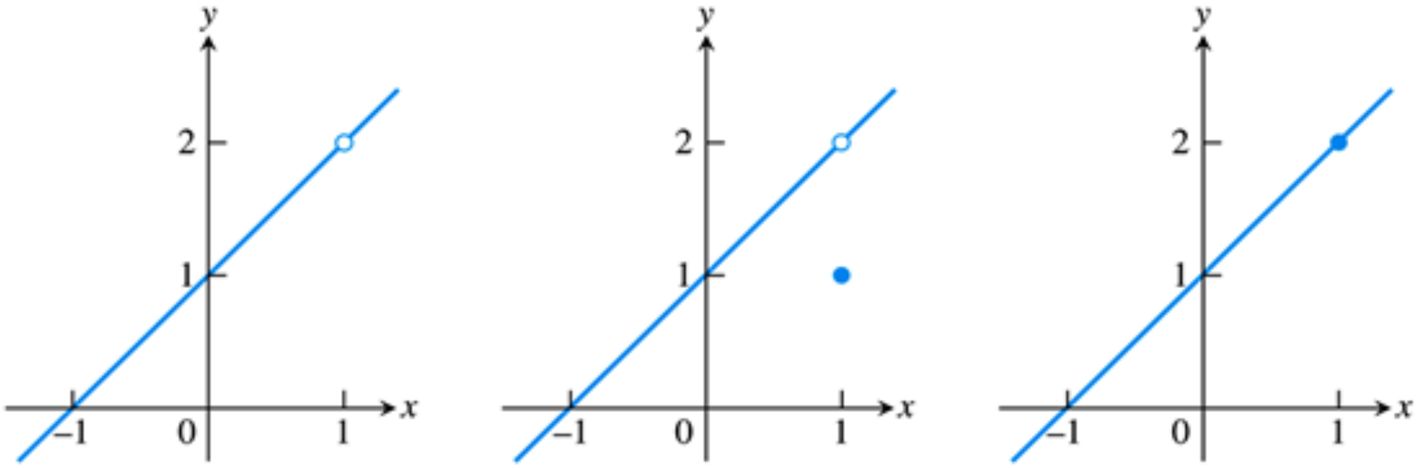
(2) $x = 1$ $\quad f(1)$ DNE (does not exist)

Ex2(p77)

Q: $\lim_{x \to 1} f(x) = 2$ 極限值和求的值無關

$\lim_{x \to 1} g(x) = 2$

$\lim_{x \to 1} h(x) = 2$

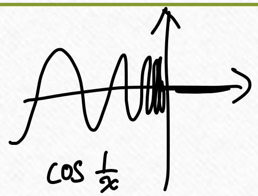


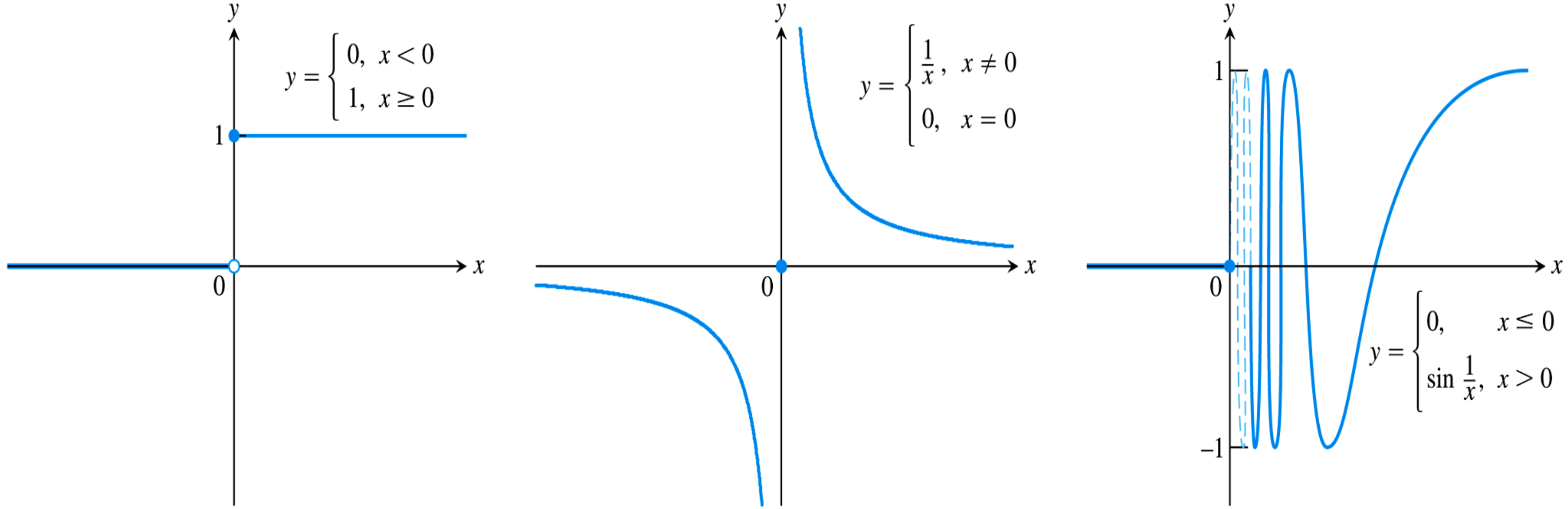
(a) $f(x) = \dfrac{x^2-1}{x-1}$

(b) $g(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

(c) $h(x) = x + 1$

Ex4(p78): The types that the limits do not exist.





(a) Unit step function $U(x)$ (b) $g(x)$ (c) $f(x)$

FIGURE 2.10 None of these functions has a limit as x approaches 0 (Example 4).

※The Limit Law

Suppose that c is a constant and the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exists.

常數

(1) $\lim_{x \to a}\left[f(x) \pm g(x)\right] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

(2) $\lim_{x \to a}\left[c\, f(x)\right] = c \lim_{x \to a} f(x)$

(3) $\lim_{x \to a}\left[f(x) g(x)\right] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

(4) $\lim_{x \to a}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

(5) $\lim_{x \to a}\left[f(x)\right]^n = \left[\lim_{x \to a} f(x)\right]^n$

(6) $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$

(7) $\lim_{x \to a} c = c$, $\lim_{x \to a} x = a$

(8) $\lim_{x \to a} x^n = a^n$

(9) $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$

Ex5(p79): Find (c) $\lim_{x \to -2} \sqrt{4x^2 + 3}$

$$\lim_{x \to -2} \sqrt{4x^2+3} = \sqrt{\lim_{x \to -2} (4x^2+3)}$$

$$= \sqrt{\left(\lim_{x \to -2} 4x^2\right) + \left(\lim_{x \to -2} 3\right)}$$

$$= \sqrt{4\left(\lim_{x \to -2} x\right)^2 + 3}$$

$$= \sqrt{4(-2)^2+3} = \sqrt{19}$$

rational function 有理函數 $R(x) = \frac{p(x)}{q(x)}$

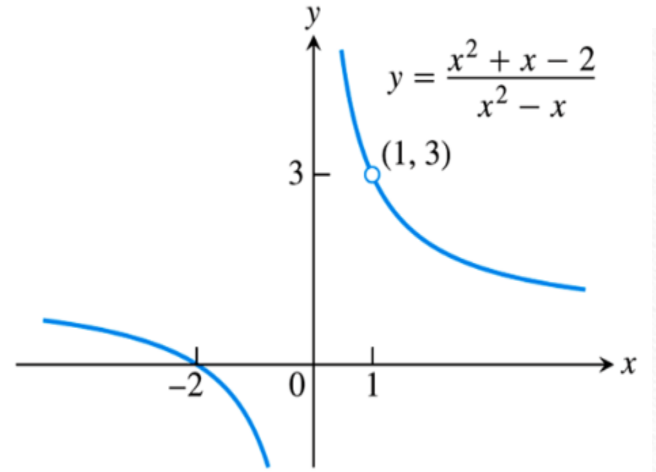
where p, q are polynomials.

Theorem: If f is a polynomial, a rational function, an nth root of a polynomial, or an nth root of a rational function, and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$.

Ex7(p81): Evaluate $\lim_{x\to 1} \frac{x^2+x-2}{x^2-x}$

法① $= \lim_{x\to 1} \frac{2x+1}{2x-1} = \frac{3}{1} = 3$

法② $= \lim_{x\to 1} \frac{(x+2)\cancel{(x-1)}}{x\cancel{(x-1)}} = \frac{3}{1} = 3$



※The Sandwich Theorem 夾擠定理

(Squeeze Theorem)

If $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a) and

$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$ then $\lim_{x \to a} g(x) = L$

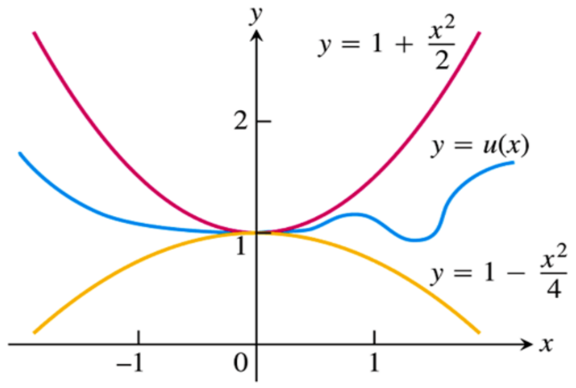
Ex10(p83): Given a function u that satisfies $1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2}$ for all $x \ne 0$, find $\lim_{x \to 0} u(x)$.

$\lim_{x \to 0} 1 - \frac{x^2}{4} = 1$

$\lim_{x \to 0} 1 + \frac{x^2}{2} = 1$

By Sandwich Theorem

$\lim_{x \to 0} u(x) = 1$



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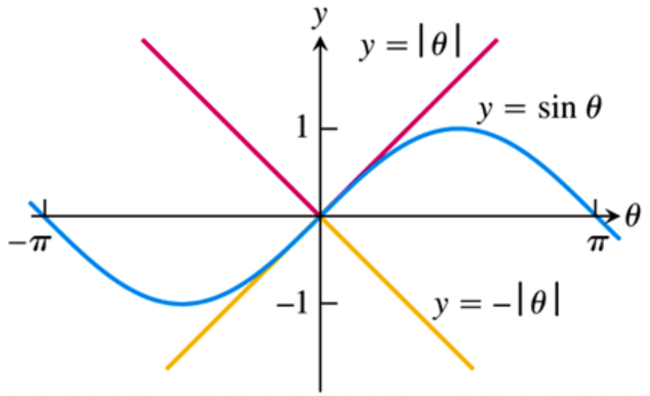
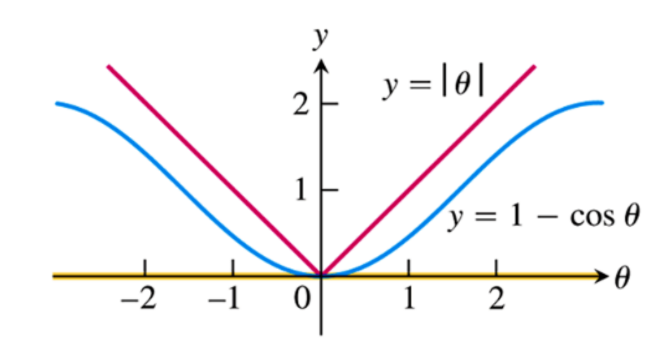
Ex11(p83): Show that (a) $\lim_{\theta\to 0}\sin\theta = 0$ (b) $\lim_{\theta\to 0}\cos\theta = 1$

(c) For any function f, $\lim_{x\to c}|f(x)| = 0$ implies $\lim_{x\to c} f(x) = 0$.





(a) $\sin\theta \to -|\theta| \le \sin\theta \le |\theta|$

$\lim_{\theta\to 0}\sin\theta = 0$

$\lim_{\theta\to 0} -|\theta| = 0$, $\lim_{\theta\to 0}|\theta| = 0$

By Sandwich Theorem

$\lim_{\theta\to 0}\sin\theta = 0$

(b) $\cos\theta \to 0 \le 1-\cos\theta \le |\theta|$

$\lim_{\theta\to 0} 0 = 0$, $\lim_{\theta\to 0}|\theta| = 0$

$\Rightarrow \lim_{\theta\to 0}(1-\cos\theta) = 0$

$\lim_{\theta\to 0}\cos\theta = \lim_{\theta\to 0} 1-(1-\cos\theta)$

$= \lim_{\theta\to 0} 1 - (\lim_{\theta\to 0} 1 - \lim_{\theta\to 0}\cos\theta) = 1$

(c) $\lim_{x\to c}|f(x)| = 0$

$\lim_{x\to c} -|f(x)| = 0$

by Sandwich Theorem

$\lim_{x\to c} f(x) = 0$

HW2-2

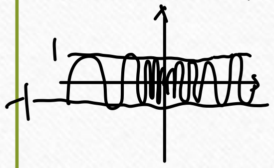
- **HW: 14,18,24,36,47,54,66,80,81**

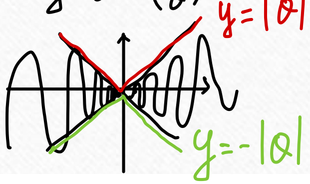
106本

2. (18 分) 計算下列極限值。

$$\frac{((7x)^{\frac{1}{2}}-7)'}{(7-x)'} = \frac{(\frac{7}{2}x)^{-\frac{1}{2}}}{-1} = -\frac{1}{\sqrt{\frac{7}{2}\times 7}} = -\frac{\sqrt{2}}{7} \#$$

V (a) $\lim_{\theta\to 0} \theta \sin(\frac{1}{\theta})$ (b) $\lim_{x\to 7} \frac{\sqrt{7x}-7}{7-x}$ (c) $\lim_{t\to-\infty} \frac{5-4t^3}{3t^2+2}$





$-1 \le \sin\frac{1}{\theta} \le 1$

$-|\theta| \le \theta\sin\frac{1}{\theta} \le |\theta|$

$\lim_{\theta\to 0} -|\theta| = 0 = \lim_{\theta\to 0} |\theta|$

By Sandwich Theorem

$\lim_{\theta\to 0} \theta\sin(\frac{1}{\theta}) = 0$

107

1. (25 pts) Find the limit (極限).

(a) $\lim_{\theta\to 0+} \operatorname{arccot}(\frac{1}{\theta})\sin^2(\frac{1}{\theta})$

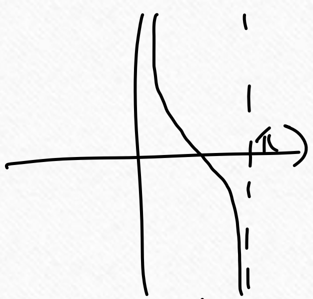
$0 \le \sin^2(\frac{1}{\theta}) \le 1$

$0 \le \cot^{-1}(\frac{1}{\theta})\sin^2(\frac{1}{\theta}) \le |\cot^{-1}(\frac{1}{\theta})|$

$\lim_{\theta\to 0^+} 0 = 0 = \lim_{\theta\to 0^+} |\cot^{-1}(\frac{1}{\theta})|$

By Sandwich Theorem

$\lim_{\theta\to 0^+} \operatorname{arccot}(\frac{1}{\theta})\sin^2(\frac{1}{\theta}) = 0$



$\theta\to 0^+$ $\cot\theta\to\infty$

$\frac{1}{\theta}\to\infty \Rightarrow \cot^{-1}(\frac{1}{\theta})\to 0$