2-6 Limits Involving Infinity; Asymptotes of Graphs

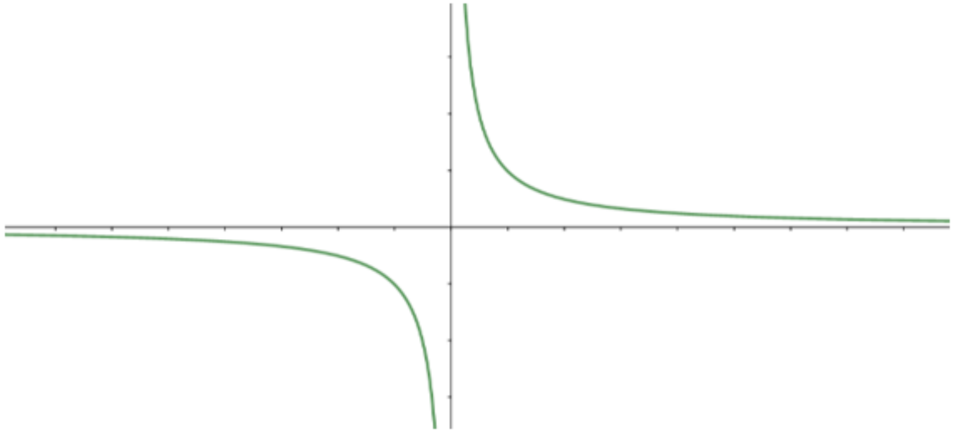
師大工教一

Observe $\lim_{x \to a} f(x) = L$ (i) $a = \pm\infty$ (ii) $L = \pm\infty$

※Finite Limits as $x \to \pm\infty$

(0與±∞互為倒數)

Ex1(p116): Show that (a) $\lim_{x \to \infty} \frac{1}{x} = 0$ (b) $\lim_{x \to -\infty} \frac{1}{x} = 0$



Theorem: Assume that c is a constant, $k > 0$, then $\lim\limits_{x \to \pm\infty} \frac{c}{x^k} = 0$.

Ex3(p117): Find (a) $\lim\limits_{x \to \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2}$ (b) $\lim\limits_{x \to -\infty} \frac{11x + 2}{2x^3 - 1}$

(a) $= \lim\limits_{x \to \infty} \dfrac{5 + \frac{8}{x} - \frac{3}{x^2}}{3 + \frac{2}{x^2}}$

$= \frac{5}{3}$

(b) $\lim\limits_{x \to -\infty} \dfrac{\frac{11}{x^2} + \frac{2}{x^3}}{2 - \frac{1}{x^3}}$

$= 0$

水平 漸近線

※Horizontal Asymptotes

Definition: A line $y=b$ is a **horizontal asymptote** of the graph of a function $y=f(x)$ if either $\lim_{x\to\infty} f(x)=b$ or $\lim_{x\to-\infty} f(x)=b$.

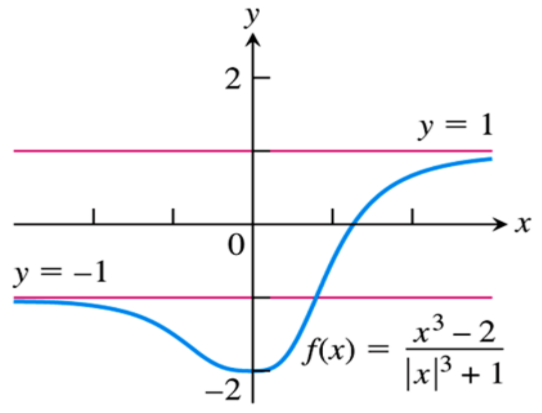
$y=b$ $\lim_{x\to\pm\infty} f(x)=b$

Ex4(p118) Find the horizontal asymptotes of of the graph of $f(x)=\dfrac{x^3-2}{|x|^3+1}$.

$$\lim_{x\to\infty}\frac{x^3-2}{|x|^3+1}=\lim_{x\to\infty}\frac{1-\frac{2}{x^3}}{1+\frac{1}{x^3}}=1$$

$$\lim_{x\to-\infty}\frac{x^3-2}{|x|^3+1}=\lim_{x\to-\infty}\frac{1-\frac{2}{x^3}}{-1+\frac{1}{x^3}}=-1$$

$\therefore$ $y=1$ and $y=-1$ are horizontal asym.



※Oblique Asymptotes (slant line asymptotes)

斜漸近線(用在有理函數，分子次數=分母次數)

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an **oblique** or **slant line asymptote**.

Ex10(p120) Find the oblique asymptote of the graph of $f(x) = \dfrac{x^2-3}{2x-4}$.

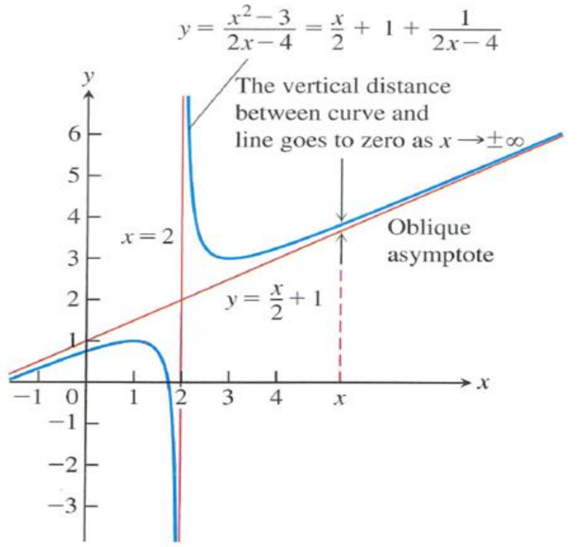
$$\begin{array}{r|rrr} & & \frac{1}{2} & +1 \\ 2 \;\; -4 & 1 & +0 & -3 \\ & 1 & -2 & \\ \hline & & 2 & -3 \\ & & 2 & -4 \\ \hline & & & 1 \end{array}$$

$$\frac{x^2-3}{2x-4} = \frac{1}{2}x+1+\frac{1}{2x-4}$$

$$\lim_{x\to\infty} f(x)-\left(\frac{1}{2}x+1\right) = \lim_{x\to\infty}\frac{1}{2x-4} = 0$$

同理 $\lim_{x\to\infty} f(x)-\left(\frac{1}{2}x+1\right)=0$

$\therefore$ $y=\frac{1}{2}x+1$ is a slant line asym.

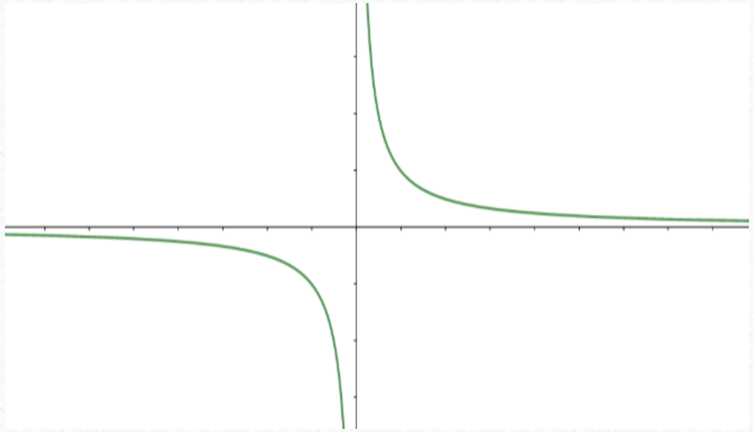


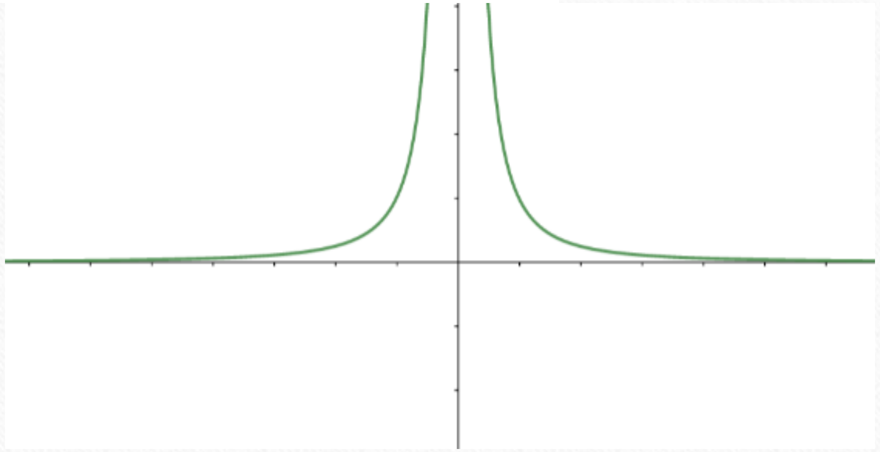
※Infinite Limits

Ex: Find the following limits:

(i) $\lim_{x\to 0^+}\frac{1}{x} = \infty$ (ii) $\lim_{x\to 0^-}\frac{1}{x} = -\infty$ (iii) $\lim_{x\to 0}\frac{1}{x}$ DNE.

(iv) $\lim_{x\to 0^+}\frac{1}{x^2} = \infty$ (v) $\lim_{x\to 0^-}\frac{1}{x^2} = \infty$ (vi) $\lim_{x\to 0}\frac{1}{x^2} = \infty$





※Vertical Asymptotes (分母 = 0 就有鉛直 Asym.)

鉛直 漸近線

A line $x = a$ is a **vertical asymptote** of the graph of the function $y = f(x)$ if

either $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$.

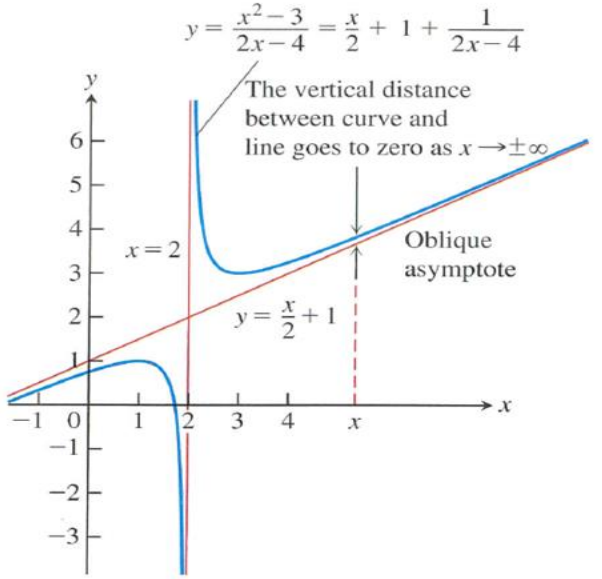
※Dominant Terms

Ex10: $f(x)=\dfrac{x^2-3}{2x-4}=\left(\dfrac{x}{2}+1\right)+\left(\dfrac{1}{2x-4}\right)$

We say that $\dfrac{x}{2}+1$ dominates when x approach ∞ or $-\infty$.

控制

We say that $\dfrac{1}{2x-4}$ dominates when x approach 2.



HW2-6

- **HW: 13,14,28,42,53,65,67,80,105,108**

111 2. (18 pts) Find the limit if it exists.

(a) $\lim_{x\to 0} \dfrac{\sin 3x - 3x + x^2}{\sin x \, \sin 2x}$ V (b) $\lim_{x\to\infty} \dfrac{3x}{5x + 2\sin x}$ (c) $\lim_{x\to 0} \left(\dfrac{1}{x\sqrt{1+x}} - \dfrac{1}{x} \right)$

(b) $\lim_{x\to\infty} \dfrac{\frac{3x}{x}}{\frac{5x}{x} + \frac{2\sin x}{x}} = \lim_{x\to\infty} \dfrac{3}{5 + 2\cdot\frac{\sin x}{x}} = \dfrac{3}{5}$

$-1 \leq \sin x \leq 1$

$\dfrac{-1}{|x|} \leq \dfrac{\sin x}{x} \leq \dfrac{1}{|x|}$

$\lim_{x\to\infty} \dfrac{-1}{|x|} = 0 = \lim_{x\to\infty} \dfrac{1}{|x|}$

By sandwich Theorem $\lim_{x\to\infty} \dfrac{\sin x}{x} = 0$

103 4 Let $f(x) = \dfrac{2x^3}{x^2-1}$

(有斜 asym. 就不會有水平 asym.)

(a) (10 points) find horizontal asymptotes (水平漸近線), vertical asymptotes (垂直漸近線) and slant line asymptotes (斜漸近線) of $f(x)$,

(b) (10 points) find all critical points (臨界點) of $f(x)$.

(a) $\dfrac{2x^3}{x^2-1} = 2x + \dfrac{2x}{x^2-1}$

$\lim_{x\to\pm\infty} \left(\dfrac{2x^3}{x^2-1} - 2x \right) = \lim_{x\to\pm\infty} \dfrac{2x}{x^2-1} = 0$

$\therefore$ $y = 2x$ is oblique asymptote.

$x^2-1 = (x+1)(x-1)$ $\lim_{x\to 1^+} \dfrac{2x^3}{x^2-1} = \infty \ \left(\dfrac{2}{0^+}\right)$

$\lim_{x\to(-1)^+} \dfrac{2x^3}{x^2-1} = \infty \ \left(\dfrac{2}{0^-}\right)$

$\Rightarrow$ $x=1$, $x=-1$ are vertical asymptotes.

105分 6. Let

$$f(x) = \ln \frac{3(x^2 - x - 2)}{x^2 - 4}.$$

(a) (6 points) Find the domain of the function $f(x)$. $(-\infty, -2) \cup (-1, 2) \cup (2, \infty)$

(b) (12 points) Find the vertical and horizontal asymptotes for the graph of the function $f(x)$.

$y = \ln 3$ horizontal asym.

$x = -2$, $x = -1$, vertical asymptotes.