5-4 The Fundamental Theorem of Calculus

師大工教一

$$F'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$x - a = h$$
$$x = a + h$$

$$F'(x) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$F'(x)\, dh = f(a+dh) - f(a)$$

$$f(x)\, dx = F(a+dx) - F(a)$$

Theorem 3 The Mean Value Theorem for Definite Integrals

If f is continuous on $[a,b]$, then at some point c in $[a,b]$,
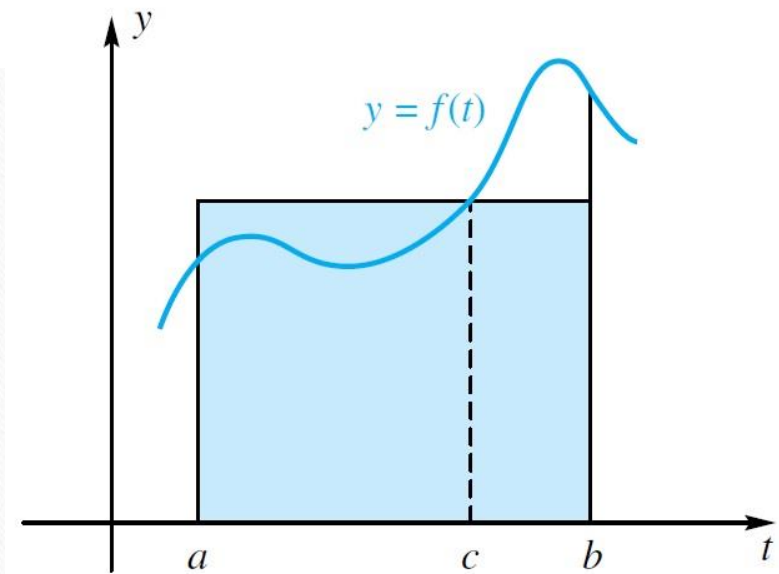
$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Figure 3

Theorem 4 F.T.O.C. PartI: If f is continuous on $[a,b]$, then

$F(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$ and differentiable on (a,b), and its

derivative is $f(x)$: $F'(x) = \dfrac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

Ex2(p340) Use the Fundamental Theorem to find $\dfrac{dy}{dx}$ if

(a) $y = \int_a^x (t^3 + 1)\,dt$ (b) $y = \int_x^5 3t\sin t\,dt$ (c) $y = \int_1^{x^2} \cos t\,dt$ (d) $y = \int_{1+3x^2}^4 \dfrac{1}{2+e^t}\,dt$

$\dfrac{dy}{dx} = x^3 + 1$

$\dfrac{dy}{dx} = -3x\sin x$

$\dfrac{dy}{dx} = \cos(x^2)\cdot 2x - \cos(1)\cdot 0$
$= 2x\cos x^2$

$\dfrac{dy}{dx} = -\dfrac{1}{2+e^{1+3x^2}}\cdot 6x$

(handwritten note, right side):
$F(x) = \int_a^x f(t)\,dt$. We want to find
$\dfrac{d}{dx}\left[\int_{h(x)}^{g(x)} f(t)\,dt\right]$
$\int_{h(x)}^{g(x)} f(t)\,dt = \int_a^{g(x)} f(t)\,dt - \int_a^{h(x)} f(t)\,dt$
$= F(g(x)) - F(h(x))$
$\dfrac{d}{dx}\left[\int_{h(x)}^{g(x)} f(t)\,dt\right] = F'(g(x))\cdot g'(x) - F'(h(x))\cdot h'(x)$
$= f(g(x))\cdot g'(x) - f(h(x))\cdot h'(x)$

Ex(#2,105 年考古題) 已知 $\int_0^{x^5} f\left(2t^2 - t + 1\right) dt = \ln x$，球 $f(2017)$ 之值。

$$f\left(2x^{10} - x^5 + 1\right) \cdot \left(5x^4\right) = \frac{1}{x}$$

$$x = 2 \text{ 代入 } \quad f\left(2 \cdot 2^{10} - 2^5 + 1\right)\left(5 \cdot 2^4\right) = \frac{1}{2}$$

$$f(2017) \cdot 80 = \frac{1}{2}$$

$$f(2017) = \frac{1}{160}$$

Theorem 4 F.T.O.C. Part II: If f is continuous over $[a,b]$, and F is any

antiderivative of f on $[a,b]$, then $\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$.

Ex3(p342) Find the following definite integrals:

(a) $\int_0^\pi \cos x\,dx$ (b) $\int_{-\frac{\pi}{4}}^0 \sec x \tan x\,dx$ (c) $\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx$

(d) $\int_0^1 \frac{1}{x+1}\,dx$ (e) $\int_0^1 \frac{1}{x^2+1}\,dx = \arctan x\Big|_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

(a) $\int_0^\pi \cos x\,dx$

$= \sin x\Big|_0^\pi$

$= 0$

(b) $= \sec x\Big|_{-\frac{\pi}{4}}^0$

$= 1 - \frac{1}{\cos(-45°)} = 1 - \sqrt{2}$

(c) $= x^{\frac{3}{2}} + 4x^{-1}\Big|_1^4$

$= 8 + 1 - 1 - 4 = 4$

(d) $= \ln|x+1|\Big|_0^1$

$= \ln 2 - \ln 1 = \ln 2$

Total Area

To find the area between the graph of $y = f(x)$ and the $x-$axis over the

interval $[a,b]$:

1. Subdivide $[a,b]$ at the zeros of f.

2. Integrate f over each subintervals.

3. Add the absolute values of the integrals.
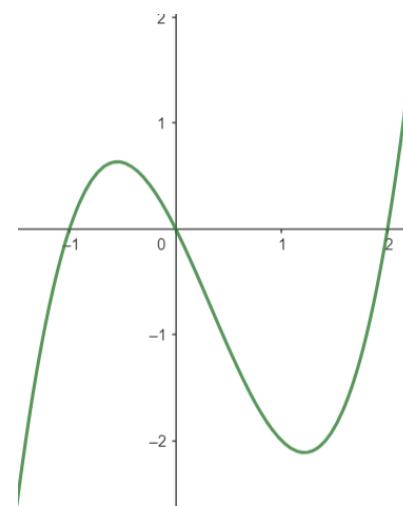
Ex8(p346) Find the area of the region and the $x-$axis and the graph of

$f(x) = x^3 - x^2 - 2x, -1 \le x \le 2$.

$\text{Total area} = \int_{-1}^{0} (x^3 - x^2 - 2x)dx - \int_{0}^{2} (x^3 - x^2 - 2x)dx$

$= \left(\frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2\Big|_{-1}^{0}\right) - \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2\Big|_{0}^{2}\right) = -\left(\frac{1}{4} + \frac{1}{3} - 1\right) - 4 + \frac{8}{3} + 4$

$= \frac{-1}{4} - \frac{1}{3} + 1 + \frac{8}{3} = -\frac{1}{4} + \frac{7}{3} + 1 = \frac{37}{12}$

HW5-4

- **HW:1,6,7,10,17,22,29,39,43,47,48,83.**