拉格朗日乘子

13-8 Lagrange Multipliers

師大工教一

Theorem 12—The Orthogonal Gradient Theorem

Suppose that $f(x,y,z)$ is differentiable in a region whose interior contains a smooth curve $C:\vec{r}(t)=x(t)\vec{i}+y(t)\vec{j}+z(t)\vec{k}$. If P_0 is a point on C where f has a local maximum or minimum relative to its values on C, then ∇f is orthogonal to the curve's tangent vector $\vec{r}'$ at P_0.

Corollery At the points on a smooth curve $\vec{r}(t)=x(t)\vec{i}+y(t)\vec{j}$ where a differentiable function $f(x,y)$ takes on its local maxima or minima relative to its values on the curve, we have $\nabla f\cdot\vec{r}'=0$.

The Method of Lagrange Multipliers

$$\begin{cases} \nabla f = \lambda \cdot \nabla g \\ g = k \end{cases} \Rightarrow$$

$(x_1,y_1,z_1), \dots, (x_n,y_n,z_n)$

比較 $f(x_1,y_1,z_1), \dots, f(x_n,y_n,z_n)$

最大為絕對極大值

小 …… 小

Suppose that $f(x,y,z)$ and $g(x,y,z)$ are differentiable and $\nabla g \neq 0$ when $g(x,y,z)=0$. To find the local maximum and minimum values of f subject to the constraint (限制) $g(x,y,z)=0$ (I these exist), find the values of $x, y, z,$ and λ that simultaneously (同時) satisfy (滿足) the equations $\nabla f = \lambda \nabla g$ and $g(x,y,z)=0$. For functions of two independent variables, the condition is similar, but without the variable z.

題目問法：求 $f(x,y,z)$ 之絕對極值

在 $g(x,y,z)=k$ 的限制下

☆ 雙變數 $\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = k \end{cases}$ 　三變數 $\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g(x,y,z) = k \end{cases}$

Ex3(p785) Find the largest and smallest values that the function $f(x,y) = xy$ takes on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$.

$\begin{cases} y = \lambda \cdot \frac{x}{4} & -① \\ x = \lambda \cdot y & -② \\ \frac{x^2}{8} + \frac{y^2}{2} = 1 & -③ \end{cases}$

$\frac{①}{②} \Rightarrow \frac{y}{x} = \frac{\frac{x}{4}}{y} \Rightarrow x^2 = 4y^2 \Rightarrow x = \pm 2y$

(i) $x = 2y$ 代入③：$y^2 = 1 \Rightarrow y = \pm 1$
$\Rightarrow (2,1), (-2,-1)$

(ii) $x = -2y$ 代入③ $y^2 = 1 \Rightarrow y = \pm 1$
$\Rightarrow (-2,1), (2,-1)$

$f(2,1) = 2$, $f(-2,-1) = 2$ > abs. max. value

$f(-2,1) = -2$, $f(2,-1) = -2$ > abs. min. value

Ex4(p786) Find the maximum and minimum values of the function

$f(x,y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

$\begin{cases} 3 = \lambda \cdot 2x & -① \\ 4 = \lambda \cdot 2y & -② \\ x^2 + y^2 = 1 & -③ \end{cases}$

$\frac{①}{②} \Rightarrow \frac{3}{4} = \frac{x}{y}$

$x = \frac{3}{4}y$ 代入③ $\Rightarrow \frac{9}{16}y^2 + y^2 = 1$ 　$y^2 = \frac{16}{25}$ 　$y = \pm \frac{4}{5}$

(i) $y = \frac{4}{5} \Rightarrow (\frac{3}{5}, \frac{4}{5})$ 　$f(\frac{3}{5}, \frac{4}{5}) = \frac{9}{5} + \frac{16}{5} = 5 \rightarrow$ abs. max. value

(ii) $y = -\frac{4}{5} \Rightarrow (-\frac{3}{5}, -\frac{4}{5})$ 　$f(-\frac{3}{5}, -\frac{4}{5}) = -\frac{9}{5} - \frac{16}{5} = -5 \rightarrow$ abs. min. value

HW13-8

- **HW: 1,4,19**

(102 年#7) Find the maximum and minimum values of x^2+y^2 subject to the constraint $x^2-2x+y^2-4y=0$

106本 6. (10 pts) Find the minimum distance from $P(1,2,0)$ to the surface $z=\sqrt{x^2+y^2}$.

$f(x,y,z)=(x-1)^2+(y-2)^2+z^2$ $g(x,y,z)=z-\sqrt{x^2+y^2}=0$ 記得距離$=\sqrt{f(x,y,z)}$

107 7. (10 pts) Find any extrema of the function $f(x,y)=e^{-\frac{xy}{4}}$ under the constraint $x^2+y^2=1$.

108 5. (10 points) Find the minimum distance from the surface $x^2-y^2-z^2=1$ to the origin.

$f(x,y,z)=x^2+y^2+z^2$

$g(x,y,z)=x^2-y^2-z^2=1$

111 3. (a) (10 points) Find all local maxima, local minima and saddle points(if any exists) of

$$f(x,y)=x^2y+x^2-2y.$$

(b) (15 points) Determine the maximum and minimum points of $f(x,y)=x^2+y^2+\frac{3}{2}x+1$ on the set $G=\{(x,y)|4x^2+y^2=1\}$. (不限高中或大學做法, Hint: G 是一橢圓)

111. (a) $f_x = 2yx + 2x \overset{\text{let}}{=} 0$ $\quad 2x(y+1)=0$ $\begin{cases} x=0 \\ y=-1 \end{cases}$ $\quad \therefore (\sqrt{2},-1)、(-\sqrt{2},-1)$ are critical point

$f_y = x^2 - 2 \overset{\text{let}}{=} 0 \quad x^2=2 \quad x=\pm\sqrt{2}$

$f_{xx} = 2y+2 \qquad D(\sqrt{2},-1) = 0(0)-8<0 \qquad \therefore f$ has saddle points at $(\sqrt{2},-1),(-\sqrt{2},-1)$

$f_{xy} = 2x \qquad D(-\sqrt{2},-1) = 0(0)-8<0$

$f_{yy} = 0$

(b) $\nabla f = \lambda \nabla g$ $\quad \begin{cases} 2x+\frac{3}{2} = \lambda \cdot 8x & -① \Rightarrow \lambda=1 \Rightarrow x=\frac{1}{4} \\ 2y = \lambda \cdot 2y & -② \Rightarrow \lambda=1 \text{ or } y=0 \\ 4x^2+y^2=1 & -③ \end{cases}$

$x=\frac{1}{4}$代入 $\quad y^2=\frac{3}{4} \quad y=\pm\frac{\sqrt{3}}{2} \Rightarrow (\frac{1}{4},\frac{\sqrt{3}}{2}),(\frac{1}{4},-\frac{\sqrt{3}}{2})$

$y=0$代入 $\quad 4x^2=1 \quad x=\pm\frac{1}{2} \Rightarrow (\frac{1}{2},0),(-\frac{1}{2},0)$

$f(\frac{1}{4},\frac{\sqrt{3}}{2}) = f(\frac{1}{4},-\frac{\sqrt{3}}{2}) = \frac{1}{16}+\frac{3}{4}+\frac{3}{8}+1 = \frac{1+12+6+16}{16} = \frac{35}{16}$

$f(\frac{1}{2},0) = \frac{1}{4}+\frac{3}{4}+1 = 2 \qquad f(-\frac{1}{2},0) = \frac{1}{4}-\frac{3}{4}+1 = \frac{1}{2}$

max. point at $(\frac{1}{4},\frac{\sqrt{3}}{2})$ & $(\frac{1}{4},-\frac{\sqrt{3}}{2})$

$f_{max} = \frac{35}{16}$

min. point at $(-\frac{1}{2},0)$

$f_{min} = \frac{1}{2}$