13-2 Limits and Continuity in Higher Dimensions

師大工教一

Limits for Functions of Two Variables

Definition Suppose that every open circular disk centered at (x_0, y_0) contains a point in the domain of f other than (x_0, y_0) itself. We say that a function $f(x,y)$ approaches the **limit** L as (x,y) approaches (x_0, y_0), and write $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if, for every $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x,y) in the domain of f, $|f(x,y) - L| < \varepsilon$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$.

$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$

$\overset{\Delta}{\Leftrightarrow} \forall \varepsilon > 0, \exists \delta > 0$ such that

$|f(x,y) - L| < \varepsilon$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

Claim: $\lim\limits_{(x,y)\to(x_0,y_0)} x = x_0 \cdots (1)$, $\lim\limits_{(x,y)\to(x_0,y_0)} y = y_0 \cdots (2)$, $\lim\limits_{(x,y)\to(x_0,y_0)} k = k \cdots (3)$

For every $\varepsilon > 0$, let $\delta = \varepsilon$, if $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta = \varepsilon \Rightarrow |x - x_0| < \varepsilon$.

Claim: $\lim\limits_{(x,y)\to(x_0,y_0)} g(x) = \lim\limits_{x\to x_0} g(x) \cdots (4)$, $\lim\limits_{(x,y)\to(x_0,y_0)} h(y) = \lim\limits_{y\to y_0} h(y) \cdots (5)$

Theorem 1-Properties of Limits of Functions of Two Variables

If $L, M, k \in R$, and $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ and $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$

1. *Sum Rule*: $\lim_{(x,y)\to(x_0,y_0)} \left[f(x,y) + g(x,y) \right] = L + M$

2. *Difference Rule*: $\lim_{(x,y)\to(x_0,y_0)} \left[f(x,y) - g(x,y) \right] = L - M$

3. *Constant Multiple Rule*: $\lim_{(x,y)\to(x_0,y_0)} k\, f(x,y) = k\, L$

4. *Product Rule*: $\lim_{(x,y)\to(x_0,y_0)} \left[f(x,y) \cdot g(x,y) \right] = L \cdot M$

5. *Quotient Rule*: $\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}, M \neq 0$

6. *Power Rule*: $\lim_{(x,y)\to(x_0,y_0)} \left[f(x,y) \right]^n = L^n$

7. *Root Rule*: $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L}$

8. *Composition Rule*: $\lim_{(x,y)\to(x_0,y_0)} h(f(x,y)) = h(L)$ if $h(z)$ is continuous at L

Ex1(p725) Find the limits:

(a) $\lim\limits_{(x,y)\to(0,1)} \dfrac{x - xy + 3}{x^2 y + 5xy - y^3}$ (b) $\lim\limits_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2}$ (c) $\lim\limits_{(x,y)\to\left(\frac{\pi}{2},0\right)} \left(\dfrac{x}{\sin x} - \dfrac{\sin y}{y}\right)$

$= \dfrac{3}{-1} = -3$

$= 5$

$= \dfrac{\frac{\pi}{2}}{1} - 1 = \frac{\pi}{2} - 1$

Ex2(p725) Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$.

$= \lim\limits_{(x,y)\to(0,0)} \dfrac{\overset{x}{\cancel{(x^2 - xy)}}(\sqrt{x} + \sqrt{y})}{\cancel{x - y}}$

$= \lim\limits_{(x,y)\to(0,0)} x\sqrt{x} + x\sqrt{y} = 0$

Ex3(p726) Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2+y^2}$ if it exists.

Ex4(p727) If $f(x,y)=\dfrac{y}{x}$, does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist?

Ex 3

Analysis: $\left|\dfrac{4xy^2}{x^2+y^2}-0\right|<\varepsilon$

$|4x|<\varepsilon$

$|x|<\dfrac{\varepsilon}{4}$

$\sqrt{(x-0)^2+(y-0)^2}<\dfrac{\varepsilon}{4}$

$\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2+y^2}$

$\forall \varepsilon>0$, let $\delta=\dfrac{\varepsilon}{4}>0$

As $0<\sqrt{(x-0)^2+(y-0)^2}<\delta=\dfrac{\varepsilon}{4}$

$\Rightarrow \sqrt{x^2+y^2}<\dfrac{\varepsilon}{4}$

$\Rightarrow |x|<\dfrac{\varepsilon}{4}$

$\Rightarrow |4x|<\varepsilon$

$\left|\dfrac{4xy^2}{x^2+y^2}-0\right|<\varepsilon \left(\because \dfrac{y^2}{x^2+y^2}\le 1\right)$

$\therefore \lim\limits_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2+y^2}=0$

Ex3(p726) Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2+y^2}$ if it exists.

Ex4(p727) If $f(x,y)=\dfrac{y}{x}$, does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist?

Ex4

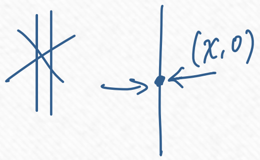
$\lim\limits_{(x,y)\to(0,0)} \dfrac{y}{x}$

along x-axis

$= \lim\limits_{x\to 0} \dfrac{0}{x} = \lim\limits_{x\to 0} 0 = 0$

$\lim\limits_{(x,y)\to(0,0)} \dfrac{y}{x} = \lim\limits_{x\to 0} \dfrac{x}{x}$

$= \lim\limits_{x\to 0} 1 = 1$



$\therefore \lim\limits_{(x,y)\to(0,0)} \dfrac{y}{x}$ DNE

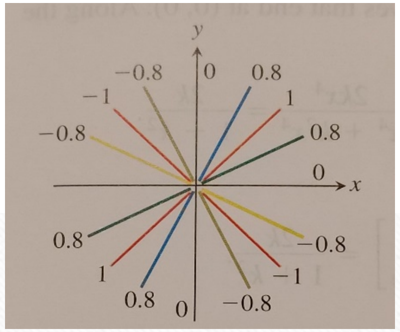
Continuity

Definition Suppose that every open circular disk centered at (x_0, y_0) contains a point in the domain of f other than (x_0, y_0) itself. Then a function $f(x, y)$ is **continuous at the point** (x_0, y_0) if

1. f is defined at (x_0, y_0).
2. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists.
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

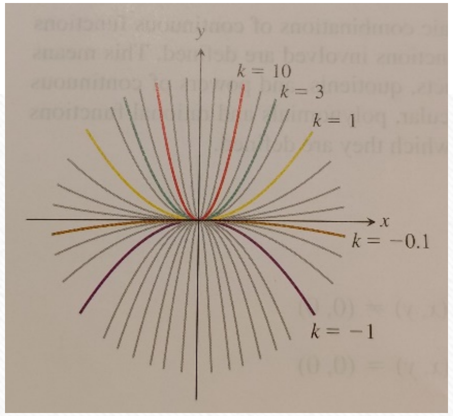
Ex5(p727) Show that $f(x,y)=\begin{cases}\dfrac{2xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$ is continuous at every point except the origin.



Two-Path Test for Nonexistence of a Limit

If a function $f(x,y)$ has different limits along two different paths in the domain of f as (x,y) approaches (x_0,y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.

Ex6(p728) Show that the function $f(x,y)=\dfrac{2x^2y}{x^4+y^2}$ has no limit as (x,y) approaches $(0,0)$.



Conclusion: Having the same limit along all straight lines approaching (x_0,y_0) does not implies that a limit exists at (x_0,y_0).

Continuity of Compositions

If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f((x_0, y_0))$, then the composition $h = g \circ f$ defined by $h(x,y) = g(f(x,y))$ is also continuous at (x_0, y_0).

Functions of More Than Two Variables

The definition of limit and continuity for functions of two variables and the conclusions about limits and continuity for sums, products, quotients, and compositions all extend to functions of three or more variables.

Extreme Values of Continuous Functions on Closed, Bounded Sets

We have similar results like Extreme Value Theorem for functions with several variables. “A function with two or more variables takes on an absolute maximum(minimum) values at some point in a closed and bounded region”.

HW13-2

- **HW: 1,11,14,15,32,34,44,49**

(103) 2. Evaluate $\lim_{(x,y)\to(1,1)} \frac{xy^2-1}{y-1}$ if it exists. If the limit does not exist, explain why. (10 pts)

105分 11. Find $h(x,y)=g(f(x,y))$ and the set on which h is continuous. (5 points)

$$g(t)=t+\ln t \quad \text{and} \quad f(x,y)=\frac{1-xy}{1+x^2y^2}.$$

105本 8. (加分題，5 分) 計算極限 $\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2}$ 。

106分 1. Find the limit, if it exists, or show that the limit does not exists.

(a) (5pts) $\lim_{(x,y)\to(0,0)} \left(\frac{x^2-y^2}{x^2+y^2}\right)^2$.

(b) (5pts) $\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2}$ (Hint: you may use polar coordinates).

(107) 1. Find the following limits (if it exists). If the limit does not exist, explain why.

(a) (5 pts) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$ (b) (5 pts) $\lim_{(x,y)\to(0,0)} \frac{5xy^2}{x^2+2y^4}$