3-6 The Chain Rule

師大工教一

The Chain Rule If $f(u)$ is differentiable at $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x and $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$. In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$, where $\frac{dy}{du}$ is evaluated at $u = g(x)$.

Ex1(p173): Differentiate $y = \left(3x^2 + 1\right)^2$.

$y' = 2(3x^2+1)(6x) = 36x^3 + 12x$

Formula: (1)Power Chain Rule: $\left[\left(f(x)\right)^n\right]' = n\cdot\left(f(x)\right)^{n-1}\cdot f'(x)$

(2) $\left[\sin\left(f(x)\right)\right]' = \cos\left(f(x)\right)\cdot f'(x)$

$[\cos(f(x))]' = -\sin(f(x))\cdot f'(x)$

$[\tan(f(x))]' = \sec^2(f(x))\cdot f'(x)$

$[\cot(f(x))]' = -\csc^2(f(x))\cdot f'(x)$

$[\sec(f(x))]' = \sec(f(x))\cdot\tan(f(x))\cdot f'(x)$

$[\csc(f(x))]' = -\csc(f(x))\cdot\cot(f(x))\cdot f'(x)$

Ex3(p175)Differentiate $y = \sin\left(x^2 + e^x\right)$.

$y' = (2x + e^x)\cos(x^2 + e^x)$

Formula: (3) $\left[e^{f(x)}\right]' = e^{f(x)} \cdot f'(x)$

Ex4(p175) Differentiate $y = e^{\cos x}$.

$y' = e^{\cos x}(-\sin x) = -e^{\cos x}\sin x$

Repeated Use of the Chain Rule

Ex: Differentiate $y = \cos\sqrt{\sin(\tan \pi x)}$. $[\sin(\tan\pi x)]^{\frac{1}{2}}$

$$y' = -\sin\left(\sin(\tan\pi x)\right)^{\frac{1}{2}} \cdot \frac{1}{2}\left(\sin(\tan\pi x)\right)^{-\frac{1}{2}} \cdot \cos(\tan\pi x) \cdot (\tan\pi x)'$$

$$= \frac{-\sin\sqrt{\sin(\tan\pi x)}}{2\sqrt{\sin(\tan\pi x)}} \cdot \cos(\tan\pi x) \cdot \sec^2(\pi x) \cdot \pi$$

$$= \frac{\pi\sec^2(\pi x) \cdot \cos(\tan\pi x) \cdot \left(-\sin\sqrt{\sin(\tan\pi x)}\right)}{2\sqrt{\sin(\tan\pi x)}}$$

Ex6(p176) The Power Chain Rule simplifies computing the derivative of a power of an expression.

(a) $\frac{d}{dx}\left(5x^3 - x^4\right)^7$ (b) $\frac{d}{dx}\left(\frac{1}{3x-2}\right)$ (c) $\frac{d}{dx}\left(\sin^5 x\right)$ (d) $\frac{d}{dx}\left(e^{\sqrt{3x+1}}\right)$

(a)
$$= 7(5x^3 - x^4)^6 \cdot (15x^2 - 4x^3)$$

(b)
$$= \frac{-3}{(3x-2)^2}$$

(c)
$$= \frac{d}{dx}(\sin x)^5$$
$$= 5(\sin^4 x) \cdot \cos x$$

(d)
$$= e^{\sqrt{3x+1}} \cdot \left[(3x+1)^{\frac{1}{2}}\right]'$$
$$= e^{\sqrt{3x+1}} \cdot \frac{1}{2}(3x+1)^{-\frac{1}{2}} \cdot 3$$
$$= \frac{3e^{\sqrt{3x+1}}}{2\sqrt{3x+1}}$$

HW3-6

- **HW: 10,14,17,21,33,38,43,48,61,74,87**

104 (d) (5 points) $y = \sqrt{2 + \sqrt{2 + \sqrt{x}}}$

$$\frac{d}{dx}\left(\sqrt{2+\sqrt{2+\sqrt{x}}}\right) = \frac{d}{dx}\left(2+(2+x^{\frac{1}{2}})^{\frac{1}{2}}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}\left(2+(2+x^{\frac{1}{2}})^{\frac{1}{2}}\right)^{-\frac{1}{2}} \cdot \frac{1}{2}(2+x^{\frac{1}{2}})^{-\frac{1}{2}} \cdot \frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{1}{8} \cdot \frac{1}{\sqrt{2+\sqrt{2+\sqrt{x}}}} \cdot \frac{1}{\sqrt{2+\sqrt{x}}} \cdot \frac{1}{\sqrt{x}}$$

$$= \frac{1}{8} \cdot \frac{1}{\sqrt{(2+\sqrt{2+\sqrt{x}})(2+\sqrt{x})(x)}} = \frac{1}{8\sqrt{(2+\sqrt{2+\sqrt{x}})(2+\sqrt{x})(x)}}$$

$$y = e^{3x}\sin(\cos^2(3x))$$

$$\frac{d}{dx}\left(e^{3x}\sin(\cos^2(3x))\right)$$

$$= (e^{3x})'\left(\sin(\cos^2(3x))\right) + (e^{3x})\left(\sin(\cos^2(3x))\right)'$$

$$= 3e^{3x}\left(\sin(\cos^2(3x))\right) + e^{3x}\left(\cos(\cos^2(3x))\right)\left[(\cos(3x))^2\right]'$$

$$= 3e^{3x}\left(\sin(\cos^2(3x))\right) + e^{3x}\left(\cos(\cos^2(3x))\right)\left(2(\cos(3x))3(-\sin 3x)\right)$$

$$= 3e^{3x}\left(\sin(\cos^2(3x))\right) - 6e^{3x}\left(\cos(\cos^2(3x))\right)(\cos 3x \sin 3x)$$