1. Find the Area

Theorem. **(兩函數之間的面積)** 給定連續函數 $f(x)$、$g(x)$，並且 $f(x) \geq g(x)$，那麼由 $y = f(x)$、$y = g(x)$、$x = a$、$x = b$ 所圍成的區域面積為：

$$A = \int_a^b \underbrace{[f(x) - g(x)]}_{\text{高}} \underbrace{\mathrm{d}x}_{\text{底}}$$

而由 $x = f(y)$、$x = g(y)$、$y = c$、$y = d$，其中 $f(y) \geq g(x)$，所圍成區域的面積則是：

$$A = \int_a^b \underbrace{[f(y) - g(y)]}_{\text{高}} \underbrace{\mathrm{d}y}_{\text{底}}$$

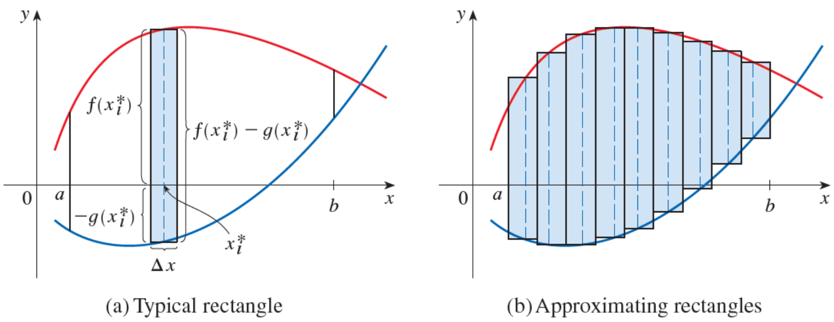


Figure 1: 兩函數間的面積計算：以類似黎曼和的過程逼近兩曲線之間的面積

Remark 只要畫圖確定好兩函數的大小關係，再依照黎曼和的觀點累積每個分割出的長方形即可.

Question Solve the following question.

▷ Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.
解聯立找交點：$x^2 = 2x - x^2 \Rightarrow 2x^2 - 2x = 0 \Rightarrow x(x-1) = 0$
交點為 $(0,0)$ 和 $(1,1)$
區域上下界：$y_T = 2x - x^2$, $y_B = x^2$, 積分區間 $x \in [0,1]$

$$\begin{aligned} A &= \int_0^1 (y_T - y_B)\,\mathrm{d}x \\ &= \int_0^1 \left[(2x - x^2) - x^2\right]\,\mathrm{d}x \\ &= \int_0^1 (2x - 2x^2)\,\mathrm{d}x \\ &= \left[x^2 - \tfrac{2x^3}{3}\right]_0^1 = \frac{1}{3} \end{aligned}$$

- Find the area of the region enclosed by the parabolas $y^2 = 2x+6$ and $y = x-1$.
  將 $y^2 = 2x+6$ 改寫為 $x = \frac{1}{2}y^2 - 3$, 另一條曲線為 $x = y+1$
  解聯立找交點：$\frac{1}{2}y^2 - 3 = y+1 \Rightarrow \frac{1}{2}y^2 - y - 4 = 0 \Rightarrow y = -2, 4$
  區域左右界：$x_L = \frac{1}{2}y^2 - 3$, $x_R = y+1$, 積分區間 $y \in [-2, 4]$

$$\begin{aligned}A &= \int_{-2}^{4} (x_R - x_L)\,\mathrm{d}y \\ &= \int_{-2}^{4} \left[(y+1) - \left(\tfrac{1}{2}y^2 - 3\right)\right]\,\mathrm{d}y \\ &= \int_{-2}^{4} \left(-\tfrac{1}{2}y^2 + y + 4\right)\,\mathrm{d}y \\ &= \left[-\tfrac{1}{6}y^3 + \tfrac{1}{2}y^2 + 4y\right]_{-2}^{4} = 18\end{aligned}$$

2. Find the Volume

2.1. Disk Method

Theorem. **(圓盤法計算旋轉體體積)** 設 $f(x)$ 在 $[a, b]$ 上是一個非負的連續函數，若將 $y = f(x)$ 圖形以 x 軸為中心軸旋轉一圈，則所得柱體體積為：

$$V = \int_a^b \underbrace{\pi[f(x)]^2}_{\text{底面積}} \underbrace{\mathrm{d}x}_{\text{高}}$$

設 $g(y)$ 在 $[c, d]$ 上是一個非負的連續函數，若將 $x = g(y)$ 圖形以 y 軸為中心軸旋轉一圈，則所得柱體體積為：

$$V = \int_c^d \underbrace{\pi[g(y)]^2}_{\text{底面積}} \underbrace{\mathrm{d}y}_{\text{高}}$$

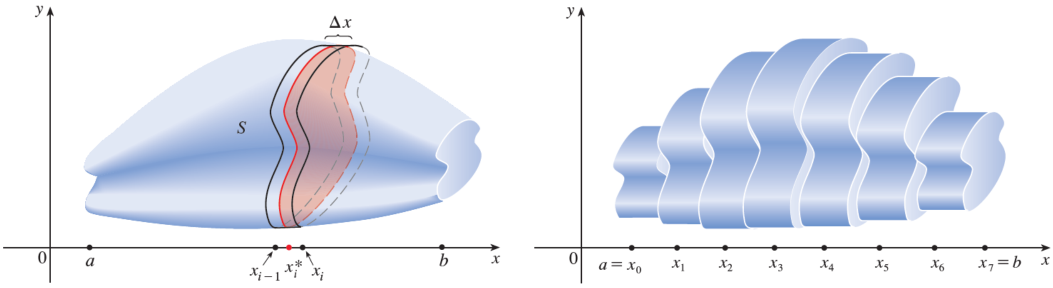


Figure 2: 使用積分來計算立體體積：將該立體沿著垂直於 x 方向切割成多塊，每塊體積使用柱體來近似

Remark 圓盤法即是由垂直於旋轉軸的方向切分旋轉體，並按照切割區間使用無數個柱體來近似，只需要畫圖確定好每一個小柱體的半徑與高即可列出積分式並求解.

Question Solve the following question.

▷ Find the volume of the solid obtained by rotating the region enclosed by $y = x^2$ and $y = x$ about the line $y = 2$.

找交點：$x^2 = x \Rightarrow x(x-1) = 0 \Rightarrow x = 0, 1$

外半徑：$R_{\text{outer}}(x) = 2 - x^2$, 內半徑：$R_{\text{inner}}(x) = 2 - x$

繞 $y = 2$ 旋轉：

$$\begin{aligned} V &= \pi \int_0^1 \left[(2-x^2)^2 - (2-x)^2\right] \mathrm{d}x \\ &= \pi \int_0^1 (x^4 - 5x^2 + 4x) \, \mathrm{d}x \\ &= \pi \left[\tfrac{x^5}{5} - \tfrac{5x^3}{3} + 2x^2\right]_0^1 = \frac{8\pi}{15} \end{aligned}$$

▶ (師大微乙 (一)112#6) Let Ω be the region bounded by the curves $y = x^2$ and $y = x + 2$. Find the volume of the solid generated by revolving Ω about the x-axis.

求交點：$x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow x = -1, 2$

外半徑：$R_{\text{outer}}(x) = x + 2$, 內半徑：$R_{\text{inner}}(x) = x^2$

繞 x $(y = 0)$ 軸旋轉：

$$\begin{aligned} V &= \pi \int_{-1}^2 \left[(x+2)^2 - (x^2)^2\right] \mathrm{d}x \\ &= \pi \int_{-1}^2 (-x^4 + x^2 + 4x + 4) \, \mathrm{d}x \\ &= \pi \left[-\tfrac{x^5}{5} + \tfrac{x^3}{3} + 2x^2 + 4x\right]_{-1}^2 = \frac{72\pi}{5} \end{aligned}$$

2.2. Shells Method

Theorem. (**剝殼法計算旋轉體體積**) 設 $f(x)$ 在 $[a, b]$ 上是一個非負的連續函數，若將 $y = f(x)$ 圖形以 y 軸為中心軸旋轉一圈，則所得柱體體積為：

$$V = \int_a^b \underbrace{(2\pi x)}_{\text{圓周長}} \cdot \underbrace{(f(x))}_{\text{高}} \underbrace{\mathrm{d}x}_{\text{厚}}$$

設 $g(y)$ 在 $[c, d]$ 上是一個非負的連續函數，若將 $x = g(y)$ 圖形以 x 軸為中心軸旋轉一圈，則所得柱體體積為：

$$V = \int_c^d \underbrace{(2\pi y)}_{\text{圓周長}} \cdot \underbrace{(g(y))}_{\text{高}} \underbrace{\mathrm{d}y}_{\text{厚}}$$

Remark 剝殼法即是將旋轉體由內到外一層一層加總，只要畫圖確定好範圍，依照每個殼的體積是圓周 $\times$ 高 $\times$ 厚度做積分即可.

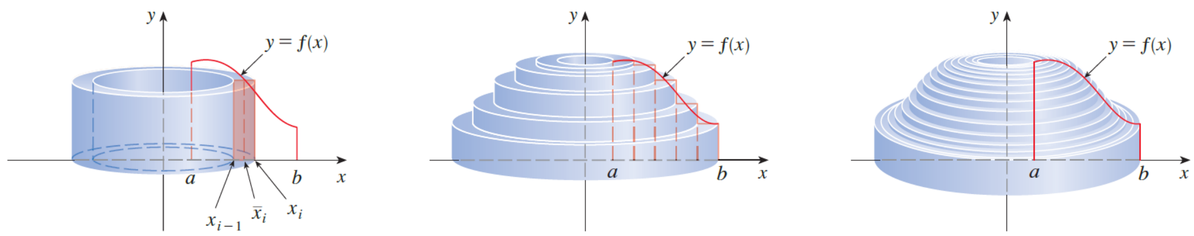


Figure 3: 撥殼法的圖示：將區間 $[a,b]$ 分割為無數個子區間，並使用無數的「殼」來近似每一塊的體積

Question Solve the following question.

▷ Find the volume of the solid obtained by rotating the region enclosed by $y = 2x^2 - x^3$ and $y = 0$ about the line y-axis.
找出區域在 x 軸上的範圍：$2x^2 - x^3 = x^2(2-x) = 0 \Rightarrow x = 0,\ 2$
於 $x \in [0,2]$ 上，曲線 $y = 2x^2 - x^3$ 在 $y = 0$ 之上
圓柱殼半徑 $= x$，高度 $= y = 2x^2 - x^3$

$$\begin{aligned} V &= \int_0^2 2\pi(\text{半徑})(\text{高度})\ \mathrm{d}x \\ &= 2\pi \int_0^2 x(2x^2 - x^3)\ \mathrm{d}x \\ &= 2\pi \int_0^2 (2x^3 - x^4)\ \mathrm{d}x \\ &= 2\pi \left[\tfrac{2x^4}{4} - \tfrac{x^5}{5}\right]_0^2 = 2\pi \cdot \tfrac{8}{5} = \frac{16\pi}{5} \end{aligned}$$

▶ (師大微乙 (一)112#3) Let $\mathcal{R}$ be the region bounded by the curves $x^3 - x + y = 0$, $y = 0$, $x = 0$ and $1 - x = 0$. Find the volume of the solid generated by revolving $\mathcal{R}$ about the y-axis.
將 $x^3 - x + y = 0$ 改寫為 $y = -x^3 + x$.
找出區域在 x 軸上的範圍：$-x^3 + x = x(1 - x^2) = 0 \Rightarrow x = 0, 1$.
於 $x \in [0,1]$ 上，曲線 $y = -x^3 + x$ 在 $y = 0$ 之上
圓柱殼半徑 $= x$, 高度 $= x - x^3$.

$$\begin{aligned} V &= \int_0^1 2\pi(\text{半徑})(\text{高度})\ \mathrm{d}x \\ &= 2\pi \int_0^1 x(x - x^3)\ \mathrm{d}x = 2\pi \int_0^1 (x^2 - x^4)\ \mathrm{d}x \\ &= 2\pi \left[\tfrac{x^3}{3} - \tfrac{x^5}{5}\right]_0^1 = \frac{4\pi}{15} \end{aligned}$$

計算旋轉體體積時，選擇圓盤法或剝殼法的關鍵在於判斷哪種方法能讓積分變得更簡單. 核心原則是根據旋轉軸與積分變數的關係來決定：

- 圓盤法 (Disk/Washer Method)：將物體切成與旋轉軸垂直的圓盤或圓環.
  - 積分變數與旋轉軸相同.

- 當函數容易寫成 $y = f(x)$ 且繞 x 軸旋轉，或寫成 $x = g(y)$ 且繞 y 軸旋轉時，適合使用圓盤法.

- 剝殼法 (Cylindrical Shell Method)：將物體切成與旋轉軸平行的圓柱殼.
  - 積分變數與旋轉軸相異.
  - 當函數容易寫成 $y = f(x)$ 且繞 y 軸旋轉，或寫成 $x = g(y)$ 且繞 x 軸旋轉時，適合使用剝殼法.

在計算前，務必先繪製圖形. 透過分析旋轉軸與函數形式的關係，選擇能讓積分式最簡潔的方法，這就是最適合的選擇.

3. Find the Arc Length

Theorem. **(曲線弧長)** 給定定義在 $[a, b]$ 上的曲線 $y = f(x)$，若 $f'(x)$ 在 $[a, b]$ 上是連續的，則我們說曲線 $y = f(x)$ 從 $(a, f(a))$ 到 $(b, f(b))$ 的弧長是：

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, \mathrm{d}x$$

或是曲線可以等價的被表示為 $x = g(y)$，$c \leq y \leq d$，且 $g'(x)$ 是連續的，則曲線弧長也可以被表示為：

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\, \mathrm{d}y$$

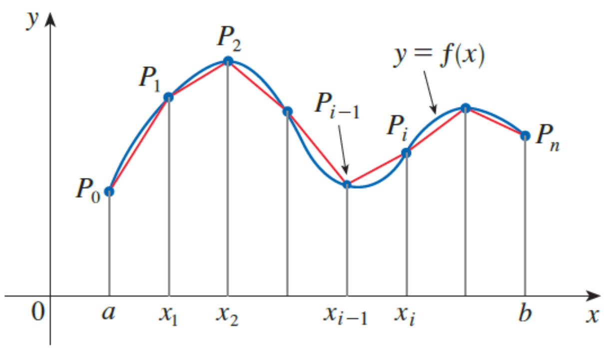


Figure 4: 曲線弧長的計算方式：使用多個直線段來逼近真實的曲線弧長

Question Solve the following question.

▷ 求曲線 $y = \int_{\pi/4}^{x} \tan t \, \mathrm{d}t$ 在 $0 \leq x \leq \dfrac{\pi}{4}$ 範圍內的長度 (arc length).

$$\begin{aligned}
y' &= \frac{\mathrm{d}}{\mathrm{d}x}\int_{\pi/4}^{x} \tan t\ \mathrm{d}t = \tan x \\
L &= \int_0^{\pi/4} \sqrt{1+(y')^2}\ \mathrm{d}x = \int_0^{\pi/4} \sqrt{1+\tan^2 x}\ \mathrm{d}x \\
&= \int_0^{\pi/4} \sec x\ \mathrm{d}x = \Big[\ln|\sec x + \tan x|\Big]_0^{\pi/4} \\
&= \ln(\sec(\pi/4)+\tan(\pi/4)) - \ln(\sec 0 + \tan 0) \\
&= \ln(\sqrt{2}+1) - \ln(1+0) \\
&= \ln(\sqrt{2}+1)
\end{aligned}$$

► (師大微乙 (一)110#4) 求曲線 $y^2 = \frac{4}{9}(x+1)^3$ 由 $x=2$ 到 $x=7$ 的弧長.

$$\begin{aligned}
y &= \frac{2}{3}(x+1)^{3/2}, \quad y' = (x+1)^{1/2} \\
L &= \int_2^7 \sqrt{1+(y')^2}\ \mathrm{d}x = \int_2^7 \sqrt{1+(x+1)}\ \mathrm{d}x \\
&= \int_2^7 \sqrt{x+2}\ \mathrm{d}x = \left[\tfrac{2}{3}(x+2)^{3/2}\right]_2^7 \\
&= \frac{2}{3}(9^{3/2}-4^{3/2}) = \frac{2}{3}(27-8) = \frac{38}{3} \Rightarrow L_T = 2L = \frac{76}{3}
\end{aligned}$$

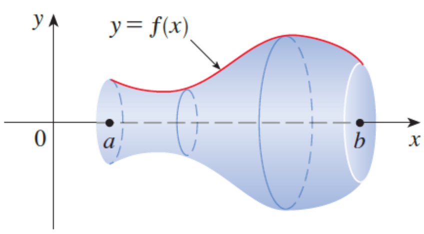
4. Find the Surface Area

Theorem. (**旋轉體表面積**) 給定函數 $f(x) \geq 0$ 在 $[a,b]$ 上可微並導函數連續，曲線 $y=f(x)$ 繞行 x 軸一週後形成的曲面面積為：

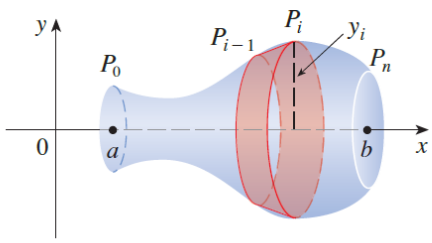
$$A = \int_a^b \underbrace{2\pi f(x)}_{\text{圓周長}} \cdot \underbrace{\sqrt{1+(f'(x))^2}\ \mathrm{d}x}_{\text{寬}}$$

給定函數 $g(y) \geq 0$ 在 $[c,d]$ 上可微並導函數連續，曲線 $x=f(y)$ 繞行 y 軸一週後形成的曲面面積為：

$$A = \int_c^d \underbrace{2\pi g(y)}_{\text{圓周長}} \cdot \underbrace{\sqrt{1+(g'(y))^2}\ \mathrm{d}y}_{\text{寬}}$$



(a) Surface of revolution



(b) Approximating band

Figure 5: 曲線旋轉而成的曲面: 每個分割出來的曲面可以用圓筒帶來近似

Question Solve the following question.

▷ (師大微乙 (一)105#4) 求線段 $y = 1 - \frac{x^2}{4}$, $0 \le x \le 2$ 對 y 軸旋轉所得到的旋轉體表面積.

$y' = -\frac{x}{2}$

$$\begin{aligned} S =& 2\pi \int_0^2 x\sqrt{1+(y')^2}\,\mathrm{d}x = 2\pi \int_0^2 x\sqrt{1+\tfrac{x^2}{4}}\,\mathrm{d}x \\ =& 2\pi \int_0^2 x\sqrt{\tfrac{x^2+4}{4}}\,\mathrm{d}x = \pi \int_0^2 x\sqrt{x^2+4}\,\mathrm{d}x, \qquad \text{令 } u = x^2+4,\ \mathrm{d}u = 2x\,\mathrm{d}x \\ =& \tfrac{\pi}{2}\int_4^8 \sqrt{u}\,\mathrm{d}u = \tfrac{\pi}{2}\left[\tfrac{2}{3}u^{3/2}\right]_4^8 \\ =& \tfrac{\pi}{3}\left(8^{3/2} - 4^{3/2}\right) = \tfrac{\pi}{3}(16\sqrt{2} - 8) \end{aligned}$$

► (師大微乙 (一)104#3) 線段 $y = 2\sqrt{x}$, $1 \le x \le 2$ 對 x-軸旋轉的表面積.

$y' = \dfrac{1}{\sqrt{x}}$

$$\begin{aligned} S =& 2\pi \int_1^2 y\sqrt{1+(y')^2}\,\mathrm{d}x = 2\pi \int_1^2 2\sqrt{x}\sqrt{1+\tfrac{1}{x}}\,\mathrm{d}x \\ =& 4\pi \int_1^2 \sqrt{x}\sqrt{\tfrac{x+1}{x}}\,\mathrm{d}x = 4\pi \int_1^2 \sqrt{x+1}\,\mathrm{d}x \\ =& 4\pi\left[\tfrac{2}{3}(x+1)^{3/2}\right]_1^2 = \tfrac{8\pi}{3}\left(3^{3/2} - 2^{3/2}\right) \end{aligned}$$

5. Find the Average Value

Theorem. (**函數平均值**) 函數 $f(x)$ 在 $[a, b]$ 區間上的平均值定義為：

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

Question Solve the following question.

▷ (師大微乙 (一)112#5) Find the average value of $g(x) = \dfrac{1}{e^x + e^{-x}}$ on the interval $[-\ln 3, 0]$.

$$\begin{aligned} f_{avg} &= \frac{1}{0-(-\ln 3)}\int_{-\ln 3}^0 \frac{1}{e^x+e^{-x}}\,\mathrm{d}x = \frac{1}{\ln 3}\int_{-\ln 3}^0 \frac{1}{e^x+e^{-x}}\,\mathrm{d}x \\ &= \frac{1}{\ln 3}\int_{-\ln 3}^0 \frac{e^x}{1+e^{2x}}\,\mathrm{d}x, \qquad \text{Let } u = e^x, \quad \mathrm{d}u = e^x\mathrm{d}x \\ &= \frac{1}{\ln 3}\int_{1/3}^1 \frac{\mathrm{d}u}{1+u^2} \\ &= \frac{1}{\ln 3}\left[\tan^{-1} u\right]_{1/3}^1 = \frac{1}{\ln 3}\left(\frac{\pi}{4} - \tan^{-1}\frac{1}{3}\right) \end{aligned}$$

▶ (師大微乙 (一)113 暑 #4) Find the average value of $g(x) = |x| - 1$ on the interval $[-1, 3]$.

$$f_{avg} = \frac{1}{3-(-1)}\int_{-1}^{3}(|x|-1)\,\mathrm{d}x = \frac{1}{4}\left(\int_{-1}^{0}(-x-1)\,\mathrm{d}x + \int_{0}^{3}(x-1)\,\mathrm{d}x\right)$$
$$= \frac{1}{4}\left(\left[-\frac{x^2}{2}-x\right]_{-1}^{0} + \left[\frac{x^2}{2}-x\right]_{0}^{3}\right) = \frac{1}{4}$$

overview

- 定積分與微積分基本定理
  - 定積分定義-黎曼和: $\displaystyle\int_a^b f(x)\mathrm{d}x = \lim_{||P||\to 0}\sum_{i=1}^{n} f(x_i^*)\Delta x_i$
  - 微積分基本定理 (F.T.C. I): $\displaystyle\int_a^b f(x)\mathrm{d}x = F(b) - F(a), \quad F'(x) = f(x)$
  - 微積分基本定理 (F.T.C. II): $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(x)\,\mathrm{d}x = f(x)$
- 積分技巧
  - 變數代換：設 $u = g(x)$, 則 $\mathrm{d}u = g'(x)\mathrm{d}x$
  - 三角代換：遇到 $a^2 \pm x^2$ 依情形設 $x = \sin x$、$x = \tan x$、$x = \sec x$
  - 部分分式：多項式除法拆分成基本分式
  - 分部積分：依據微分除法規則有 $\displaystyle\int$ 小大 $=$ 大大 $-\displaystyle\int$ 另一組
- 積分應用
  - 算面積：$\displaystyle A = \int_a^b \underbrace{[f(x) - g(x)]}_{\text{高}}\,\underbrace{\mathrm{d}x}_{\text{底}}$
  - 旋轉體體積-圓盤法：$\displaystyle V = \int_a^b \underbrace{\pi[f(x)]^2}_{\text{底面積}}\,\underbrace{\mathrm{d}x}_{\text{高}}$
  - 旋轉體體積-剝殼法：$\displaystyle V = \int_a^b \underbrace{(2\pi x)}_{\text{圓周長}}\cdot\underbrace{(f(x))}_{\text{高}}\,\underbrace{\mathrm{d}x}_{\text{厚}}$
  - 曲線弧長：$\displaystyle L = \int_a^b \sqrt{1 + (f'(x))^2}\,\mathrm{d}x$
  - 旋轉體表面積：$\displaystyle A = \int_a^b \underbrace{2\pi f(x)}_{\text{圓周長}}\cdot\underbrace{\sqrt{1 + (f'(x))^2}\,\mathrm{d}x}_{\text{寬}}$
  - 函數平均值：$\displaystyle f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$

Final Exam: 12/17