1. Inverse Laplace Transform

傅立葉變換與傅立葉逆變換是：

$$f(t) \underset{\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega u}\,\mathrm{d}\omega}{\overset{\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,\mathrm{d}t}{\rightleftarrows}} F(\omega)$$

類似地，**拉普拉斯逆變換 (Inverse Laplace Transform)** 使用積分定義為：

$$f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\alpha-iT}^{\alpha+iT} F(s)e^{st}\,\mathrm{d}s$$

但這是一個在複平面上的線積分，計算起來非常複雜. 因此我們更偏向於湊出我們已知函數的拉普拉斯變換：

Question Find the inverse Laplace transform of following functions.

▷ $\mathcal{L}^{-1}\left\{\frac{1}{s-3}+\frac{12}{(s-1)^4}+\frac{s+1}{s^2+9}+5\right\}$.

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{1}{s-3}+\frac{12}{(s-1)^4}+\frac{s+1}{s^2+9}+5\right\}\\
=&\mathcal{L}^{-1}\left\{\frac{1}{s}\Big|_{s\to s-3}+\frac{3!}{s^4}\Big|_{s\to s-1}+\frac{s}{s^2+3^2}+\frac{1}{3}\frac{3}{s^2+3^2}+5\cdot 1\right\}\\
=&u(t)\cdot e^{3t}+t^3e^t+\cos 3t+\frac{1}{3}\sin 3t+5\delta(t)
\end{aligned}$$

► $\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s+2}+\frac{1}{s}e^{-s}+\frac{1}{s-2}e^{-3s}\right\}$.

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s+2}+\frac{1}{s}e^{-s}+\frac{1}{s-2}e^{-3s}\right\}\\
=&\mathcal{L}^{-1}\left\{\left(\frac{1}{s+1}-\frac{1}{s+2}\right)+\frac{1}{s}e^{-s}+\frac{1}{s-2}e^{-3s}\right\}\\
=&e^{-t}-e^{-2t}+u(t-1)+e^{2(t-3)}u(t-3)
\end{aligned}$$

1.1. Supplement：Partial Fraction Decomposition

Case 1：相異一次式

若分式具有相異的一次因子：

$$F(s) = \frac{N(s)}{(s-a)(s-b)} = \frac{A}{s-a} + \frac{B}{s-b}$$

- 求 A：把 A 的分母遮住，即計算 $A = (s-a)F(s)\Big|_{s=a}$
- 求 B：把 B 的分母遮住，即計算 $B = (s-b)F(s)\Big|_{s=b}$

Question Decompose the following fraction as partial fraction.

▷ $\dfrac{17x-53}{x^2-2x-15}$.

Let $\dfrac{17x-53}{x^2-2x-15} = \dfrac{17x-53}{(x-5)(x+3)} = \dfrac{A}{x-5} + \dfrac{B}{x+3}$,

then $17x-53 = A(x+3) + B(x-5)$.

Therefore：$\begin{cases} A = \dfrac{17x-53}{x+3}\Big|_{x=5} = 4 \\ B = \dfrac{17x-53}{x-5}\Big|_{x=-3} = 13 \end{cases}$

Hence, $\dfrac{17x-53}{x^2-2x-15} = \dfrac{4}{x-5} + \dfrac{13}{x+3}$.

▶ $\dfrac{-9x^2+4x+125}{(x-1)(x+3)(x+4)}$.

Let $\dfrac{-9x^2+4x+125}{(x-1)(x+3)(x+4)} = \dfrac{A}{x-1} + \dfrac{B}{x+3} + \dfrac{C}{x+4}$,

then $-9x^2+4x+125 = A(x+3)(x+4) + B(x-1)(x+4) + C(x-1)(x+3)$.

Therefore：$\begin{cases} A = \dfrac{-9x^2+4x+125}{(x+3)(x+4)}\Big|_{x=1} = 6 \\ B = \dfrac{-9x^2+4x+125}{(x-1)(x+4)}\Big|_{x=-3} = -8 \\ C = \dfrac{-9x^2+4x+125}{(x-1)(x+3)}\Big|_{x=-4} = -7 \end{cases}$

Hence, $\dfrac{-9x^2+4x+125}{(x-1)(x+3)(x+4)} = \dfrac{6}{x-1} - \dfrac{8}{x+3} - \dfrac{7}{x+4}$.

Case 2：不可約二次因子

當 $F(s)$ 的分母包含一般形式的不可約二次式 s^2+as+b 時：

$$F(s)=\frac{N(s)}{(s-p)(s^2+as+b)}=\frac{A}{s-p}+\frac{Bs+C}{s^2+as+b}$$

- 求 A：把 A 的分母遮住，即計算 $A=(s-p)F(s)\Big|_{s=p}$
- 求 B：將等式兩邊同乘 s，並取 $s\to\infty$，得到 $\lim\limits_{s\to\infty}[sF(s)]=A+B$
- 求 C：取簡單的 s 代入原恆等式

Question Decompose the following fraction as partial fraction.

▷ $\dfrac{3x^2+7x+28}{x(x^2+x+7)}$.

Let $\dfrac{3x^2+7x+28}{x(x^2+x+7)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+x+7}$,

then $3x^2+7x+28=A(x^2+x+7)+(Bx+C)x$.

$$\text{Therefore：}\begin{cases} A=\dfrac{3x^2+7x+28}{x^2+x+7}\Big|_{x=0}=4 \\ B:\lim\limits_{x\to\infty}[xF(x)]=3=A+B\implies B=-1 \\ C:\ \text{Set } x=1\implies \dfrac{38}{9}=4+\dfrac{-1+C}{9}\implies C=3 \end{cases}$$

Hence, $\dfrac{3x^2+7x+28}{x(x^2+x+7)}=\dfrac{4}{x}+\dfrac{-x+3}{x^2+x+7}$.

▶ $\dfrac{5x^2-2x+7}{(x-1)(x^2+4)}$.

Let $\dfrac{5x^2-2x+7}{(x-1)(x^2+4)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+4}$,

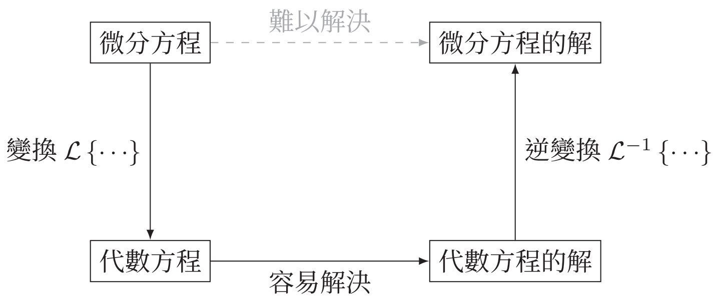
then $5x^2-2x+7=A(x^2+4)+(Bx+C)(x-1)$.

$$\text{Therefore：}\begin{cases} A=\dfrac{5x^2-2x+7}{x^2+4}\Big|_{x=1}=2 \\ B:\lim\limits_{x\to\infty}[xF(x)]=5=A+B\implies B=3 \\ C:\ \text{Set } x=0\implies \dfrac{7}{-4}=\dfrac{2}{-1}+\dfrac{C}{4}\implies C=1 \end{cases}$$

Hence, $\dfrac{5x^2-2x+7}{(x-1)(x^2+4)}=\dfrac{2}{x-1}+\dfrac{3x+1}{x^2+4}$.

2. Laplace Transform for solving DEs

正如在 Handout7 對於拉普拉斯變換特性的介紹，它能夠將在 t-domain 的 n 階微分轉化為在 s-domain 乘上 s^n、能夠將對於 t 的微分方程轉化為對於 s 的代數方程式. 因此，當瞭解拉普拉斯逆變換時，就能夠更容易的解決微分方程：



Question Solve the following initial value problems (IVP).

▷ $y'' - 3y' + 2y = 0, \quad y(0) = 1, y'(0) = 4.$

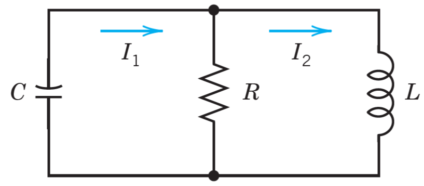
$$
\begin{aligned}
&\Rightarrow y'' - 3y' + 2y = 0\\
&\Rightarrow \mathcal{L}\{y'' - 3y' + 2y\} = \mathcal{L}\{0\}\\
&\Rightarrow (s^2Y(s) - sy(0) - y'(0)) - 3(sY(s) - y(0)) + 2Y(s) = 0\\
&\Rightarrow (s^2 - 3s + 2)Y(s) - s - 4 + 3 = 0\\
&\Rightarrow (s-1)(s-2)Y(s) = s + 1\\
&\Rightarrow Y(s) = \frac{s+1}{(s-1)(s-2)} = \frac{-2}{s-1} + \frac{3}{s-2}\\
&\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{-2}{s-1} + \frac{3}{s-2}\right\} = -2e^t + 3e^{2t}
\end{aligned}
$$

▶ $y'' + 4y = 8, \quad y(0) = 0, y'(0) = 6.$

$$
\begin{aligned}
&\Rightarrow y'' + 4y = 8\\
&\Rightarrow \mathcal{L}\{y'' + 4y\} = \mathcal{L}\{8\}\\
&\Rightarrow (s^2Y(s) - sy(0) - y'(0)) + 4Y(s) = \frac{8}{s}\\
&\Rightarrow (s^2 + 4)Y(s) - 6 = \frac{8}{s}\\
&\Rightarrow (s^2 + 4)Y(s) = \frac{8}{s} + 6 = \frac{6s+8}{s} = \frac{2}{s} - 2\frac{s}{s^2+4} + 3\frac{2}{s^2+4}\\
&\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s} - \frac{2s}{s^2+4} + \frac{6}{s^2+4}\right\} = 2 - 2\cos(2t) + 3\sin(2t)
\end{aligned}
$$

Question Solve the following IVP or BVP.

▷ **Circuit System.** Find the current I_1, I_2 in Fig. when $R = 1\Omega$, $L = 1.25\text{H}$, $C = 0.2\text{F}$ with initial current $I_1(0) = 1\text{A}$, $I_2(0) = 1\text{A}$.



$$\begin{cases} 1.25I_2' + 1(I_2 - I_1) = 0 \\ 1(I_1' - I_2') + \dfrac{1}{0.2}I_1 = 0 \end{cases}$$

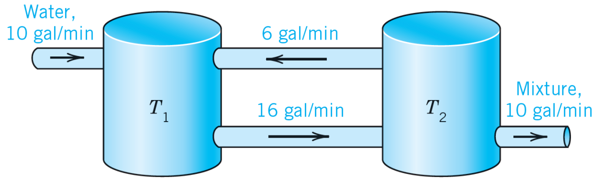
$$\Rightarrow \begin{cases} I_1' + 4.2I_1 + 0.8I_2 = 0, & I_1(0) = 1 \\ -0.8I_1 + I_2' + 0.8I_2 = 0, & I_2(0) = 1 \end{cases}$$

$$\Rightarrow \begin{cases} (s + 4.2)I_1 + 0.8I_2 = 1 \\ -0.8I_1 + (s + 0.8)I_2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} I_1(s) = \dfrac{s}{(s+1)(s+4)} = \dfrac{-1/3}{s+1} + \dfrac{4/3}{s+4} \\ I_2(s) = \dfrac{s+5}{(s+1)(s+4)} = \dfrac{4/3}{s+1} - \dfrac{1/3}{s+4} \end{cases}$$

$$\Rightarrow \begin{cases} I_1(t) = -\dfrac{1}{3}e^{-t} + \dfrac{4}{3}e^{-4t} \quad \text{(A)} \\ I_2(t) = \dfrac{4}{3}e^{-t} - \dfrac{1}{3}e^{-4t} \quad \text{(A)} \end{cases}$$

▶ **Tank Mixing problem.** Tank T_1 in Fig. initially contains 200 gal of water in which 160 lb of salt are dissolved. Tank T_2 initially contains 100 gal of pure water. Liquid is pumped through the system as indicated, and the mixtures are kept uniform by stirring. Find the amounts of salt $y_1(t)$ and $y_2(t)$ in T_1 and T_2, respectively.



$$\begin{cases} \dfrac{\mathrm{d}y_1}{\mathrm{d}t} = \left(\dfrac{6}{100}y_2\right) - \left(\dfrac{16}{200}y_1\right) \\ \dfrac{\mathrm{d}y_2}{\mathrm{d}t} = \left(\dfrac{16}{200}y_1\right) - \left(\dfrac{16}{100}y_2\right) \end{cases}$$

$$\Rightarrow \begin{cases} y_1' + 0.08y_1 - 0.06y_2 = 0, & y_1(0) = 160 \\ -0.08y_1 + y_2' + 0.16y_2 = 0, & y_2(0) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} (s + 0.08)Y_1 - 0.06Y_2 = 160 \\ -0.08Y_1 + (s + 0.16)Y_2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} Y_1(s) = \dfrac{160(s+0.16)}{(s+0.04)(s+0.2)} = \dfrac{120}{s+0.04} + \dfrac{40}{s+0.2} \\ Y_2(s) = \dfrac{12.8}{(s+0.04)(s+0.2)} = \dfrac{80}{s+0.04} - \dfrac{80}{s+0.2} \end{cases}$$

$$\Rightarrow \begin{cases} y_1(t) = 120e^{-0.04t} + 40e^{-0.2t} \quad \text{(lb)} \\ y_2(t) = 80e^{-0.04t} - 80e^{-0.2t} \quad \text{(lb)} \end{cases}$$

overview

- **級數與變換**
  - 傅立葉級數：$f(t) = c + \sum a_n \cos\dfrac{n\pi t}{l} + b_n \sin\dfrac{n\pi t}{l}$
  - 傅立葉變換：$F(\omega) = \displaystyle\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,\mathrm{d}t$
  - 拉普拉斯變換：$F(s) = \displaystyle\int_0^{\infty} f(t)e^{-st}\,\mathrm{d}t$

$f(t)$	$F(s)$	$f(t)$	$F(s)$
$\delta(t)$	1	$c_1 f(t) + c_2 g(t)$	$c_1 F(s) + c_2 G(s)$
$u(t)$	$\dfrac{1}{s}$	$f(t) * g(t)$	$F(s) \cdot G(s)$
t^n	$\dfrac{n!}{s^{n+1}}$	$e^{at} f(t)$	$F(s-a)$
e^{at}	$\dfrac{1}{s-a}$	$f(t+a)u(t+a)$	$e^{as} F(s)$
$\sin(at)$	$\dfrac{a}{s^2+a^2}$	$\dfrac{d^n}{dt^n} f(t)$	$s^n F(s) - s^{n-1} f(0) \cdots - f^{(n-1)}(0)$
$\cos(at)$	$\dfrac{s}{s^2+a^2}$	$t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n} F(s)$
$\sinh(at)$	$\dfrac{a}{s^2-a^2}$	$\displaystyle\int_0^t f(\tau)\,\mathrm{d}\tau$	$\dfrac{1}{s} F(s)$
$\cosh(at)$	$\dfrac{s}{s^2-a^2}$	$\dfrac{1}{t} f(t)$	$\displaystyle\int_s^{\infty} F(\xi)\,\mathrm{d}\xi$

- **微分方程求解**
  - 求解流程：微分方程式 $\xrightarrow{\mathcal{L}}$ 代數方程式 $\xrightarrow{\text{整理}}$ $Y(s)$ $\xrightarrow{\mathcal{L}^{-1}}$ 解 $y(t)$.
  - 部份分式展開：
    * 相異實根：$\frac{1}{(s-a)(s-b)} = \frac{A}{s-a} + \frac{B}{s-b}$ 對應 e^{at} 形式.
    * 重根：$\frac{1}{(s-a)^n}$ 對應 $t^{n-1}e^{at}$ 形式.
    * 不可約二次式：$\frac{Bs+C}{s^2+as+b}$ 需配方湊出 $\sin, \cos$ 形式.

Next Week：Final Exam 6/8-6/12.