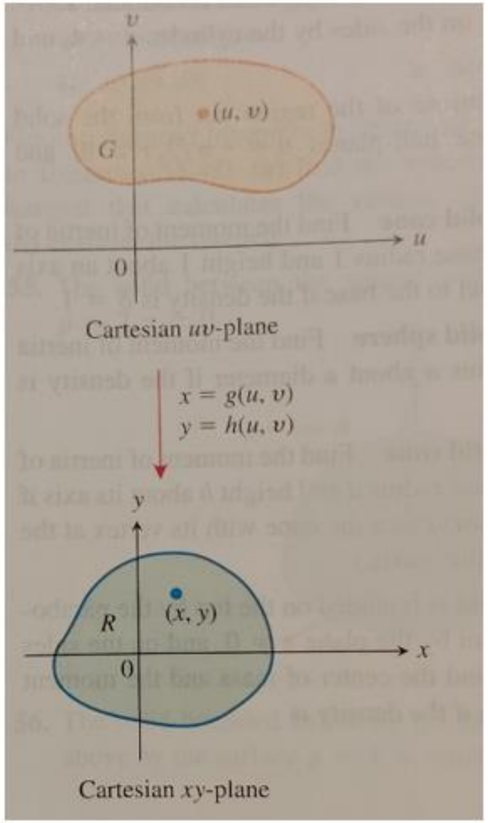
14-8 Substitutions in Multiple Integrals

師大工教一

Recall: Substitution $\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f\left(g(u)\right) g'(u)\,du$.

Substitutions in Double Integrals



$$x = g(u,v)$$
$$y = h(u,v)$$

R : **image** of G ,

G : **preimage** of R

Definition The **Jacobian determinant** or **Jacobian** of the coordinate

transformation $x = g(u,v), y = h(u,v)$ is $J(u,v) = \begin{vmatrix} \dfrac{\partial x}{du} & \dfrac{\partial x}{dv} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \dfrac{\partial x}{du}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{dv}\dfrac{\partial y}{\partial u}$.

Note: Symbol $J(u,v) = \dfrac{\partial(x,y)}{\partial(u,v)}$

※Derive the Jacobian of polar coordinates transformation

$x = r\cos\theta, y = r\sin\theta$.

Theorem 3—**Substitution for Double Integrals**

Suppose that $f(x,y)$ is continuous over the region R. Let G be the

preimage of R under the transformation $x=g(u,v), y=h(u,v)$, which is

assumed to be one-to-one on the interior of G. If the functions g and h have continuous first partial derivatives within the interior of G, then

$$\iint_R f(x,y)\,dx\,dy = \iint_G f\left(g(u,v),h(u,v)\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \qquad (2)$$

Ex1(p851) Find the Jacobian of the polar coordinate transformation $x = r\cos\theta$, $y = r\sin\theta$, and use Equation (2) to write the Cartesian integral $\iint\limits_R f(x, y)\,dxdy$ as a polar integral.

Ex2(p852) Evaluate $\int_0^4 \int_{x=\frac{y}{2}}^{x=\left(\frac{y}{2}\right)+1} \frac{2x-y}{2} dxdy$ by applying the transformation $u = \frac{2x-y}{2}$, $v = \frac{y}{2}$ and integrating over an appropriate region in the $uv-$ plane.

Ex3(p853) Evaluate $\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\, dy dx$.

Substitutions in Triple Integrals

Suppose that the region G in $uvw-$space is transformed one-to-one into the region D in $xyz-$space by $x=g(u,v,w), y=h(u,v,w), z=k(u,v,w)$. Then

$$\iiint_D F(x,y,z)\,dxdydz=\iiint_G H(u,v,w)\left|J(u,v,w)\right|dudvdw$$

, where $F\left(g(u,v,w),h(u,v,w),k(u,v,w)\right)=H(u,v,w)$.

Note: The **Jacobian determinant** $J(u,v,w)=\begin{vmatrix} \dfrac{\partial x}{du} & \dfrac{\partial x}{dv} & \dfrac{\partial x}{dw} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{dw} \\ \dfrac{\partial z}{du} & \dfrac{\partial z}{dv} & \dfrac{\partial z}{dw} \end{vmatrix}=\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$.

Derive the Jacobian of spherical coordinates transformation $x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, $z=\rho\cos\phi$.

HW14-8

- **HW: 6,9,12,13,22,23**