Advanced Engineering Mathematics | 賴以威

2nd Differential Equation with Constant Coefficient

Reference CENGAGE Learning

Homogeneous Linear DE with Constant Coefficients

Auxiliary equation to solve homogeneous DE

constraints: (1) homogeneous

(2) linear

(3) constant coefficients

$$\boxed{a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = 0}$$

$a_0,\ a_1,\ a_2,\ \cdots,\ a_n$ are constants

(the simplest case of the higher order DEs)

Solution Format

solutions has the form of e^{mx}

Example: $y''(x) - 3\,y'(x) + 2\,y(x) = 0$

Set $y(x) = e^{mx}$, $\quad m^2 e^{mx} - 3m\, e^{mx} + 2\, e^{mx} = 0$

$$m^2 - 3m + 2 = 0 \qquad \text{solve } m$$

Replace <u>n times derivative</u> by m^n .Transform it into a polynomial sequence called <u>auxiliary equation</u> .

$$\boxed{a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = 0}$$

Step 1-1
auxiliary function

$$\boxed{a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0}$$

Step 1-1 Find n roots , $m_1,\ m_2,\ m_3,\ \cdots.,\ m_n$

(If $m_1,\ m_2,\ m_3,\ \cdots.,\ m_n$ are distinct)

Step 1-2 n independent solutions $e^{m_1 x}$, $e^{m_2 x}$, $e^{m_3 x}$, $\cdots$, $e^{m_n x}$

(**3 cases**)

Step 1-3

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ \cdots \ + c_n e^{m_n x}}$$

Three Cases for Roots (2nd Order DE)

$$\boxed{a_2 y''(x) + a_1 y'(x) + a_0 y(x) = 0}$$

$$\downarrow$$

$$\boxed{a_2 m^2 + a_1 m + a_0 = 0}$$

roots $\quad m_1 = \dfrac{-a_1 + \sqrt{a_1^2 - 4a_2 a_0}}{2a_2} \qquad m_2 = \dfrac{-a_1 - \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$

Three Cases for Roots (2nd Order DE)

Case 1 $m_1 \neq m_2$, m_1, m_2 are <u>real</u>

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

Case 2 $m_1 = m_2$

Case 3 $m_1 \neq m_2$, m_1 and m_2 are <u>conjugate and complex</u>

Case 2 $m_1 = m_2$

First solution: $\boxed{y_1 = e^{m_1 x}}$

Second solution: using the method of "**Reduction of Order**"

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx$$

$$= e^{m_1 x}\int e^{-2m_1 x} e^{-\int a_1/a_2 dx} dx$$

$$= e^{m_1 x}\int e^{(-2m_1 - \frac{a_1}{a_2})x} dx \qquad m_1 = \frac{-a_1}{2a_2}$$

$$= e^{m_1 x}\int dx = e^{m_1 x}(x + c)$$

$$\boxed{y_2(x) = xe^{m_1 x}}$$

$$\boxed{y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}}$$

Case 3 $m_1 \neq m_2$, m_1 and m_2 are conjugate and complex

$$m_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2a_0}}{2a_2} = \alpha + j\beta \qquad m_2 = \alpha - j\beta$$

$$\alpha = -a_1/2a_2, \qquad \beta = \sqrt{4a_2a_0 - a_1^2}/2a_2$$

Solution: $\boxed{y = C_1e^{\alpha x + j\beta x} + C_2e^{\alpha x - j\beta x}}$

Another form:

$$y = e^{\alpha x}\left(C_1e^{j\beta x} + C_2e^{-j\beta x}\right)$$

$$= e^{\alpha x}\left(C_1\cos\beta x + jC_1\sin\beta x + C_2\cos\beta x - jC_2\sin\beta x\right)$$

set $c_1 = C_1 + C_2$ and $c_2 = jC_1 - jC_2$

$\boxed{y = e^{\alpha x}\left(c_1\cos\beta x + c_2\sin\beta x\right)}$ c_1 and c_2 are some constant

Example

(a) $\boxed{2y'' - 5y' - 3y = 0}$

$2m^2 - 5m - 3 = 0, \quad m_1 = -1/2, \ m_2 = 3$

$\boxed{y = c_1 e^{-x/2} + c_2 e^{3x}}$

(b) $\boxed{y'' - 10y' + 25y = 0}$

(c) $\boxed{y'' + 4y' + 7y = 0}$

Higher Order DE with Constant Coeff.

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = 0$$

auxiliary function: $a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$

roots: $m_1, m_2, m_3, \cdots, m_n$

(1) If $m_p \neq m_q$ for $p = 1, 2, \cdots, n$ and $p \neq q$

(meaning that there is only one root at m_q)

then $\boxed{e^{m_q x}}$ is a solution of the DE.

(2) If the multiplicities of m_q is,

$\boxed{e^{m_q x},\ x e^{m_q x},\ x^2 e^{m_q x},\ \cdots,\ x^{k-1} e^{m_q x}}$ are the solutions of the DE.

Higher Order DE with Constant Coeff.

(3) If both $\alpha + j\beta$ and $\alpha - j\beta$ are the roots of the auxiliary function,

then $\boxed{e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x)}$

are the solutions of the DE.

(4) If the multiplicities of $\alpha + j\beta$ is k and the multiplicities of $\alpha - j\beta$ is also k,

then

$$\boxed{\begin{array}{l} e^{\alpha x}\cos(\beta x)\ ,\ xe^{\alpha x}\cos(\beta x)\ ,\ x^2e^{\alpha x}\cos(\beta x)\ ,\ \cdots,\ x^{k-1}e^{\alpha x}\cos(\beta x) \\ e^{\alpha x}\sin(\beta x)\ ,\ xe^{\alpha x}\sin(\beta x)\ ,\ x^2e^{\alpha x}\sin(\beta x)\ ,\ \cdots\ ,\ x^{k-1}e^{\alpha x}\sin(\beta x) \end{array}}$$

are the solutions of the DE.

Note: If $\alpha + j\beta$ is a root of a real coefficient polynomial, then $\alpha - j\beta$ is also a root of the polynomial.

$$a_n(\alpha + j\beta)^n + a_{n-1}(\alpha + j\beta)^{n-1} + \cdots + a_1(\alpha + j\beta) + a_0 = 0$$

$a_0, a_1, a_2, \dots, a_n$ are real

Example3 (text page 135)

Solve $\boxed{y''' + 3y'' - 4y = 0}$

Step 1-1 $m^3 + 3m^2 - 4 = 0$

$$(m-1)(m^2 + 4m + 4) = 0 \quad \boxed{m_1 = 1, \quad m_2 = m_3 = -2}$$

Step 1-2 3 independent solutions: e^x , e^{-2x} , xe^{-2x}

Step 1-3 general solution: $\boxed{y = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x}}$

Example3

Solve $\boxed{y^{(4)}(x)+2y''(x)+y(x)=0}$

Step 1-1 $m^4+2m^2+1=0$

$(m^2+1)^2=0$ $\boxed{\text{four roots: } i,\ i, -i, -i}$

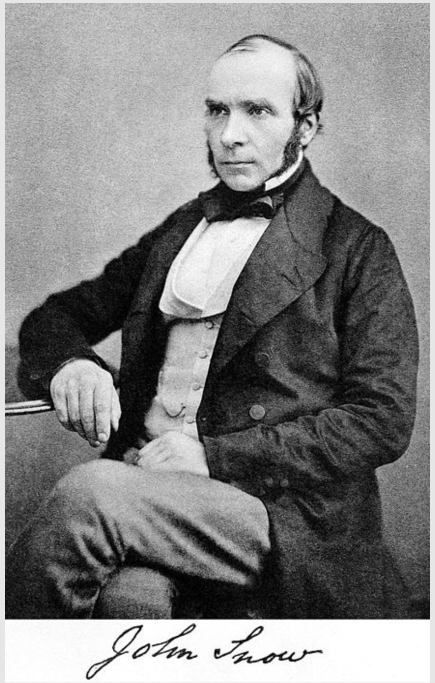
Step 1-2 4 independent solutions: $\cos x$, $x\cos x$, $\sin x$, $x\sin x$

Step 1-3 general solution:

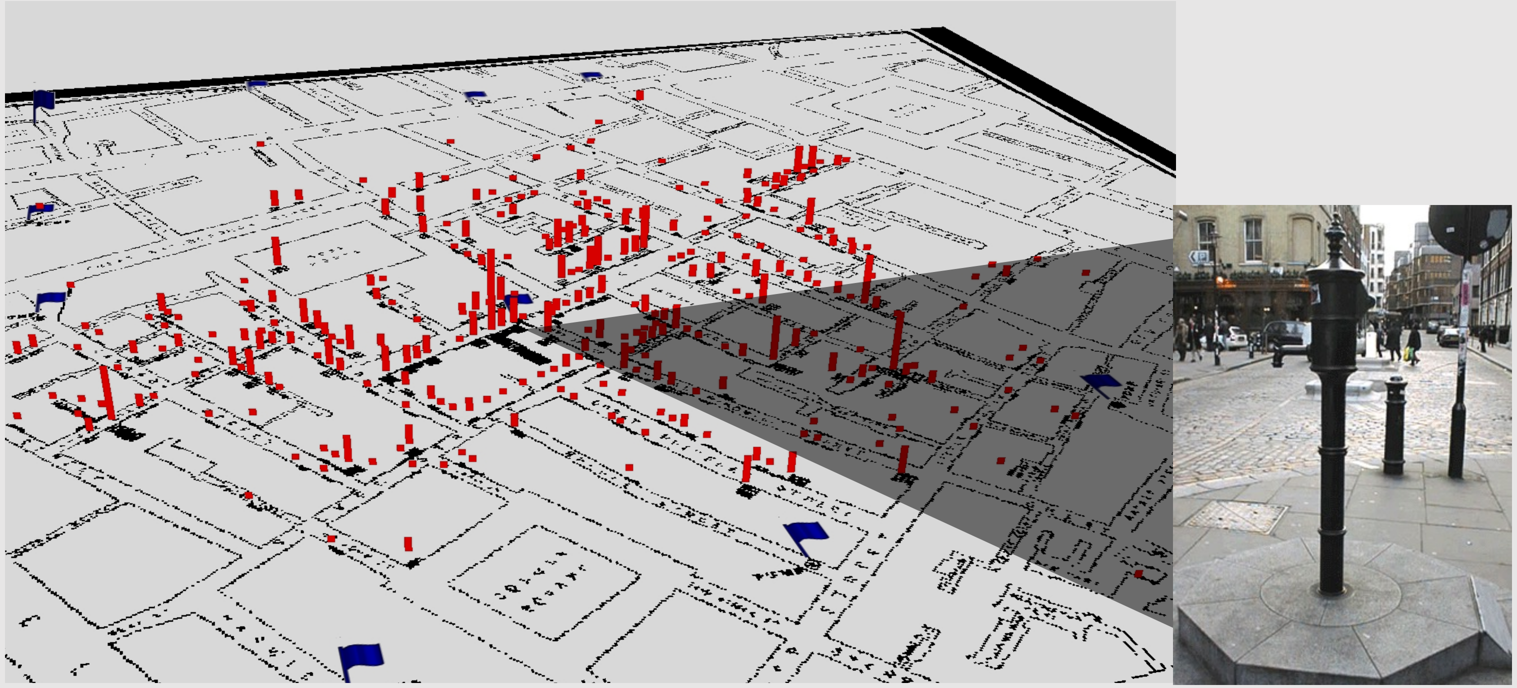
$$\boxed{y=c_1\cos x+c_2x\cos x+c_3\sin x+c_4x\sin x}$$

Math that saved millions

Dr. Snow

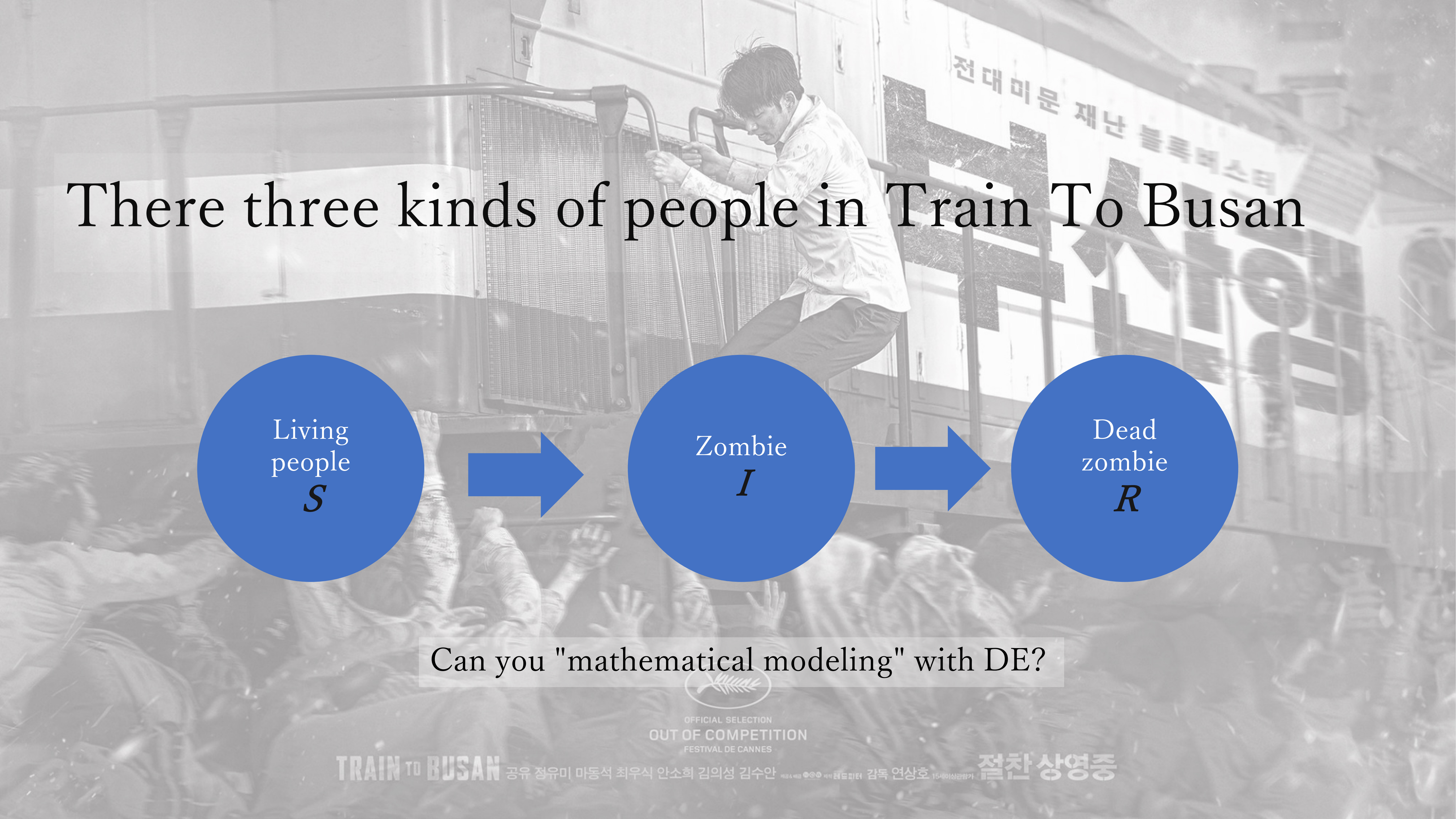


1854 Broad Street cholera outbreak



In cholera, people are divided into three categories

$\boxed{\text{Can you "mathematical modeling" with DE?}}$



There three kinds of people in Train To Busan

Living people S → Zombie I → Dead zombie R

Can you "mathematical modeling" with DE?

How are zombies infected?

- ➢infection growth rate?
- ➢infection growth rate related to…? $\frac{dS}{dt} = -\frac{\beta SI}{N}$
- ➢death growth rate?
- ➢death growth rate related to…? $\frac{dR}{dt} = \gamma I$

More equations

- Total Number of people $\mathrm{N} = \mathrm{S(t)} + \mathrm{I(t)} + R(t)$

- Changing rate $\dfrac{dI}{dt} = \dfrac{\beta SI}{N} - \gamma I$

Exercise

1. $\frac{dy^2}{dx^2} - 2\frac{dy}{dx} + 2y(x) = 0$

2. $\frac{dy^2}{dx^2} - 10\frac{dy}{dx} + 25y = 0$

Exercise

3. $\dfrac{dy^3}{dx^3} - 2\dfrac{dy^2}{dx^2} + 2y(x) = 0$

4. $y^{(4)} + 2y''' + 11y'' + 2y' + 10y = 0$

Exercise

5. $y'' + 4y' + 4y = 0$

6. $y''' - y'' - 8y' + 12 = 0$

7. $y''(x) + ay'(x) + by = 0, y(0) = 2, y'(0) = -3$

$and\ y(x) = e^{-x}\left(c \cos\sqrt{2}\, x + d \sin\sqrt{2}\, x\right)$

$a, b, c, d \in R, what\ are\ the\ values\ of\ a, b, c, d?$