13-7 Extreme Values and Saddle Points 鞍點

師大工教一

Definition Let $f(x,y)$ be defined on a region R containing the point (a,b).

1. $f(a,b)$ is a **local maximum** value of f if $f(a,b) \ge f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b).

局部

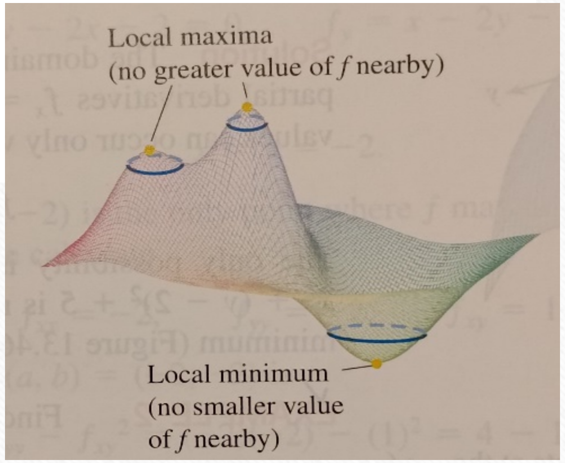
disk

若 $\exists r>0$ 使得

$f(x,y) \le f(a,b), \forall (x,y) \in D_r(a,b)$

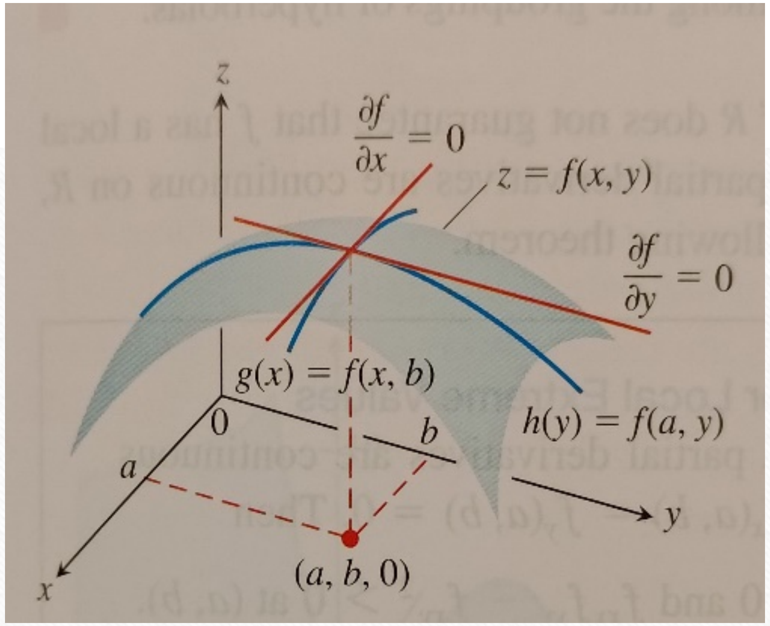
$= \{(x,y) \mid (x-a)^2+(y-b)^2 \le r^2\}$

2. $f(a,b)$ is a **local minimum** value of f if $f(a,b) \le f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b).



Theorem 10— First Derivative Theorem for Local Extreme Values

If $f(x,y)$ has a local maximum or minimum value at an interior point (a,b) of its domain and if the first derivatives exist there, then $f_x(a,b)=0$ and $f_y(a,b)=0$.



Definition An interior point of the domain of a function $f(x,y)$ where both f_x and f_y are zero or one or both of f_x and f_y do not exist is a **critical point** of f.

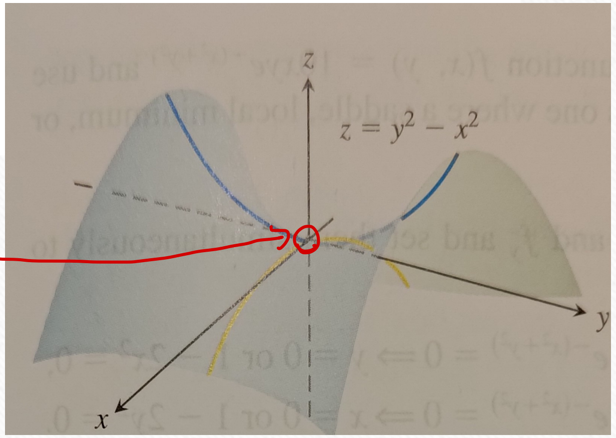
$f_x(a,b)=f_y(a,b)=0$ 或 $(f_x(a,b)$ $\overset{\text{one}}{\underset{\text{both}}{\text{or}}}$ $f_y(a,b)$ DNE$)$

$\Rightarrow (a,b)$ 是臨界點

Definition A differentiable function $f(x,y)$ has a **saddle point** at a critical point (a,b) if in every open disk centered at (a,b) there are domain points (x,y) where $f(x,y) > f(a,b)$ and domain points (x,y) where $f(x,y) < f(a,b)$. The corresponding point $(a,b,f(a,b))$ on the surface $z = f(x,y)$ is called a saddle point of the surface.

鞍點：1.臨界點 ↗鄰域
2.在任何$D_r(a.b)$
不是極大也不是極小

(a.b.f(a.b))



Ex1(p774) Find the local extreme values of $f(x,y)=x^2+y^2-4y+9$.

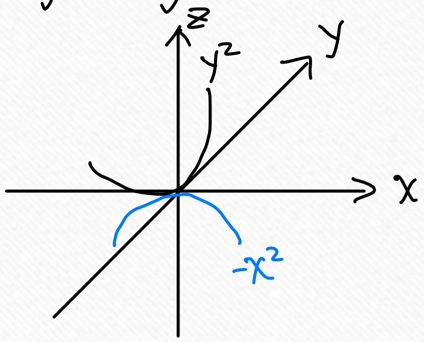
$f(x,y)=x^2+(y-2)^2+5$ $\Rightarrow f(0,2)=5$ is a local min. value

(f has local min. at $(0,2)$)

$\begin{cases} f_x=2x \\ f_y=2y-4 \end{cases} \xRightarrow{\text{Let}=0} \begin{cases} x=0 \\ y=2 \end{cases} \Rightarrow (0,2)$ is a critical point

Ex2(p774) Find the local extreme values (if any) of $f(x,y)=y^2-x^2$.

$\left.\begin{matrix} f_x=-2x \overset{\text{Let}}{=} 0 \\ f_y=2y \overset{\text{Let}}{=} 0 \end{matrix}\right. \Rightarrow \begin{cases} x=0 \\ y=0 \end{cases} \Rightarrow (0,0)$ is a critical point



$x=0 \Rightarrow f(x,y)=y^2$

$y=0 \Rightarrow f(x,y)=-x^2$

$\Rightarrow$ f has a saddle point at $(0,0)$

Theorem 11—Second Derivative Test for Local Extreme Values

Suppose that $f(x,y)$ and its first and second partial derivatives are

critical point

continuous throughout a disk centered at (a,b) and that

$f_x(a,b) = f_y(a,b) = 0$. Then   Let $D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$

i) f has a **local maximum** at (a,b) if $f_{xx} < 0$, $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b).

類似 $f(x)<0$ 有 local max.

ii) f has a **local minimum** at (a,b) if $f_{xx} > 0$, $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a,b).

iii) f has a **saddle point** at (a,b) if $f_{xx} f_{yy} - f_{xy}^2 < 0$ at (a,b).

iv) the test is inconclusive at (a,b) if $f_{xx} f_{yy} - f_{xy}^2 = 0$ at (a,b).

Note: The expression $f_{xx}f_{yy}-f_{xy}^{\ 2}$ is **the Hessian** of f.

$$f_{xx}f_{yy}-f_{xy}^{\ 2}=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

Ex3(p775) Find the local extreme values of $f(x,y)=xy-x^2-y^2-2x-2y+4$.

$f_x = y-2x-2$

$f_y = x-2y-2$

$f_{xx}=-2$

$f_{xy}=1$

$f_{yy}=-2$

Let $f_x=0$, $f_y=0$

$\begin{cases} -2x+y-2=0 \\ x-2y-2=0 \end{cases}$

$2x-4y-4=0$

$-3y=6$

$y=-2$

$x=-2$

$(-2,-2)$ is a critical point

$f(-2,-2)=\cancel{4}\cancel{-4}\cancel{-4}\cancel{+4}+4+4=8$ #

$D(a,b)=(-2)(-2)-1^2>0$ $\therefore f(x,y)$ has a local max. at $(-2,-2)$

$f_{xx}=-2<0$

Ex4(p775) Find the local extreme values of $f(x,y)=3y^2-2y^3-3x^2+6xy$.

$f_x=-6x+6y$

$f_y=6y-6y^2+6x$

$f_{xx}=-6$

$f_{xy}=6$

$f_{yy}=6-12y$

Let $f_x=0$, $f_y=0$ $\Rightarrow \begin{cases} x=y \\ 6x^2-12x=0 \end{cases} \Rightarrow x(x-2)=0$

$(0,0)$, $(2,2)$ are critical point

$D(0,0)=-6(6)-6^2<0$

$D(2,2)=-6(-18)-6^2>0$ $f_{xx}<0$

$\therefore f$ has a saddle point at $(0,0)$

f has a local max. at $(2,2)$

$f(2,2)=12-16\cancel{-12}\underset{12}{\cancel{+24}}=8$ #

Absolute Maxima and Minima on Closed Bounded Regions

To search the absolute extrema of a continuous function $f(x, y)$ on a closed and bounded region R.

1. *List the interior points of* R where f may have local extrema and evaluate f at these points. These are the critical points.
2. *List the boundary points of* R where f may have local extrema and evaluate f at these points.
3. *Look through the list.*

If $f(a,b) \geq f(x,y)$ ($\leq$), $\forall (x,y) \in \text{Domain}(f)$
then we say $f(a,b)$ is the absolute maximum (minimum) value of f

Extreme Value Thm.

If f is continuous on a closed and bounded set, then f attain its absolute maximum value & absolute minimum value.

Ex6(p777) Find the absolute maximum and absolute minimum values of $f(x, y) = 2 + 2x + 4y - x^2 - y^2$ on the triangular region in the first quadrant bounded by the line $x = 0$, $y = 0$, and $y = 9 - x$.

HW11-7

- **HW: 1,14,30**