1. Substitution Rule

將被積函數的一部分打包成一個新變數 $u = u(x)$，有機會使積分難度降低. 打包時，若成功湊出 $u'(x)\,\mathrm{d}x$ 則積分會變得更加容易. 而定積分在使用變數變換時，需要注意上下限範圍. 最後記得換回最一開始的變數.

Question Calculate the following integral.

▷
$$\begin{aligned}\int 2x\sqrt{x^2+1}\,\mathrm{d}x &= \int \sqrt{u}\,\mathrm{d}u \quad ,\text{令 } u = x^2+1,\text{ 則 } \mathrm{d}u = 2x\mathrm{d}x\\ &= \int u^{1/2}\,\mathrm{d}u\\ &= \frac{u^{3/2}}{3/2} + C\\ &= \frac{2}{3}(x^2+1)^{3/2} + C\end{aligned}$$

▶
$$\begin{aligned}\int x^2\sqrt{x^3+1}\,\mathrm{d}x &= \int \sqrt{u}\frac{\mathrm{d}u}{3} \quad ,\text{令 } u = x^3+1,\text{ 則 } \mathrm{d}u = 3\mathrm{d}x\\ &= \frac{1}{3}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C\\ &= \frac{2}{9}(x^3+1)^{\frac{3}{2}} + C\end{aligned}$$

▷
$$\begin{aligned}\int \sec x\,\mathrm{d}x &= \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,\mathrm{d}x\\ &= \int \frac{u'}{u}\,\mathrm{d}x \quad ,\text{令 } u = \sec x + \tan x,\text{ 則 } \mathrm{d}u = (\sec x\tan x + \sec^2 x)\mathrm{d}x\\ &= \ln|u| + C\\ &= \ln|\sec x + \tan x| + C\end{aligned}$$

▶
$$\begin{aligned}\int \tan x\,\mathrm{d}x &= \int \frac{\sin x}{\cos x}\,\mathrm{d}x\\ &= \int -\frac{1}{u}\,\mathrm{d}u \quad ,\text{令 } u = \cos x,\text{ 則 } \mathrm{d}u = -\sin x\mathrm{d}x\\ &= -\ln|u| + C\\ &= -\ln|\cos x| + C\end{aligned}$$

▷
$$\begin{aligned}\int x^2e^{x^3+1}\,\mathrm{d}x &= \int e^u\frac{\mathrm{d}u}{3} \quad ,\text{令 } u = x^3+1,\text{ 則 } \mathrm{d}u = 3x^2\mathrm{d}x\\ &= \frac{1}{3}\int e^u\,\mathrm{d}u\\ &= \frac{1}{3}e^u + C\\ &= \frac{1}{3}e^{x^3+1} + C\end{aligned}$$

► $$\int \frac{1}{x\ln x}\,\mathrm{d}x = \int \frac{1}{u}\,\mathrm{d}u \quad ,\text{令 } u = \ln x,\ \text{則 } \mathrm{d}u = \frac{\mathrm{d}x}{x}$$
$$= \ln|u| + C$$
$$= \ln|\ln x| + C$$

2. Trigonometric Substitution

三角代換算式變數變換的一種特殊方法，三角代換是為了使用一些三角恆等式將 x 直接替換成 $\sin\theta$、$\tan\theta$、$\sec\theta$：

積分中的表達式	代換方法	代換結果
$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta,\ 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Remark 只要記住「根號內的數值必須大於等於零」

Question Calculate the following integral.

▷ $$\int \frac{1}{1+x^2}\,\mathrm{d}x = \int \cos^2 u \sec^2 u\,\mathrm{d}u \quad ,\text{令 } x = \tan u,\ \text{則 } \mathrm{d}x = \sec^2 u\mathrm{d}u$$
$$= \int 1\,\mathrm{d}u$$
$$= u + C$$
$$= \tan^{-1} x + C$$

► $$\int \frac{1}{\sqrt{4-x^2}}\,\mathrm{d}x = \int \frac{\sec u}{2}\cdot 2\cos u\,\mathrm{d}u \quad ,\text{令 } x = 2\sin u,\ \text{則 } \mathrm{d}x = 2\cos u\mathrm{d}u$$
$$= \int 1\,\mathrm{d}u$$
$$= u + C$$
$$= \arcsin\frac{x}{2} + C$$

▷ $$\int \frac{1}{\sqrt{x^2+2x}}\,\mathrm{d}x = \int \frac{1}{\sqrt{(x+1)^2-1}}\,\mathrm{d}x \quad ,\text{令 } x+1=\sec u,\ \text{則 } \mathrm{d}x=\sec u\tan u\mathrm{d}u$$
$$= \int \frac{\sec u\tan u}{\tan u}\,\mathrm{d}u$$
$$= \int \sec u\,\mathrm{d}u$$
$$= \ln|\sec u+\tan u|+C$$
$$= \ln\left|(x+1)+\sqrt{(x+1)^2-1}\right|+C$$

▶ $$\int \frac{1}{x^2+2x+5}\,\mathrm{d}x = \int \frac{1}{(x+1)^2+2^2}\,\mathrm{d}x \quad ,\text{令 } x+1=\tan u,\ \text{則 } \mathrm{d}x=\sec^2 u\mathrm{d}u$$
$$= \int \frac{1}{4}\cos^2 u\cdot 2\cos^2 u\,\mathrm{d}u$$
$$= \int \frac{1}{2}\,\mathrm{d}u$$
$$= \frac{u}{2}+C$$
$$= \frac{1}{2}\tan^{-1}\frac{x+1}{2}$$

3. Partial Fraction

將複雜的分式函數 $f(x)=\dfrac{P(x)}{Q(x)}$ 使用長除法與因式分解化簡成：

$$f(x)=S(x)+\frac{R(x)}{Q(x)},$$

其中 $S(x)$ 為多項式 (若 $\deg P\geq\deg Q$ 才會有 $S(x)$ 這一項)，而 $\frac{R(x)}{Q(x)}$ 為真分式 ($\deg R<\deg Q$). 對於真分式，可依據 $Q(x)$ 的因式分解進一步拆分為一次因式與不可分解的二次因式：

$$\frac{A_1}{x-a_1}+\cdots+\frac{A_m}{(x-a_m)^m}+\frac{B_1x+C_1}{x^2+b_1x+c_1}+\cdots+\frac{B_nx+C_n}{(x^2+b_nx+c_n)^n}$$

拆分後，每一項的積分通常簡單許多，從而簡化計算.

Question Decompose the following fractions into partial fractions.

▷ $$\frac{2x^4+3x^3+7x^2-5x-3}{x^3-x}$$

$$= \ 2x+3+\frac{9x^2-2x-3}{x^3-x}$$

$$= \ 2x+3+\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}$$

所以$9x^2-2x-3=A(x-1)(x+1)+Bx(x+1)+Cx(x-1)$

對照係數：$\begin{cases} A+B+C=9, \\ B-C=-2, \\ -A=-3 \end{cases} \Rightarrow (A,B,C)=(3,2,4)$

$$= \ 2x+3+\frac{3}{x}+\frac{2}{x-1}+\frac{4}{x+1}.$$

▶ $\dfrac{3x^4+x^3+3x^2-3x-2}{x^3-x}$

$$= \ 3x+1+\frac{6x^2-2x-2}{x^3-x}$$

$$= \ 3x+1+\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}$$

所以$6x^2-2x-2=A(x-1)(x+1)+Bx(x+1)+Cx(x-1)$

對照係數：$\begin{cases} A+B+C=6, \\ B-C=-2, \\ -A=-2 \end{cases} \Rightarrow (A,B,C)=(2,1,3)$

$$= \ 3x+1+\frac{2}{x}+\frac{1}{x-1}+\frac{3}{x+1}.$$

Question Calculate the following integral.

▷ $\displaystyle\int \frac{2x^4+3x^3+7x^2-5x-3}{x^3-x}\,\mathrm{d}x$

$$= \int \left(2x+3+\frac{3}{x}+\frac{2}{x-1}+\frac{4}{x+1}\right)\,\mathrm{d}x$$

$$= x^2+3x+3\ln|x|+2\ln|x-1|+4\ln|x+1|+C$$

▶ $\displaystyle\int \frac{3x^4+x^3+3x^2-3x-2}{x^3-x}\,\mathrm{d}x$

$$= \int \left(3x+1+\frac{2}{x}+\frac{1}{x-1}+\frac{3}{x+1}\right)\,\mathrm{d}x$$

$$= \frac{3}{2}x^2+x+2\ln|x|+\ln|x-1|+3\ln|x+1|+C$$

4. Integration by part

微分乘法法則告訴我們：

$$\frac{\mathrm{d}}{\mathrm{d}x}(F(x)\cdot G(x)) = F(x)g(x) + f(x)G(x)$$

其中 $F'(x) = f(x)$、$G'(x) = g(x)$. 同時對兩邊積分可以得到：

$$F(x)\cdot G(x) = \int F(x)g(x)\ \mathrm{d}x + \int f(x)G(x)\ \mathrm{d}x$$

移項後有：

$$\int F(x)g(x)\ \mathrm{d}x = F(x)\cdot G(x) - \int f(x)G(x)\ \mathrm{d}x$$

這即是分部積分的基礎，它幫助我們處理有關函數乘積的積分.

Question Calculate the following integrals by parts.

▷ $\displaystyle \int x\ln x\ \mathrm{d}x = x^2\frac{\ln x}{2} - \int \frac{x}{2}\ \mathrm{d}x = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$

▶ $\displaystyle \int \ln x\ \mathrm{d}x = x\ln x - \int 1\ \mathrm{d}x = x\ln x - x + C$

▷ $\displaystyle \int xe^x\ \mathrm{d}x = xe^x - \int e^x\ \mathrm{d}x = e^x(x-1) + C$

▶ $\displaystyle \int x^2e^x\ \mathrm{d}x = x^2e^x - \int 2xe^x\ \mathrm{d}x = x^2e^x - 2(xe^x - \int e^x\ \mathrm{d}x) = e^x(x^2-2x+2) + C$

$$\triangleright \int e^x \sin x \, \mathrm{d}x = e^x \sin x - \int e^x \cos x \, \mathrm{d}x$$
$$= e^x \sin x - \left(e^x \cos x - \int e^x (-\sin x) \, \mathrm{d}x \right)$$
$$= e^x (\sin x - \cos x) - \int e^x \sin x \, \mathrm{d}x$$
$$\Rightarrow \ 2 \int e^x \sin x \, \mathrm{d}x = e^x (\sin x - \cos x)$$
$$\therefore \int e^x \sin x \, \mathrm{d}x = \frac{e^x}{2} (\sin x - \cos x) + C$$

$$\blacktriangleright \int e^x \cos x \, \mathrm{d}x = e^x \cos x + \int e^x \sin x \, \mathrm{d}x$$
$$= e^x \cos x + \left(e^x \sin x - \int e^x \cos x \, \mathrm{d}x \right)$$
$$= e^x (\sin x + \cos x) - \int e^x \cos x \, \mathrm{d}x$$
$$\Rightarrow \ 2 \int e^x \cos x \, \mathrm{d}x = e^x (\sin x + \cos x)$$
$$\therefore \int e^x \cos x \, \mathrm{d}x = \frac{e^x}{2} (\sin x + \cos x) + C$$

5. Improper Integrals

Definition 若積分上、下限中有一端為無窮，則稱該積分屬於**第一類瑕積分**：

- $\displaystyle\int_a^\infty f(x) \, \mathrm{d}x = \lim_{t \to \infty} \int_a^t f(x) \, \mathrm{d}x$
- $\displaystyle\int_{-\infty}^b f(x) \, \mathrm{d}x = \lim_{t \to -\infty} \int_t^b f(x) \, \mathrm{d}x$
- $\displaystyle\int_{-\infty}^\infty f(x) \, \mathrm{d}x = \int_{-\infty}^a f(x) \, \mathrm{d}x + \int_a^\infty f(x) \, \mathrm{d}x$

Remark 對應的極限若存在，稱該積分收斂 (Convergent)；若不存在，則稱發散 (Divergent).

Question Calculate the following improper integrals.

$$\triangleright \int_1^\infty 2x^{-3} \, \mathrm{d}x = \lim_{t \to \infty} \int_1^t 2x^{-3} \, \mathrm{d}x$$
$$= \lim_{t \to \infty} \left[-x^{-2} \right]_1^t$$
$$= \lim_{t \to \infty} (-t^{-2} + 1) = 1$$

▶ $$\int_{-\infty}^{-1} \frac{1}{\sqrt[3]{x}}\,\mathrm{d}x = \lim_{t\to-\infty}\int_t^{-1} x^{-1/3}\,\mathrm{d}x$$
$$= \lim_{t\to-\infty}\left[\frac{3}{2}x^{2/3}\right]_t^{-1}$$
$$= \frac{3}{2}(-1)^{2/3} - \lim_{t\to-\infty}\frac{3}{2}t^{2/3} \to -\infty \qquad \Rightarrow \text{該積分發散 (Diverges)}$$

▷ $$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x = \int_{-\infty}^{0}\frac{1}{x^2+1}\,\mathrm{d}x + \int_0^{\infty}\frac{1}{x^2+1}\,\mathrm{d}x$$
$$= \lim_{t\to-\infty}\int_t^0 \frac{1}{x^2+1}\,\mathrm{d}x + \lim_{s\to\infty}\int_0^s \frac{1}{x^2+1}\,\mathrm{d}x$$
$$= \lim_{t\to-\infty}(\tan^{-1}0 - \tan^{-1}t) + \lim_{s\to\infty}(\tan^{-1}s - \tan^{-1}0)$$
$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

▶ $$\int_{-\infty}^{\infty} \frac{1}{x^2+9}\,\mathrm{d}x = \int_{-\infty}^{0}\frac{1}{x^2+9}\,\mathrm{d}x + \int_0^{\infty}\frac{1}{x^2+9}\,\mathrm{d}x$$
$$= \lim_{t\to-\infty}\int_t^0 \frac{1}{x^2+9}\,\mathrm{d}x + \lim_{s\to\infty}\int_0^s \frac{1}{x^2+9}\,\mathrm{d}x$$
$$= \lim_{t\to-\infty}\frac{1}{3}(\tan^{-1}(0/3) - \tan^{-1}(t/3)) + \lim_{s\to\infty}\frac{1}{3}(\tan^{-1}(s/3) - \tan^{-1}(0/3))$$
$$= \frac{1}{3}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \frac{\pi}{3}$$

Definition 若被積函數在積分區間內某點不連續或無界，則稱該積分屬於**第二類瑕積分**：

- $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to b^-}\int_a^t f(x)\,\mathrm{d}x$ (當 $f(x)$ 在 $x=b$ 不連續)
- $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to a^+}\int_t^b f(x)\,\mathrm{d}x$ (當 $f(x)$ 在 $x=a$ 不連續)
- $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$ (當 $f(x)$ 在 (a,b) 內某點 c 不連續)

Question Calculate the following improper integrals.

▷ $$\int_0^1 \frac{1}{x}\,\mathrm{d}x = \lim_{t\to0^+}\int_t^1 \frac{1}{x}\,\mathrm{d}x$$
$$= \lim_{t\to0^+}[\ln x]_{x=t}^{x=1}$$
$$= \lim_{t\to0^+}\ln 1 - \ln t$$
$$= \lim_{t\to0^+} -\ln t \to \infty \qquad \Rightarrow \text{該積分發散 (Diverges)}$$

▶ $\displaystyle\int_{-1}^{0}\frac{1}{x^2}\,\mathrm{d}x = \lim_{t\to 0^-}\int_{-1}^{t} x^{-2}\,\mathrm{d}x$

$$\begin{aligned}
&= \lim_{t\to 0^-}\left[-x^{-1}\right]_{x=-1}^{x=t} \\
&= \lim_{t\to 0^-}(-t^{-1}) - ((-1)^{-1}) \\
&= \lim_{t\to 0^-} -t^{-1} + 1 \to \infty \qquad \Rightarrow \text{該積分發散 (Diverges)}
\end{aligned}$$

Theorem. (**直接比較審斂法 (Direct Comparison Theorem)**) 給定連續函數 $f(x)$、$g(x)$，對於任意 $x \geq a$ 有 $f(x) \geq g(x) \geq 0$，那麼：

1. 若 $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x$ 收斂，則 $\displaystyle\int_a^\infty g(x)\,\mathrm{d}x$ 收斂.
2. 若 $\displaystyle\int_a^\infty g(x)\,\mathrm{d}x$ 發散，則 $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x$ 發散.

Question Determine whether the following integrals converge.

▷ $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,\mathrm{d}x$ (Hint: $0 \leq \sin^2 x \leq 1, \forall x \in \mathbb{R}$)

由於 $0 \leq \sin^2 x \leq 1$, 我們有 $0 \leq \dfrac{\sin^2 x}{x^2} \leq \dfrac{1}{x^2}, \quad \forall x \geq 1$

並且積分 $\displaystyle\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x = 1$ 收斂

由直接比較審斂法，$\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,\mathrm{d}x$ 亦收斂.

▶ $\displaystyle\int_1^\infty \frac{\ln x}{x^3}\,\mathrm{d}x$ (Hint: $\ln x \leq x, \forall x \geq 1$)

由於 $\ln x \leq x$ 對所有 $x \geq 1$, 我們有 $0 \leq \dfrac{\ln x}{x^3} \leq \dfrac{x}{x^3} = \dfrac{1}{x^2}$

並且積分 $\displaystyle\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x = 1$ 收斂

由直接比較審斂法，$\displaystyle\int_1^\infty \frac{\ln x}{x^3}\,\mathrm{d}x$ 亦收斂.

Next week: 算面積、算體積、算弧長、算表面積、算函數平均值