定積分

5-3 The Definite Integrals

師大工教一

Definition of the Definite Integrals

Definition Let $f(x)$ be a function defined on a close interval $[a,b]$. We say that a number J is the **definite integral of** f **over** $[a,b]$ and that J is the limit of the Riemann sum $\sum_{k=1}^{n} f(c_k)\Delta x_k$ if the following condition is satisfied:

Given any number $\varepsilon > 0$, there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, x_2, \cdots, x_{n-1}, x_n\}$ of $[a,b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have $\left|\sum_{k=1}^{n} f(c_k)\Delta x_k - J\right| < \varepsilon$.

If the limit $\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$ exists, we say that the definite integral exists and we write $J = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$.

When the definite integral exists, we say that the Riemann sums of f on $[a,b]$ converge to the definite integral $J = \int_a^b f(x)\,dx$ and that f is **integrable** over $[a,b]$.

$\lim_{\|\mathcal{P}\| \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$ exists,

f is integrable over $[a,b]$

$\int_a^b f(x)\,dx = \lim_{\|\mathcal{P}\|} \sum_{k=1}^{n} f(c_k)\Delta x_k$

可積分

Consider the case that the subintervals all have the equal width $\Delta x = \dfrac{b-a}{n}$. If the definite integral exists, then

$$\boxed{J = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right).}$$

$\mathcal{P}$是$[a,b]$ n等分

The Definite Integral as a Limit of Riemann Sums with Equal-Width Subintervals

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right)$$

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Ex(102 考古題 #8 加分題) 求 $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sqrt{\left(1+\frac{k}{n}\right)}$. $= \lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\left(1+\frac{k}{n}\right)}\cdot\left(\frac{1}{n}\right)$

(將里曼和(Riemann sum)之極限表成定積分，再求出定積分的值。)

$P = \left\{1, 1+\frac{1}{n}, 1+\frac{2}{n}, \cdots, 1+\frac{n}{n}=2\right\}$, n 等分

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$R(f, P) = \sum_{k=1}^{n} f\left(1+\frac{k}{n}\right)\cdot\frac{1}{n}$

$= \sum_{k=1}^{n}\sqrt{\left(1+\frac{k}{n}\right)}\cdot\frac{1}{n}$

$\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\left(1+\frac{k}{n}\right)}\cdot\frac{1}{n} = \int_1^2\sqrt{x}\,dx = \frac{4\sqrt{2}-2}{3}$

Note: $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$: dummy variable.

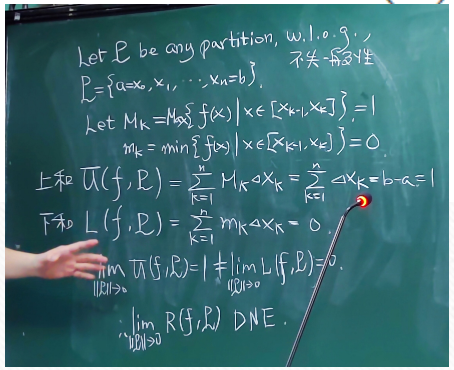
Theorem 1-Integrability of Continuous Functions

可積性定理

＊ 只要連續，就可積分

If a function f is continuous over the interval $[a,b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)\,dx$ exists and f is integrable over $[a,b]$.

Ex1(p328) Show that the function $f(x)=\begin{cases}1, & \text{if } x \text{ is rational}\\ 0, & \text{if } x \text{ is irrational}\end{cases}$ is not Riemann integrable over $[0,1]$.



Properties of Definite Integrals

Theorem 2 When f, g are integrable over the interval $[a, b]$, the definite integral satisfies the following rules.

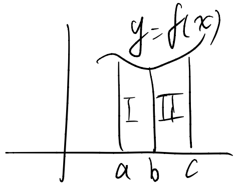
1. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

2. $\int_a^a f(x)\,dx = 0$

3. $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

4. $\int_a^b \left(f(x) \pm g(x)\right)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$



$\int_a^c f(x)\,dx = \text{area}\,(\mathrm{I} + \mathrm{II})$

$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \text{area}\,\mathrm{I} + \text{area}\,\mathrm{II}$

6. If f has maximum value $\max f$ and minimum value $\min f$ on $[a,b]$, then

$$(\min f)\cdot(b-a) \le \int_a^b f(x)\,dx \le (\max f)\cdot(b-a)$$

7. If $f(x) \ge g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.

If $f(x) \ge 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \ge 0$.

Ex2(p331) Suppose that $\int_{-1}^{1} f(x)\,dx = 5$, $\int_{1}^{4} f(x)\,dx = -2$, and $\int_{-1}^{1} h(x)\,dx = 7$,

find 1. $\int_{4}^{1} f(x)\,dx$ 2. $\int_{-1}^{1} [2f(x) + 3h(x)]\,dx$ 3. $\int_{-1}^{4} f(x)\,dx$

$= 2$ $= 31$ $= 3$

Area Under the Graph of a Nonnegative Function

Definition If $y = f(x)$ is nonnegative and integrable over a closed interval $[a,b]$, then the area under the curve $y = f(x)$ over $[a,b]$ is the integral of f from a to b, $A = \int_a^b f(x)\,dx$.

Ex4(p331) Compute $\int_0^b x\,dx$ and find the area A under $y=x$ over the interval $[0,b]$.

$\int_0^b x\,dx = \frac{1}{2} \times b \times b = \frac{b^2}{2}$

Extension: 1. $\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}$.

2. $\int_a^b c\,dx = c(b-a)$.

3. $\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}$.

Average Value of a Continuous Function Revisited

Definition If f is integrable on $[a,b]$, then its average value on $[a,b]$, which is also called its mean, is $av(f)=\frac{1}{b-a}\int_a^b f(x)\,dx$.

Ex5(p334) Find the average value of $f(x)=\sqrt{4-x^2}$ on $[-2,2]$.

HW5-3

- **HW:3,6,10,16,18,41,49,55,61,65,69**

102.1 (4) $\int_0^{2/3} \sqrt{4-9x^2}\, dx$ → 橢圓

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Area $=\pi ab$

$y=\sqrt{4-9x^2}$

$y^2+9x^2=4$

$\frac{x^2}{\frac{4}{9}}+\frac{y^2}{4}=1$

$a=\frac{2}{3}$

$b=2$

$\frac{1}{4}\int_0^{\frac{2}{3}}\sqrt{4-9x^2}\,dx = \frac{1}{4}\left(\pi\cdot\frac{2}{3}\cdot 2\right)=\frac{\pi}{3}$