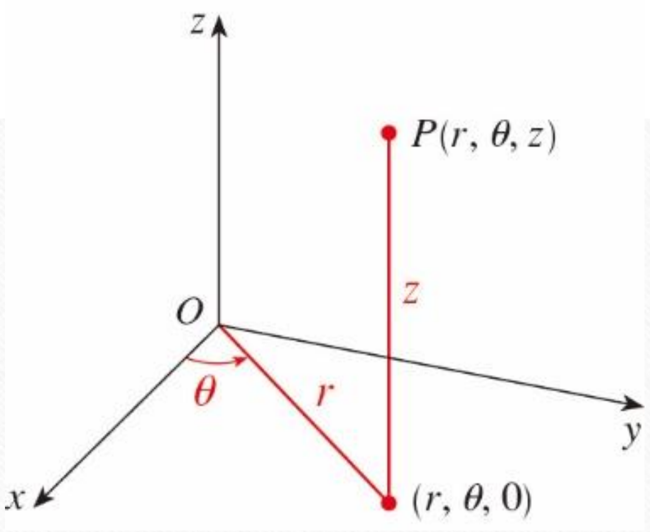
14-7 Triple Integrals in Cylindrical and Spherical Coordinates

師大工教一

Integration in Cylindrical Coordinates

Definition **Cylindrical coordinates** represent a point P in space by ordered triples (r,θ,z) in which

1. r and θ are polar coordinates for the vertical projection of P on the $xy-$plane, with $r \geq 0$, and
2. z is the rectangular vertical coordinate.



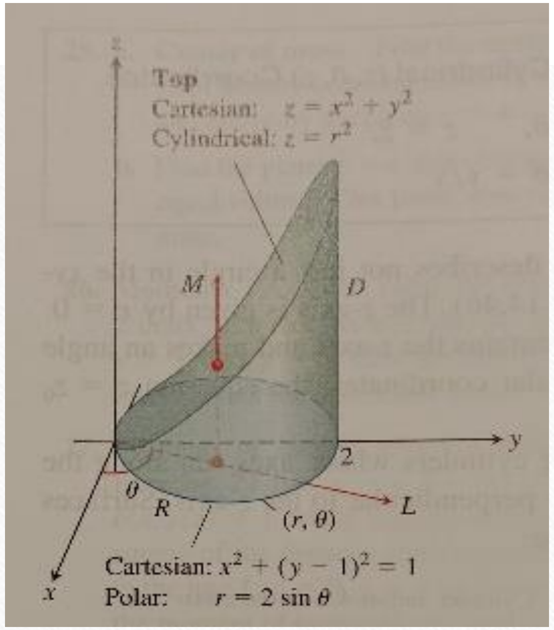
Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates

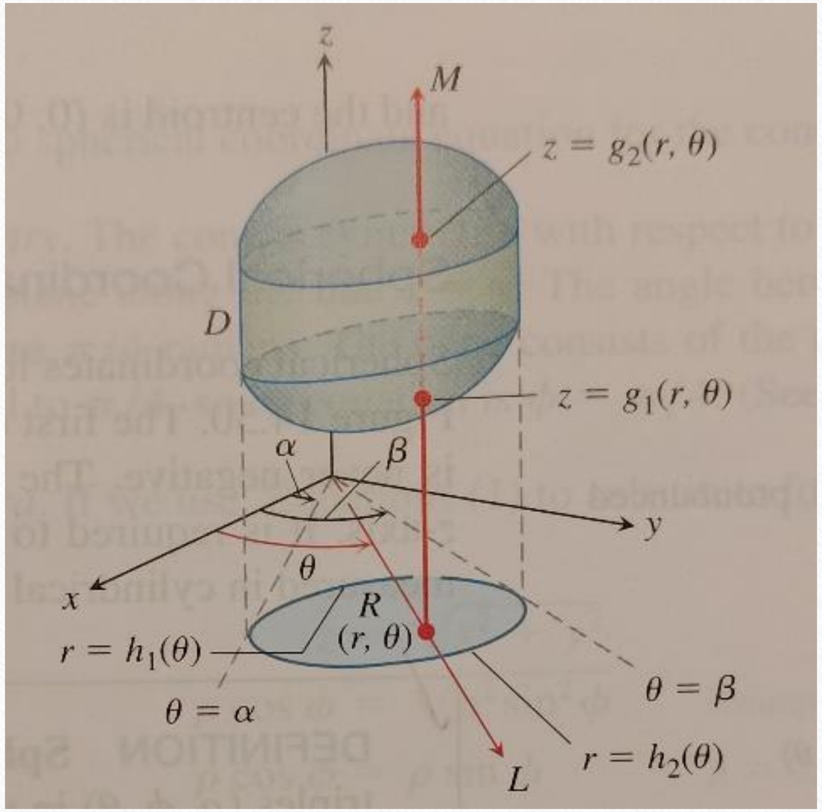
$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$

Formula: $\iiint\limits_D f\,dV = \iiint\limits_D f\,r\,dz\,dr\,d\theta$

Ex1(p839) Find the limits of integration in cylindrical coordinates for integrating a function $f(r,\theta,z)$ over the region D bounded below by the plane $z=0$, laterally by the circular cylinder $x^2+(y-1)^2=1$, and above by the paraboloid $z=x^2+y^2$.



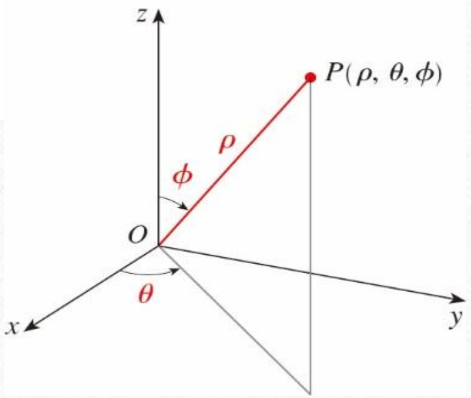


Formula: $\iiint\limits_D f(r,\theta,z)\,dV = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r=h_2(\theta)}\int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r,\theta,z)\,r\,dz\,dr\,d\theta$

Spherical Coordinates and Integration

Definition Spherical coordinates represent a point P in space by ordered triples (ρ,ϕ,θ) in which

1. ρ is the distance from P to the origin($\rho \geq 0$).
2. ϕ is the angle $\overrightarrow{OP}$ makes with the positive $z-$axis($0 \leq \phi \leq \pi$) .
3. θ is the angle from cylindrical coordinates.



Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$
\begin{aligned}
&r = \rho \sin \phi, \quad x = r \cos \theta = \rho \sin \phi \cos \theta, \\
&z = \rho \cos \phi, \quad y = r \sin \theta = \rho \sin \phi \sin \theta, \\
&\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}
\end{aligned}
$$

Ex3(p842) Find a spherical coordinate equation for the sphere

$x^2 + y^2 + (z-1)^2 = 1$.

Ex4(p843) Find a spherical coordinate equation for the cone $z = \sqrt{x^2 + y^2}$.

Formula: $\iiint\limits_D f(\rho,\phi,\theta)\,dV = \iiint\limits_D f(\rho,\phi,\theta)\,\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$.(Not natural!)

Ex5(p844) Find the volume of the “ice cream cone” D bounded above by the sphere $\rho = 1$ and bounded below by the cone $\phi = \dfrac{\pi}{3}$.

Coordinate Conversion Formulas

Cylindrical To Rectanglar	**Spherical To Rectangular**	**Spherical To Cylindrical**
$x = r\cos\theta$ $y = r\sin\theta$ $z = z$	$x = \rho\sin\phi\cos\theta$ $y = \rho\sin\phi\sin\theta$ $z = \rho\cos\phi$	$r = \rho\sin\phi$ $z = \rho\cos\phi$ $\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dxdydz \\ &= rdzdrd\theta \\ &= \rho^2 \sin\phi d\rho d\phi d\theta \end{aligned}$$

HW14-7

- **HW:7,14,15,27,28,33,43,47,55,59**