1. n-th order linear NON-homogeneous ODEs with Constant Coefficients

1.1. Method of Undetermined Coefficients

對於二階或更高階的線性非齊次微分方程式 (NON-homogeneous ODEs)：

$$L[y] = y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = r(x), \quad r(x) \neq 0$$

待定係數法即是依照 $r(x)$ 的樣子，猜測特解的形式，並將其代回原 ODEs 解出常數：

$r(t)$ 的項	猜測 $y_p(x)$ 的形式
e^{ax}	Ce^{ax}
x^n	$C_n x^n + C_{n-1}x^{n-1} + \cdots + C_1 x + C_0$
$\cos\omega x,\ \sin\omega x$	$C_1\cos\omega x + C_2\sin\omega x$
$e^{ax}\cos\omega x,\ e^{ax}\sin\omega x$	$e^{ax}(C_1\cos\omega x + C_2\sin\omega x)$

在使用此方法時，必須遵循以下規則：

1. 若 $r(x)$ 為上表中左欄所述之函數，則選擇右欄對應的形式作為特解 y_p，並代入原式求出係數.
2. 若猜測的 y_p 中有任何一項正好是齊次 ODE 的解（即與 y_h 重複），則必須將該項乘以 x. 若該解對應到特徵方程式的重根，則需乘以 x^2，以此類推.
3. 若非齊次項 $r(x)$ 是由多個不同類型的函數相加而成，即 $r(x) = r_1(x) + r_2(x) + \cdots + r_m(x)$，則特解 y_p 亦為各個對應特解之和：

$$y_p = y_{p1} + y_{p2} + \cdots + y_{pm}$$

4. 若 $r(x)$ 是由多項式 $P_n(x)$、指數函數 e^{ax} 與三角函數 ($\sin\omega x$ 或 $\cos\omega x$) 相乘而成，則猜測的 y_p 必須包含該乘積所有可能的導數形式.

Question Find the general solution of following non-homogeneous equations.

▷ $y'' + 5y' + 4y = 10e^{-3x}$

齊次解 $y_h : \lambda^2 + 5\lambda + 4 = 0 \Rightarrow (\lambda+1)(\lambda+4) = 0 \Rightarrow \lambda = -1, -4$

$\Rightarrow y_h = C_1 e^{-x} + C_2 e^{-4x}$

令特解 $y_p = Ae^{-3x} \Rightarrow y_p' = -3Ae^{-3x}, y_p'' = 9Ae^{-3x}$

代回：$9Ae^{-3x} + 5(-3Ae^{-3x}) + 4(Ae^{-3x}) = 10e^{-3x}$

$\Rightarrow -2A = 10 \Rightarrow A = -5 \Rightarrow y_p = -5e^{-3x}$

$\Rightarrow y = C_1 e^{-x} + C_2 e^{-4x} - 5e^{-3x}$

▶ $y'' + 3y' + 2y = 12x^2$

齊次解 $y_h : \lambda^2 + 3\lambda + 2 = 0 \Rightarrow (\lambda+1)(\lambda+2) = 0 \Rightarrow \lambda = -1, -2$

$\Rightarrow y_h = C_1 e^{-x} + C_2 e^{-2x}$

令特解 $y_p = Ax^2 + Bx + C \Rightarrow y_p' = 2Ax + B, y_p'' = 2A$

代回：$2A + 3(2Ax + B) + 2(Ax^2 + Bx + C) = 12x^2$

$\Rightarrow 2Ax^2 + (6A + 2B)x + (2A + 3B + 2C) = 12x^2$

$\Rightarrow A = 6, B = -18, C = 21 \Rightarrow y_p = 6x^2 - 18x + 21$

$\Rightarrow y = C_1 e^{-x} + C_2 e^{-2x} + 6x^2 - 18x + 21$

▷ $y'' + 2y' = e^{-x}\cos x + x^2$

齊次解 $y_h : \lambda^2 + 2\lambda = 0 \Rightarrow \lambda(\lambda+2) = 0 \Rightarrow \lambda = 0, -2$

$\Rightarrow y_h = C_1 + C_2 e^{-2x}$

令特解的第一部分 $y_{p1} = e^{-x}(A\cos x + B\sin x)$

代回：$\Rightarrow -2A\cos x e^{-x} - 2B\sin x e^{-x} = e^{-x}\cos x$

$\Rightarrow A = -1/2, B = 0 \Rightarrow y_{p1} = -\frac{1}{2}e^{-x}\cos x$

令特解的第二部分 $y_{p2} = x(Dx^2 + Ex + F)$ (Note: 常數項與 y_h 重複)

$y_{p2}' = 3Dx^2 + 2Ex + F, \quad y_{p2}'' = 6Dx + 2E$

代回：$(6Dx + 2E) + 2(3Dx^2 + 2Ex + F) = x^2$

$\Rightarrow 6Dx^2 + (6D + 4E)x + (2E + 2F) = x^2$

$\Rightarrow D = 1/6, E = -1/4, F = 1/4$

$\Rightarrow y_p = -\frac{1}{2}e^{-x}\cos x + \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{4}x$

$\Rightarrow y = C_1 + C_2 e^{-2x} - \frac{1}{2}e^{-x}\cos x + \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{4}x$

▶ $y'' - 16y = 10e^{4x} + 30e^{x}$

齊次解 $y_h : \lambda^2 - 16 = 0 \Rightarrow \lambda = 4, -4 \Rightarrow y_h = C_1 e^{4x} + C_2 e^{-4x}$

令特解 $y_p = Axe^{4x} + Be^{x}$ (Note: e^{4x} 與齊次解重複)

$y_p' = A(1 + 4x)e^{4x} + Be^{x}, y_p'' = A(8 + 16x)e^{4x} + Be^{x}$

代回：$A(8 + 16x)e^{4x} + Be^{x} - 16(Axe^{4x} + Be^{x}) = 10e^{4x} + 30e^{x}$

$\Rightarrow 8Ae^{4x} - 15Be^{x} = 10e^{4x} + 30e^{x} \Rightarrow A = 5/4, B = -2$

$\Rightarrow y = C_1 e^{4x} + C_2 e^{-4x} + \frac{5}{4}xe^{4x} - 2e^{x}$

1.2. Method of Variation of Parameters

當非齊次項 $r(x)$ 較為複雜，不屬於待定係數法所能涵蓋的類型，如 $\tan x, \sec x, \ln x$ 等，我們可以使用更具一般性的**參數變換法**. 考慮二階線性非齊次方程式，必須先確保其為標準形式（即 y'' 的係數為 1）：

$$y'' + p(x)y' + q(x)y = r(x)$$

若已求得齊次解為 $y_h = c_1y_1(x) + c_2y_2(x)$，參數變換法的核心思想是令特解形式為：

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

為了求出兩個未知函數 u_1, u_2，我們需要建立兩個方程式. 根據推導，這兩個函數的一階導數必須滿足以下聯立方程組：

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = r(x) \end{cases} \iff \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ r(x) \end{bmatrix}$$

利用克拉瑪公式 (Cramer's Rule) 解上述方程組，我們可以得到：

$$u_1' = \frac{\det \begin{vmatrix} 0 & y_2 \\ r(x) & y_2' \end{vmatrix}}{\det \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}, \quad u_2' = \frac{\det \begin{vmatrix} y_1 & 0 \\ y_1' & r(x) \end{vmatrix}}{\det \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

積分求得 u_1, u_2：

$$u_1(x) = -\int \frac{y_2(x)r(x)}{W(x)}\, dx, \quad u_2(x) = \int \frac{y_1(x)r(x)}{W(x)}\, dx$$

Remark 在套用公式前，務必確認方程式已化為 $y'' + p(x)y' + q(x)y = r(x)$ 的形式. 若原方程式為 $ay'' + by' + cy = r(x)$，則公式中的 $r(x)$ 應取為 $r(x)/a$.

對於三階的情況，我們令 $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + u_3(x)y_3(x)$，並且 $u_1(x)$、$u_2(x)$、$u_3(x)$ 滿足：

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3 \\ y_1'' & y_2'' & y_3'' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ r(x) \end{bmatrix}$$

同樣使用克拉瑪公式解出 $u_1'(x)$、$u_2'(x)$、$u_3'(x)$，再進而得到特解. 對於高階情況同理.

Question Find the general solution of following non-homogeneous equations.

- $y'' + 9y = 12\sec 3x$

  齊次解 $y_h : \lambda^2 + 9 = 0 \Rightarrow \lambda = \pm 3i \Rightarrow y_h = C_1 \cos 3x + C_2 \sin 3x$

  令特解 $y_p = u_1 \cos 3x + u_2 \sin 3x$，其中 u_1, u_2 滿足：

$$\begin{bmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 12\sec 3x \end{bmatrix}$$

  由克拉瑪公式得：

$$u_1' = \frac{\det \begin{vmatrix} 0 & \sin 3x \\ 12\sec 3x & 3\cos 3x \end{vmatrix}}{\det \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}} = \frac{-12\sec 3x \sin 3x}{3} = -4\tan 3x \Rightarrow u_1 = \frac{4}{3}\ln|\cos 3x|$$

$$u_2' = \frac{\det \begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & 12\sec 3x \end{vmatrix}}{\det \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}} = \frac{12\cos 3x \sec 3x}{3} = 4 \Rightarrow u_2 = 4x$$

  $\Rightarrow y_p = \frac{4}{3}\cos 3x \ln|\cos 3x| + 4x\sin 3x$

  $\Rightarrow y = C_1 \cos 3x + C_2 \sin 3x + \frac{4}{3}\cos 3x \ln|\cos 3x| + 4x \sin 3x$

- $y'' + y = \tan x$

  齊次解 $y_h : \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \Rightarrow y_h = C_1 \cos x + C_2 \sin x$

  令特解 $y_p = u_1 \cos x + u_2 \sin x$，其中 u_1, u_2 滿足：

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \tan x \end{bmatrix}$$

  由克拉瑪公式得：

$$u_1' = \frac{\det \begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix}}{\det \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\sin x \tan x}{1} = \frac{-\sin^2 x}{\cos x} = \cos x - \sec x$$

  $\Rightarrow u_1 = \sin x - \ln|\sec x + \tan x|$

$$u_2' = \frac{\det \begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix}}{\det \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\cos x \tan x}{1} = \sin x \Rightarrow u_2 = -\cos x$$

  $\Rightarrow y_p = (\sin x - \ln|\sec x + \tan x|)\cos x + (-\cos x)\sin x = -\cos x \ln|\sec x + \tan x|$

  $\Rightarrow y = C_1 \cos x + C_2 \sin x - \cos x \ln|\sec x + \tan x|$

1.3. IVP & BVP for NON-homogeneous ODEs

Question Solve the following IVP or BVP.

▷ A series RLC circuit has $L = 1\text{H}, R = 3\Omega$, and $C = 0.5\text{F}$. A voltage source is applied such that the circuit is governed by $i'' + 3i' + 2i = 10e^{-3t}$. Given the initial state $i(0) = 1$ A and $i'(0) = 0$ A, find the current $i(t)$.

齊次解 $i_h : \lambda^2 + 3\lambda + 2 = 0 \Rightarrow (\lambda+1)(\lambda+2) = 0 \Rightarrow i_h = C_1e^{-t} + C_2e^{-2t}$

$\boxed{\text{Solving } i_p \text{ by variation of parameters method}}$

令特解 $i_p = u_1e^{-t} + u_2e^{-2t}$，其中 u_1, u_2 滿足：

$$\begin{bmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 10e^{-3t} \end{bmatrix}$$

由克拉瑪公式得：

$$u_1' = \frac{\det \begin{vmatrix} 0 & e^{-2t} \\ 10e^{-3t} & -2e^{-2t} \end{vmatrix}}{\det \begin{vmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{vmatrix}} = \frac{-10e^{-5t}}{-e^{-3t}} = 10e^{-2t} \Rightarrow u_1 = -5e^{-2t}$$

$$u_2' = \frac{\det \begin{vmatrix} e^{-t} & 0 \\ -e^{-t} & 10e^{-3t} \end{vmatrix}}{\det \begin{vmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{vmatrix}} = \frac{10e^{-4t}}{-e^{-3t}} = -10e^{-t} \Rightarrow u_2 = 10e^{-t}$$

那麼特解為：$i_p = (-5e^{-2t})e^{-t} + (10e^{-t})e^{-2t} = 5e^{-3t}$

$\boxed{\text{Solving } i_p \text{ by undetermined coefficients method}}$

令特解 $i_p = Ae^{-3t} \Rightarrow i_p' = -3Ae^{-3t}, i_p'' = 9Ae^{-3t}$

代回：$9Ae^{-3t} + 3(-3Ae^{-3t}) + 2(Ae^{-3t}) = 10e^{-3t}$

$\Rightarrow (9A - 9A + 2A)e^{-3t} = 10e^{-3t}$

$\Rightarrow 2A = 10 \Rightarrow A = 5$

$\Rightarrow i_p = 5e^{-3t}$

因此通解為：$i(t) = C_1e^{-t} + C_2e^{-2t} + 5e^{-3t}$

代入初始條件：

$i(0) = C_1 + C_2 + 5 = 1 \Rightarrow C_1 + C_2 = -4$

$i'(t) = -C_1e^{-t} - 2C_2e^{-2t} - 15e^{-3t} \Rightarrow i'(0) = -C_1 - 2C_2 - 15 = 0 \Rightarrow C_1 + 2C_2 = -15$

解得 $C_1 = 7, C_2 = -11 \Rightarrow i(t) = 7e^{-t} - 11e^{-2t} + 5e^{-3t}$

- Consider a series RLC circuit where the current $i(t)$ satisfies the differential equation $i'' + 3i' + 2i = 12\sin 2t$. Given the initial conditions $i(0) = 0$ A and $i'(0) = 0$ A/s, determine the current $i(t)$.

齊次解 $i_h : \lambda^2 + 3\lambda + 2 = 0 \Rightarrow (\lambda+1)(\lambda+2) = 0 \Rightarrow i_h = C_1 e^{-t} + C_2 e^{-2t}$

Solving i_p by variation of parameters method

令特解 $i_p = u_1 e^{-t} + u_2 e^{-2t}$，其中 u_1, u_2 滿足：

$$\begin{bmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 12\sin 2t \end{bmatrix}$$

由克拉瑪公式得：

$$u_1' = \frac{\det\begin{vmatrix} 0 & e^{-2t} \\ 12\sin 2t & -2e^{-2t} \end{vmatrix}}{\det\begin{vmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{vmatrix}} = \frac{-12e^{-2t}\sin 2t}{-e^{-3t}} = 12e^{t}\sin 2t$$

$\Rightarrow u_1 = \frac{12}{5}e^{t}(\sin 2t - 2\cos 2t)$

$$u_2' = \frac{\det\begin{vmatrix} e^{-t} & 0 \\ -e^{-t} & 12\sin 2t \end{vmatrix}}{\det\begin{vmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{vmatrix}} = \frac{12e^{-t}\sin 2t}{-e^{-3t}} = -12e^{2t}\sin 2t$$

$\Rightarrow u_2 = -3e^{2t}(\sin 2t - \cos 2t)$

那麼特解為：$i_p = \frac{12}{5}(\sin 2t - 2\cos 2t) - 3(\sin 2t - \cos 2t) = -\frac{3}{5}\sin 2t - \frac{9}{5}\cos 2t$

Solving i_p by undetermined coefficients method

令特解 $i_p = A\cos 2t + B\sin 2t$

$\Rightarrow i_p' = -2A\sin 2t + 2B\cos 2t, \quad i_p'' = -4A\cos 2t - 4B\sin 2t$

代回原方程式整理可得：$(-2A + 6B)\cos 2t + (-6A - 2B)\sin 2t = 12\sin 2t$

比較係數得：$A = -\frac{9}{5},\ B = -\frac{3}{5}$

$\Rightarrow i_p = -\frac{3}{5}\sin 2t - \frac{9}{5}\cos 2t$

因此通解為：$i(t) = C_1 e^{-t} + C_2 e^{-2t} - \frac{3}{5}\sin 2t - \frac{9}{5}\cos 2t$

代入邊界條件：

$i(0) = C_1 + C_2 - \frac{9}{5} = 0 \Rightarrow C_1 + C_2 = \frac{9}{5}$

$i'(t) = -C_1 e^{-t} - 2C_2 e^{-2t} - \frac{6}{5}\cos 2t + \frac{18}{5}\sin 2t$

$i'(0) = -C_1 - 2C_2 - \frac{6}{5} = 0 \Rightarrow C_1 + 2C_2 = -1.2$ 解得 $C_1 = 4.8,\ C_2 = -3$

$\Rightarrow i(t) = 4.8e^{-t} - 3e^{-2t} - \frac{3}{5}\sin 2t - \frac{9}{5}\cos 2t$

Next Week: DE system.