13-6 Tangent Plane and Differentials
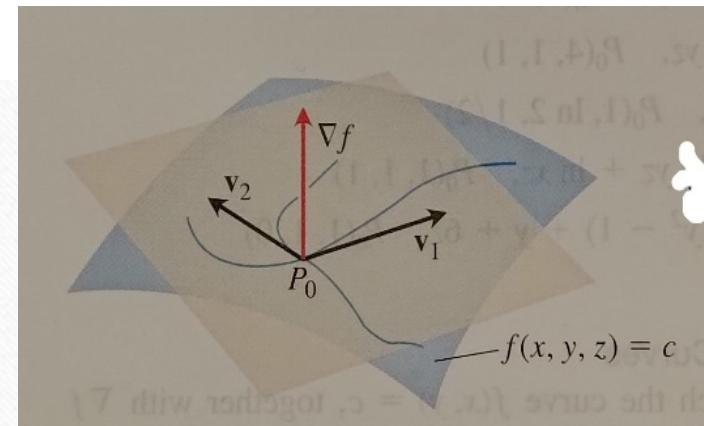
師大工教一

If $\bar{r}(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$ is a smooth curve on the surface $f(x,y,z) = c$

of a differential function f, $\dfrac{d}{dt}f(\bar{r}(t)) = \nabla f(\bar{r}(t)) \cdot \bar{r}'(t) = 0$, which implies the

gradient ∇f is orthogonal to the curve's velocity vector $\bar{r}'$.

Restrict attention to the curves that pass through P_0. All the velocity vectors at P_0 are orthogonal to ∇f at P_0, so the curve's tangent lines all lie in the plane through P_0 normal to ∇f.

Definition The **tangent plane** to the level surface $f(x, y, z) = c$ of a differentiable function f at a point P_0 where the gradient is ot zero is the plane through P_0 normal to $\nabla f\big|_{P_0}$. The **normal line** of the surface at P_0 is the line through P_0 parallel to $\nabla f\big|_{P_0}$.

Tangent Plane to $f(x,y,z) = c$ **at** $P_0(x_0, y_0, z_0)$**:**

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

Normal Line to $f(x,y,z) = c$ **at** $P_0(x_0, y_0, z_0)$**:**

$$x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t$$

Ex1(p763) Find the tangent plane and normal line of the level surface

$$f(x,y,z) = x^2 + y^2 + z - 9 = 0 \text{ at the point } P_0(1,2,4).$$

$$\nabla f = 2x\vec{i} + 2y\vec{j} + \vec{k}$$

$$\nabla f \Big|_{P_0} = 2\vec{i} + 4\vec{j} + \vec{k}$$

Equation of tangent plane

$$2(x-1) + 4(y-2) + (z-4) = 0$$

$$2x + 4y + z = 14$$

Equation of nomal line

對稱式 $\rightarrow$ $\dfrac{x-1}{2} = \dfrac{y-2}{4} = \dfrac{z-4}{1} = t$

Tangent Plane to a Surface $z = f(x, y)$ **at** $(x_0, y_0, f(x_0, y_0))$

The plane tangent to the surface $z = f(x, y)$ of a differentiable function f at

the point $P_0(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

$$z = f(x, y)$$

$$F(x, y, z) = f(x, y) - z$$

$$\nabla F = (f_x, f_y, -1)$$

Analysis: $z = f(x, y)$ is equivalent to $f(x, y) - z = 0$. The surface $z = f(x, y)$

is the zero level surface of the function $F(x, y, z) = f(x, y) - z$. The formula

$F_x(P_0)(x - x_0) + F_y(P_0)(y - y_0) + F_z(P_0)(z - z_0) = 0$ is therefore reduced to

$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$.

Ex2(p764) Find the plane tangent to the surface $z = x\cos y - ye^x$ at $(0,0,0)$.

$f_x = \cos y - ye^x$

$f_y = -x\sin y - e^x \big|_{(0,0)} = 0 - 1 = -1$

$f_x(0,0) = 1 - 0 = 1$

$\Rightarrow 1(x-0) - 1(y-0) - (z-0) = 0$

Ex3(p764) The surfaces $f(x,y,z) = x^2 + y^2 - 2 = 0$ and $g(x,y,z) = x + z - 4 = 0$

椭圆图

meet in an ellipse E. Find parametric equations for the line tangent to E at

the point $P_0(1,1,3)$.

$f(x,y,z) = 0 \ , \quad g(x,y,z) = 0$

$\nabla f = 2x\vec{\imath} + 2y\vec{\jmath}$

$\nabla g = \vec{\imath} + \vec{k}$

$\vec{V}_{tn} \perp \nabla f, \nabla g$

$\nabla f \times \nabla g \ /\!/ \ \vec{V}_{tn}$
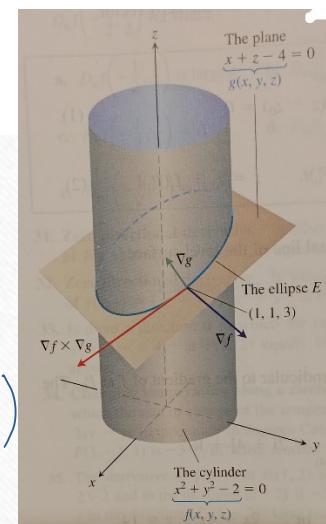
$\nabla f \times \nabla g = \left(\begin{vmatrix} 2y & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 2x \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 2x & 2y \\ 1 & 0 \end{vmatrix} \right)$

$= (2y, -2x, -2y)$

$x = 1 + 2t$

$y = 1 - 2t \qquad z = 3 - 2t$

The plane $x + z - 4 = 0$ $g(x, y, z)$
The ellipse E (1, 1, 3)
∇g
$\nabla f \times \nabla g$ ∇f
The cylinder $x^2 + y^2 - 2 = 0$ $f(x, y, z)$

Estimaing the Change in f in the direction $\bar{u}$

To estimate the change in the value of a differentiable function f when we

move a small distance ds from a point P_0 in a particular direction $\bar{u}$, use

this formula: $df = \left(\nabla f \big|_{P_0} \cdot \bar{u} \right) ds$.

Ex4(p765) Estimate how much the value of $f(x,y,z) = y \sin x + 2yz$ will

change if the point $P(x,y,z)$ moves 0.1 unit from $P_0(0,1,0)$ straight toward

$P_1(2,2,-2)$.

HW13-6

- **HW:1,9,13,19,23**

6. Find a parametric equation for the line tangent to the curve of intersection of the sufaces $x + y^2 + 2z - 4 = 0$ and $x = 1$ at the point $(1, 1, 1)$. (10 pts)

3. Let $z = ye^{2xy}$ be a surface and $P(0, 2, 2)$ be a point in its domain. (5 pts each)

(a) Find an equation of the tangent plane at P.
(b) Find the normal line to the surface at P.

102.b.

$$f(x, y, z) = x + y^2 + 2z - 4$$
$$g(x, y, z) = x - 1$$

$$\nabla f = (1, 2y, 2) \qquad \nabla f(1,1,1) = (1, 2, 2)$$

$$\nabla g = (1, 0, 0) \qquad \nabla g(1,1,1) = (1, 0, 0)$$

$$\nabla f(1,1,1) \times \nabla g(1,0,0) = (0, 2, -2)$$

$$x = 1$$
$$y = 1 + 2t$$
$$z = 1 - 2t$$

103. 3. $f(x, y, z) = ye^{2xy} - z$

$$f_x = 2y^2 e^{2xy}$$
$$f_y = e^{2xy} + 2xy e^{2xy}$$
$$f_z = -1$$

$$\nabla f = (2y^2 e^{2xy})\vec{i} + (e^{2xy} + 2xy e^{2xy})\vec{j} - \vec{k}$$

$$\nabla f(0,2,2) = 8\vec{i} + \vec{j} - \vec{k}$$

$$8(x-0) + (y-2) - (z-2) = 0$$

$$\begin{cases} x = 0 + 8t \\ y = 2 + t \\ z = 2 - t \end{cases}$$