12-1 Curves in Space and Their Tangents

師大工教一

When a particle move through space during a time interval I , we think of the particle's coordinates as functions defined on I :

$$x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in I$$

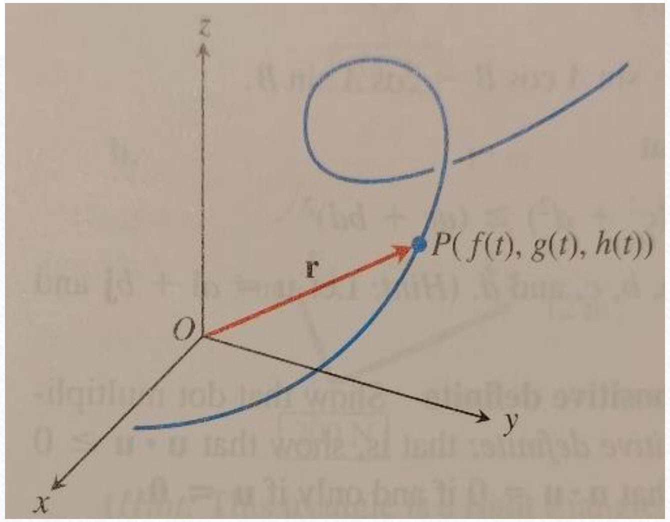
The points $(x, y, z) = (f(t), g(t), h(t)), t \in I$, make up the **curve** in space that we call the particle's **path**.

向量

Vector Form

$$\vec{r}(t)=\overrightarrow{OP}=f(t)\vec{i}+g(t)\vec{j}+h(t)\vec{k} \qquad (2)$$

The above equation defines $\vec{r}$ as a **vector-valued function** or **vector function** on a domain set D is a rule that assigns a vector in space to each element in D. The functions f,g,h are called the **component functions** (or components) of the position function.



Ex1(p681) Graph the vector function $\vec{r}(t)=(\cos t)\vec{i}+(\sin t)\vec{j}+t\vec{k}$. This graph is called the helix(from the ancient Greek word for “spiral”).

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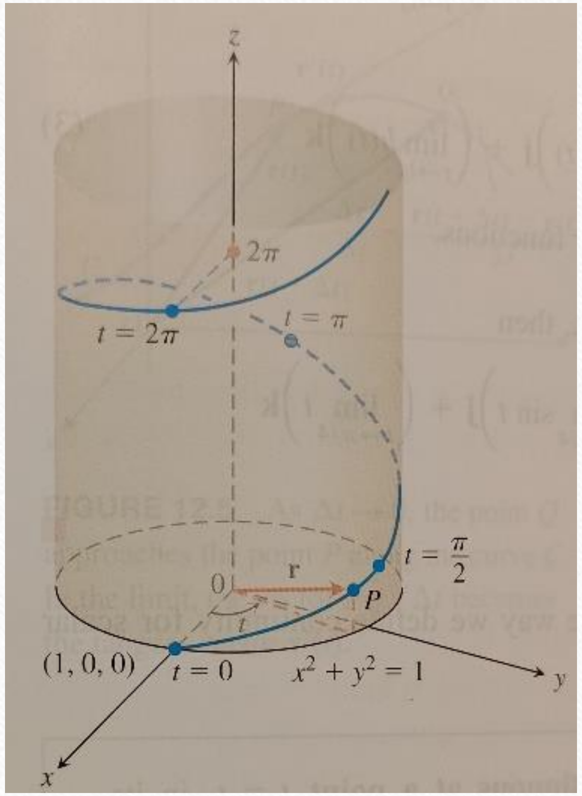


Limits and Continuity

Definition Let $\vec{r}(t)=f(t)\vec{i}+g(t)\vec{j}+h(t)\vec{k}$ be a vector function with domain D, and let $\vec{L}$ be a vector. We say that $\vec{r}$ has **limit** $\vec{L}$ as t approaches t_0 and write $\lim_{t\to t_0}\vec{r}(t)=\vec{L}$, if , for every number $\varepsilon>0$, there exists a corresponding number $\delta>0$ such that, for all $t\in D$, $\left|\vec{r}(t)-\vec{L}\right|<\varepsilon$ whenever $0<\left|t-t_0\right|<\delta$. Or $\lim_{t\to t_0}\vec{r}(t)=\left(\lim_{t\to t_0}f(t)\right)\vec{i}+\left(\lim_{t\to t_0}g(t)\right)\vec{j}+\left(\lim_{t\to t_0}h(t)\right)\vec{k}$.

Ex2(p682) If $\vec{r}(t)=(\cos t)\vec{i}+(\sin t)\vec{j}+t\vec{k}$, find $\lim_{t\to\frac{\pi}{4}}\vec{r}(t)$.

$$\lim_{t\to\frac{\pi}{4}}\vec{r}(t) = \left(\lim_{t\to\frac{\pi}{4}}\cos t\right)\vec{i}+\left(\lim_{t\to\frac{\pi}{4}}\sin t\right)\vec{j}+\left(\lim_{t\to\frac{\pi}{4}} t\right)\vec{k}$$

$$= \frac{\sqrt{2}}{2}\vec{i}+\frac{\sqrt{2}}{2}\vec{j}+\frac{\pi}{4}\vec{k}$$

Definition A vector function $\vec{r}(t)$ is **continuous at a point** $t=t_0$ in its domain if $\lim_{t\to t_0}\vec{r}(t)=\vec{r}(t_0)$. The function is **continuous** if it is continuous at every point in its domain.

Ex3(p682) (a) If each component function is continuous, then the space curve is also continuous.

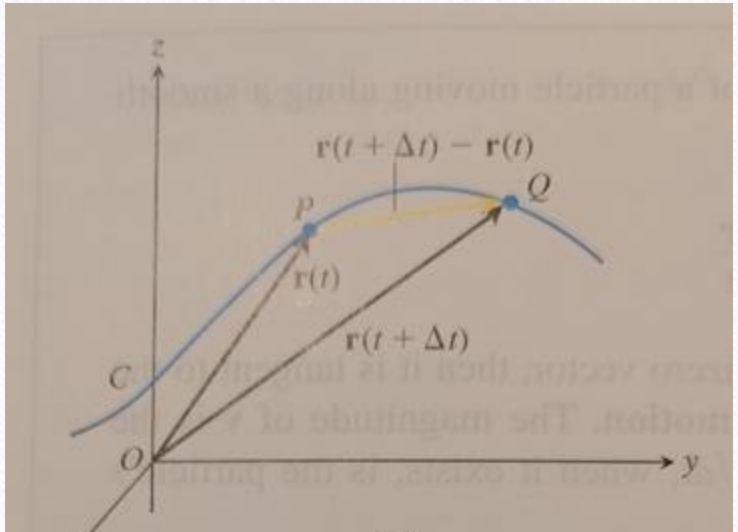
(b) The function $\vec{g}(t)=(\cos t)\vec{i}+(\sin t)\vec{j}+[t]\vec{k}$ is discontinuous at every integer, where $[t]$ is the greatest integer function.

Derivatives and Motion

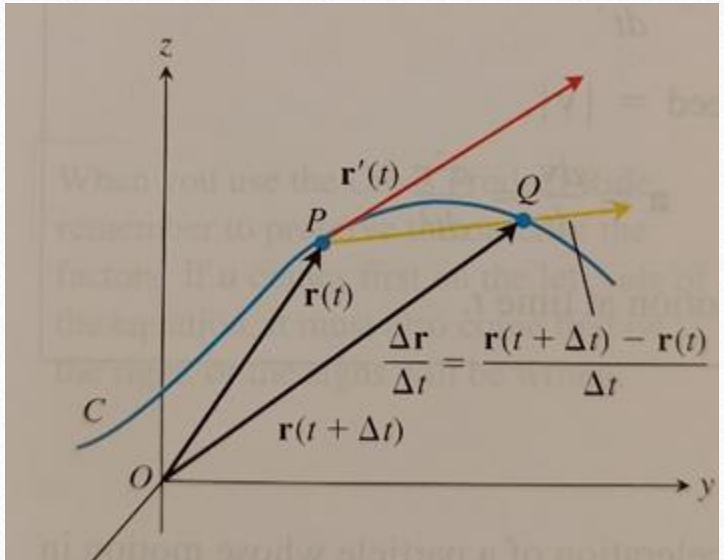
$$
\begin{aligned}
\Delta\vec{r} &= \vec{r}(t+\Delta t) - \vec{r}(t) \\
&= \left[f(t+\Delta t)\vec{i} + g(t+\Delta t)\vec{j} + h(t+\Delta t)\vec{k} \right] - \left[f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k} \right] \\
&= \left[f(t+\Delta t) - f(t) \right]\vec{i} + \left[g(t+\Delta t) - g(t) \right]\vec{j} + \left[h(t+\Delta t) - h(t) \right]\vec{k}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} &= \left[\lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} \right]\vec{i} + \left[\lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t} \right]\vec{j} + \left[\lim_{\Delta t \to 0} \frac{h(t+\Delta t) - h(t)}{\Delta t} \right]\vec{k} \\
&= \left[\frac{df}{dt} \right]\vec{i} + \left[\frac{dg}{dt} \right]\vec{j} + \left[\frac{dh}{dt} \right]\vec{k}
\end{aligned}
$$

Definition The vector function $\vec{r}(t)=f(t)\vec{i}+g(t)\vec{j}+h(t)\vec{k}$ has a **derivative (is differentiable) at** t if f,g,h have derivatives at t. The derivative is the vector function $\vec{r}\,'(t)=\dfrac{d\vec{r}}{dt}=\lim\limits_{\Delta t\to 0}\dfrac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t}=\dfrac{df}{dt}\vec{i}+\dfrac{dg}{dt}\vec{j}+\dfrac{dh}{dt}\vec{k}$.



(a)



(b)

A vector function $\vec{r}$ is **differentiable** if it is differentiable at every point in its domain. The curve traced by $\vec{r}$ is **smooth** if $\dfrac{d\vec{r}}{dt}$ is continuous and never $\vec{0}$, that is, if f, g, h have continuous first derivatives that are not simultaneously 0. A curve that is made up of a finite number of smooth curves pieced together in a contiuous fashion is called **piecewise smooth**.

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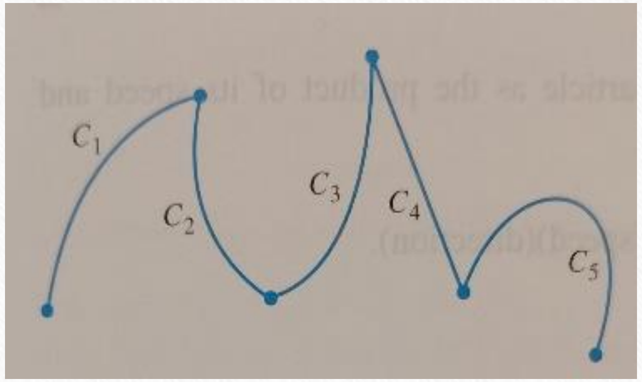


Definition If $\vec{r}$ is the position vector of a particle along a smooth curve in space, then $\vec{v}(t) = \dfrac{d\vec{r}}{dt}$ is the particle's **velocity vector**. If $\vec{v}$ is a nonzero

速度

vector, then it is tangent to the curve, and its direction is the **direction of motion**. The magnitude of $\vec{v}$ is the particle's **speed**, and the derivative $\vec{a} = \dfrac{d\vec{v}}{dt}$, when it exists, is the particle's **acceleration vector**. In summary,

1. Velocity is the derivative of position: $\vec{v} = \dfrac{d\vec{r}}{dt}$.

2. Speed is the magnitude of velocity: Speed=$\left|\vec{v}\right|$.

3. Acceleration is the derivative of velocity: $\vec{a} = \dfrac{d\vec{v}}{dt} = \dfrac{d^2\vec{r}}{dt^2}$.

4. The unit vector $\dfrac{\vec{v}}{\left|\vec{v}\right|}$ is the direction of motion at time t.

Ex4(p684) Find the velocity, speed, and acceleration of a particle whose motion in space is given by the motion vector $\vec{r}(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j} + 5\cos^2 t\,\vec{k}$.

Sketch the velocity vector $\vec{v}\left(\dfrac{7\pi}{4}\right)$.

$\bar{r}(t)$

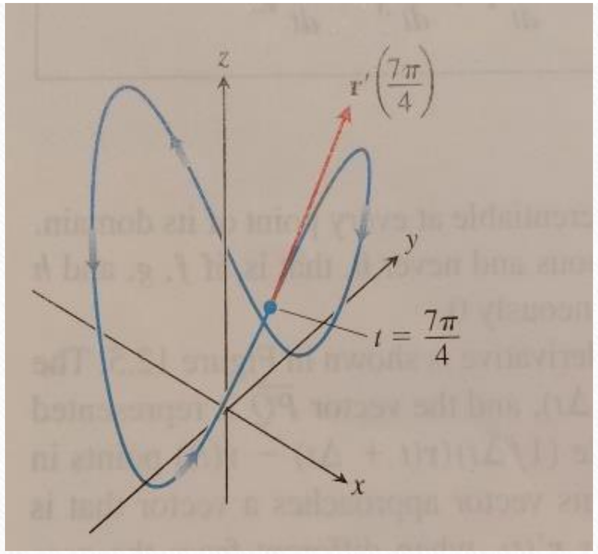
$\vec{V}(t) = \vec{r}\,'(t) = -2\sin t\,\vec{i} + 2\cos t\,\vec{j} - 10\cos t\sin t\,\vec{k}$

$|\vec{V}(t)| = \sqrt{4\sin^2 t + 4\cos^2 t + 100\cos^2 t\sin^2 t} = \sqrt{4 + 100\sin^2 t\cos^2 t}$

$\bar{a}(t) = \vec{v}\,'(t) = -2\cos t\,\vec{i} - 2\sin t\,\vec{j} - 10(-\sin^2 t + \cos^2 t)\vec{k}$

$= -2\cos t\,\vec{i} - 2\sin t\,\vec{j} + (10\sin^2 t - 10\cos^2 t)\vec{k}$

$= -2\cos t\,\vec{i} - 2\sin t\,\vec{j} + 10\cos 2t\,\vec{k}$



1. *Constant Function Rule*: $\frac{d}{dt}\vec{C} = \vec{0}$.

2. *Scalar Multiple Rule*: $\frac{d}{dt}\left[c\vec{u}(t)\right] = c\vec{u}'(t)$,

$$\frac{d}{dt}\left[f(t)\vec{u}(t)\right] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$$

3. *Dot Product Rule*: $\frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$

4. *Cross Product Rule*: $\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$

5.*Chain Rule*: $\frac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t)\vec{u}'(f(t))$

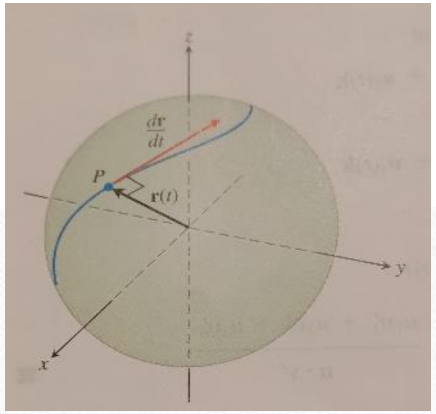
When a particle moves on a sphere centered at the origin, the position vector has a constant length equal to the radius of the sphere.

$$\vec{r}(t)\cdot\vec{r}(t)=\left|\vec{r}(t)\right|^2=c^2$$

$$\frac{d}{dt}\left[\vec{r}(t)\cdot\vec{r}(t)\right]=\vec{r}'(t)\cdot\vec{r}(t)+\vec{r}(t)\cdot\vec{r}'(t)=2\vec{r}'(t)\cdot\vec{r}(t)=0$$

If $\vec{r}$ is differentiable vector function of t and the length of $\vec{r}(t)$ is constant,

then $\vec{r}\cdot\dfrac{d\vec{r}}{dt}=0$.



HW12-1

- **HW: 1,4,9,12,16,18**