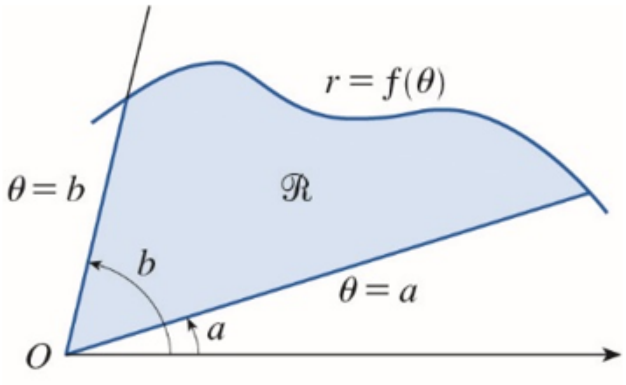
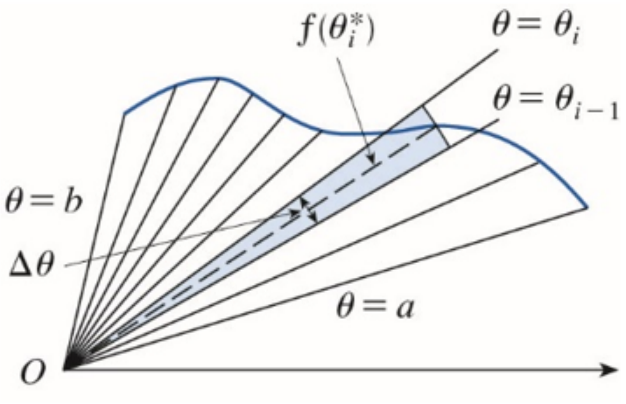
10-5 Area and Length in Polar Coordinates

師大工教一

Q: Area(R) =?





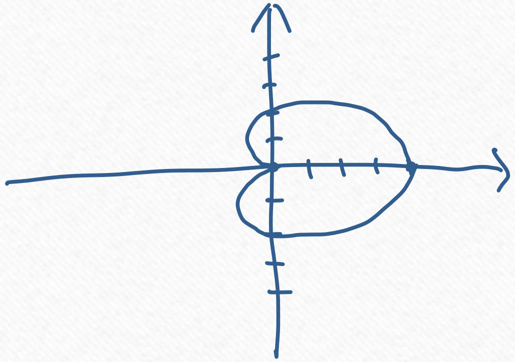
$$A_i = \frac{1}{2} r_i^2 \Delta\theta_i = \frac{1}{2}\left(f\left(\theta_i^*\right)\right)^2 \Delta\theta_i$$

$$\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \frac{1}{2}\left(f\left(\theta_i^*\right)\right)^2 \Delta\theta_i$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}\left(f\left(\theta_i^*\right)\right)^2 \Delta\theta_i = \int_a^b \frac{1}{2}\left(f\left(\theta\right)\right)^2 d\theta$$

Ex1(p625) Find the area of the region in the $xy-$plane enclosed by the cardioid $r = 2(1+\cos\theta)$.

θ	0	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π
r	4	2	0	2	4



Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta), \alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} {r_2}^2 d\theta - \int_{\alpha}^{\beta} \frac{1}{2} {r_1}^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2}\left({r_2}^2 - {r_1}^2\right) d\theta$$

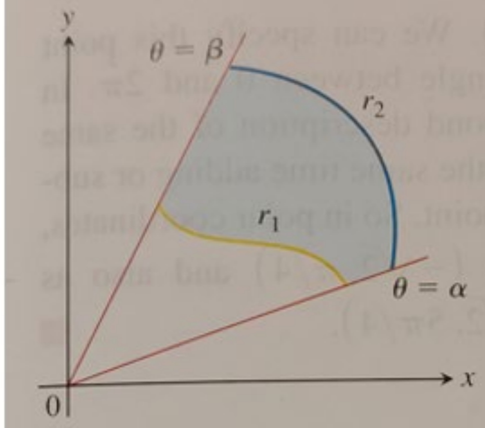


FIGURE 10.35 The area of the shaded region is calculated by subtracting the area of the region between r_1 and the origin from the area of the region between r_2 and the origin.

Ex(103 年, #5) (1) Graph the circle $r = 3\sin\theta$ and the cardioid $r = 1 + \sin\theta$. Indicate the intersection points of the two curves.

(2) Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

Length of a Plane Curve

$$r = f(\theta), \alpha \le \theta \le \beta$$

$$x = r\cos\theta = f(\theta)\cos\theta, y = r\sin\theta = f(\theta)\sin\theta, \alpha \le \theta \le \beta$$

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\cos\theta + r(-\sin\theta)\right)^2 + \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2}\, d\theta$$

$$= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Ex4(p627) Find the length of a cardioid $r = 1 - \cos\theta$.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	0	1	2	1	0

$\sin\theta$

$$L = 2\int_0^\pi \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2}\, d\theta$$

$$= 2\int_0^\pi \sqrt{(1-\cos\theta)^2 + \sin^2\theta}\, d\theta$$

$$= 2\int_0^\pi \sqrt{2 - 2\cos\theta}\, d$$

$$= 2\int_0^\pi \sqrt{2(1-\cos\theta)}\, d\theta$$

$$= 2\sqrt{2}\int_0^\pi \sqrt{1-\cos\theta}\, d\theta = 2\sqrt{2}\int_0^\pi \sqrt{\frac{1-\cos^2\theta}{1+\cos\theta}}\, d\theta$$

$$= 2\sqrt{2}\int_0^{\pi}\sqrt{\frac{1-\cos^2\theta}{1+\cos\theta}}\,d\theta$$

$$= 2\sqrt{2}\int_0^{\pi}\sqrt{\frac{\sin^2\theta}{1+\cos\theta}}\,d\theta$$

Let $u = 1+\cos\theta$

$du = -\sin\theta d\theta$

$-du = \sin\theta d\theta$

$$= 2\sqrt{2}\int_2^{0}\frac{-du}{\sqrt{u}}$$

$$= -2\sqrt{2}\left(2\sqrt{u}\Big|_2^0\right)$$

$= 8$ #

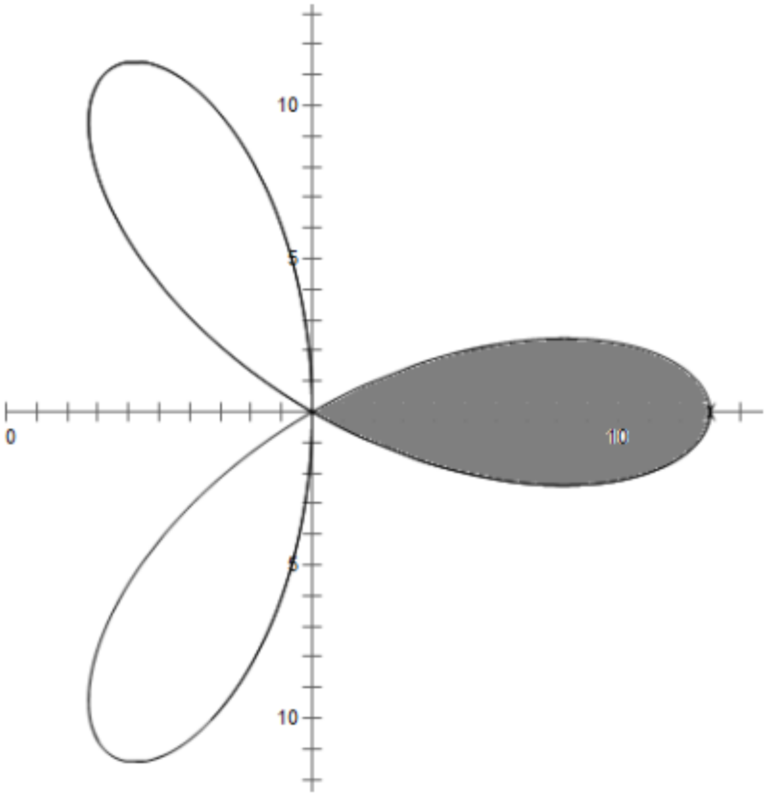
HW10-5

- **HW: 1,21,23**

106本 5. (10 pts) Find the area of the polar region (極坐標區域) enclosed by $r = 2\sin(3\theta)$

$r = \sin a\theta$
- a 奇數, a 瓣玫瑰線
- a 偶數, $2a$ 瓣玫瑰線

110 6. Find the area of one petal of $r = 13\cos 3\theta$. (7 points)



109 3. Let C be the polar curve given by $r = 2\sqrt{\cos(2\theta)}$, $-\pi/4 \le \theta \le \pi/4$.

(a) (10 pts) Sketch the polar curve C.

(b) (10 pts) Find the area enclosed by C.

θ	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
r	0	$\sqrt{2}$	2	$\sqrt{2}$	0

$$A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}\left[2\sqrt{\cos 2\theta}\right]^2 d\theta$$