# 9-8 Taylor and Maclaurin Series

師大工教一

## Series Representations

Let $f(x)=\sum_{n=0}^{\infty}a_n(x-a)^n=a_0+a_1(x-a)+a_2(x-a)^2+\cdots+a_n(x-a)^n+\cdots$

$$\Rightarrow f'(x)=1\cdot a_1+2\cdot a_2(x-a)+3a_3(x-a)^2+\cdots+na_n(x-a)^{n-1}+\cdots$$

$$\Rightarrow f''(x)=1\cdot 2\cdot a_2+2\cdot 3\cdot a_3(x-a)+3\cdot 4\cdot a_4(x-a)^2+\cdots$$

$$\Rightarrow f'''(x)=1\cdot 2\cdot 3\cdot a_3+2\cdot 3\cdot 4\cdot a_4(x-a)+3\cdot 4\cdot 5\cdot a_5(x-a)^2+\cdots$$

$$\Rightarrow f'(a)=a_1,\ f''(a)=1\cdot 2\cdot a_2,\ f'''(a)=1\cdot 2\cdot 3\cdot a_3,\cdots,\ f^{(n)}(a)=n!a_n,\cdots$$

$$\Rightarrow a_2 = \frac{f''(a)}{2!}, a_3 = \frac{f'''(a)}{3!}, \cdots, a_n = \frac{f^{(n)}(a)}{n!}, \cdots$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \longrightarrow \text{Taylor series}$$

## Taylor and Maclaurin Series

Definition Let $f$ be a function with derivatives of all orders throughout some interval containing $a$ as an interior point. Then the **Taylor series generated by** $f$ **at** $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

The **Maclaurin series of** $f$ is the Taylor series generated by $f$ at $x = 0$, or

↳ Taylor series 一種特例

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

Ex1(p573) Find the Taylor series generated by $f(x)=\dfrac{1}{x}$ at $x=2$. Where, if anywhere, does the series converge to $\dfrac{1}{x}$?

1°

$$f(x)=\frac{1}{x}=x^{-1} \Rightarrow f(2)=\frac{1}{2}$$

$$f'(x)=(-1)x^{-2} \Rightarrow f'(2)=(-1)\left(\tfrac{1}{2}\right)^{2}$$

$$f''(x)=2x^{-3} \Rightarrow f''(2)=(-1)(-2)\left(\tfrac{1}{2}\right)^{3}$$

$$\vdots$$

$$f^{(n)}(x)=(-1)(-2)\cdots(-n)x^{-n-1} \Rightarrow f^{(n)}(2)=(-1)\cdots(-n)\left(\tfrac{1}{2}\right)^{n+1}=(-1)^n n!\left(\tfrac{1}{2}\right)^{n+1}$$

2°

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(2)}{n!}(x-2)^n$$

Taylor series formular

$$=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot\cancel{n!}\cdot\left(\frac{1}{2}\right)^{n+1}}{\cancel{n!}}\cdot(x-2)^n$$

代入 $f^{(n)}(2)$

$$=\sum_{n=0}^{\infty}(-1)^n\cdot\left(\tfrac{1}{2}\right)^{n+1}(x-2)^n$$

$$=\frac{1}{2}-\frac{1}{4}(x-2)+\frac{1}{8}(x-2)^2-\cdots$$

$$\lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n+1}}\cdot\cancel{\left(\frac{1}{2}\right)^{n+2}}(x-2)^{n+1}}{\cancel{(-1)^{n}}\cancel{\left(\frac{1}{2}\right)^{n+1}}\cancel{(x-2)^{n}}}\right|=\left|\frac{x-2}{2}\right|<1$$

$\Rightarrow |x-2|<2$ converges absolutely $\rightarrow \dfrac{1}{x}$

## Taylor Polynomials

泰勒多項式

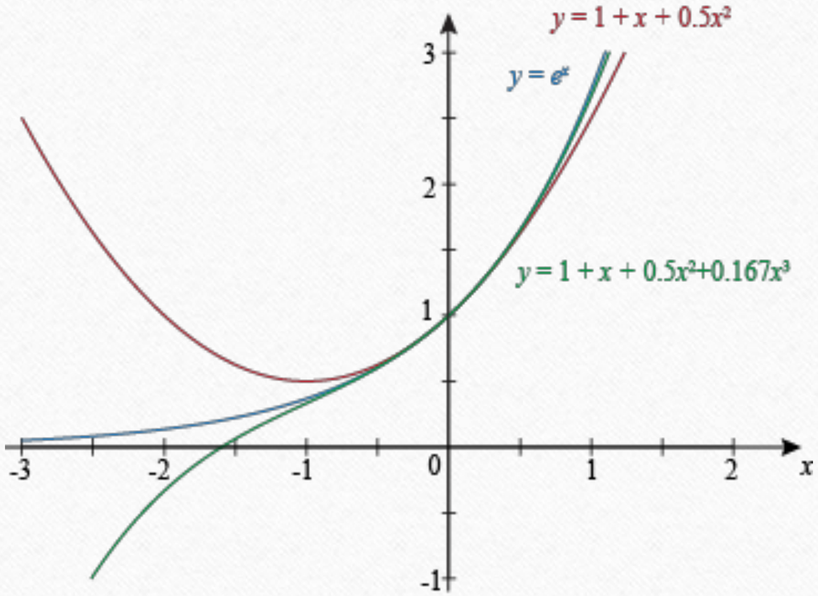
Definition Let $f$ be a function with derivatives of orders $k$ for $k = 1, 2, \cdots, N$ in some interval containing $a$ as an interior point. Then, for any integer $n$ from $0$ through $N$, the **Taylor polynomial of order** $n$ generated by $f$ at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

出現在 $|x|$、有斷点 or 尖点 function

Ex2(p574) Find the Taylor series and Taylor polynomials generated by

$f(x) = e^x$ at $x = 0$.




$f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

$\vdots$

$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = 1$

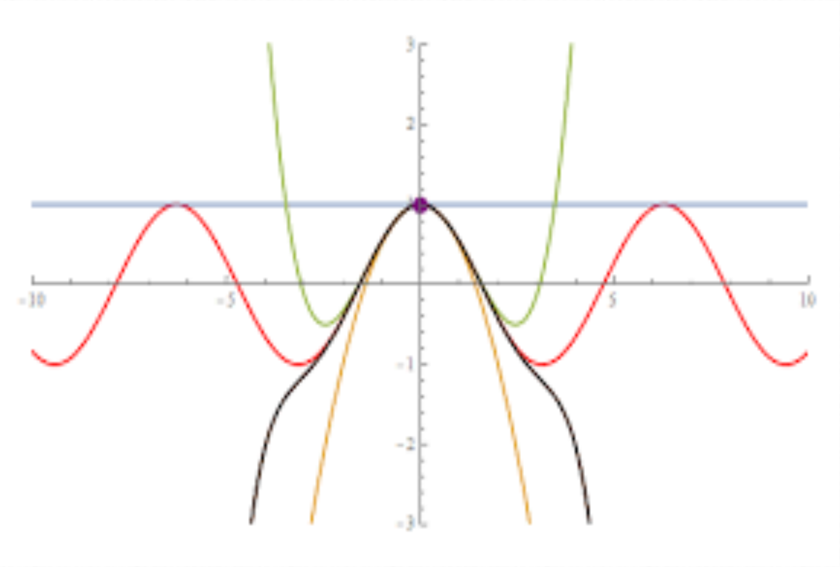
$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$

$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$P_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$

Ex3(p575) Find the Taylor series and Taylor polynomials generated by

$f(x) = \cos x$ at $x = 0$.




$f(x) = \cos x \Rightarrow f(0) = 1$

$f'(x) = -\sin x \Rightarrow f'(0) = 0$

$f''(x) = -\cos x \Rightarrow f''(0) = -1$

$f'''(x) = \sin x \Rightarrow f'''(0) = 0$

$f^{(4)}(x) = \cos x \Rightarrow f^{(4)}(0) = 1$

$\vdots$

$f^{(n)}(x) = \cos x \approx 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

$P_{2n}(x) = P_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k}$

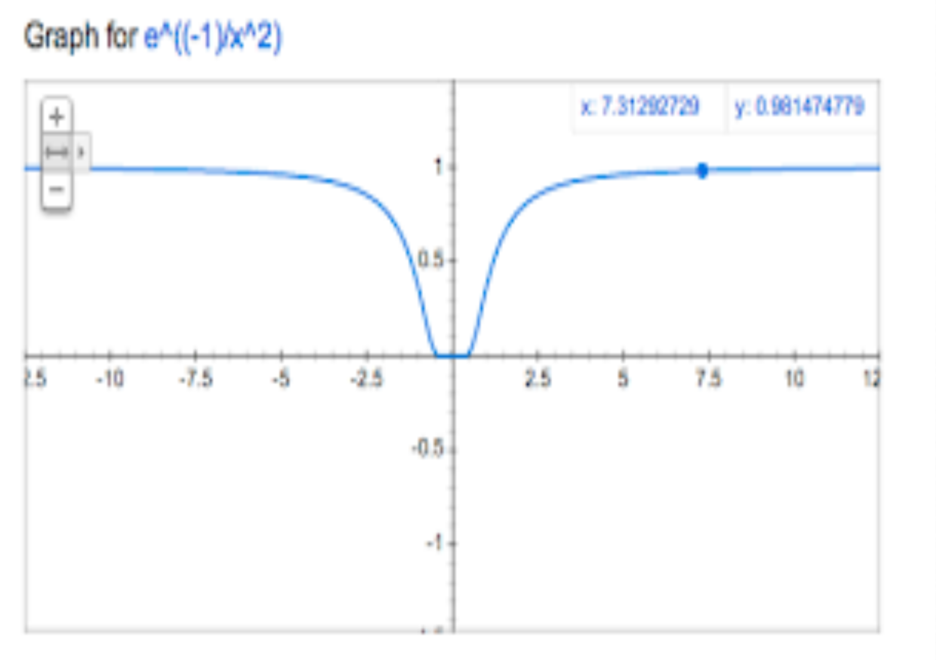
Note: By Ex4(p575). Consider $f(x)=\begin{cases}0, & x=0\\ e^{-\frac{1}{x^2}}, & x\neq 0\end{cases}$. It can be shown that

$f^{(n)}(0)=0$ for all $n$. Thus, its Taylor series generated by $f$ at $x=0$ is

$$f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots$$

$$=0+0\cdot x+0\cdot x^2+\cdots+0\cdot x^n+\cdots=0+0+0+\cdots+0+\cdots$$

The series converges for every $x$ but converges to $f(x)$ only at $x=0$.




Thus, the Taylor series generated by $f(x)$ is not equal to the function $f(x)$ over the entire interval of convergence.

Two questions still remain.

1. For what values of $x$ can we expect a Taylor series to converge to its generating function?

2. How accurately do a function's Taylor polynomials approximate the function on a given interval?

# HW9-8

- **HW:2,3,13,29,32**