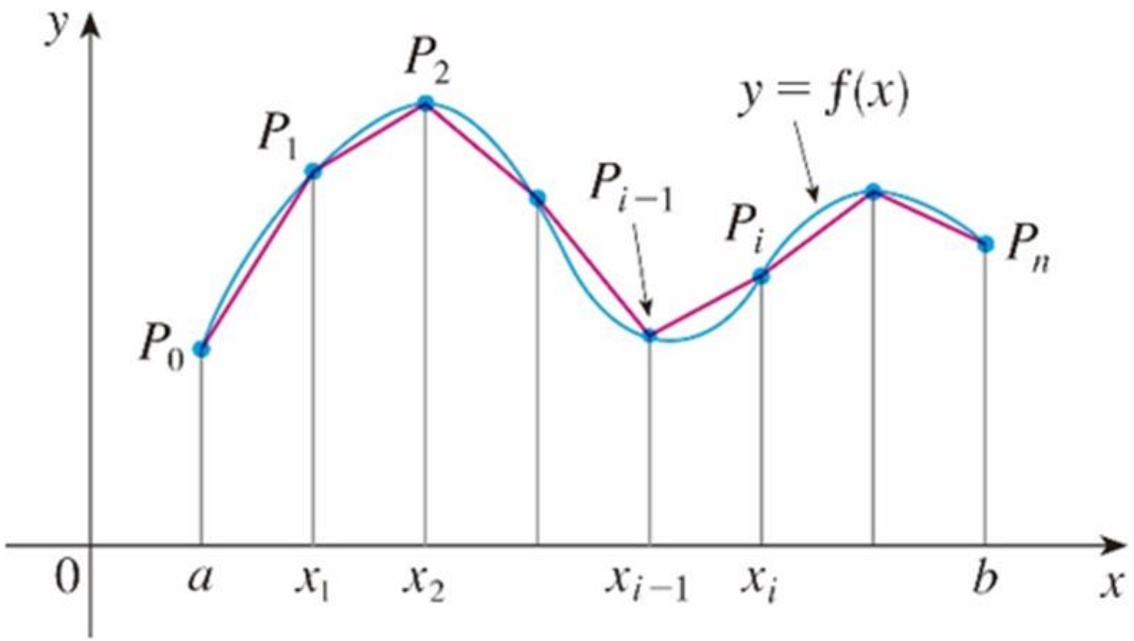
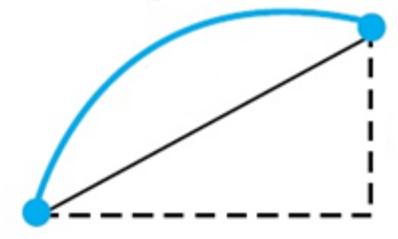
6-3 Arc Length

師大工教一

Length of a curve $y = f\left(x\right)$

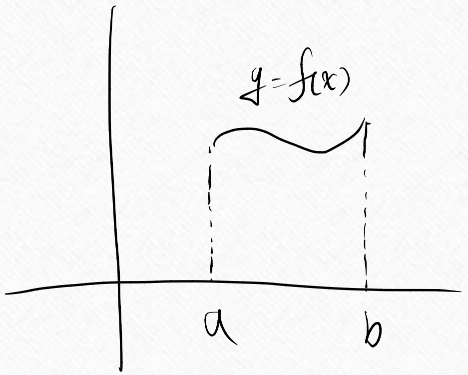




$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

$$\sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + \left(f'(c_k)\,\Delta x_k\right)^2} = \sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2}\,(\Delta x_k)$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} L_k = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2}\,(\Delta x_k) = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

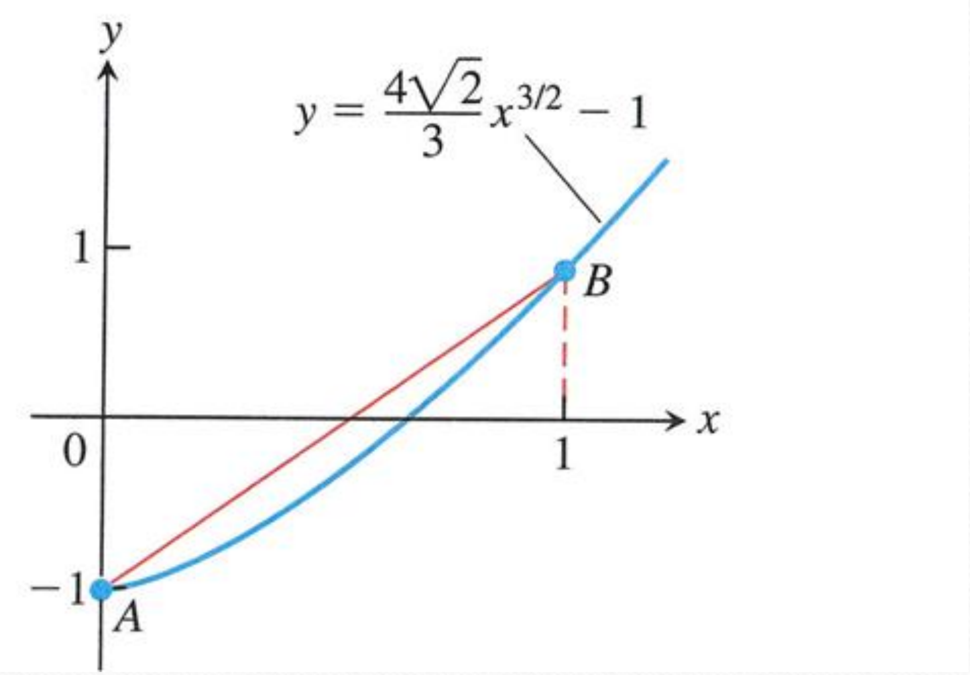


$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

Definition If f' is continuous on $[a,b]$, then the **length** (**arc length**) of the curve $y=f(x)$ from the point $A=(a,f(a))$ to the point $B=(b,f(b))$ is the value of the integral $L=\int_a^b \sqrt{1+[f'(x)]^2}\,dx=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$.

Ex1(p394) Find the length of the curve shown in Figure 6.24, which is the graph of the function $y=\frac{4\sqrt{2}}{3}x^{\frac{3}{2}}-1, \quad 0\le x\le 1$.

$$\frac{dy}{dx}=2\sqrt{2}\,x^{\frac{1}{2}}=2\sqrt{2x}$$



$$L=\int_0^1\sqrt{1+8x}\,dx$$

Let $u=1+8x$

$du=8dx$

$\frac{du}{8}=dx$

$$=\int_1^9\sqrt{u}\,\frac{du}{8}$$

$$=\frac{1}{8}\left[\frac{2}{3}u^{\frac{3}{2}}\Big|_1^9\right]=\frac{1}{8}\left[\frac{2}{3}(1+8x)^{\frac{3}{2}}\Big|_0^1\right]$$

$$=\frac{1}{8}\cdot\left(\frac{2}{3}\times 27-\frac{2}{3}\right)=\frac{1}{12}\times 26=\frac{13}{6}$$ #

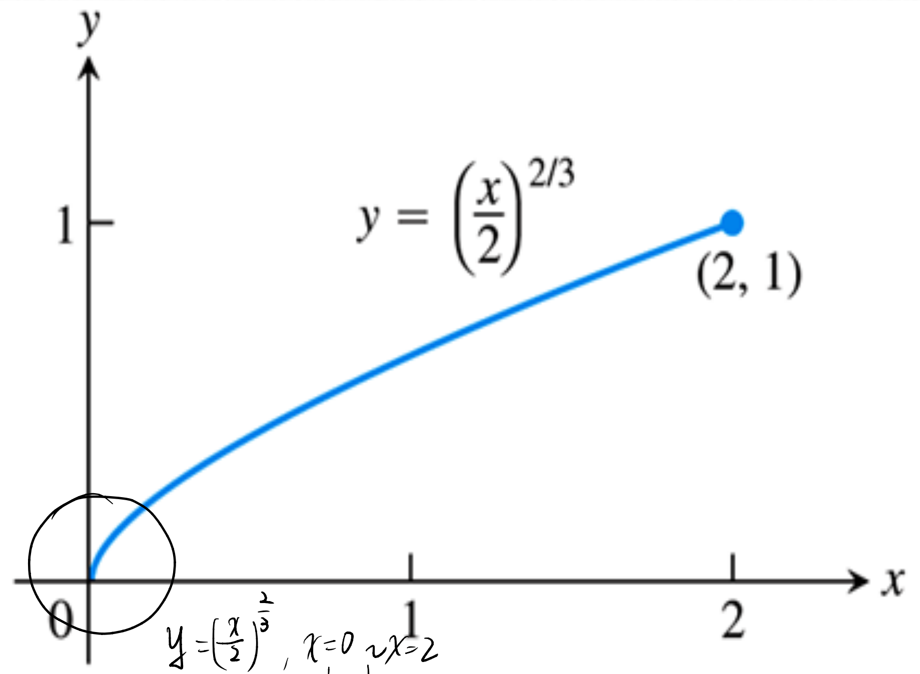
Dealing with Discontinuities in $\dfrac{dy}{dx}$

Formula for the length of $x = g(y), c \leq y \leq d$

If g' is continuous on $[c, d]$, the length of the curve $x = g(y)$ from the point $A = (g(c), c)$ to the point $B = (g(d), d)$ is

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy .$$

Ex4(p396) Find the length of the curve $y=\left(\frac{x}{2}\right)^{\frac{2}{3}}$ from $x=0$ to $x=2$.



$y=\left(\frac{x}{2}\right)^{\frac{2}{3}}$, $x=0 \sim x=2$

$y'=\frac{2}{3}\left(\frac{x}{2}\right)^{-\frac{1}{3}}\cdot\frac{1}{2}$

$y'|_{x=0}$ DNE.

$y^{\frac{3}{2}}=\frac{x}{2}$

$x=2y^{\frac{3}{2}}$ $\quad x=0 \Rightarrow y=0$ $\quad \frac{dx}{dy}=3y^{\frac{1}{2}}$

$x=2 \Rightarrow y=1$

Let $u=1+9y$ $\quad du=9dy$

$L=\int_0^1\sqrt{1+9y}\,dy=\int_1^{10}\sqrt{u}\,\frac{du}{9}$

$=\frac{2}{3}\times\frac{1}{9}\left[u^{\frac{3}{2}}\Big|_1^{10}\right]$

$=\frac{2}{27}\left(10^{\frac{3}{2}}-1\right)$

$\frac{dy}{dx} = \sqrt{\cos 4x}$

Ex(102 年, #3) 求曲線 $y = \int_0^x \sqrt{(\cos 4t)}\,dt, 0 \le x \le \frac{\pi}{4}$ 的長度。

$\cos^2(2x) = \frac{1+\cos(4x)}{2}$

$L = \int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{2(\cos^2(2x))}\,dx = \sqrt{2}\int_0^{\frac{\pi}{4}} |\cos(2x)|\,dx = \sqrt{2}\cdot\frac{1}{2}\sin(2x)\Big|_0^{\frac{\pi}{4}} = \frac{1}{\sqrt{2}}(1-0) = \frac{1}{\sqrt{2}}$

Ex(106 年, #4) Find the arc length of the graph of the function $y = \ln(\cos x)$ over $\left[0, \frac{\pi}{3}\right]$.

$\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$

Let $u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x)dx$

$= \sec x(\tan x + \sec x)dx$

$L = \int_0^{\frac{\pi}{3}} \sqrt{1+\tan^2 x}\,dx$

$= \int_0^{\frac{\pi}{3}} \sec x\,dx$

$= \int_0^{\frac{\pi}{3}} \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,dx$

$= \int_1^{2+\sqrt{3}} \frac{1}{u}\,du = \ln|u|\Big|_1^{2+\sqrt{3}} = \ln(2+\sqrt{3})$

HW6-3

- **HW: 2,4,15,16**