Advanced Engineering Mathematics | 賴以威

2nd Differential Equation And Reduction of Order

Reference CENGAGE Learning

The nth Order Initial Value Problem

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad y''(x_0) = y_2, \quad \ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; y^{(n-1)}(x_0) = y_{n-1}$$

initial condition

Boundary Value Problem (BVP)

BVP: At different points

Initial conditions are specified at the same points

Example: $a_2(x)y'' + a_1(x)y' + a_0(x) = g(x)$

subject to $y(a) = y_0, \quad y(b) = y_1$

or $y'(a) = y_0, \quad y(b) = y_1$

or $\begin{cases} \alpha_1 y(a) + \beta_1 y'(a) = \gamma_1 \\ \alpha_2 y(b) + \beta_2 y'(b) = \gamma_2 \end{cases}$

An n^{th} order linear DE with n boundary conditions may have a <u>unique</u> solution, <u>no</u> s olution, or <u>infinite</u> number of solutions.

Example

$$y'' + 16y = 0$$

solution: $y = c_1 \cos(4x) + c_2 \sin(4x)$

(1) $y(0) = 0 \qquad y(\pi/2) = 0$

$y = c_2 \sin(4x) \qquad c_2$ is any constant (infinite number of solutions)

(2) $y(0) = 0 \qquad y(\pi/8) = 0$

$y = 0 \qquad$ (unique solution)

Homogeneous Equations

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

$g(x) = 0 \longrightarrow$ homogeneous

$g(x) \neq 0 \longrightarrow$ nonhomogeneous

- Important Term: Associated homogeneous equation

The associated homogeneous equation of a nonhomogeneous DE:

Setting $g(x) = 0$

- Review: solving the 1st order nonhomogeneous linear DE

Solution of the Homogeneous Equation

For an n^{th} order homogeneous linear DE $L(y) = 0$, if

① $y_1(t), y_2(t), \cdots\ldots, y_n(t)$ are the solutions of $L(y) = 0$

② $y_1(t), y_2(t), \cdots\ldots, y_n(t)$ are linearly independent

then any solution of the homogeneous linear DE can be expressed as:

$$\boldsymbol{c_1 y_1(t) + c_2 y_2(t) \ldots . + c_n y_n(t)}$$

An n^{th} order homogeneous linear DE has n linearly independent solutions.

Find n linearly independent solutions

== Find all the solutions of an n^{th} order homogeneous linear DE

fundamental set of solutions

$$y_1(t), y_2(t), \cdots, y_n(\mathrm{t})$$

general solution of the homogenous linear DE

$$y(t) = c_1 y_1(t) + \cdots + c_n y_n(t)$$

Linear Dependence / Independence

If there is no solution other than $c_1 = c_2 = \cdots\cdots. = c_n = 0$ for the following equality

then $y_1(t), y_2(t), \cdots.., y_n(t)$ are said to be linearly independent.

Otherwise, they are linearly dependent.

One way to determine linearly independent property: Wronskian

Wronskian

$$W(y_1, y_2, \dots, y_n) = det\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ {y_1}' & {y_2}' & \cdots & {y_n}' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{bmatrix}$$

$W(y_1, y_2, \dots, y_n) \neq 0$ $\longrightarrow$ linearly independent

Example

$$y''' - 6y'' + 11y' - 6y = 0$$

$y_1 = e^x$, $y_2 = e^{2x}$, and $y_3 = e^{3x}$ are three of the solutions

Since

$$\det\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} = \begin{bmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{bmatrix} = e^{x+2x+3x}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} = 2e^{6x} \neq 0$$

Therefore, y_1, y_2, and y_3 are linear independent for any x

general solution:

$$\boxed{y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}} \qquad x \in (-\infty, \infty)$$

General Solution of a Nonhomogeneous Linear DE

$$y = y_c + y_p$$

general solution of the associ
ated homogeneous function
(complementary function)

particular solution (any solution)
of the nonhomogeneous linear
DE

$\oplus$

general solution of the nonho
mogeneous linear DE

Example

$$\boxed{y''' - 6y'' + 11y' - 6y = 3x}$$

$$y''' - 6y'' + 11y' - 6y = 0$$

Three linearly independent solutions

$$e^x, \quad e^{2x}, \quad e^{3x}$$

$$y_p = -\frac{11}{12} - \frac{1}{2}x$$

Check by Wronskian (Example 9)

$$\begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x}$$

General solution: $\boxed{y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} - \frac{11}{12} - \frac{1}{2}x}$

Reduction of Order

Reduction of Order

(1) (2) (3)

Suitable for the 2nd order linear homogeneous DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

(4) One of the solution $y_1(x)$ has been known.

Example

$$\boxed{y'' - y = 0}$$

We have known that $y_1 = e^x$ is one of the solutions

$$\boxed{y_2(x) = u(x)\,y_1(x)}$$

General solution:

$$\boxed{y_2(x) = e^{-x}}$$

$$\boxed{y(x) = c_1 e^{x} + c_2 e^{-x}}$$

Second Solution

Assume $\boxed{y_2(x) = u(x)\, y_1(x)}$

First change DE into Standard form

$$y'' + P(x)\, y' + Q(x)\, y = 0$$

If $\boxed{y(x) = u(x)\; y_1(x)}$,

$$y' = uy_1' + u'y_1 \qquad\qquad y'' = uy_1'' + 2u'y_1' + u''y_1$$

$$uy_1'' + 2u'y_1' + u''y_1 + P(x)uy_1' + P(x)u'y_1 + Q(x)uy_1 = 0$$

$$\underbrace{u(y_1'' + P(x)\, y_1' + Q(x)\, y_1)}_{\text{zero}} + 2u'y_1' + u''y_1 + P(x)u'y_1 = 0$$

$$u''y_1 + u'(2y_1' + P(x)y_1) = 0$$

set $w = u'$

$$\frac{dw}{dx}y_1 + w\left(2\frac{dy_1}{dx} + P(x)y_1\right) = 0$$

multiplied by $dx/(y_1 w)$

$$\frac{dw}{w} + 2\frac{dy_1}{y_1} + P(x)dx = 0$$

separable variable
(with 3 variables)

$$\int\frac{dw}{w} + 2\int\frac{dy_1}{y_1} + \int P(x)dx = 0$$

$$\ln|w| + c_3 + 2\ln|y_1| + c_4 = -\int P(x)dx$$

$$\ln|w| + 2\ln|y_1| = \ln|w| + \ln|y_1|^2 = \ln|w||y_1|^2 = \ln|wy_1^2|$$

$$\ln|wy_1^2| = -\int P(x)dx + c$$

$$\ln\left|w y_1^2\right| = -\int P(x)\,dx + c$$

$$w y_1^2 = \pm e^{-\int P(x)dx + c}$$

$$w = c_1 e^{-\int P(x)dx} / y_1^2$$

$$\boxed{u = \int w\,dx = c_1 \int \frac{e^{-\int P(x)dx}}{y_1^2}\,dx + c_2}$$

$$\boxed{y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)}\,dx}$$

We can set $c_1 = 1$ and $c_2 = 0$

(We calculate $u(x)$ only to find ano
ther solution which is independent
to $y_1(x)$)

Example

$$\boxed{y'' - y = 0}$$

We have known that $y_1 = e^x$ is one of the solutions

$P(x) = 0 \quad y_2(x) = e^x \int ce^{-2x} dx = -\frac{1}{2} ce^{-x}$

Specially, set $c = -2$, (c can be assigned any value as long as $y_2(x)$ is independent of $y_1(x)$)

$$\boxed{y_2(x) = e^{-x}}$$

General solution: $\boxed{y(x) = c_1 e^x + c_2 e^{-x}}$

1. The following formula in the interval are linear dependent or not? please explain.

(a) $e^{2x}, e^{-2x}; (-\infty < x < \infty)$

(b) $x + 1, x - 1; (0 < x < 1)$

(c) $\ln x, \ln x^2; (x > 0)$

Exercise

2. $Assumed\ a_1 = -2 - 4x - 4x^2 - 2x^3, a_2 = -2x^2 - x^3\ and$
$a_3 = 2 + 4x^2 - 3x^3\ , prove\ that\ a_1, a_2\ and\ a_3\ are\ linear\ independent.$

3. $Prove\ that\ \{x^{-2}\ , x^{-2}\ \ln x\ , x^{-2}\ (\ln x)^2\}\ is\ linear\ independent$
$when\ x > 0.$

Exercise

4. $y'' - 3y' + 2y = 0$, with general solution: $y(x) = c_1 e^{x} + c_2 e^{2x}$, *please solve the initial value problem with* $y(0) = -2, y'(0) = 3$

5. $y_1(x) = e^{-2x}$ *is one solution of* $y'' + 4y' + 4y = 0$.

   Find the general solution.

Exercise

6. $y_1(x) = x^2$ *is one solution of* $y'' - \left(\frac{3}{x}\right)y' + \left(\frac{4}{x^2}\right)y = 0$, when $\mathrm{x} > 0$.
   Find the general solution.