# Laplace Transform for DE

Reference CENGAGE Learning

# Operational Property I

Important properties that can simplify the Laplace transform.

First Translation Theorem (translation for $s$)

$$\boxed{L\{e^{at}f(t)\} = F(s-a)}$$

Second Translation Theorem (translation for $t$)

$$\boxed{L\{f(t-a)u(t-a)\} = e^{-as}F(s)}$$

$u(t)$: the step function

# First Translation Theorem (Translation for s)

$$\boxed{L\{e^{at}f(t)\} = F(s-a)}$$

Proof:

$$L\{e^{at}f(t)\} = \int_0^{\infty} e^{-st}e^{at}f(t)dt = \int_0^{\infty} e^{-(s-a)t}f(t)dt = F(s-a)$$

# Example

(a) $L\{e^{5t}t^3\} = \quad L\{t^3\}\Big|_{s\to s-5} = \dfrac{3!}{s^4}\Bigg|_{s\to s-5} = \dfrac{6}{(s-5)^4}$

(b) $L\{e^{-2t}\cos 4\,t\} =$

# Example

(a) $$L^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = L^{-1}\left\{\frac{2(s-3)+11}{(s-3)^2}\right\} = 2L^{-1}\left\{\frac{1}{s-3}\right\} + 11L^{-1}\left\{\frac{1}{(s-3)^2}\right\}$$
$$= 2e^{3t} + 11te^{3t}$$

(b) $$L^{-1}\left\{\frac{s/2+5/3}{s^2+4s+6}\right\} =$$

# Example

$$y'' - 6y' + 9y = t^2 e^{3t} \qquad y(0) = 2, \quad y'(0) = 17$$

$$\begin{array}{lll} 1\times & 2 & 17 \\ -6\times & & 2 \end{array} = \begin{array}{rr} 2 & 17 \\ & -12 \\ \hline 2s & 5 \end{array}$$

$$L\left\{t^2 e^{3t}\right\} = L\left\{t^2\right\}\Big|_{s\to s-3} = \frac{2}{s^3}\Bigg|_{s\to s-3} = \frac{2}{(s-3)^3}$$

$$(s^2 - 6s + 9)Y(s) = 2s + 5 + \frac{2}{(s-3)^3}$$

$$(s^2 - 6s + 9)Y(s) = 2s + 5 + \frac{2}{(s-3)^3}$$

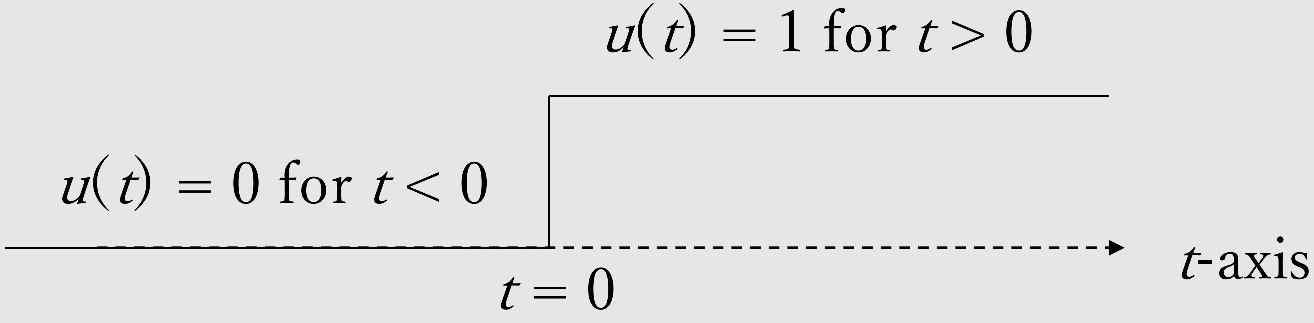
$$Y(s) = \frac{2s+5}{(s-3)^2} + \frac{2}{(s-3)^5} = \frac{2(s-3)+11}{(s-3)^2} + \frac{2}{(s-3)^5}$$

$$= \frac{2}{s-3} + \frac{11}{(s-3)^2} + \frac{1}{12}\frac{4!}{(s-3)^5}$$

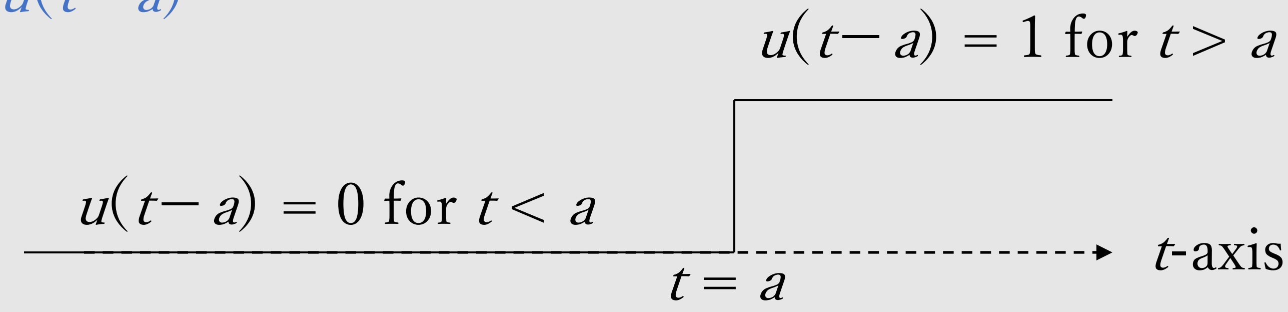
$$y(t) = 2e^{3t} + 11te^{3t} + \frac{1}{12}t^4e^{3t}$$

# Step Function

$u(t)$: unit step function




$u(t-a)$




The unit step function acts as a switch.

- Any piecewise continuous function can be expressed as the unit step function for $t \geq 0$

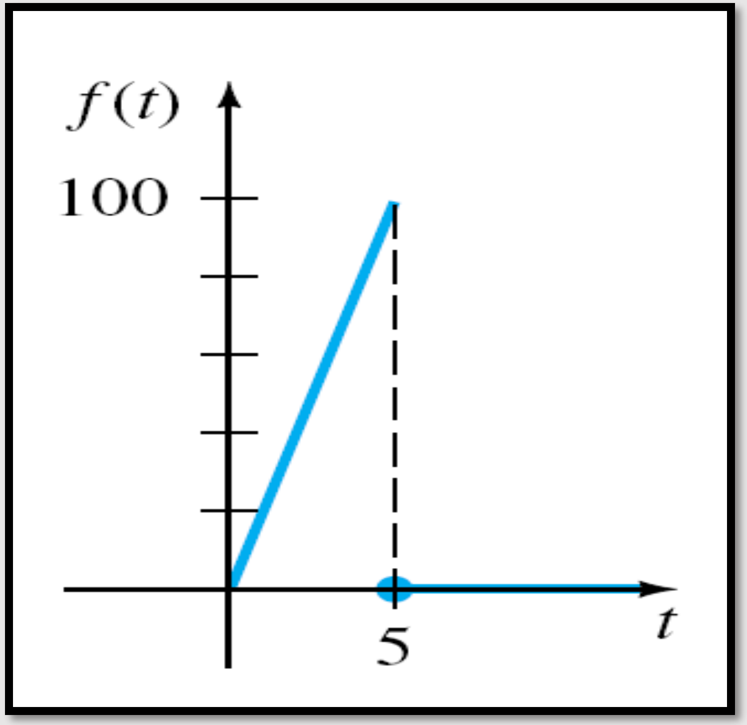



Fig. 7.3.5

$$f(t) = \begin{cases} 20t & \text{for } 0 \leq t < 5 \\ 0 & \text{for } t > 5 \end{cases}$$

$$f(t) = 20t \cdot u(t) - 20t \cdot u(t-5)$$

In general,

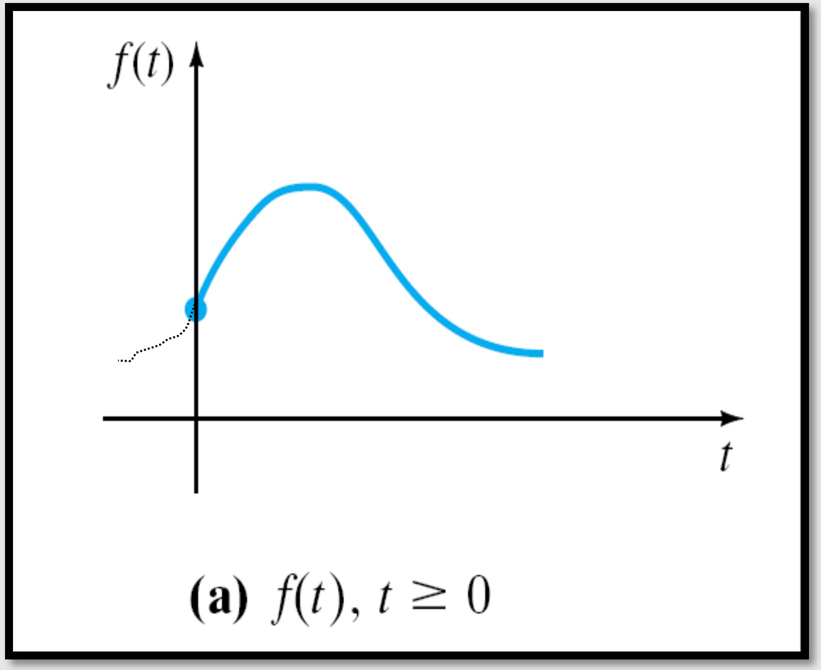
$$f(t) = \begin{cases} h_1(t) & \text{for } 0 \leq t < a \\ h_2(t) & \text{for } t > a \end{cases}$$

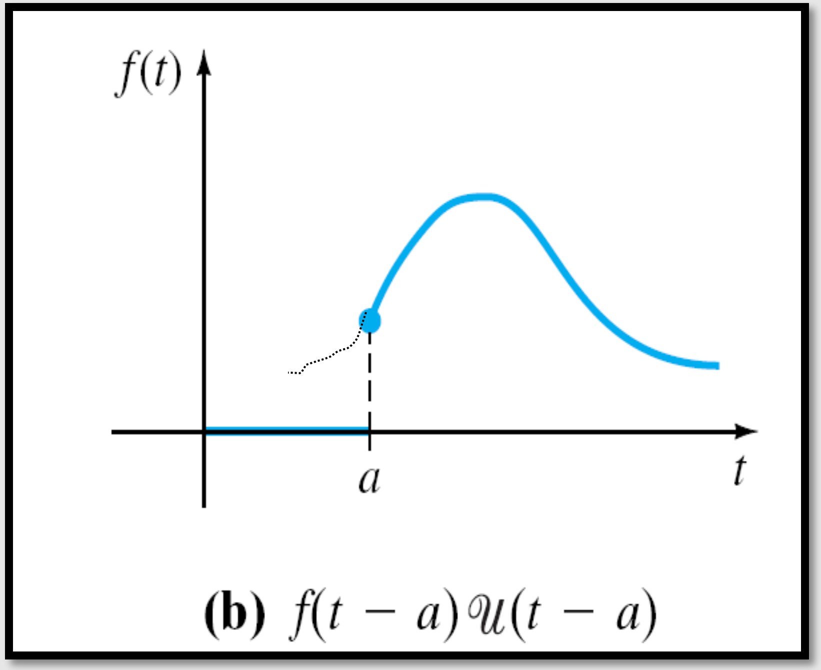
$$f(t) = h_1(t) \cdot u(t) + (h_2(t) - h_1(t)) \cdot u(t-a)$$

# Second Translation Theorem (Translation for t)

$$\boxed{L\{f(t-a)u(t-a)\} = e^{-as}F(s)} \qquad a > 0$$

or $$\boxed{L\{g(t)u(t-a)\} = e^{-as}L\{g(t+a)\}} \qquad a > 0$$



Proof:

$$L\{f(t-a)u(t-a)\} = \int_0^\infty e^{-st} f(t-a)u(t-a)dt = \int_a^\infty e^{-st} f(t-a)dt$$

$$= \int_0^\infty e^{-s(t_1+a)} f(t_1) dt_1$$ let $t_1 = t - a$

$$= e^{-as} \int_0^\infty e^{-st_1} f(t_1) dt_1 = e^{-as} F(s)$$

# Example

$$\boxed{L^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\}}$$

$$\boxed{L\{\cos t \cdot u(t-\pi)\}}$$

$$L\{\cos(t+\pi)\} = -L\{\cos(t)\} = -\frac{s}{s^2+1}$$

$$L\{\cos t \cdot u(t-\pi)\} = -\frac{s}{s^2+1}e^{-\pi s}$$

# Example

$$y' + y = f(t) \qquad y(0) = 5 \qquad f(t) = \begin{cases} 0 & \text{for } 0 \le t < \pi \\ 3\cos t & \text{for } t \ge \pi \end{cases}$$

$$f(t) = 3\cos t \cdot u(t-\pi)$$

$$L\{\cos(t+\pi)\} = -L\{\cos(t)\} = -\frac{s}{s^2+1}$$

$$L\{3\cos t \cdot u(t-\pi)\} = -\frac{3s}{s^2+1}e^{-\pi s}$$

$$(s+1)Y(s) = 5 - 3\frac{3s}{s^2+1}e^{-\pi s}$$

$$Y(s) = \frac{5}{s+1} - \frac{3}{2}\left[-\frac{1}{s+1} + \frac{1}{s^2+1} + \frac{s}{s^2+1}\right]e^{-\pi s}$$

$$Y(s) = \frac{5}{s+1} - \frac{3}{2}\left[-\frac{1}{s+1} + \frac{1}{s^2+1} + \frac{s}{s^2+1}\right]e^{-\pi s}$$

$$\underbrace{\left[-\frac{1}{s+1} + \frac{1}{s^2+1} + \frac{s}{s^2+1}\right]}_{-e^{-t} + \sin(t) + \cos(t)}$$

$$y(t) = 5e^{-t} + \frac{3}{2}\left[e^{-(t-\pi)} - \sin(t-\pi) - \cos(t-\pi)\right]u(t-\pi)$$

$$= 5e^{-t} + \frac{3}{2}\left[e^{-(t-\pi)} + \cos(t) + \sin(t)\right]u(t-\pi)$$

# Derivatives of Transforms

$$\boxed{L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)}$$

Comparison：

$$L\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots\cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

differential $\xrightarrow{\text{Laplace}}$ Multiply by $s^n$

Multiply by $t^n$ $\xrightarrow{\text{Laplace}}$ differential

Proof of the Theorem of Derivatives of Transforms:

$$F(s)=\int_0^{\infty} e^{-st} f(t)dt$$

$$\frac{d}{ds}F(s)=\frac{d}{ds}\int_0^{\infty} e^{-st}f(t)dt=\int_0^{\infty}\frac{d}{ds}[e^{-st}]f(t)dt=-\int_0^{\infty} e^{-st}tf(t)dt=-L\{tf(t)\}$$

$$\frac{d^n}{ds^n}F(s)=\frac{d^n}{ds^n}\int_0^{\infty} e^{-st}f(t)dt=\int_0^{\infty}\frac{d^n}{ds^n}[e^{-st}]f(t)dt=\int_0^{\infty} e^{-st}(-t)^n f(t)dt$$
$$=L\{(-t)^n f(t)\}=(-1)^n L\{t^n f(t)\}$$

Example $\boxed{L\{t \sin k t\}}$

$$L\{\sin kt\} = \frac{k}{s^2 + k^2} \qquad L\{t \sin k t\} = -\frac{d}{ds}\frac{k}{s^2 + k^2} = \frac{2ks}{(s^2 + k^2)^2}$$

practice：Why $L\{t \cos k t\} = \dfrac{s^2 - k^2}{(s^2 + k^2)^2}$

# Convolution

Definition of convolution:

$$\boxed{f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau}$$ (textbook definition of convolution)

* means convolution

$$\boxed{f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau}$$ (another definition of convolution)

When $f(t) = 0$ for $t < 0$ and $g(t) = 0$ for $t < 0$ ,

the 2nd definition can be simplified to the 1st one.

## The physical meaning of Convolution (important)

when $y(t) = \int_0^t f(\tau) g(t-\tau) d\tau$

The influence of input $f(\tau)$ to output $y(t)$ is $g(t-\tau)$

$g(t-\tau)$ is only effected by the difference of $t$ and $\tau$.

The effect of the input $f(\tau)$ on the output $y(t)$ is dependent on the difference of t and $\tau$.

# Convolution Theorem

$$\boxed{L\{f(t)*g(t)\} = L\{f(t)\}L\{g(t)\} = F(s)G(s)}$$

Convolution $\longrightarrow$ Multiplication

Proof:

$$F(s)G(s) = \left(\int_0^\infty e^{-s\tau} f(\tau)d\tau\right)\left(\int_0^\infty e^{-s\beta} g(\beta)d\beta\right)$$

$$= \int_0^\infty \int_0^\infty e^{-s(\tau+\beta)} f(\tau)g(\beta)d\beta\, d\tau$$

note (A)
see the next page

let $t = \tau + \beta$

$$= \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)g(t-\tau)dtd\tau$$

note (B )
see the next page

$$= \int_0^\infty e^{-st}[\int_0^t f(\tau)g(t-\tau)\, d\tau]dt = L[f*g]$$

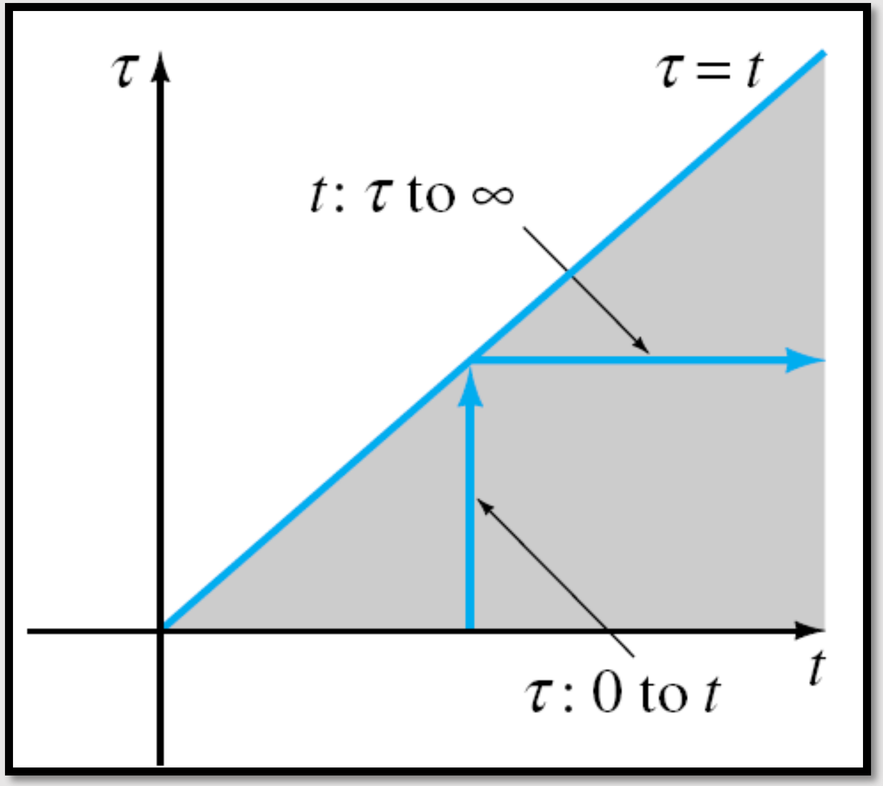



Fig. 7.4.1

note (A)

theorem : $\iint \cdots\cdots\cdots dxdy = \iint \cdots\cdots\cdots C^{-1} dwdv$

$$C = \left| \det \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \right|$$

note (B)

Change of integration interval :

$$\left| \det \begin{bmatrix} \frac{\partial t}{\partial \beta} & \frac{\partial t}{\partial \tau} \\ \frac{\partial \tau}{\partial \beta} & \frac{\partial \tau}{\partial \tau} \end{bmatrix} \right| = \left| \det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right| = 1$$

## Example

$$L\left\{\int_0^t e^{\tau}\sin(t-\tau)\,d\tau\right\} = L\left\{e^t * \sin t\right\} = \frac{1}{s-1}\,\frac{1}{s^2+1}$$

## Example

$$L^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} = \frac{1}{k^2}\left(\sin kt * \sin kt\right) = \frac{1}{k^2}\int_0^t \sin k\tau \sin k(t-\tau)\,d\tau$$

# Integration

$$L\left\{\int_0^t f(\tau)d\tau\right\} = L\{f(t) * 1\} = \frac{F(s)}{s}$$

Example:

$$L^{-1}\left\{\frac{1}{s(s^2+1)}\right\}=\int_0^t \sin\tau\, d\tau=-\cos t+1$$

Example:

$$L_1\frac{di(t)}{dt}+Ri(t)+\frac{Q(t)}{C}=E(t)$$

$$L_1\frac{di(t)}{dt}+Ri(t)+\frac{Q(0)}{C}+\frac{1}{C}\int_0^t i(\tau)d\tau=E(t)$$

$$L_1 sI(s)-L_1 i(0)+RI(s)+\frac{Q(0)}{C}\frac{1}{s}+\frac{1}{C}\frac{I(s)}{s}=L\{E(t)\}$$

# Exercise

1. $L\{e^{at}\}, using\ shifting\ theorem.$
2. $L^{-1}\{\frac{1}{s(s^2+12)}\}$

# Exercise

$$3.\ L\left\{e^{-2t}\int_0^t e^{2w}\cos(3w)\,dw\right\}, using\ the\ shifting\ in\ s\ variable\ theorem$$

# Exercise

5. $f(t) = \begin{cases} 1, & 0 \leq t < 3 \\ -5, & 3 \leq t < 7 \\ 2t + 1, & t \geq 7 \end{cases}$

6. $L\{\mathrm{t} \int_0^t \sin t \, dt\}$

# Exercise

7. $f(t) = t^2 \sin wt$

8. $solve\ t^2 * t^2 * t^2$