9-5 Absolute Convergence: The Ratio and Root Tests

比值　根式

師大工教一

Definition A series $\sum a_n$ **converges absolutely**(is **absolutely convergent**)

if the corresponding series of absolute value, $\sum |a_n|$, converges.

Theorem 12—The Absolute Convergence Test

If $\sum |a_n|$ converges, then $\sum a_n$ converges.

Ex1 Determine if the following series are absolutely convergent:

(a) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$ (b) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series, $p=2>1$)

$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$ converges absolutely.

consider $\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right|$

$\because 0 \le \frac{|\sin n|}{n^2} \le \frac{1}{n^2}$

$\sum \frac{1}{n^2}$ conv.

By Direct comparison Test

$\sum \frac{|\sin n|}{n^2}$ conv

$\Rightarrow \sum \frac{\sin n}{n^2}$ converges absolutely

The Ratio Test

Theorem 13—The Ratio Test

Let $\sum a_n$ be any series and suppose that $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \rho$.

$\rho > 1$ diverges

$\rho < 1$ converges absolutely

$\rho = 1$?

Then (a) the series *converges absolutely* if $\rho < 1$, (b) the series *diverges* if $\rho > 1$ or ρ is infinite, and (c) the series is *inconclusive* if $\rho = 1$.

Ex2 Investigate the convergence of the following series

階乘通常用 ratio test

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{2^n+5}{3^n}$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{2n!}{n!n!}$ (c) $\displaystyle\sum_{n=1}^{\infty}\frac{4^n n!n!}{(2n)!}$

$$\lim_{n\to\infty}\left|\frac{\frac{2^{n+1}+5}{3^{n+1}}}{\frac{2^n+5}{3^n}}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}+5}{2^n+5}\cdot\frac{\cancel{3^n}}{\cancel{3^{n+1}}_{\frac{1}{3}}}\right| = \lim_{n\to\infty}\left|\frac{2+\frac{5}{2^n}}{1+\frac{5}{2^n}}\cdot\frac{1}{3}\right| = \frac{2}{3} < 1$$

$\therefore$ By Ratio Test, $\sum_{n=0}^{\infty}\frac{2^n+5}{3^n}$ conv. absolutely.

$$\text{(c)}\ \lim_{n\to\infty}\frac{\frac{4^{n+1}\cancel{(n+1)!(n+1)!}}{\cancel{(2n+2)!}}}{\frac{\cancel{4^n n!n!}}{\cancel{2n!}}} = \lim_{n\to\infty}\frac{4(n+1)(n+1)}{(2n+1)(2n+2)} = 4\lim_{n\to\infty}\frac{n^2+2n+1}{4n^2+6n+2} = 4\lim_{n\to\infty}\frac{1+\frac{2}{n}+\frac{1}{n^2}}{4+\frac{6}{n}+\frac{2}{n^2}} = 1$$

$$= \frac{(2)^{2n}(1\cdot2\cdots n)(1\cdot2\cdots n)}{1\cdot2\cdot3\cdots2n} = \frac{(2\cdot4\cdots2n)(2\cdot4\cdots2n)}{1\cdot2\cdot3\cdots2n} = \frac{2\cdot4\cdot6\cdots2n}{1\cdot3\cdot5\cdots(2n-1)} \geq 2$$

$$\lim_{n\to\infty}\frac{4^n n!n!}{(2n)!} \neq 0 \Rightarrow \sum\frac{4^n n!n!}{(2n)!}\ \text{div.}$$

The Root Test

Eg. The ratio test fails $a_n = \begin{cases} \frac{n}{2^n}, & n \text{ odd} \\ \frac{1}{2^n}, & n \text{ even} \end{cases}$.

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ DNE

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \ell$

$\ell < 1$ conv. absolutely

$\ell > 1$ diverges

$\ell = 1$?

Theorem 14—The Root Test

Let $\sum a_n$ be any series and suppose that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho$.

Then (a) the series *converges absolutely* if $\rho < 1$, (b) the series *diverges* if $\rho > 1$ or ρ is infinite, and (c) the series is *inconclusive* if $\rho = 1$.

Ex3(p552) Consider the series with terms $a_n = \begin{cases} \dfrac{n}{2^n}, & n \text{ odd} \\ \dfrac{1}{2^n}, & n \text{ even} \end{cases}$. Does $\sum a_n$ converge?

$\frac{1}{2^n} \le a \le \frac{n}{2^n}$

$\lim_{n\to\infty} \sqrt[n]{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$

$\lim_{n\to\infty} \sqrt[n]{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{2} = \frac{1}{2}$

By Sandwich Them $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{2} < 1 \Rightarrow \sum a_n$ conv. absolutely.

Ex4(p552) Which of the following series converge, and which diverge?

(a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ (b) $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ (c) $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

$\lim_{n\to\infty} \sqrt[n]{n} = 1$

$\lim_{x\to\infty} x^{\frac{1}{x}} \overset{L'H}{=} 1$

$\lim_{n\to\infty} \sqrt[n]{\frac{n^2}{2^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}} = \frac{1}{2} < 1$

$\therefore \sum \frac{n^2}{2^n}$ conv.

HW9-5

- **HW:1,4,7,9,11,19,28,46**

106分

2. (5 points×6) Determine the convergence of the series.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^{0.3}}$ (b) $\sum_{n=1}^{\infty} \frac{4n}{2n^2+1}$ (c) $\sum_{n=1}^{\infty} \sin\frac{1}{n}$

(d) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ ratio (e) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n!}{1\cdot 3\cdot 5\cdots(2n-1)}$ root (f) $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$