

13-4 Chain Rule

師大工教一

Recall: If $w = f(x)$ is a differentiable function of x , and $x = g(t)$ is a differentiable function of t , then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{dw}{dx} \frac{dx}{dt}.$$

Theorem 5—Chain Rule for Functions of One Independent Variable and Two Intermediate Variables

If $w = f(x, y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable functions of t , then the composition $w = f(x(t), y(t))$ is a differentiable

function of t and $\frac{dw}{dt} = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)$ or

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

Ex1(p746) Use the Chain Rule to find the derivative of $w = xy$ with respect to t along the path $x = \cos t, y = \sin t$. What is the derivative's value at $t = \frac{\pi}{2}$?

$$\frac{dw}{dt} = -y \sin t + x \cos t \quad \Big|_{t=\frac{\pi}{2}}$$

$$= -\sin^2 t + \cos^2 t \quad \Big|_{t=\frac{\pi}{2}}$$

$$= -1$$

Theorem 6—Chain Rule for Functions of One Independent Variable and Three Intermediate Variables

If $w = f(x, y, z)$ is differentiable and if x, y, z are differentiable functions of t , then w is a differentiable function of t and
$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}.$$

Ex2(p746) Find $\frac{dw}{dt}$ if $w = xy + z$, $x = \cos t$, $y = \sin t$, $z = t$. What is the derivative's value at $t = 0$.

Theorem 7—Chain Rule for Functions of Two Independent Variables and Three Intermediate Variables

Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s , given by the formula

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial s}.$$

Ex3(p748) Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if

$$w = x + 2y + z^2, \quad x = \frac{r}{s}, \quad y = r^2 + \ln s, \quad z = 2r.$$

$$\frac{\partial w}{\partial r} = 1 \cdot \frac{1}{s} + 4r + 8r = \frac{1}{s} + 12r$$

$$\frac{\partial w}{\partial s} = 1 \cdot \left(-\frac{r}{s^2}\right) + 2\left(\frac{1}{s}\right) + (2z) \cdot 0 = -\frac{r}{s^2} + \frac{2}{s}$$

If $w = f(x, y)$, $x = g(r, s)$, and $y = h(r, s)$, then $\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r}$ and

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}.$$

Ex4(p748) Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if

$$w = x^2 + y^2, \quad x = r - s, \quad y = r + s.$$

Implicit Differentiation Revisited

$$w = F(x, y) = 0, \quad y = h(x)$$

$$\Rightarrow 0 = \frac{dw}{dx} = F_x + F_y \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y}$$

Theorem 8—A Formula for Implicit Differentiation

Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$

defines y as a differentiable function of x . Then, at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Ex5(p750) Find $\frac{dy}{dx}$ if $y^2 - x^2 - \sin xy = 0$.

Suppose that the equation $F(x, y, z) = 0$ defines the variable z implicitly as a function $z = f(x, y)$. Apply chain rule to $F(x, y, z) = 0$ and we have

$$0 = F_x \cdot 1 + F_y \cdot 0 + F_z \cdot \frac{\partial z}{\partial x} \Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}. \text{ Similarly, } \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

Ex6(p750) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(0, 0, 0)$ if $x^3 + z^2 + ye^{xz} + z \cos y = 0$.

HW13-4

- HW: 2,4,7,23,29,33

#3(102 年) Find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ at the point $(u, v) = \left(\frac{1}{2}, 1\right)$ if $w = xy + yz + xz$,

$$x = u + v, \quad y = u - v, \quad z = uv.$$

#2(104 年)考慮函數 $f(x, y, z) = x^2 e^{2y} \cos(3z)$ ，求在曲線 $x = \cos t$ ，

$y = \ln(t + 2)$ ， $z = t$ 上一點 $(1, \ln 2, 0)$ 處的導數 $\frac{df}{dt}$ 值。

#6(104 年)方程式 $xz + y \ln x - x^2 + 4 = 0$ 中 x 是由獨立變數 y 與 z 所組成的函數，

假設偏導數(partial derivative)均存在，試求偏導數 $\frac{\partial x}{\partial z}$ 在點 $(1, -1, -3)$ 處的值。

#5(105 年分部) Differentiate implicitly to find the first partial derivatives of z .

$$x \ln y + y^2 z + z^2 = 8.$$

#3(105 年本部) 考慮函數 $f(x, y) = \sqrt{x^2 + y^2}$ ，求在曲線 $x = \cos t$ ， $y = e^t$ 上一點

$(1, 1)$ 處的導數 $\frac{df}{dt}$ 值。

#6(105 年本部) 方程式 $x \ln y + y^2 z + z^2 = 6$ 中 z 是由獨立變數 x 與 y 所組成的函數，假設偏導數(partial derivative)均存在，試求偏導數 $\frac{\partial z}{\partial y}$ 在點 $(0, 1, 2)$ 處的值