1. Definite Integral

微分描述函數的變化率，而積分則累積函數隨變量的量. 對單變數函數 $f(x)$，在閉區間 $[a,b]$ 上的定積分可視為曲線下與 x 軸圍成的面積.

若 f 在 $[a,b]$ 上片段連續[1]，選任意分割 $P=\{x_0,\dots,x_n\}$，$\Delta x_k = x_k - x_{k-1}$，則：

- $R_{f,[a,b],P} = \sum_{k=1}^{n} f(x_k^*)\Delta x_k$ 為 **黎曼和 (Riemann Sum)**，其中 $x_k^* \in [x_{k-1}, x_k]$
- $U_{f,[a,b],P} = \sum_{k=1}^{n} M_k \Delta x_k$ 為 **上和 (Upper Sum)**，其中 $M_k = \max_{x\in[x_{k-1},x_k]} f(x)$
- $L_{f,[a,b],P} = \sum_{k=1}^{n} m_k \Delta x_k$ 為 **下和 (Lower Sum)**，其中 $m_k = \min_{x\in[x_{k-1},x_k]} f(x)$

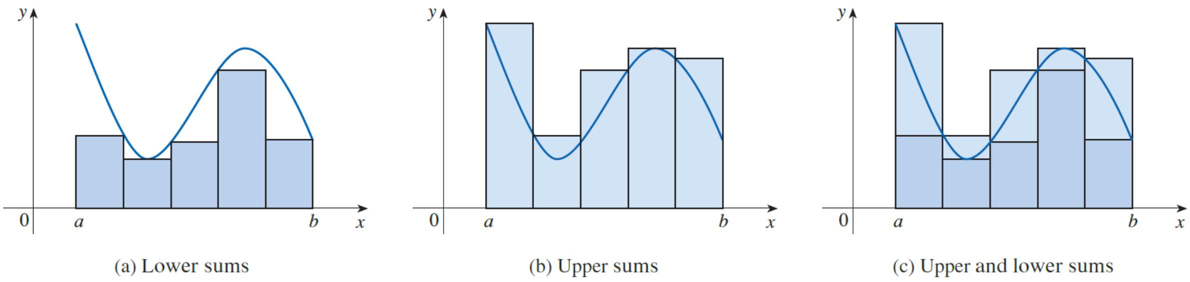

Figure 1: 上下和與黎曼和關係：上和由最大值決定，下和由最小值決定，黎曼和由任意取樣值決定，因此 $L_{f,[a,b],P} \le R_{f,[a,b],P} \le U_{f,[a,b],P}$.

當分割最長區間 $||P|| \to 0$，若上、下和極限相等為 A，則黎曼和極限亦趨向 A. 此時稱 f 在 $[a,b]$ 上**可積**，定義：

$$\int_a^b f(x)\,\mathrm{d}x = A$$

Definition 給定 f 在 $[a,b]$ 上，取任意分割 $P=\{x_0,\dots,x_n\}$，$x_i^* \in [x_{i-1}, x_i]$，若極限存在：

$$\lim_{||P||\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

則稱其為 f 的**定積分 (Definite Integral)**：

$$\lim_{||P||\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \int_a^b f(x)\,\mathrm{d}x$$

並說 f 在 $[a,b]$ 上**可積 (integrable)**。

[1] 片段連續 (Piecewise continuous) 指不連續點只有可數個

Question Sketch the graphs of the following functions and calculate their definite integrals if they are integrable.

▷ $f(x) = 1$, on $x \in [a, b]$

▷ $f(x) = \begin{cases} 1 & , x \in \mathbb{Q} \\ -1 & , x \notin \mathbb{Q} \end{cases}$, on $x \in [a, b]$

Question Rewrite the following limit as an integral equivalent using the Riemann sum.

▷ $\lim\limits_{n\to\infty} \dfrac{1}{n} \sum\limits_{k=1}^{n} \sqrt{\left(1 + \dfrac{k}{n}\right)} = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \sqrt{\left(1 + \dfrac{k}{n}\right)} \dfrac{1}{n} = \displaystyle\int_0^1 \sqrt{1+x}\ \mathrm{d}x$

▶ $\lim\limits_{n\to\infty} \dfrac{\sqrt{n^2+1} + \sqrt{n^2+4} + \sqrt{n^2+9} + \cdots + \sqrt{n^2+(n-1)^2} + \sqrt{n^2+n^2}}{n^2}$

$= \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \dfrac{\sqrt{n^2+k^2}}{n^2}$

$= \lim\limits_{n\to\infty} \dfrac{1}{n} \sum\limits_{k=1}^{n} \sqrt{1 + \left(\dfrac{k}{n}\right)^2}$

$= \displaystyle\int_0^1 \sqrt{1+x^2}\ \mathrm{d}x$

Property 給定 $f(x)$, $g(x)$ 皆是在 $[a, b]$ 上可積的函數，且係數 $c \in \mathbb{R}$ 為一常數. 那麼：

- $\displaystyle\int_a^b c \cdot f(x)\ \mathrm{d}x = c \cdot \int_a^b f(x)\ \mathrm{d}x$
- $\displaystyle\int_a^b \big(f(x) + g(x)\big)\ \mathrm{d}x = \int_a^b f(x)\ \mathrm{d}x + \int_a^b g(x)\ \mathrm{d}x$
- $\displaystyle\int_a^b f(x) \cdot g(x)\ \mathrm{d}x = \neq \int_a^b f(x)\ \mathrm{d}x \cdot \int_a^b g(x)\ \mathrm{d}x$
- $\displaystyle\int_a^b \frac{f(x)}{g(x)}\ \mathrm{d}x = \neq \frac{\int_a^b f(x)\ \mathrm{d}x}{\int_a^b g(x)\ \mathrm{d}x}$
- $\displaystyle\int_a^c f(x)\ \mathrm{d}x = \int_a^b f(x)\ \mathrm{d}x + \int_b^c f(x)\ \mathrm{d}x$
- 若 $f(x)$ 為奇函數，$\displaystyle\int_{-a}^a f(x)\ \mathrm{d}x = 0$
- 若 $f(x)$ 為偶函數，$\displaystyle\int_{-a}^a f(x)\ \mathrm{d}x = 2\int_0^a f(x)\ \mathrm{d}x$

2. Fundamental Theorem of Calculus, F.T.C.

Theorem. **(微積分基本定理第一部分, F.T.C. I)** 給定在區間 $[a, b]$ 上的可積函數 $f(x)$，若 $f(x)$ 在 $[a, b]$ 上連續，且 $F'(x) = f(x)$，那麼：

$$\int_a^b f(x)\,\mathrm{d}x = F(x)\Big|_{x=a}^{x=b} = F(b) - F(a)$$

並且我們稱 $F(x)$ 是 $f(x)$ 的**反導函數 (Anti-Derivative)**.

Question Calculate the folloing integral.

- ▷ $\displaystyle\int_a^b x^p\,\mathrm{d}x = \frac{1}{p+1}x^{p+1}\Big|_{x=a}^{x=b} = \frac{b^{p+1}}{p+1} - \frac{a^{p+1}}{p+1}$
- ► $\displaystyle\int_0^5 2x + 5\,\mathrm{d}x = x^2 + 5x\Big|_{x=0}^{x=5} = (5^2 + 5\cdot 5) - (0^2 + 5\cdot 0) = 50$
- ▷ $\displaystyle\int_0^1 2^x \ln 2\,\mathrm{d}x = 2^x\Big|_{x=0}^{x=1} = 2^1 - 2^0 = 2 - 1 = 1$
- ► $\displaystyle\int_0^1 3^x\,\mathrm{d}x = \frac{3^x}{\ln 3}\Big|_{x=0}^{x=1} = \frac{3^1}{\ln 3} - \frac{3^0}{\ln 3} = \frac{2}{\ln 3}$
- ▷ $\displaystyle\int_0^{\frac{\pi}{2}} \cos x\,\mathrm{d}x = \sin x\Big|_{x=0}^{x=\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$
- ► $\displaystyle\int_0^{\frac{\pi}{2}} \sin x\,\mathrm{d}x = -\cos x\Big|_{x=0}^{x=\frac{\pi}{2}} = \Big(-\cos\frac{\pi}{2}\Big) - \Big(-\cos 0\Big) = (-0) - (-1) = 1$

Theorem. **(微積分基本定理第二部分, F.T.C. II)** 給定在區間 $[a, b]$ 上的連續函數 $f(x)$，那麼：

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$$

更一般的，我們有：

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{g(x)}^{h(x)} f(t)\,\mathrm{d}t = f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)$$

Question Calculate the following expression.

- ▷ $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{x^2} e^{5y}\,\mathrm{d}y = e^{5x^2}\cdot(x^2)' - e^{5x} = 2xe^{5x^2} - e^{5x}$

- ▶ $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_{e^1}^{e^x} \ln u \,\mathrm{d}u = \ln e^x \cdot (e^x)' - 0 = xe^x$

- ▷ Given $\displaystyle\int_{x^2}^{2} f(t^3)\,\mathrm{d}t = \frac{1}{2}\sin\frac{\pi}{2}x$, find $f(1)$.

  $\overset{\frac{\mathrm{d}}{\mathrm{d}x}}{\Rightarrow} 0 - f\left((x^2)^3\right)\cdot 2x = \dfrac{\pi}{4}\cos\dfrac{\pi}{2}x$

  $\Rightarrow -f(1)\cdot 2 = \dfrac{\pi}{4}\cos\dfrac{\pi}{2} = 0,\ \text{for } x = 1$

  $\Rightarrow f(1) = 0$

- ▶ Given $\displaystyle\int_0^{x^5} f(2t^2 - t + 1)\,\mathrm{d}t = \ln x$, find $f(2)$.

  $\overset{\frac{\mathrm{d}}{\mathrm{d}x}}{\Rightarrow} f\left(2(x^5)^2 - x^5 + 1\right)\cdot 5x^4 - 0 = \dfrac{1}{x}$

  $\Rightarrow f\left(2x^{10} - x^5 + 1\right) = \dfrac{1}{5x^5}$

  $\Rightarrow f\left(2t^2 - t + 1\right) = \dfrac{1}{5t}$

  $\Rightarrow f\left(2\right) = \dfrac{1}{5},\ \text{for } t = 1$

3. Indefinite Integral & Anti-Derivative

為了表示所有反導函數，我們使用 **不定積分（Indefinite Integral）**：

$$\int f(x)\,\mathrm{d}x = F(x) + C \quad\iff\quad F'(x) = f(x),$$

其中 C 為積分常數.

Remark 定積分的結果是數值、不定積分的結果是一系列的函數、而反導函數是不定積分之中的單一個函數.

Remark 不定積分記得 $+C$、記得 $+C$、記得 $+C$、記得 $+C$、記得 $+C$、記得 $+C$.

Question Calculate the following integral.

- ▷ $\displaystyle\int \mathrm{d}x = \int 1\,\mathrm{d}x = x + C$

- ▶ $\displaystyle\int t^5 + \sqrt{t}\,\mathrm{d}t = \frac{t^6}{6} + \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C$

- ▷ $\displaystyle\int \sec^2 x\,\mathrm{d}x = \tan x + C$

- ▶ $\displaystyle\int \sec x\tan x\,\mathrm{d}x = \sec x + C$

- ▷ $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \sin^{-1} x + C$
- ▶ $\displaystyle\int \frac{1}{1+x^2}\,\mathrm{d}x = \tan^{-1} x + C$
- ▷ $\displaystyle\int \frac{2x}{x^2}\,\mathrm{d}x = \int \frac{(x^2)'}{x^2}\,\mathrm{d}x = \ln|x^2| + C$
- ▶ $\displaystyle\int \frac{\cos x}{1+\sin x}\,\mathrm{d}x = \frac{(1+\sin x)'}{1+\sin x}\,\mathrm{d}x = \ln|1+\sin x| + C$

4. L'Hôpital's rule

Theorem. (洛必達法則, L'Hôpital's rule) 設 $f(x), g(x)$ 在 $[a,b]$ 上連續、在 (a,b) 上可微，若 $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = 0$，且 $g(x) \neq 0$ 則：

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

Question Calculate the following limit.

- ▷ $$\begin{aligned}\lim_{x\to 0}\frac{\tan x - x}{x^3} &\overset{L}{=} \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2}\\ &= \frac{1}{3}\lim_{x\to 0}\frac{\tan^2 x}{3x^2}\\ &= \frac{1}{3}\lim_{x\to 0}\frac{\sin^2 x}{x^2}\cdot\frac{1}{\cos^2 x} = \frac{1}{3}\end{aligned}$$
- ▶ $$\begin{aligned}\lim_{x\to 0^+}\frac{e^x - 1 - x}{x^2} &\overset{L}{=} \lim_{x\to 0^+}\frac{e^x - 1}{2x}\\ &= \lim_{x\to 0^+}\frac{e^x}{2} = \frac{1}{2}\end{aligned}$$
- ▷ $$\begin{aligned}\lim_{x\to 0^+} x\ln x &= \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}\\ &\overset{L}{=} \lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{-1}{x^2}}\\ &= \lim_{x\to 0^+} -x = 0\end{aligned}$$
- ▶ $$\begin{aligned}\lim_{x\to 0^+} x^2\ln x &= \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x^2}}\\ &\overset{L}{=} \lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{-2}{x^3}}\\ &= \lim_{x\to 0^+} -\frac{1}{2}x^2 = 0\end{aligned}$$

$$\triangleright \lim_{x\to\infty}\left(1+\frac{\pi}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{\pi}{x}\right)^x}$$
$$= e^{\lim_{x\to\infty}\ln\left(1+\frac{\pi}{x}\right)^x}$$
$$= e^{\lim_{x\to\infty}\frac{\ln\left(1+\frac{\pi}{x}\right)}{\frac{1}{x}}}$$
$$\overset{L}{=} e^{\lim_{x\to\infty}\frac{\frac{-\pi/x^2}{1+\pi/x}}{-1/x^2}}$$
$$= e^{\lim_{x\to\infty}\frac{\pi}{1+\pi/x}} = e^{\pi}$$

$$\blacktriangleright \lim_{x\to 0^+}\left(e^{2x}+2x\right)^{\frac{2}{x}} = \lim_{x\to 0^+} e^{\ln\left(e^{2x}+2x\right)^{\frac{2}{x}}}$$
$$= e^{\lim_{x\to 0^+}\ln\left(e^{2x}+2x\right)^{\frac{2}{x}}}$$
$$= e^{\lim_{x\to 0^+}\frac{2\ln\left(e^{2x}+2x\right)}{x}}$$
$$\overset{L}{=} e^{\lim_{x\to 0^+}\frac{2\cdot\frac{2e^{2x}+2}{e^{2x}+2x}}{1}}$$
$$= e^{\lim_{x\to 0^+}\frac{4(e^{2x}+1)}{e^{2x}+2x}}$$
$$= e^{\frac{4(1+1)}{1+0}} = e^{8}$$

$$\triangleright \lim_{x\to 0}\frac{\int_{\sqrt{x}}^{0}\sin^2 t\,\mathrm{d}t}{\sqrt{x^3}} = \lim_{x\to 0}\frac{-\int_0^{\sqrt{x}}\sin^2 t\,\mathrm{d}t}{\sqrt{x^3}}$$
$$\overset{L}{=} \lim_{x\to 0}\frac{-\sin^2\left(\sqrt{x}\right)\cdot\frac{1}{2\sqrt{x}}}{\frac{3}{2}\sqrt{x}}$$
$$= \lim_{x\to 0}\frac{-\sin^2\left(\sqrt{x}\right)}{3x}$$
$$= \lim_{x\to 0}-\frac{1}{3}\left(\frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}}\right) = -\frac{1}{3}$$

$$\blacktriangleright \lim_{x\to -\infty}\frac{\int_{x^2}^{0}\tan^{-1} t\,\mathrm{d}t}{x^2} = \;= \lim_{x\to -\infty}\frac{-\arctan(x^2)\cdot 2x}{2x}$$
$$= \lim_{x\to -\infty}\left(-\arctan(x^2)\right) = \frac{\pi}{2}$$

Next week: 變數變換、三角代換、部分分式、分部積分、瑕積分