$f(x) = 5e^x - e^{2x}$

$f'(x) = 5e^x - 2e^{2x}$ +3 *

$= e^x(5-2e^x)$

$[-1, 2]$

$f'(-1) = 5e^{-1} - 2e^{-2}$

$f'(2) = 5e^2 - 2e^4$

In close interval $[-1,2]$,
the absolute Max value is $5e^{-1}-2e^{-2}$
and the absolute min value is $5e^2-2e^4$

We see that $x = \ln\frac{5}{2}$ is the only critical number in $(-1,2)$

| $x$ | $-1$ | | $\ln\frac{5}{2}$ | | $2$ |
|---|---|---|---|---|---|
| $f'(x)$ | + | + | | − | − |
| I/D | ↗ | ↗ | | ↘ | ↘ |

∴ $f(\ln\frac{5}{2}) = \frac{25}{4}$ is a relative maximum and hence the absolute maximum on $[-1,2]$.

∵ $f(-1) = 5e^{-1} - 2e^{-2}$

$f(2) = 5e^2 - 2e^4$

∴ absolute minimum on $[-1,2]$ is $5e^2 - 2e^4$

2)

(a)

$$\lim_{x\to\infty} \frac{5-2x^{\frac{3}{2}}}{3x^2-4} = \underline{0}$$

$$\lim_{x\to\infty} \frac{\frac{5-2x^{\frac{3}{2}}}{x^2}}{\frac{3x^2-4}{x^2}} = \lim_{x\to\infty} \frac{\frac{5}{x^2} - \frac{2}{x^{\frac{1}{2}}}}{3-\frac{4}{x^2}} = 0$$

(b)

$$\lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4} \times \frac{\sqrt{x+5}+3}{\sqrt{x+5}+3} = \lim_{x\to 4} \frac{(x-4)}{(x-4)(\sqrt{x+5}+3)}$$

$$= \frac{1}{\sqrt{4+5}+3} = \underline{\frac{1}{6}}$$

~~(c)~~

$$\lim_{\theta\to 0} \theta^2 \cos\left(\frac{1}{\theta}\right)$$

∵ when $\theta\to 0$, $\frac{1}{\theta}\to\infty$

∴ $\cos\frac{1}{\theta}$ 在 $-1, 1$ 來回振動

故極限值不存在

since $-1 \le \cos\frac{1}{\theta} \le 1$.

則 $-\theta^2 \le \theta^2\cos\frac{1}{\theta} \le \theta^2$ for $\theta \ne 0$.

又 $\lim_{\theta\to 0}(-\theta^2) = 0 = \lim_{\theta\to 0}\theta^2$

∴ by Squeeze Theorem

$$\underline{\lim_{\theta\to 0}\theta^2\cos\left(\frac{1}{\theta}\right) = 0}$$

(d)

$$\lim_{t\to 0} \frac{1-\frac{1}{e^t}}{e^t-1}$$

令 $e^t = k$,

$$\text{則式} = \lim_{k\to 1} \frac{1-\frac{1}{k}}{k-1}$$

$$= \lim_{k\to 1} \frac{\frac{k-1}{k}}{k-1} = \lim_{k\to 1} \frac{\cancel{k-1}}{k} \times \frac{1}{\cancel{k-1}}$$

$$= \lim_{k\to 1} \frac{1}{k} = \lim_{t\to 0} e^{-t} = \underline{1}$$

3.1

(a) $f(x) = \frac{x^2+2}{x^2-2}$

$f'(x) = \frac{(x^2-2)(2x)-(x^2+2)(2x)}{(x^2-2)^2}$

$= \frac{2x(-4)}{(x^2-2)^2} = \frac{-8x}{(x^2-2)^2}$

∴ the critical number is $0, \sqrt{2}, -\sqrt{2}$

| $x$ | | $-\sqrt{2}$ | | $0$ | | $\sqrt{2}$ | |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | + | | + | | − | | − |
| I/D | ↗ | | ↗ | | ↘ | | ↘ |

(−2, −1.4, 1.4)

$f(x)$ is increasing on $(\infty, -\sqrt{2}), (-\sqrt{2}, 0)$

$f(x)$ is decreasing on $(0, \sqrt{2}), (\sqrt{2}, \infty)$

(b). $f(x)$ has a relative max at $(0, -1)$

(c). $f''(x) = \frac{(x^2-2)^2(-8)+8x\cdot 2(x^2-2)\cdot 2x}{(x^2-2)^4}$

$= \frac{-8(x^2-2)^2+32x^2(x^2-2)}{(x^2-2)^4} = \frac{-8(x^2-2)[(x^2-2)-4x^2]}{(x^2-2)^4} = \frac{-8(-3x^2-2)}{(x^2-2)^3}$

$= \frac{8(3x^2+2)}{(x^2-2)^3}$

the critical number is $\pm\sqrt{2}$,

| $x$ | | $-\sqrt{2}$ | | $\sqrt{2}$ | |
|---|---|---|---|---|---|
| $f''(x)$ | + | | − | | + |
| U/D | U | | D | | U |

(−1.4, 1.4)

$f(x)$ concave upward on $(-\infty, -\sqrt{2}), (\sqrt{2}, \infty)$

$f(x)$ concave downward on $(-\sqrt{2}, \sqrt{2})$

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反曲點 → $f''(x)=0$

$(\sqrt{2}, 0)$

$(-\sqrt{2}, 0)$ #

∵ $x=\pm\sqrt{2}$

但代回 $f(x)$ 時，

$f(\sqrt{2})$, $f(-\sqrt{2})$ 均不存在

故沒有反曲點

(4)

$$f(x)=\frac{x^2+2}{x^2-2}=\frac{x^2+2}{(x+\sqrt{2})(x-\sqrt{2})}$$

$$\because \lim_{x\to\infty}\frac{x^2+2}{x^2-2}=1$$

$$\lim_{x\to-\infty}\frac{x^2+2}{x^2-2}=1$$

∴ there is a horizotal asymptotes $y=1$

(5)

$$f(x)=\frac{x^2+2}{(x+\sqrt{2})(x-\sqrt{2})}$$

$$\because \lim_{x\to\sqrt{2}^+}\frac{x^2+2}{x^2-2}=0 \qquad \lim_{x\to-\sqrt{2}^+}\frac{x^2+2}{x^2-2}=0$$

and

$$\lim_{x\to\sqrt{2}^-}\frac{x^2+2}{x-2}=0 \qquad \lim_{x\to-\sqrt{2}^-}\frac{x^2+2}{x^2-2}=0$$

∴ there is two veltical asymptotes

$x=\sqrt{2}$

$x=-\sqrt{2}$

4.

$$f(x)=\begin{cases}\frac{\sin 2x}{3x} & x<0\\ \frac{1}{6}a-7x & x\geq 0\end{cases}$$

因為 連續代表左極限值=右極限值

$$\lim_{x\to0^+}\frac{1}{6}a-7x=\lim_{x\to0^-}\frac{\sin 2x}{3x}$$

$$\frac{1}{6}a=\frac{2}{3}$$

$$\therefore a=\frac{2}{3}\times 6$$

$$=4$$ #

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5, <pf.> $1^\circ$ 令 $f(x)=2x^5+7x-1$

∵ $f(x)$ 為一連續函數

可找到 $f(0)=-1$

$f(1)=8$

By I.V.T. 必存在 $c\in(0,1)$

使得 $f(c)=0$

$2^\circ$ 假設 $f(x)$ 有 $x_1, x_2$ 的實數解

則 $f(x_1)=f(x_2)=0$，又 $f(x)$ 在

$(x_1, x_2)$ 均可微，By Rolle's Theorem

必存在 $c\in(x_1,x_2)$，使得 $f'(c)=0$

但 $f'(x)=10x^4+7$ 為恆正，故不可能

有 $f(x)=0$，可得知與 Rolle's Theorem

矛盾，故 $f(x)$ 恰有一實數解

6,

$$x^2+xy+y^2=4$$

find $\frac{dy}{dx}$

$$\Rightarrow 2x+y+x\frac{dy}{dx}+2y\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}(x+2y)=-2x-y$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(2x+y)}{x+2y}$$ 即為斜率

代入 $(2,0)$，$m=\frac{-4}{2}=-2$

$y=-2x+4$ ※

$$g(x)=\begin{cases} x\sin\left(\frac{1}{x}\right), & x\neq 0 \\ 0, & x=0\end{cases}$$

可微分→極限值=函數值

(a)

let $\frac{1}{x}=t$

$$\lim_{x\to 0^+} x\cdot\sin\frac{1}{x}=\lim_{t\to\infty}\frac{\sin t}{t}=\infty$$

$$\lim_{x\to 0^-} x\cdot\sin\frac{1}{x}=\lim_{t\to\infty}\frac{\sin t}{t}=\infty$$

$\because \lim_{x\to 0} g(x)\neq g(0)$

$\therefore$ g is not differentiable at $x=0$ → g isn't differentiable at $x=0$

(a),

$$g'(0)=\lim_{x\to 0}\frac{g(x)-g(0)}{x-0}$$

$=\lim_{x\to 0}\sin\frac{1}{x}$ doesn't exist

$$\because \sin\frac{1}{x}=\begin{cases} 0 & \text{if } x=\frac{1}{n\pi}\ n\in\mathbb{Z}\setminus\{0\} \\ 1 & \text{if } x=\frac{1}{\frac{\pi}{2}+2n\pi}\ n\in\mathbb{Z}\end{cases}$$

(b).

$$g(x)=x\cdot\sin\frac{1}{x}$$

$$g'(x)=1\cdot\sin\frac{1}{x}+x\cdot\cos\frac{1}{x}\cdot(-1)x^{-2}$$ ☆

$\forall x\in\mathbb{R}$ 但 $x\neq 0$.

8, (a)

$$y=\ln(4-x^2)^{\frac{1}{2}}$$

$$dy=\left[\frac{1}{\sqrt{4-x^2}}\cdot\frac{1}{2}\cdot(4-x^2)^{-\frac{1}{2}}\cdot(-2x)\right]\cdot dx$$ ☆

(b).

$$y=\tan^{-1}(x-2)$$

$$dy=\left[\frac{1}{1+(x-2)^2}\cdot 1\right]\cdot dx$$

仍需化簡嗎?

☆ 9 check

證明 $\forall\, \varepsilon>0,\ \exists\, \delta=4\varepsilon$

s.t. $|x-(-2)|<\delta=4\varepsilon$

$|(4x+5)-(-3)|<\varepsilon$

$|4x+8|<\varepsilon$

$4|x+2|<\varepsilon$

$\therefore 4\delta<\varepsilon$

$\delta<\frac{\varepsilon}{4}$

$\varepsilon$ $\delta$ > $|f(x)-L|<\varepsilon$

$4|\delta|<\varepsilon$

Given $\varepsilon>0$, choose $\boxed{\delta=\frac{\varepsilon}{4}}$.

s.t. $|x-(-2)|<\delta$

then $|4x+5-(-3)|=|4x+8|$

$=4|x+2|<4\delta=\varepsilon$

$\forall x\in(-2-\delta,-2+\delta)$

$\therefore \lim_{x\to -2}(4x+5)=-3$ ☆

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