5-5 Indefinite Integrals and the Substitution Method

代換法

師大工教一

Recall: In Section 4.8, we have defined the indefinite integral

$\int f(x)\,dx = F(x) + C$, where F is any antiderivative of f.

Substitution: Running the Chain Rule Backwards

E.g. $\dfrac{d}{dx}\left(\dfrac{1}{2}\ln|2\sec x - 1|\right) = \dfrac{\sec x \tan x}{2\sec x - 1} \Rightarrow \int \dfrac{\sec x \tan x}{2\sec x - 1}\,dx = \dfrac{1}{2}\ln|2\sec x - 1| + C$

Ex1(p350) Find the integral $\int (x^3+x)^5(3x^2+1)\,dx$. $= \int u^5\,du$

$= \frac{u^6}{6} + C$

$= \frac{(x^3+x)^6}{6} + C$

Let $u = x^3+x$

$du = (3x^2+1)\,dx$

$[f(g(x))]' = f'(g(x)) \cdot g'(x)$

$\Rightarrow \int f'(g(x)) \cdot g'(x) = f(g(x)) + C$

Theorem 6-The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is in an interval I, and

f is continuous on I, then $\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du$.

The Substitution Method to evaluate $\int f(g(x))\cdot g'(x)\,dx$

1. Substitute $u = g(x)$ and $du = \left(\frac{du}{dx}\right)dx = g'(x)\,dx$ to obtain $\int f(u)\,du$.

2. Integrate with respect to u.

3. Replace u by $g(x)$.

Ex4(p352) Find $\int \cos(7\theta+3)\,d\theta$.

Let $u = 7\theta + 3$

$du = 7\,d\theta$

$d\theta = \frac{du}{7}$

origin $= \int \cos(u)\frac{du}{7} = \frac{1}{7}\sin u + C = \frac{1}{7}\sin(7\theta+3) + C$

Ex(105#1(4)) Find $\int \frac{\sec x \tan x}{2\sec x - 1} dx$.

Let $u = 2\sec x - 1$

$du = 2\sec x \tan x \, dx$

$\frac{du}{2} = \sec x \tan x \, dx$

$\int \frac{\frac{du}{2}}{u} = \frac{1}{2}\int \frac{1}{u} du = \frac{1}{2} \ln|u| + C$

$= \frac{1}{2} \ln|(2\sec x - 1)| + C$

Ex6(p353) Evaluate $\int x\sqrt{2x+1}\,dx$.

Let $u = 2x+1$

$du = 2dx$

$dx = \frac{du}{2}$

$x = \frac{u-1}{2}$

origin $= \int \frac{u-1}{2}\sqrt{u}\,\frac{du}{2}$

$= \frac{1}{4}\int (u-1)\sqrt{u}\,du$

$= \frac{1}{4}\int u\sqrt{u} - \sqrt{u}\,du$

$= \frac{1}{4}\int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du$

$= \frac{1}{4}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) + C$

$= \frac{u^{\frac{5}{2}}}{10} - \frac{u^{\frac{3}{2}}}{6} + C$

$= \frac{(2x+1)^{\frac{5}{2}}}{10} - \frac{(2x+1)^{\frac{3}{2}}}{6} + C$

Ex7(p354) Evaluate the following integrals:

(a) $\int \sin^2 x\,dx$ (b) $\int \cos^2 x\,dx$ (c) $\int \tan x\,dx$

倍角公式

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1-\sin^2\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\sin^2\theta = \cos^2\theta - \cos 2\theta$

$= 1-\sin^2\theta - \cos 2\theta$

$2\sin^2\theta = 1-\cos 2\theta$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

(a) $= \int \frac{1-\cos 2x}{2}\,dx$

$= \frac{1}{2}\int (1-\cos 2x)\,dx$

$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$

(b) $= \frac{1}{2}\int (1+\cos 2x)\,dx$

$= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$

(c) $= \int \frac{\sin x}{\cos x}\,dx$

Let $u = \cos x$

$du = -\sin x\,dx$

$= \int \frac{-du}{u} = -\ln|u| + C$

$= -\ln|\cos x| + C$

$= \ln|\sec x| + C$

Ex8(p354) Evaluate the following integrals:

(a) $\int \frac{dx}{e^x + e^{-x}}$ (b) $\int \sec x\, dx$

(a) $= \int \frac{dx}{e^x + e^{-x}} \cdot \frac{e^x}{e^x}$

$= \int \frac{e^x dx}{e^{2x}+1}$

Let $u = e^x$

$du = e^x dx$

$= \int \frac{du}{u^2+1} = \tan^{-1} u + C$

$= \tan^{-1} e^x + C$

(b) $= \int \sec x dx \cdot \frac{\sec x + \tan x}{\sec x + \tan x}$

Let $u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x) dx$

$= \sec x(\sec x + \tan x) dx$

$= \int \frac{du}{u} = \ln|u| + C$

$= \ln|\sec x + \tan x| + C$

Formulas

$$\int \tan x\,dx = \ln\left|\sec x\right| + C$$

$$\int \sec x\,dx = \ln\left|\sec x + \tan x\right| + C$$

$$\int \cot x\,dx = -\ln\left|\csc x\right| + C$$

$$\int \csc x\,dx = -\ln\left|\csc x + \cot x\right| + C$$

HW5-5

- **HW: 2,3,4,11,16,24,38,55,57,67**