2-4 One-Sided Limits

師大工教一

2-4 One-Sided Limits

(1)We write $\lim_{x \to a^+} f(x) = L$ and say the **right-hand limit** of $f(x)$, as x approaches a, is equal to L (or the limit of $f(x)$, as x approaches a **from the right**) is equal to L if we can make the values of $f(x)$ arbitrarily 任意地 close to L (as close to L as we like) by restricting x to be sufficiently close to a **with** x **greater than** a.

※The Relationships between One-Sided Limits and Limits

Assume that f is defined on an interval (b,c), $a \in (b,c)$.

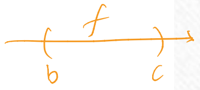
(1) $\lim\limits_{x \to a} f(x) = L \Leftrightarrow \lim\limits_{x \to a^+} f(x) = L = \lim\limits_{x \to a^-} f(x)$

(2) $\lim\limits_{x \to b} f(x) = \lim\limits_{x \to b^+} f(x)$

(3) $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c^-} f(x)$

$a \in (b,c)$ $\lim\limits_{x \to a} f(x) = L$

$\Leftrightarrow \lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^-} f(x) = L$



Ex1(p97)

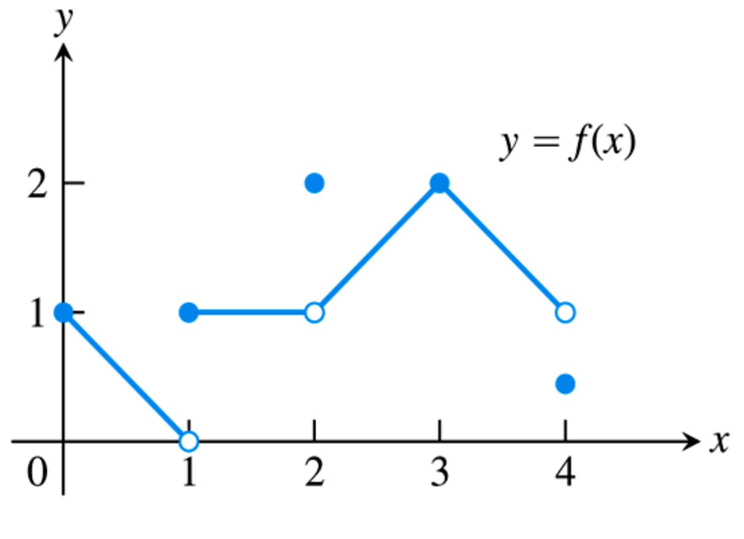
(1) $\lim_{x\to 0^+} f(x) = 1 = \lim_{x\to 0} f(x)$, $\lim_{x\to 0^-} f(x)$ DNE

(2) $\lim_{x\to 1^+} f(x) = 1$, $\lim_{x\to 1^-} f(x) = 0$. so $\lim_{x\to 1} f(x)$ DNE

(3) $\lim_{x\to 2^+} f(x) = 1$, $\lim_{x\to 2^-} f(x) = 1$, $\lim_{x\to 2} f(x) = 1$

(4) $\lim_{x\to 3^+} f(x) = 2$, $\lim_{x\to 3^-} f(x) = 2$, $\lim_{x\to 3} f(x) = 2$

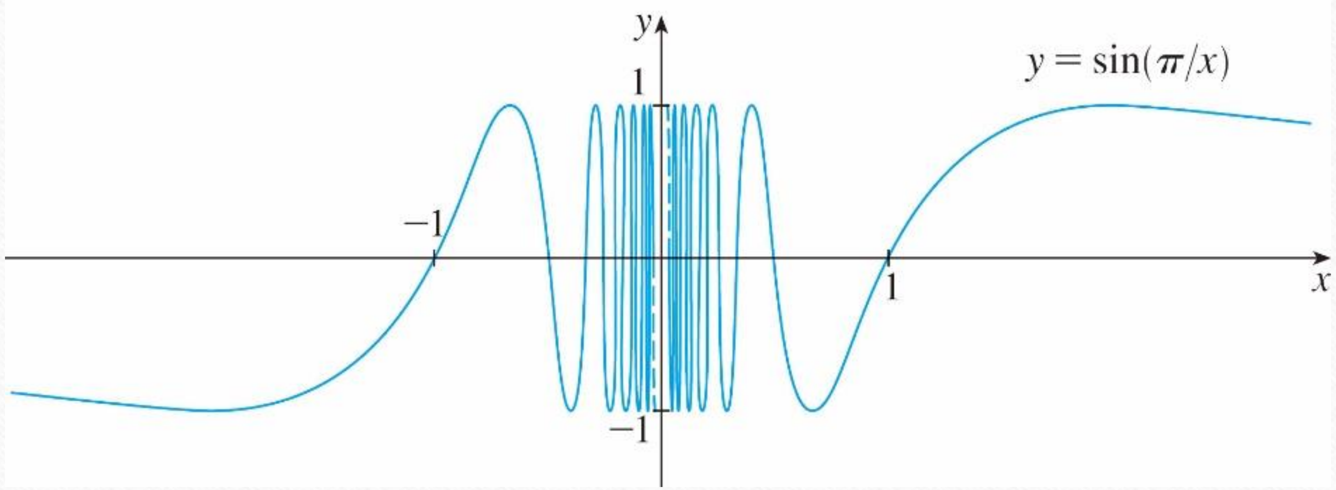
(5) $\lim_{x\to 4^+} f(x)$ DNE , $\lim_{x\to 4^-} f(x) = 1 = \lim_{x\to 4} f(x)$



Ex4(p99): Investigate $\lim_{x \to 0} \sin \frac{1}{x}$

$\lim_{x \to 0^+} \sin \frac{1}{x}$ DNE

so is $\lim_{x \to 0^-} \sin \frac{1}{x}$, $\lim_{x \to 0} \sin \frac{1}{x}$



Theorem: $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$ (θ in radians)

$\lim_{x\to 0}\frac{\sin 2x}{2x}=1$

Let $2x=\theta$

$\lim_{x\to 0}\frac{\sin 2x}{2x}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$

Ex5(p100): Show that (1) $\lim_{y\to 0}\frac{\cos y-1}{y}=0$ (2) $\lim_{x\to 0}\frac{\sin 2x}{5x}=\frac{2}{5}$

(1) $=\lim_{y\to 0}\frac{(\cos y-1)(\cos y+1)}{y(\cos y+1)}=\lim_{y\to 0}\frac{\cos^2 y-1}{y(\cos y+1)}=\lim_{y\to 0}\frac{-\sin^2 y}{y(\cos y+1)}$

$=\lim_{y\to 0}\frac{-\sin y}{y}\cdot\frac{\sin y}{\cos y+1}=-\lim_{y\to 0}\frac{\sin y}{y}\times\lim_{y\to 0}\frac{\sin y}{\cos y+1}=-1\times\frac{0}{1+1}=0$

(2) $=\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\frac{2x}{5x}=\lim_{x\to 0}\frac{\sin 2x}{2x}\times\lim_{x\to 0}\frac{2\cancel{x}}{5\cancel{x}}$

$=1\times\frac{2}{5}=\frac{2}{5}$ ※

HW2-4

- **HW: 1,5,6,20,22,25,31,36**

$$\lim_{\theta \to \infty} \frac{\sin\theta}{\theta} = 0$$

$$-1 \le \sin\theta \le 1$$

$$\frac{-1}{|\theta|} \le \frac{\sin\theta}{\theta} \le \frac{1}{|\theta|}$$

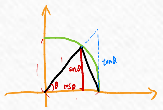
$$\lim_{\theta \to \infty} \frac{-1}{|\theta|} = 0$$

$$\lim_{\theta \to \infty} \frac{1}{|\theta|} = 0$$

by Sandwich Theorem

$$\lim_{\theta \to \infty} \frac{\sin\theta}{\theta} = 0$$

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$



$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

$$\lim_{\theta \to 0} \cos\theta = 1$$

$$\lim_{\theta \to 0} 1 = 1$$

By Sandwich Theorem

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

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2. Find the limit if it exists. Notice that L'Hôpital's Rule is forbidden. (20 pts)

(a) $\displaystyle\lim_{x\to 2}\frac{\sqrt{x+7}-3}{x-2}$ V (b) $\displaystyle\lim_{x\to 0}\frac{\tan(5x)}{\sin(3x)}$ (c) $\displaystyle\lim_{x\to -\infty}\left(x+\sqrt{x^2+3}\right)$

(d) $\displaystyle\lim_{x\to\infty}\frac{\sqrt[3]{8x^6+2x^3+1}}{\sqrt{4x^4+3x+1}}$

(b) $\displaystyle\lim_{x\to 0}\frac{\tan(5x)}{\sin(3x)}$

$\displaystyle=\lim_{x\to 0}\left(\frac{\sin 5x}{\cos 5x}\times\frac{1}{\sin 3x}=\lim_{x\to 0}\frac{\sin 5x}{5x}\times\frac{3x}{\sin 3x}\times\frac{5x}{3x}\times\frac{1}{\cos 5x}\right)$

$\displaystyle=\lim_{x\to 0}\left(\frac{\sin 5x}{5x}\times\frac{1}{\frac{\sin 3x}{3x}}\times\frac{5}{3}\times\frac{1}{\cos x}\right)$

$\displaystyle=1\times 1\times\frac{5}{3}\times 1=\frac{5}{3}$ #

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4. (28 pts) Find the limit (極限).

(a) $\displaystyle\lim_{x\to -\infty}\left(2x+\sqrt{4x^2+1}\right)$ (b) $\displaystyle\lim_{x\to 0^+}\frac{\sqrt{x+1}-1}{\sqrt{x^2+1}-\sqrt{x+1}}$

V (c) $\displaystyle\lim_{\theta\to 0}\theta^2\cot(7\theta)\csc(4\theta)$ (d) $\displaystyle\lim_{t\to 0}\ln\sqrt{(1+t)^{\frac{1}{3t}}}$

(c) $\displaystyle\lim_{\theta\to 0}\theta^2\cot(7\theta)\csc(4\theta)$

$\displaystyle=\lim_{\theta\to 0}\theta^2\cdot\frac{\cos 7\theta}{\sin 7\theta}\cdot\frac{1}{\sin 4\theta}$

$\displaystyle=\lim_{\theta\to 0}\frac{7\theta}{\sin 7\theta}\cdot\frac{4\theta}{\sin 4\theta}\cdot\frac{1}{28}$

$\displaystyle=\lim_{\theta\to 0}\frac{1}{\frac{\sin 7\theta}{7\theta}}\cdot\frac{1}{\frac{\sin 4\theta}{4\theta}}\cdot\frac{1}{28}$

$\displaystyle=1\times 1\times\frac{1}{28}=\frac{1}{28}$