Introduction of Fourier Series

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Periodic Function

- Fourier series is basic tool for representing periodic functions

$$f(x+p) = f(x) \qquad \forall x.$$

- Example:
  - cosine
  - sine
  - tangent
  - cotangent
- Counterexample:
  - x, x^2, x^3
  - e^x
  - $\cosh x$
  - $\ln x$.

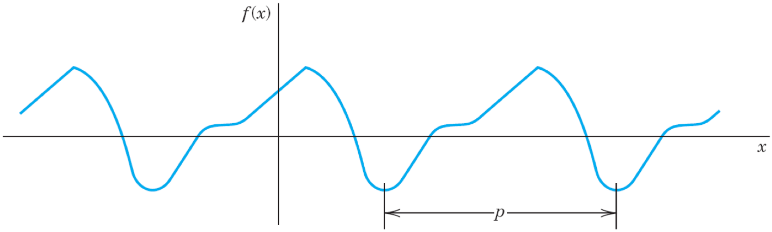
Periodic Function

- If $f(x)$ has period p, it also has the period $2p$

$$f(x+np) = f(x) \qquad \forall x.$$

- If $f(x)$ and $g(x)$ have period p
    - $af(x) + bg(x)$ with any constants a and b also has the period p.

Periodic Function



- Suppose that $f(x)$ is a given function of period 2π and is such that it can be **represented** by a series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx, \qquad \text{(a)}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx, \qquad n = 1,\ 2, \cdots, \qquad \text{(b)}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx, \qquad n = 1,\ 2, \cdots. \qquad \text{(c)}$$

Orthogonality of the Trigonometric System

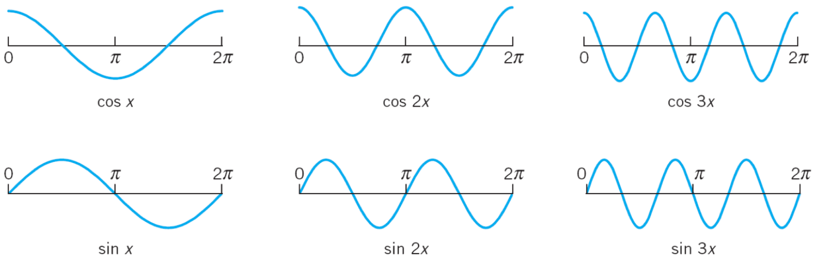
- The integral of the product of any two basis functions in over that interval is 0, for any integers n and m

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0 \qquad (n \neq m), \tag{a}$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0 \qquad (n \neq m), \tag{b}$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0 \qquad (n \neq m \text{ or } n = m). \tag{c}$$

Cosine and sine functions having the period 2π



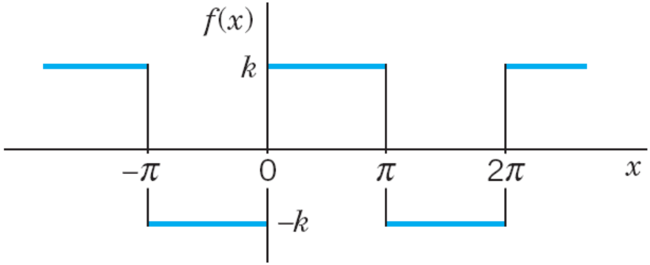
Example: Periodic Rectangular Wave

- Find the Fourier coefficients of the periodic function $f(x)$

$$f(x) = \begin{cases} -k & \text{if} \quad -\pi < x < 0 \\ k & \text{if} \quad 0 < x < \pi \end{cases} \quad \text{and} \quad f(x+2\pi) = f(x).$$

- Functions of this kind occur as external forces acting on mechanical systems, electromotive forces in electric circuits, etc.
    - The value of $f(x)$ at a single point does not affect the integral; hence we can leave $f(x)$ undefined at $x = 0$ and $x = \pm\pi$.

Example: Periodic Rectangular Wave



The given function $f(x)$ (Periodic rectangular wave)

Solution

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\,dx + \int_{0}^{\pi} k\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-k\,\frac{\sin nx}{n}\Big|_{-\pi}^{0} + k\,\frac{\sin nx}{n}\Big|_{0}^{\pi}\right] = 0,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[k\,\frac{\cos nx}{n}\Big|_{-\pi}^{0} - k\,\frac{\cos nx}{n}\Big|_{0}^{\pi}\right].$$

Solution

Since $\cos(-\alpha) = \cos\alpha$ and $\cos 0 = 1$, this yields

$$b_n = \frac{k}{n\pi}\,[\,\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0\,] = \frac{2k}{n\pi}\,(1 - \cos n\pi).$$

Now, $\cos\pi = -1, \cos 2\pi = 1, \cos 3\pi = -1$, etc.; in general,

$$\cos n\pi = \begin{cases} -1 & \text{for odd } n, \\ 1 & \text{for even } n, \end{cases} \quad \text{and thus} \quad 1 - \cos n\pi = \begin{cases} 2 & \text{for odd } n, \\ 0 & \text{for even } n. \end{cases}$$

Hence the Fourier coefficients b_n of our function are

$$b_1 = \frac{4k}{\pi}, \quad b_2 = 0, \quad b_3 = \frac{4k}{3\pi}, \quad b_4 = 0, \quad b_5 = \frac{4k}{5\pi}, \quad \cdots .$$

Solution

Since the a_n are zero, the Fourier series of $f(x)$ is

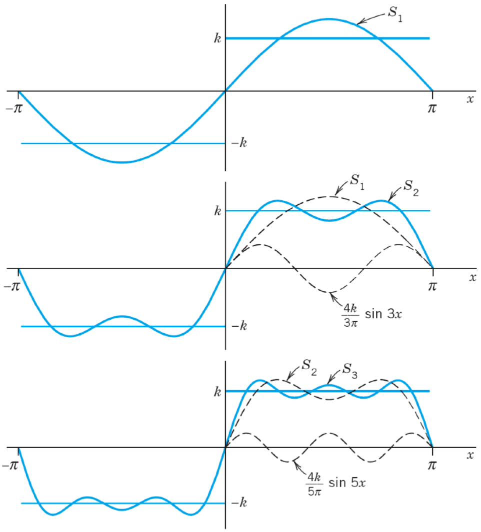
$$\frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right).$$

The partial sums are

$$S_1 = \frac{4k}{\pi}, \quad S_2 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right), \quad \text{etc.}$$

Their graphs in Fig. 261 seem to indicate that the series is convergent and has the sum $f(x)$, the given function.

Solution (Textbook page 478, Fig. 261.)



The first three partial sums of the corresponding Fourier series

Arbitrary Period

- Transition from period 2π to any period $2L$, for the function f, simply by a transformation of scale on the x-axis.

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

$$\Downarrow$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x).$$

Arbitrary Period

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx, \qquad n = 1,\ 2, \cdots,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx, \qquad n = 1,\ 2, \cdots.$$

$$\Downarrow$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx,$$

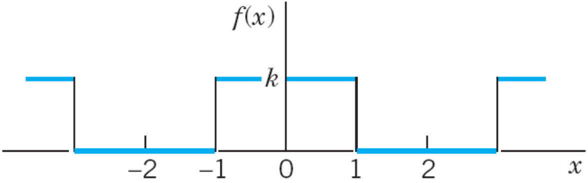
$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx, \qquad n = 1,\ 2, \cdots,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx, \qquad n = 1,\ 2, \cdots.$$

Example

- Find the Fourier series of the function

$$f(x) = \begin{cases} 0 & \text{if} \quad -2 < x < -1, \\ k & \text{if} \quad -1 < x < 1, \\ 0 & \text{if} \quad 1 < x < 2, \end{cases} \qquad p = 2L = 4, \ L = 2.$$



Solution

$a_0 = \frac{k}{2}$ (verify!).

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}\,dx = \frac{2k}{n\pi}\sin\frac{n\pi}{2}.$$

Thus $a_n = 0$ if n is even and

$$a_n = \frac{2k}{n\pi} \quad \text{if} \quad n = 1,\ 5,\ 9,\cdots, \qquad\qquad a_n = \frac{-2k}{n\pi} \quad \text{if} \quad n = 3,\ 7,\ 11,\cdots.$$

$b_n = 0$ for $n = 1,\ 2,\cdots$. Hence the Fourier series is

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - + \cdots\right).$$

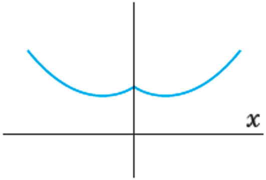


Fig. 266.
Even function

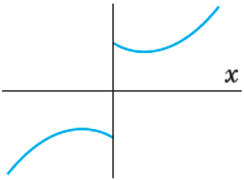


Fig. 267.
Odd function

How about cosine and sine?

Even Function Simplification

- For an even function, its Fourier series representation can be simplified as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx,$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\, dx,$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx, \quad n = 1,\ 2, \cdots .$$

Odd Function Simplification

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx, \quad n = 1, \ 2, \cdots .$$

Example

- Find the Fourier series of the function

$$f(x) = x + \pi \quad \text{if} \quad -\pi < x < \pi \quad \text{and} \quad f(x + 2\pi) = f(x).$$

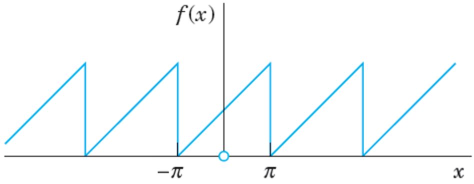


Fig. 268. The function $f(x)$. Sawtooth wave

Solution

- $f = f_1 + f_2$,
    - where $f_1 = x$ and $f_2 = \pi$.

- The Fourier coefficients of f_2 are zero, except for the first one (the constant term), which is π.

- Fourier coefficients a_n, b_n are those of f_1, except for a_0, which is π. Since f_1 is odd, $a_n = 0$ for $n = 1,\ 2, \cdots$, and

$$b_n = \frac{2}{\pi} \int_0^\pi f_1(x) \sin nx \, dx = \frac{2}{\pi} \int_0^\pi x \sin nx \, dx.$$

Solution

- Integrating by parts, we obtain

$$b_n = \frac{2}{\pi}\left[\left.\frac{-x\cos nx}{n}\right|_0^{\pi} + \frac{1}{n}\int_0^{\pi} \cos nx \, dx \right] = -\frac{2}{n}\cos n\pi.$$

- Hence $b_1 = 2, b_2 = -\frac{2}{2}, b_3 = \frac{2}{3}, b_4 = -\frac{2}{4}, \cdots$, and the Fourier series of $f(x)$ is

$$f(x) = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - + \cdots\right). \quad \text{(Fig. 269)}$$

Solution (Textbook page 488, Fig. 269.)

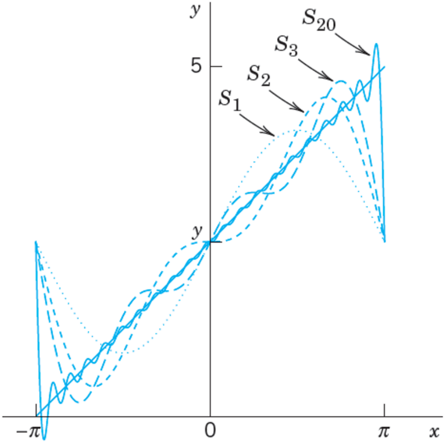


Fig. 269. Partial sums S_1, S_2, S_3, S_{20} in Example 5

Exercise 1

Find the Fourier series of the given function $f(x)$, which is assumed to have the period 2π. Show the details of your work. Sketch or graph the partial sums up to that including $\cos 5x$ and $\sin 5x$.

$$f(x) = |x|.$$

Exercise 2

Find the Fourier series of the given function $f(x)$, which is assumed to have the period 2π. Show the details of your work. Sketch or graph the partial sums up to that including $\cos 5x$ and $\sin 5x$.

$$f(x) = \begin{cases} x & \text{if} \quad -\pi < x < 0 \\ \pi - x & \text{if} \quad 0 < x < \pi. \end{cases}$$

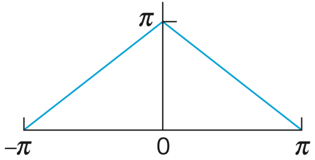
Exercise 3

Find the Fourier series of the given function $f(x)$, which is assumed to have the period 2π. Show the details of your work. Sketch or graph the partial sums up to that including $\cos 5x$ and $\sin 5x$.

$$f(x) = x^2 \quad (0 < x < 2\pi).$$

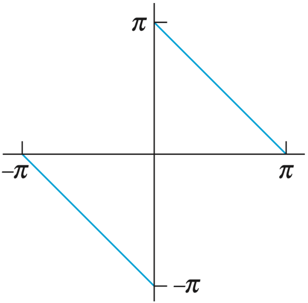
Exercise 4

Find the Fourier series of the given function $f(x)$, which is assumed to have the period 2π. Show the details of your work. Sketch or graph the partial sums up to that including $\cos 5x$ and $\sin 5x$.



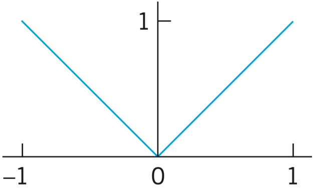
Exercise 5

Find the Fourier series of the given function $f(x)$, which is assumed to have the period 2π. Show the details of your work. Sketch or graph the partial sums up to that including $\cos 5x$ and $\sin 5x$.



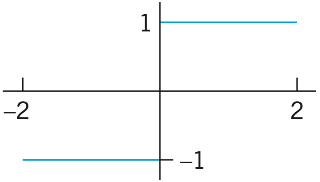
Exercise 6

Is the given function even or odd or neither even nor odd? Find its Fourier series. Show details of your work.



Exercise 7

Is the given function even or odd or neither even nor odd? Find its Fourier series. Show details of your work.



Exercise 8

Is the given function even or odd or neither even nor odd? Find its Fourier series. Show details of your work.

$$f(x) = x^2 \quad (-1 < x < 1), \quad p = 2.$$

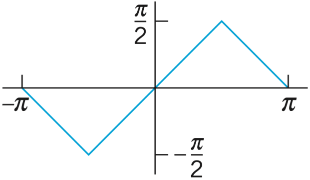
Exercise 9

Is the given function even or odd or neither even nor odd? Find its Fourier series. Show details of your work.

$$f(x) = \cos \pi x \quad (-\frac{1}{2} < x < \frac{1}{2}), \quad p = 1.$$

Exercise 10

Is the given function even or odd or neither even nor odd? Find its Fourier series. Show details of your work.



Reference

- **Textbook**
  - E. Kreyszig, et al., "Advanced Engineering Mathematics", Wiley, Hoboken, NJ, Tenth edition, (2011)
- **Videos**
  - But what is a Fourier series? From heat flow to drawing with circles - 3Blue1Brown
  - Fourier Transform, Fourier Series, and frequency spectrum - Physics Videos by Eugene Khutoryansky
  - What is a Fourier Series? (Explained by drawing circles) - SmarterEveryDay