3-2 The Derivative as a Function

師大工教一

Definition: The **derivative** of the function $f(x)$ is $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

provided the limit exists. The process of calculating a derivative is called **differentiation**.

Ex1(p139) Differentiate $f(x)=\frac{x}{x-1}$.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{x+h}{x+h-1}-\frac{x}{x-1}}{h}=\lim_{h\to 0}\frac{\frac{(x+h)(x-1)-x(x+h-1)}{(x+h-1)(x-1)}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{(x+h)(\cancel{x-1})-x(\cancel{x-1})-xh}{(x+h-1)(\cancel{x-1})}}{h}=\lim_{h\to 0}\frac{x+h-x-xh}{x+h-1}\times\frac{1}{h}$$

$$=\lim_{h\to 0}\frac{h(x-1)-xh}{h(x-1+h)(x-1)}=\lim_{h\to 0}\frac{-1}{(x-1+h)(x-1)}=\frac{-1}{(x-1)^2}$$

Ex2(p139)

(a) Find the derivative of $f(x)=\sqrt{x}$ for $x>0$.

(b) Find the tangent line to the curve $f(x)=\sqrt{x}$ at $x=4$.

(a) $$f'(x)=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{1}{(\sqrt{x+h}+\sqrt{x})}=\frac{1}{2\sqrt{x}}$$

(b) 點斜式，過 (x_0,y_0)，斜率 m 直線方程式：$y-y_0=m(x-x_0)$

$x=4$ $f(4)=\sqrt{4}=2$，過 $(4,2)$

$m_{tan}=f'(4)=\frac{1}{2\sqrt{4}}=\frac{1}{4}$

$\therefore$ Eq. of tangent line

$y-2=\frac{1}{4}(x-4)$

$4y-8=x-4$

$x-4y+4=0$.

Notation: $f'(x) = y' = \dfrac{dy}{dx} = \dfrac{df}{dx} = \dfrac{d}{dx} f(x) = D(f)(x) = D_x f(x)$

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx} f(x)\right|_{x=a}$$

A function $y = f(x)$ is **differentiable on an open interval** (finite or infinite) if it has a derivative at each point of the interval. It is **differentiable on a closed interval** $[a, b]$ if it is differentiable on the interval (a, b) and if the limit

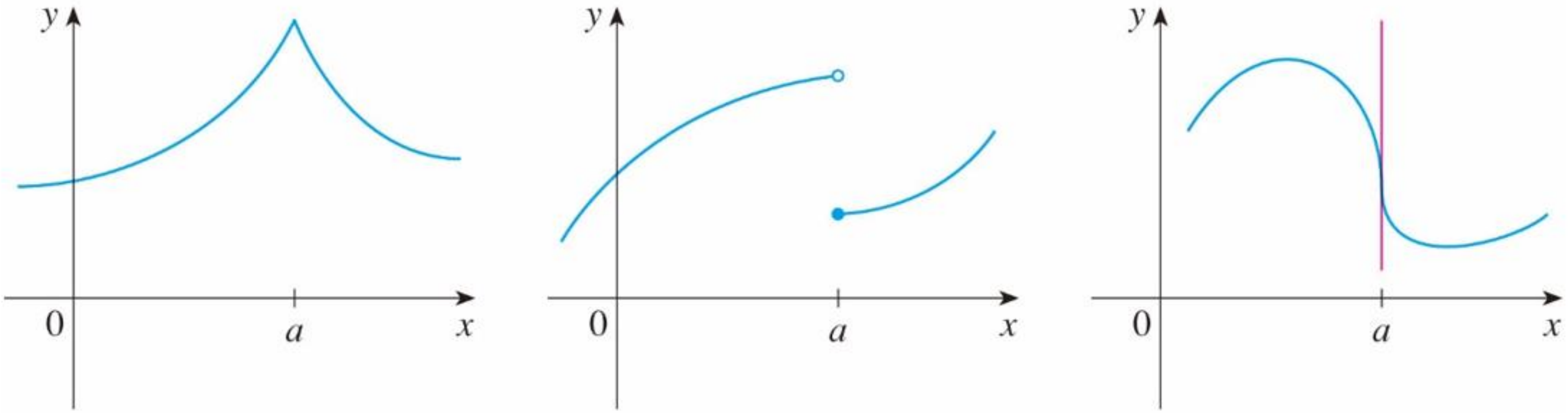
$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$ **Right-hand derivative at** a

$$\lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$$ **Left-hand derivative at** b

Ex4(p141) Show that the function $f(x)=|x|$ is differntiable on $(-\infty,0)$ and on $(0,\infty)$ but has no derivative at $x=0$.

Note: A differentiable function is **smooth**.

When does a function not have a derivative at a point?



(a) A corner (b) A discontinuity (c) A vertical tangent

Theorem 1-Differentiability Implies Continuity If f has a derivative at $x=c$, then f is continuous at $x=c$.

$$f(x)-f(c)=\frac{f(x)-f(c)}{x-c}\cdot(x-c)$$

$$f(x)=f(c)+\frac{f(x)-f(c)}{x-c}\cdot(x-c)$$

$$\lim_{x\to c} f(x)=\lim_{x\to c} f(c)+\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\cdot\lim_{x\to c}(x-c)$$

$$\lim_{x\to c} f(x)=f(c)+f'(c)\cdot c=f(c)$$

$\therefore f$ is conti. at $x=c$.

Caution: The converse of Theorem 1 is false. A function need not have a derivative at a point where it is continuous.

HW3-2

- **HW: 2,4,8,9,27~30,38,39,59**