凹向性

4-4 Concavity and Curve Sketching

師大工教一

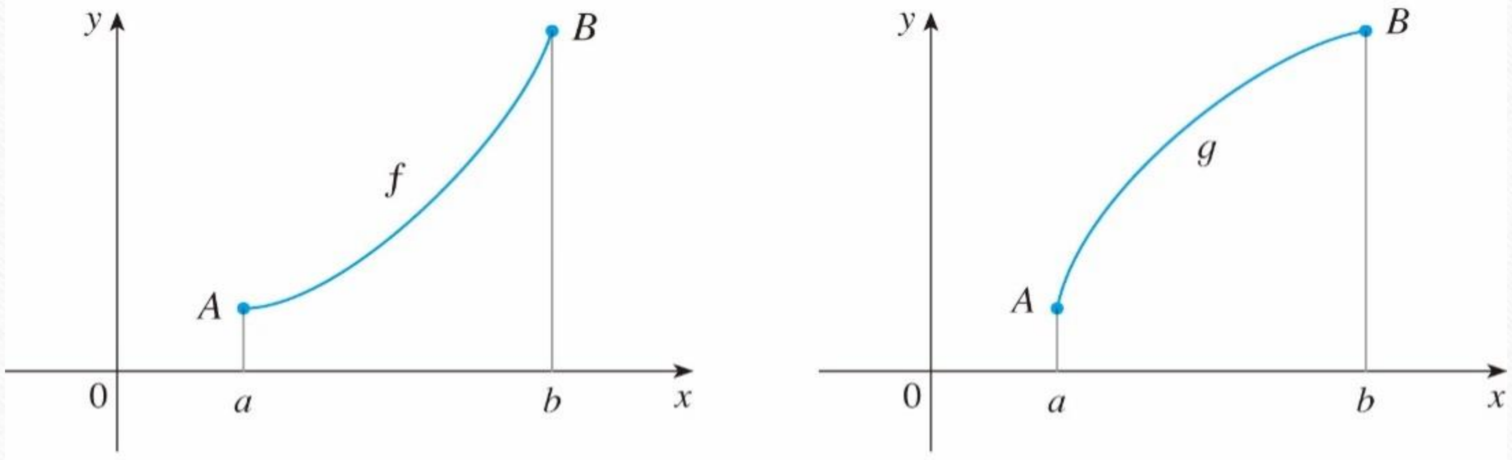
Concavity

Definition The graph of a differentiable function $y = f(x)$ is

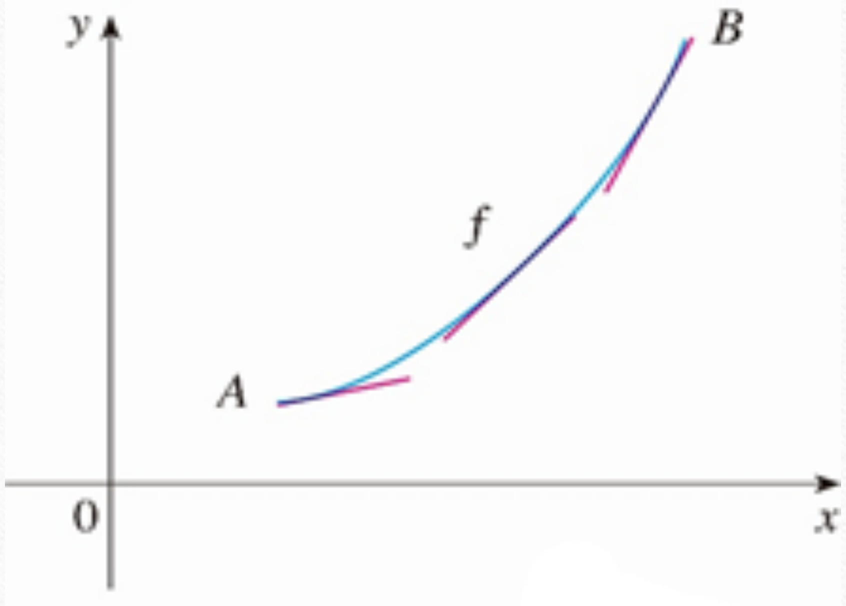
(凹向上)

(a) **concave up** (upward) on an interval I if f' is increasing on I.

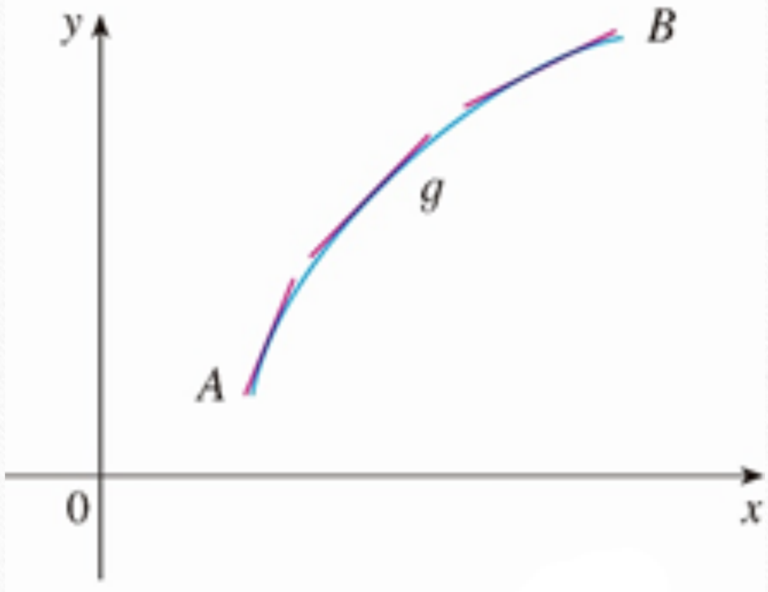
(b) **concave down** (downward) on an interval I if f' is decreasing on I.



(a) (b)



(a) Concave up



(b) Concave down

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

Ex1(p252) Discuss the concavity of the functions (a) $y = x^3$ (b) $y = x^2$

(a) $y' = 3x^2$

$y'' = 6x \overset{\text{let}}{=} 0$

$x = 0$

x		0	
$f''(x)$	$-$	0	$+$

concave up: $(0, \infty)$

concave down: $(-\infty, 0)$

(b) $y' = 2x$

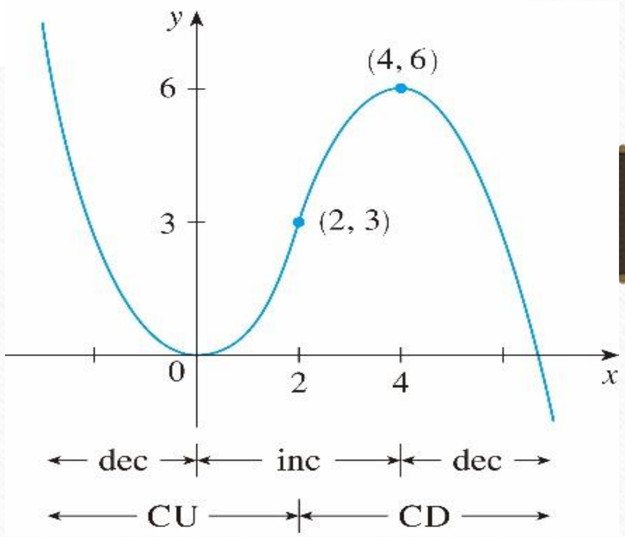
$y'' = 2$

Concave up: $(-\infty, \infty)$

concave down: None

Point of Inflection 反曲點 Inflection point

Definition A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.



At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

Ex3 Determine the concavity and find the inflection points of the function

$f(x) = x^3 - 3x^2 + 2$.

$f'(x) = 3x^2 - 6x$

$f''(x) = 6x - 6 \overset{\text{let}}{=} 0$

$x = 1$

x		1	
$f''(x)$	$-$		$+$

concave up: $(1, \infty)$

concave dowe: $(-\infty, 1)$

$(1, 0)$ is an inflection point.

Theorem 5—**Second Derivative Test for Local Extrema**

Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.

2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

HW4-4

- **HW: 11,17,35,50,52,58,61**

(110) 8. Given a function $f(x) = x^4 - 2x^2$ on the interval $[-2, 3]$.

(a) Find open intervals where f is increasing or decreasing. (6 pts)

(b) Find the relative extrema of f. (10 pts)

(c) Determine open intervals where f is concave upward or downward. (6 pts)

(a) $f'(x) = 4x^3 - 4x = 4x(x^2-1) = 4x(x+1)(x-1) \overset{\text{Let}}{=} 0$

$x = 0, -1, 1$ (critical point)

x		-1		0		1	
$f'(x)$	$-$		$+$		$-$		$+$

$\nearrow$: $(1, \infty), (-1, 0)$

$\searrow$: $(-\infty, -1), (0, 1)$

(b) $f(-1) = f(1) = -1$ is a local minimum value.

$f(0) = 0$ is a local maximum value.

(c) $f''(x) = 12x^2 - 4 \overset{\text{Let}}{=} 0$

$4(3x^2 - 1) = 0$

$4(\sqrt{3}x + 1)(\sqrt{3}x - 1) = 0$

$x = \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$

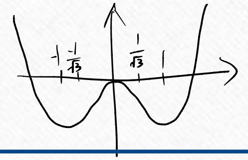
x		$\frac{-1}{\sqrt{3}}$		$\frac{1}{\sqrt{3}}$	
$f''(x)$	$+$		$-$		$+$

concave up : $(-\infty, \frac{-1}{\sqrt{3}}), (\frac{1}{\sqrt{3}}, \infty)$

concave down : $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$

(d) graphic

x		-1		$\frac{-1}{\sqrt{3}}$		0		$\frac{1}{\sqrt{3}}$		1	
$f'(x)$	$-$		$+$		$+$		$-$		$-$		$+$
$f''(x)$	$+$		$+$		$-$		$-$		$+$		$+$



Ex(108 年考古題#5) Consider the function $f(x)=x^{\frac{2}{3}}\left(x^2-1\right)$. $= x^{\frac{8}{3}} - x^{\frac{2}{3}}$

(a) Find the open intervals on which f is increasing(遞增) or decreasing(遞減).

(b) Find the open intervals on which the graph of f is concave upward(凹口向上) or concave downward(凹口向下).

(c) Locate the point of inflection(反曲點) if it exists.

(a) $f'(x) = \frac{8}{3}x^{\frac{5}{3}} - \frac{2}{3}x^{-\frac{1}{3}} \overset{\text{Let}}{=} 0$

$\frac{8}{3} \times \sqrt[3]{x^5} - \frac{2}{3} \times \frac{1}{\sqrt[3]{x}} = 0$

$\frac{8\sqrt[3]{x^5}}{3} - \frac{2}{3\sqrt[3]{x}} = 0$

$\frac{8\sqrt[3]{x^6}-2}{3\sqrt[3]{x}} = \frac{8x^2-2}{3\sqrt[3]{x}} = \frac{2(4x^2-1)}{3\sqrt[3]{x}} = \frac{2(2x+1)(2x-1)}{3\sqrt[3]{x}} = 0$

$x = -\frac{1}{2}, \frac{1}{2}, 0$

x		$-\frac{1}{2}$		0		$\frac{1}{2}$	
$f'(x)$	$-$		$+$		$-$		$+$

$\nearrow: (\frac{1}{2}, \infty), (-\frac{1}{2}, 0)$

$\searrow: (-\infty, -\frac{1}{2}), (0, \frac{1}{2})$

(b) $f''(x) = \frac{40}{9}x^{\frac{2}{3}} + \frac{2}{9}x^{-\frac{4}{3}}$

$= \frac{40}{9}\sqrt[3]{x^2} + \frac{2}{9} \cdot \frac{1}{\sqrt[3]{x^4}}$

$= \frac{40\sqrt[3]{x^2} \cdot \sqrt[3]{x^4} + 2}{9\sqrt[3]{x^4}} = \frac{40x^2+2}{9\sqrt[3]{x^4}}$

x		0	
$f''(x)$	$-$		$+$

concave up: $(-\infty, 0), (0, \infty)$

concave down: None

(c) No inflection pt.