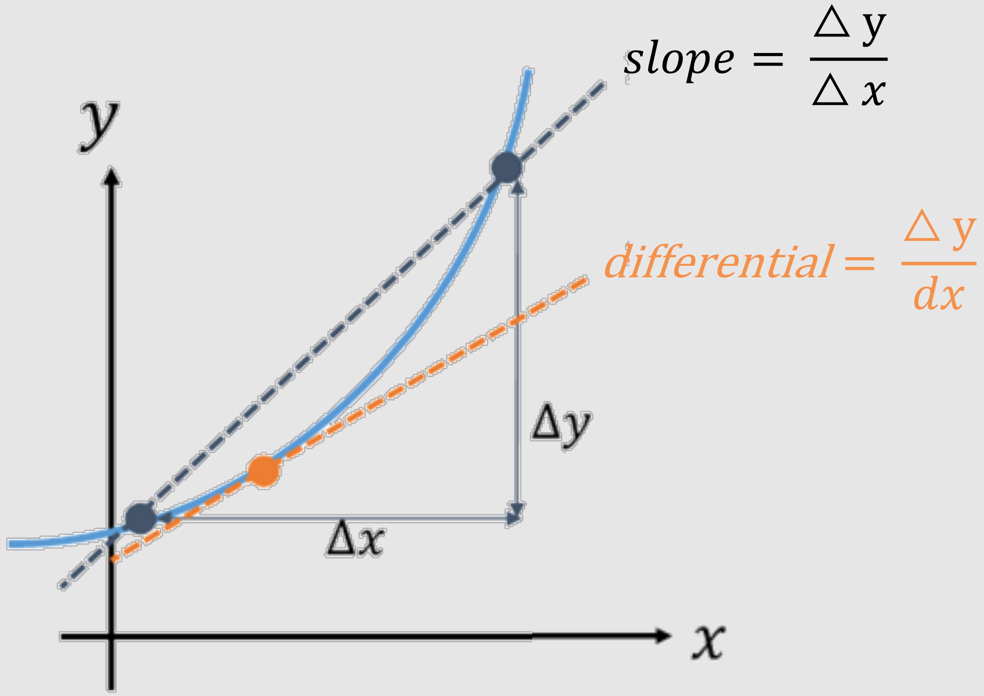
Advanced Engineering Mathematics | 賴以威

Introduction of DE
and
Separable Equation

Let's review the differential(dT/dt)

$$\frac{\mathrm{dT(t)}}{\mathrm{dt}} = -k(T(t) - R)$$



Newton's law of cooling

$dT(t)/dt$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \propto$

Wine temperature changes with time, which is proportional to the difference between the wine temperature and the room temperature at the moment

$T(t)$-R

Time: t
The temperature of the wine at time t: T(t)
Room temperature: R

$$\frac{dT(t)}{dt} = -k(T(t) - R)$$

- Being proportional = k times
- What is the minus sign stand for?

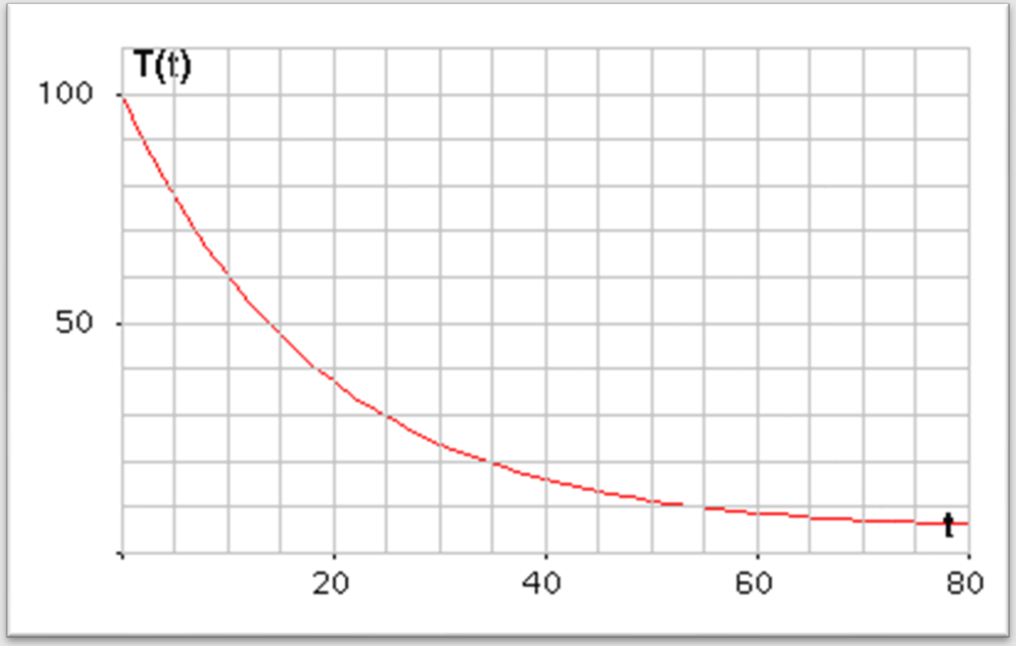
Newton's law of cooling

$$\frac{dT(t)}{dt} = -k(T(t) - R)$$

Don't wanna see

$$T(t) = \mathrm{R} + (\mathrm{T}_0 - R)e^{-kt}$$

T_0: The initial temperature



The purpose of DE:
Get rid of all the symbols associated with calculus

Separable Variables

1st order DE general form: $dy(x)/dx = f(x, y)$

If $dy(x)/dx = f(x, y)$ and $f(x, y)$ can be separate as

$$f(x, y) = g(x)\,h(y)$$

i.e., $\boxed{dy(x)/dx = g(x)\,h(y)}$

then the 1st order DE is separable (or has a separable variable).

condition : $dy(x)/dx = g(x)h(y)$

$$\frac{dy}{dx} = \cos(x)e^{x+2y}$$ Separable?

$$\frac{dy}{dx} = x + y$$ Separable?

Example 1

$$\boxed{(1+x)\,dy - y\,dx = 0}$$

$$\downarrow$$

Step 1 $\dfrac{dy}{y} = \dfrac{dx}{1+x}$

$$\downarrow \quad \downarrow$$

Step 2 $\ln|y| = \ln|1+x| + c_1$

$$\downarrow$$

$$|y| = e^{\ln|1+x|}e^{c_1} \longrightarrow y = \pm e^{c_1}e^{\ln|1+x|}$$

$$y = \pm e^{c_1}\,|1+x| = \pm e^{c_1}(1+x)$$

$$\downarrow$$

$$\boxed{y = c(1+x)} \qquad c = \pm e^{c_1}$$

Example 2

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

$$\boxed{y(4) = -3}$$

Step 1 $\quad y\,dy = -x\,dx$

Step 2 $\quad y^2/2 = -x^2/2 + c$

$$4.5 = -8 + c, \quad c = 12.5$$

$x^2 + y^2 = 25$ (implicit solution)

$y = \sqrt{25 - x^2}$ invalid

$\boxed{y = -\sqrt{25 - x^2}}$ valid

(explicit solution)

Procedure

If $\boxed{\dfrac{dy}{dx} = g(x)h(y)}$, then

$$\downarrow$$

Step 1 $\dfrac{dy}{h(y)} = g(x)dx$ Separate variables

$$\downarrow$$

$\boxed{p(y)dy = g(x)dx}$ where $\boxed{p(y) = 1/h(y)}$

$$\downarrow$$

Step 2 $\underline{\int p(y)dy} = \underline{\int g(x)dx}$ Integrate respectively

$$\downarrow \qquad \downarrow$$

$\underline{P(y) + c_1} = \underline{G(x) + c_2}$ where $\boxed{\dfrac{dP(y)}{dy} = p(y)}$ $\boxed{\dfrac{dG(x)}{dx} = g(x)}$

$$\downarrow$$

$\boxed{P(y) = G(x) + c}$

Extra Step: (a) Initial conditions

Example 3(Try!)

$$\boxed{\frac{dy}{dx} = y^2 - 4}$$

Step 1

$$\frac{dy}{y^2 - 4} = dx$$

$$\frac{1}{4}\frac{dy}{y-2} - \frac{1}{4}\frac{dy}{y+2} = dx$$

Step 2

$$\frac{1}{4}\ln|y-2| - \frac{1}{4}\ln|y+2| = x + c_1$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4x + 4c_1$$

$$\frac{y-2}{y+2} = \pm e^{4x+4c_1} = ce^{4x} \longrightarrow \boxed{y = 2\frac{1+ce^{4x}}{1-ce^{4x}}}$$

$$c = \pm e^{4c_1}$$

Extra Step (b)

check the singular solution

$$\frac{dy}{dx} = y^2 - 4$$

set $y = r$,

$0 = r^2 - 4$

$r = \pm 2,$

$y = \pm 2$

or $\boxed{y = \pm 2}$

Procedure (revisit)

If $\boxed{\dfrac{dy}{dx} = g(x)h(y)}$, then

$$\downarrow$$

Step 1 $\dfrac{dy}{h(y)} = g(x)dx$ Separate variables

$$\downarrow$$

$\boxed{p(y)dy = g(x)dx}$ where $\boxed{p(y) = 1/h(y)}$

$$\downarrow$$

Step 2 $\int p(y)dy = \int g(x)dx$ Integrate respectively

$$\downarrow$$

$P(y) + c_1 = G(x) + c_2$ where $\boxed{\dfrac{dP(y)}{dy} = p(y)}$ $\boxed{\dfrac{dG(x)}{dx} = g(x)}$

$$\downarrow$$

$\boxed{P(y) = G(x) + c}$

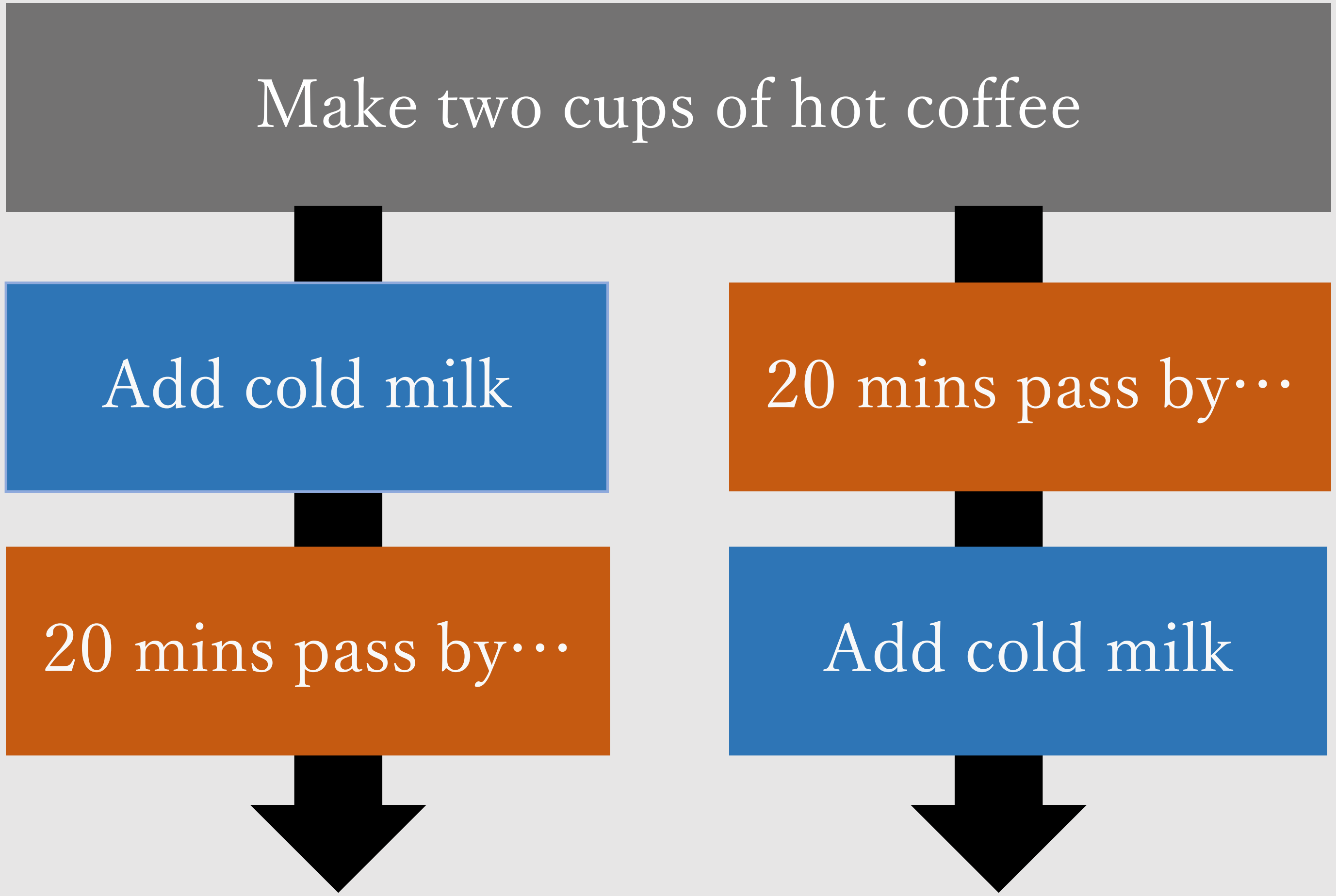
Extra Step: (a) Initial conditions
(b) Check the singular solution (i.e., the constant solution)

Back to Law of Cooling

$$\frac{\mathrm{dT(t)}}{\mathrm{dt}} = -k(T(t) - R)$$

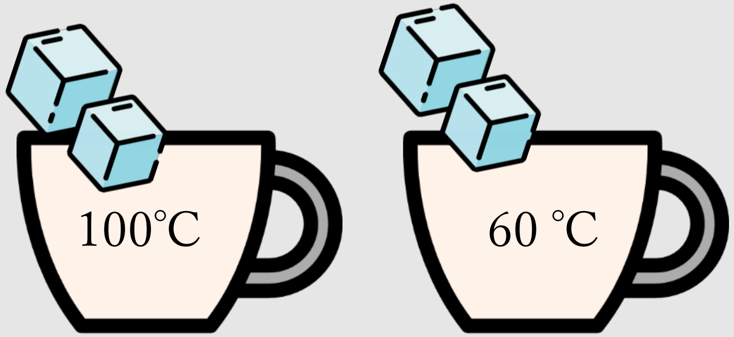
$$\Downarrow$$

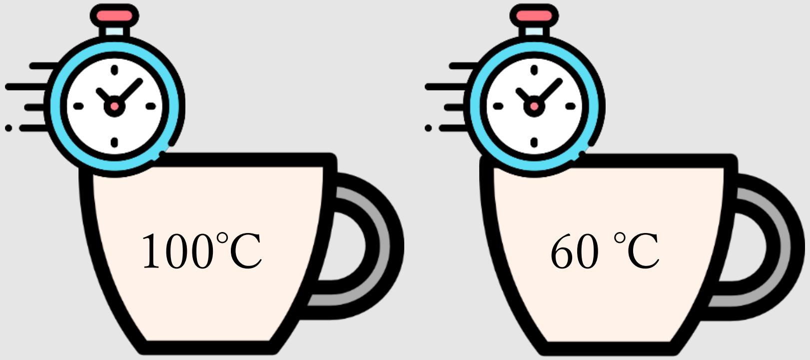
$$T(t) = \mathrm{R} + (\mathrm{T_0} - R)e^{-kt}$$



Which cup of coffee is cooler?

Influencing factor

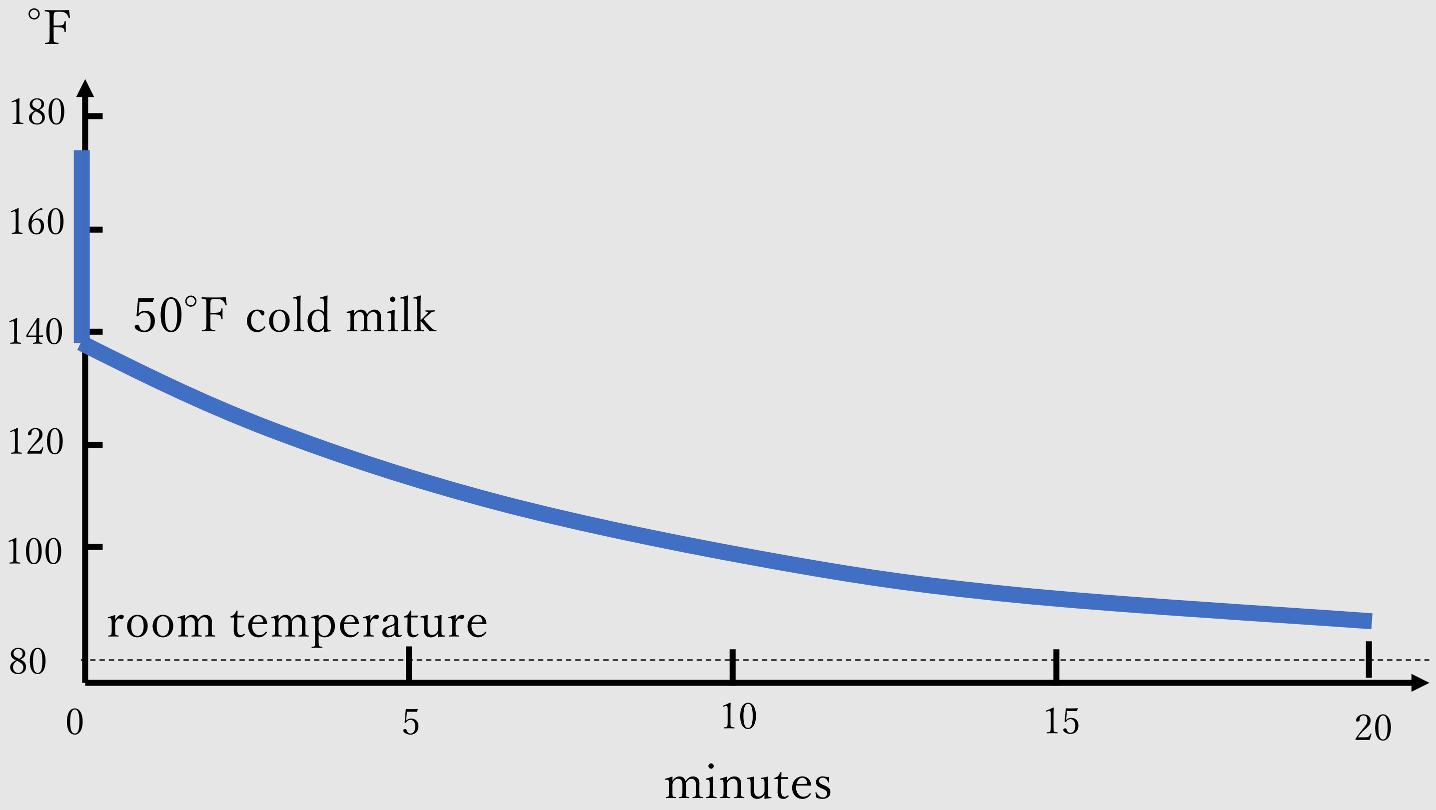




Hot water cools down more

Hot water cools down more

Pouring milk first



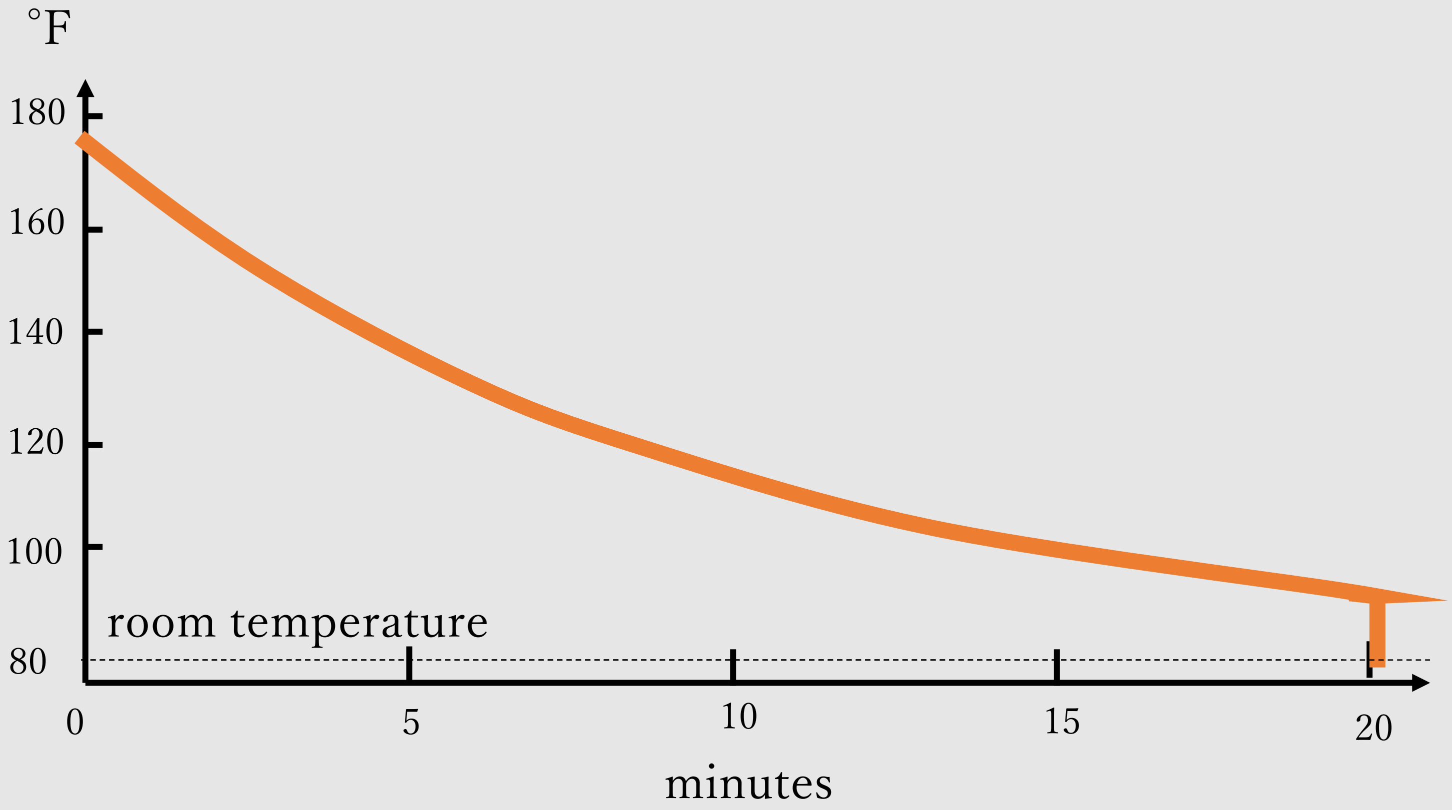
Why does the equation look like this?

$$\boldsymbol{T(T_N) = R + (rT_0 + (1 - r)C - R)e^{-kT_N}}$$

$$\frac{\mathrm{dT(t)}}{\mathrm{dt}} = -k(T(t) - R)$$

$$\mathrm{T}_0 \to r\mathrm{T}_0 + (1 - r)C$$

Pouring milk later

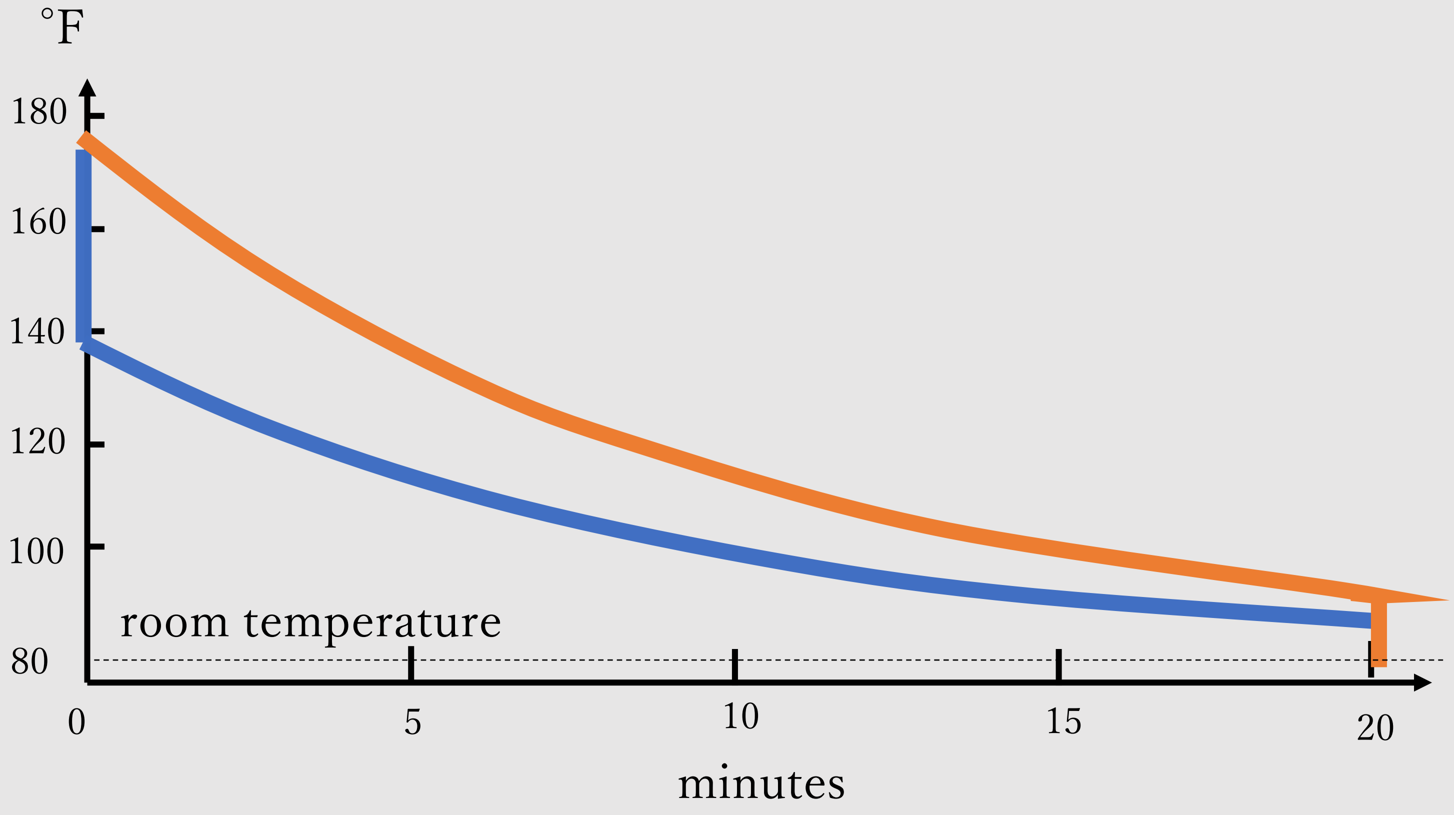


Why does the equation look like this?

$$\boldsymbol{T(T_N) = r\left(R + (T_0 - R)e^{-kT_N}\right) + (1 - r)C}$$

$$\frac{dT(t)}{dt} = -k(T(t) - R)$$

Comparison



Which Is cooler?

Milk first: $$T(T_N) = R + (rT_0 + (1-r)C - R)e^{-kT_N}$$

$$\frac{dT(t)}{dt} = -k(T(t) - R)$$

$$T(t) = R + (T_0 - R)e^{-kt}$$

Milk later: $$T(T_N) = r\left(R + (T_0 - R)e^{-kT_N}\right) + (1-r)C$$

Describe changes in phenomena

Back to Law of Cooling (Again!)

$$\frac{\mathrm{dT(t)}}{\mathrm{dt}} = -k(T(t) - R)$$

$$T(t) = \mathrm{R} + (\mathrm{T}_0 - R)e^{-kt}$$

What other phenomena are similar to this formula?

Exercise

$$1. Solve\ \frac{dy}{dx} = \sqrt{x+y}$$

$$2. Solve\ \frac{dy}{dx} - 2xy = x^2 + y^2$$

Exercise

1.solve $\frac{dy}{dx} = \sqrt{x+y}$

Let $u = x + y\,, du = dx + dy,$
make $dy = du - dx$ into the original equation,
so $\frac{du-dx}{dx} = \sqrt{u}$ or $\frac{du}{dx} = \sqrt{u}+1,$
after separate variable we can get $\frac{du}{\sqrt{u}+1} = dx,$
integrate the above formula will get $2\sqrt{u} - 2ln \mid \sqrt{u} + 1 \mid = x + c,$
make $u = x + y$ into above equation,
we can get ODE general solution $2\sqrt{x+y} - 2ln \mid \sqrt{x+y} + 1 \mid = x + c$

Exercise

2. $Solve\ \dfrac{dy}{dx} - 2xy = x^2 + y^2$

Rewritten as $\frac{dy}{dx} = (x+y)^2$,
let $u = x + y$, $du = dx + dy$, make $dy = du - dx$ into the original equation,
then $\frac{du-dx}{dx} = u^2$ or $\frac{du}{dx} = 1 + u^2$,
after separate variable we can get $\frac{du}{1+u^2} = dx$,
integrate the above formula will get $tan^{-1}u = x + c$,
make $u = x + y$ into above equation,
we can get ODE general solution $tan^{-1}(x+y) = x + c$

Exercise

3. *Solve the following first order* ordinary differential equations:

(a) $\frac{dy}{dx} = \frac{x+y}{x-y}$

(b) $\frac{dy}{dx} = \frac{2(x^3+xy)}{x^2-y}$

3. (a) $\frac{dy}{dx} = \frac{x+y}{x-y}$

Rewritten as $\frac{dy}{dx} = \frac{1+\frac{y}{x}}{1-\frac{y}{x}}$ (1) ,

let $u(x) = \frac{y}{x}$, then $y(x) = ux$, $\frac{dy}{dx} = x\frac{du}{dx} + u$ substitute into (1),

$x\frac{du}{dx} + u = \frac{1+u}{1-u}$, $x\frac{du}{dx} = -\frac{u+1}{u-1} - \mathrm{u} = \frac{-u-1-u(u-1)}{u-1} = -\frac{u^2+1}{u-1}$

after separate variable we can get $\frac{u-1}{u^2+1}du = -\frac{dx}{x}$,

integrate the above formula will get $\frac{1}{2}\ln(u^2+1) - tan^{-1}u = -\ln|x| + c$,

make $u = \frac{y}{x}$ into above equation,

we can get ODE general solution $\frac{1}{2}\ln(\frac{y^2}{x^2}+1) - tan^{-1}\frac{y}{x} = -\ln|x| + c$

3. (b) $\frac{dy}{dx} = \frac{2(x^3+xy)}{x^2-y}$

Rewritten as $\frac{dy}{dx} = \frac{2x(x^2+y)}{x^2-y}$ (1) ,
let $u = x^2$, then du $= 2xdx$ substitute into (1),
$\frac{dy}{du} = \frac{u+y}{u-y}$,
from (a) we can know that
ODE general solution is $\frac{1}{2}\ln(y^2 + u^2) - tan^{-1}\frac{y}{u} = c$,
$\frac{1}{2}\ln(y^2 + x^4) - tan^{-1}\frac{y}{x^2} = c$

Exercise

4. $Solve\ \sin y\frac{dy}{dx}+\sin x\cos y=\sin \mathrm{x}$

5. $Solve\ \frac{dy}{dx}=\frac{y-x}{y+x}$

4. $Solve\ \sin y \frac{dy}{dx} + \sin x \cos y = \sin \mathrm{x}$

Rewritten as $\sin y \frac{dy}{dx} = \sin x\,(1 - cos\ y)$,

separate variable $\frac{\sin y}{1-\cos y} dy = \sin x\, dx$,

after integrate we can get $\ln|\,1 - \cos y\,| = -\cos x + c_1 (1)$,

take the natural base index for both sides of (1) ,

$|\,1 - \cos y\,| = \pm c_2 e^{-\cos x} = c_3\ e^{-\cos x} (c_3\ = \pm c_2)$

or $\cos y = 1 - \ c_3\ e^{-\cos x}$

5. $Solve\ \frac{dy}{dx} = \frac{y-x}{y+x}$

Rewritten as $\frac{dy}{dx} = \frac{\frac{y}{x}-1}{\frac{y}{x}+1}$ (1) ,

let $u(x) = \frac{y}{x}, y(x) = ux,\ \frac{dy}{dx} = x\frac{du}{dx} + u$ substitute into (1),

$x\frac{du}{dx} = \frac{u-1}{u+1} - \mathrm{u} = \frac{u-1-u(u+1)}{u+1} = \frac{u^2+1}{u+1}$,

separate variable $\frac{u+1}{u^2+1}du = -\frac{dx}{x}$,

after integrate we can get $\frac{1}{2}\ln(u^2+1) + tan^{-1}\mathrm{u} = -\ln|x| + c$,

make $u = \frac{y}{x}$ into above equation,

we can get ODE general solution $\frac{1}{2}\ln\left(\frac{y^2}{x^2}+1\right) + tan^{-1}\frac{y}{x} = -\ln|x| + c$

6.$Find\ the\ general\ soultion\ of\ (x^2+xy+y^2)dx-x^2dy=0$

7. $\frac{dy}{dx}=\frac{e^x \sin y}{y-2e^x cosy}$

Exercise

6.$Find\ the\ general\ soultion\ of\ (x^2+xy+y^2)dx-x^2dy=0$

Rewritten as$x^2dx+xydx+y^2dx-x^2dxy=0$,

$(x^2+y^2)\mathrm{dx}+\mathrm{x}(\mathrm{ydx}-x\mathrm{dy})=0$,

so $(x^2+y^2)\mathrm{dx}+\mathrm{x}(x^2+y^2)d(tan^{-1}\frac{y}{x})=0$,

times $\frac{1}{x^2+y^2}$ on the both side,

we can get $\frac{\mathrm{dx}}{x}+d(tan^{-1}\frac{y}{x})$=0,

after integrate we can get ODE general solution $\ln|x|+tan^{-1}\frac{y}{x}=c$

Exercise

7. $\frac{dy}{dx} = \frac{e^x \sin y}{y - 2e^x cos y}$

Rewritten as $(e^x \sin y)\mathrm{dx} + (2e^x \cos y)\mathrm{dy} - \mathrm{ydy} = 0$,
$(\sin y)d(e^x) + 2e^x \mathrm{d}(\sin y) - ydy = 0$,

So $\frac{d\{(e^x)(\sin y)^2\}}{\sin y} - y\,dy = 0$, $d\{(e^x)(\sin y)^2\} - \mathrm{y} \sin \mathrm{y}\ \mathrm{dy} = 0$

after integrate we can get ODE general solution
$e^x \sin y^2 + y \cos y - \sin y = c$