

8-4 Integration of Rational Functions by Partial Fractions

用部分分式求有理函數的積分

師大工教一

Recall: Rational functions $f(x) = \frac{P(x)}{Q(x)}$, where P, Q are polynomials.

If $\deg(P(x)) \geq \deg(Q(x))$, use division and transfer $f(x)$ into a polynomial and a proper fraction. Now consider the proper fraction part.

For example: Q: Evaluate $\int \frac{x^3 - x}{x^2 + x + 1} dx$.

By polynomial long division, we have $\int \frac{x^3 - x}{x^2 + x + 1} dx = \int x - 1 + \frac{-x + 1}{x^2 + x + 1} dx$.

The Method of Partial Fractions: Express $R(x)$ as the sum of $\frac{A}{(ax+b)^i}$ or

$$\frac{Ax+B}{(ax^2+bx+c)^j}.$$

Case I $Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_nx + b_n)$

Ex1(p475) Evaluate $\int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)} dx$.

Step I : $\frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+3)}$
 $= \frac{A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)}{(x-1)(x+1)(x+3)}$

$$\therefore x^2 + 4x + 1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)$$

$$\text{Plug } x=1 \Rightarrow 1+4+1 = A \cdot 2 \cdot 4 \Rightarrow 6 = 8A \quad A = \frac{3}{4}$$

$$\text{Plug } x=-1 \Rightarrow 1-4+1 = -2 = B \times (-2)(2) \quad B = \frac{1}{2}$$

$$\text{Plug } x=-3 \Rightarrow 9-12+1 = -2 = C \times (-4)(-2) \quad C = -\frac{1}{4}$$

Step II : Origin $= \int \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} - \frac{\frac{1}{4}}{x+3} dx = \frac{3}{4} \ln|x-1| + \frac{1}{2} \ln|x+1| - \frac{1}{4} \ln|x+3| + C$

Case II $Q(x) = (a_1x + b_1)^{i_1} (a_2x + b_2)^{i_2} \cdots (a_nx + b_n)^{i_n}$

Ex2(p476) Evaluate $\int \frac{6x+7}{(x+2)^2} dx$.

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} = \frac{A(x+2)+B}{(x+2)^2}$$

$$A=6 \quad B=-5$$

$$\begin{aligned} \text{origin} &= \int \frac{6}{x+2} - \frac{5}{(x+2)^2} dx \\ &= 6 \ln|x+2| + 5(x+2)^{-1} + C \end{aligned}$$

Case III $Q(x)$ contains $ax^2 + bx + c$, where $b^2 - 4ac < 0$

Ex4(p477) Evaluate $\int \frac{-2x+4}{(x^2+1)(x-1)^2} dx$.

$$\begin{aligned}\frac{-2x+4}{(x^2+1)(x-1)^2} &= \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1} \\ &= \frac{A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2}{(x^2+1)(x-1)^2} \\ -2x+4 &= A(x^3+x^2-1) + B(x^2+1) + (Cx+D)(x^2-2x+1)\end{aligned}$$

$$\begin{aligned}&= Ax^3 - Ax^2 + Ax - A + Bx^2 + B + Cx^3 - 2Cx^2 + Cx + Dx^2 - 2Dx + D \\ &= x^3(A+C) + x^2(-A+B-2C+D) + x(A+C-2D) + (-A+B+D) \\ A+C &= 0 \\ -A+B-2C+D &= 0 \\ A+C-2D &= -2 \\ -A+B+D &= 4\end{aligned}$$

$$\begin{aligned}4-2C &= 0 \quad \text{or } g(x) = \int \left(\frac{-2}{(x-1)} + \frac{1}{(x-1)^2} + \frac{2x+1}{x^2+1} \right) dx \\ C &= 2 \\ A &= -2 \\ B &= 1\end{aligned}$$

$$\begin{aligned}\int \frac{2x}{x^2+1} dx &= \int u^{-1} du = \ln|u| + C \\ \text{Let } u &= x^2+1 \quad = \ln|x^2+1| + C \\ du &= 2x dx \\ \int \frac{1}{x^2+1} dx &= \tan^{-1} x + C\end{aligned}$$

Ex5(p478) Evaluate $\int \frac{dx}{x(x^2+1)^2}$.

$$\begin{aligned}\frac{1}{x(x^2+1)^2} &= \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} \\ &= \frac{A(x^2+1)^2 + (Bx+C)(x^2+1) + (Dx+E)x}{x(x^2+1)^2}\end{aligned}$$

Ex6(p479) Find A, B, C in the equation

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

HW8-4

- HW:10,11,17,25,29,47