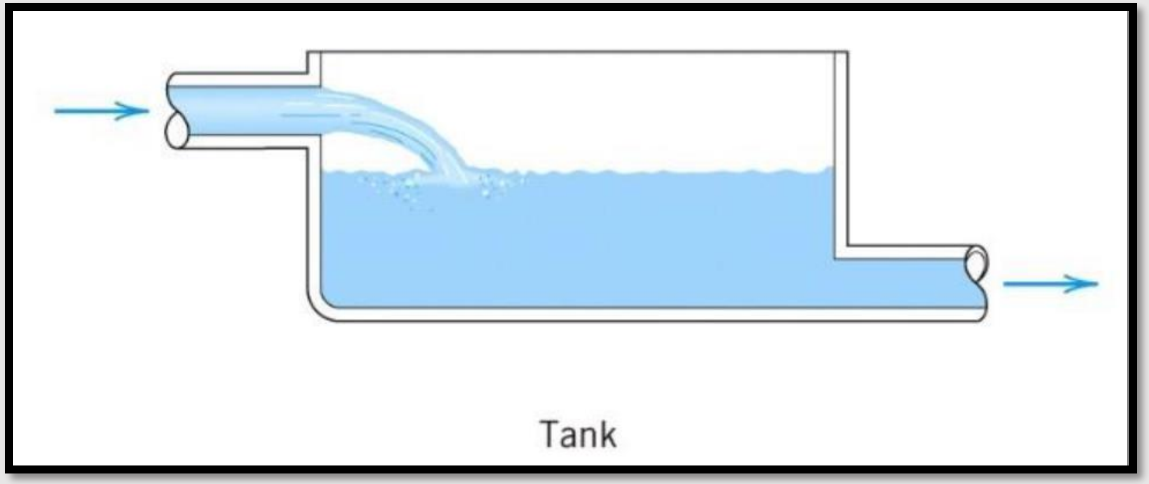
Exact Method
And
Linear Equation

Reference CENGAGE Learning

Review of Separable Method : Mixing Tank

The tank contains 1000 gal of water in which initially 100 lb of salt is dissolved. Brine runs in at a rate of 10 gal/min, and each gallon contains 5 lb of dissolved salt. The mixture in the tank is kept uniform by stirring. Brine runs out at 10 gal/min. Find the amount of salt in the tank at any time t.



Solution

- Independent and depend variables?
- y(t)= the amount of salt in the tank at time t
- Flow rate ?
- 5 lb times 10 gal gives an inflow of 50 lb of salt. Now, the outflow is 10 gal of brine.

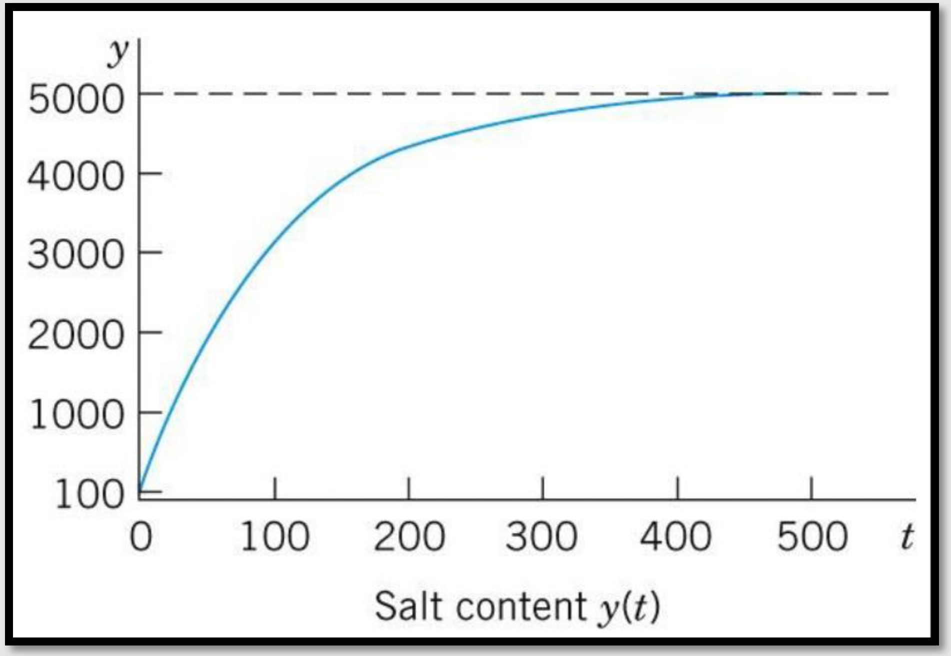
$$y' = 50 - 0.01y = -0.01(y - 5000).$$

Solution (cont'd)

$$\frac{dy}{y-5000} = -0.01\mathrm{dt}, \ln \mid y - 5000 \mid = -0.01t + c^*, y - 5000ce^{-0.01t}$$

$$100 - 5000 = ce^0 = c$$

$$y(t) = 5000 - 4900e^{-0.01t}$$



Salt content $y(t)$

Exact Method

General form of first order DE $M(x,y)dx + N(x,y)dy = 0$

Exact Equation

$$\boxed{\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y)}$$

Idea behind Exact Method

- The concept of partial differentiation

$$df(x,y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

- When $f(x, y) = c$ (constant),

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

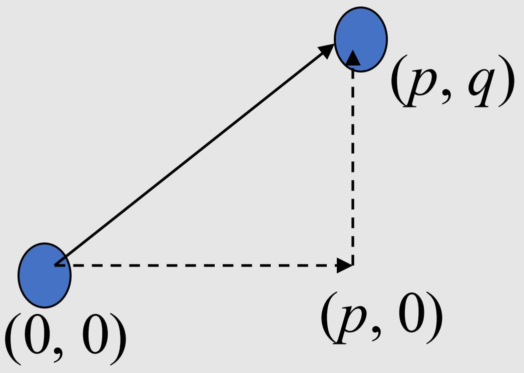
$$df(x,y,z) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$df(x_1, x_2, x_3, \cdots, x_k) = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 + \cdots\cdots + \frac{\partial f}{\partial x_k}dx_k$$

A hill ascends *a* meters for every meter he moves East. On the other hand, the hill ascends *b* meters for every meter he moves West. Now a man is standing at (0, 0). What is the relative height of the hill at coordinate (p,q), which is to the northeast of the man?

$$a \times p + b \times q$$

$$df(x,y) = \underset{a}{\boxed{\frac{\partial f(x,y)}{\partial x}}}\,\underset{p}{\boxed{dx}} + \underset{b}{\boxed{\frac{\partial f(x,y)}{\partial y}}}\,\underset{q}{\boxed{dy}}$$



[Definition: Exact Equation]

$$\boxed{M(x,y)dx + N(x,y)dy = 0}$$

with

$$\boxed{\frac{\partial f(x,y)}{\partial x} = M(x,y)} \quad \text{and} \quad \boxed{\frac{\partial f(x,y)}{\partial y} = N(x,y)} \quad ,$$

The method for checking whether the DE is an exact equation:

$$\boxed{\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}}$$

(Proof): If $\dfrac{\partial f(x,y)}{\partial x} = M(x,y)$ and $\dfrac{\partial f(x,y)}{\partial y} = N(x,y)$,

then $$\frac{\partial M(x,y)}{\partial y} = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x,y) = \frac{\partial N(x,y)}{\partial x}$$

$$M(x,y)dx + N(x,y)dy = 0$$

$$\downarrow$$

$$\frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy = 0$$

$$\downarrow$$

$$df(x,y) = 0,$$

$$\downarrow$$

$$f(x,y) = c$$

Examples - Example 1

$$\boxed{2xydx+(x^2-1)dy=0}$$

$$M(x,y)=2xy \qquad N(x,y)=x^2-1$$

Step 0: check whether it is exactly

$$\frac{\partial M}{\partial y}=2x=\frac{\partial N}{\partial x}$$

$$\frac{\partial f}{\partial x}=2xy \qquad \frac{\partial f}{\partial y}=x^2-1$$

Step 1 Step 2 Step 2

$$f(x,y)=x^2y+g(y) \qquad \frac{\partial f}{\partial y}=x^2+g'(y)=x^2-1$$

Step 3 Step 2

Step 3 $g'(y)=-1$

Step 2

$$x^2y-y=c \qquad g(y)=-y$$

Step 4

$$\boxed{y=c/(x^2-1)}$$

Quick Question: Is there another way to solve Example 1?

Solution Process

The method for solving the exact equation (A):

$$\boxed{\frac{\partial f(x,y)}{\partial x} = M(x,y)}$$

(1)

$$f(x,y) = \int M(x,y)\,dx + \underline{g(y)} = c$$

$g(y)$ is a constant for x

(2)

$$\boxed{\frac{\partial f(x,y)}{\partial y} = N(x,y)}$$

(2)

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\,dx + g'(y) = N(x,y)$$

(2)

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx$$

(2)

$$g(y) = \int g'(y)\,dy$$

Put in (3)

(3)

$$\boxed{\int M(x,y)\,dx + g(y) = c}$$

further computation (4)

Solution

Organized Key Points:

Previous Step: Check whether $\boxed{\dfrac{\partial M(x,y)}{\partial y} = \dfrac{\partial N(x,y)}{\partial x}}$ is satisfied.

Step 1: Solve $\dfrac{\partial f(x,y)}{\partial x} = M(x,y) \longrightarrow f(x,y) = \int M(x,y)\,dx + g(y)$

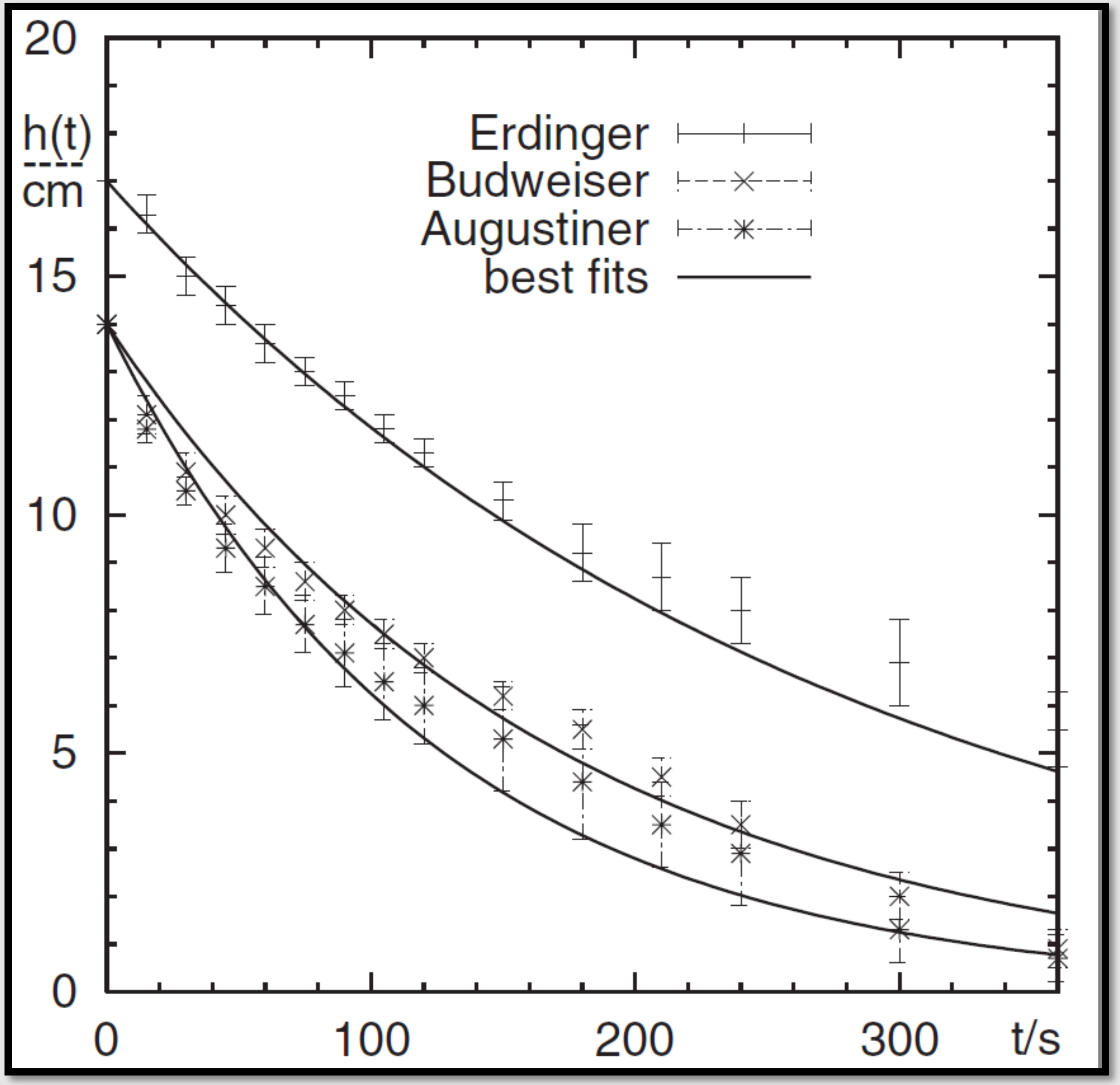
Step 2: Use $f(x, y)$ to solve $\dfrac{\partial f(x,y)}{\partial y} = N(x,y)$, get g(y)

Step 3: Substitute $g(y)$ into $f(x,y) = \int M(x,y)\,dx + g(y) = c$

Step 4: Further computation and obtain the solution

Extra Steps: (a) Consider the initial value problem

Take a break…


20
h(t)
cm
15
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100
200
300
t/s
Erdinger
Budweiser
Augustiner
best fits


ERDINGER
Weißbier
Budweiser
HELLES
LAGERBIER HELL

Modified Exact Equation Method

Technique: Use the integrating factor $\mu\ (x,\ y)$ to convert the 1st order DE into the exact equation.

$$M(x,y)dx + N(x,y)dy = 0$$

$$\downarrow$$

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

such that $$\frac{\partial\mu(x,y)M(x,y)}{\partial y} = \frac{\partial\mu(x,y)N(x,y)}{\partial x}$$

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

$$\mu_x N - \mu_y M = (M_y - N_x)\mu$$

It is hard to find μ .

$$\boxed{\mu_x N - \mu_y M = (M_y - N_x)\mu}$$

(1) When $\boxed{(M_y - N_x)/M}$ is a function of y alone:

→ We can set μ to be dependent on y alone.

Therefore,

$$-\mu_y M = (M_y - N_x)\mu$$

$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu$$

Use "separable variable"

$$\frac{d\mu}{\mu} = \frac{N_x - M_y}{M}dy$$

$$\boxed{\mu(y) = e^{\int \frac{(N_x - M_y)}{M}dy}}$$

$$\boxed{\mu_x N - \mu_y M = (M_y - N_x)\mu}$$

(1) When $\boxed{(M_y - N_x)/N}$ is a function of x alone:

→ We can set μ to be dependent on x alone.

Therefore,

$$\mu_x N = (M_y - N_x)\mu$$

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$$

Use "separable variable"

$$\frac{d\mu}{\mu} = \frac{N_x - M_y}{M}dy$$

$$\boxed{\mu(x) = e^{\int \frac{(M_y - N_x)}{N}dx}}$$

Solution Process of Modified Exact Method

Step 0:

Whether $M_y = N_x$

- Yes → **Case 1**: Use the solution process of 2-4-3
- No → Whether $\dfrac{M_y - N_x}{M}$ is independent of x
  - Yes → **Case 2**: $\mu(y) = e^{\int \frac{(N_x - M_y)}{M} dy}$
  - No → Whether $\dfrac{M_y - N_x}{N}$ is independent of y
    - Yes → **Case 3**: $\mu(x) = e^{\int \frac{(M_y - N_x)}{N} dx}$
    - No → **Case 4**: Use other methods

In Cases 2 and 3,

$$M(x,y)dx + N(x,y)dy = 0$$

$$\downarrow$$

$$\mu M(x,y)dx + \mu N(x,y)dy = 0$$

$$\downarrow$$

Using the process of 2-4-3, but $M(x, y)$ should be modified as $\mu \; M(x, y)$, $N(x, y)$ should be modified as $\mu \; N(x, y)$

Example 4

$$\boxed{xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0}$$

Step 0: $M = xy$ $\quad N = 2x^2 + 3y^2 - 20$

$$\boxed{M_y - N_x = x - 4x = -3x}$$

$$\frac{M_y - N_x}{N} = \frac{-3x}{2x^2 + 3y^2 - 20}$$

$$\boxed{\frac{M_y - N_x}{M} = -\frac{3}{y}}$$

(independent of x)
(Case 2)

$$\mu(y) = e^{\int \frac{3}{y} dy} = e^{3\ln|y|} = y^3$$

Q: Why c and $\pm$ can be neglected?

$$\boxed{xy^4\,dx + (2x^2y^3 + 3y^5 - 20y^3)\,dy = 0}$$

double N

Steps 1~4:

$$\boxed{\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = c}$$

Linear Equation

"friendly" form of DEs

2-3-1 Definition and Constraint

- [Definition 2.3.1] The first-order DE is a linear equation if it has the following form:

$$\boxed{a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)}$$

Standard form: $\boxed{\dfrac{dy}{dx}+P(x)y=f(x)}$

$$a_1(x)\frac{dy}{dx}+a_0(x)y=g(x) \longrightarrow \frac{dy}{dx}+\frac{a_0(x)}{a_1(x)}y=\frac{g(x)}{a_1(x)}$$

Many natural phenomena can be expressed as linear first order DE

Derive Solution

$$\boxed{\frac{dy}{dx}+P(x)y=f(x)}$$

Problem 1

$$\boxed{\frac{dy_c}{dx}+P(x)y_c=0}$$

Find the general solution $y_c(x)$

(homogeneous solution)

Problem 2

$$\boxed{\frac{dy_p(x)}{dx}+P(x)y_p(x)=f(x)}$$

Find any solution $y_p(x)$

(particular solution)

Solution of the DE

$$\boxed{y(x)=y_c(x)+y_p(x)}$$

- $y_c + y_p$ is a solution of the linear first order DE, since

$$\frac{d(y_c+y_p)}{dx}+P(x)(y_c+y_p)$$
$$=\left(\frac{dy_c}{dx}+P(x)y_c\right)+\left(\frac{dy_p}{dx}+P(x)y_p\right)$$
$$=0+f(x)=f(x)$$

- Any solution of the linear first order DE should have the form $y_c + y_p$.

The proof is as follows. If y is a solution of the DE, then

$$\frac{dy}{dx}+P(x)y-\left(\frac{dy_p}{dx}+P(x)y_p\right)=f(x)-f(x)=0$$
$$\frac{d(y-y_p)}{dx}+P(x)(y-y_p)=0$$

Thus, $y - y_p$ should be the solution of $\dfrac{dy_c}{dx}+P(x)y_c=0$

y should have the form of $y = y_c + y_p$

Example 1

$$\boxed{\frac{dy}{dx} - 3y = 6}$$

homogeneous sol. $y_c = ce^{3x}$

Particular sol. is associated with homogeneous sol. $y_p(x) = u(x) y_c$

particular sol. $y_p = u(x) e^{3x}$ or $u(x)\ e^{3x} + c_1$?

After lengthy calculation…$u(x) = -2e^{-3x}$

$y_p = -2$

$$\boxed{y = -2 + ce^{3x}}$$

Solving the homogeneous solution $y_c(x)$ (Problem 1)

$$\boxed{\frac{dy_c}{dx}+P(x)y_c=0}$$

separable variable

$$\frac{dy_c}{y_c}=-P(x)dx$$

$$\ln|y_c|=\int -P(x)dx+c_1$$

$$\boxed{y_c=ce^{-\int P(x)dx}}$$

Set $y_1=e^{-\int P(x)dx}$, then $y_c=cy_1$

Solving the particular solution $y_p(x)$ (Problem 2)

$$\boxed{\frac{dy_p(x)}{dx} + P(x)y_p(x) = f(x)}$$

Set $y_p(x) = u(x)\, y_1(x)$ (Assume that the particular solution and the homogeneous solution are similar)

$$u(x)\frac{dy_1(x)}{dx} + y_1(x)\frac{du(x)}{dx} + P(x)u(x)y_1(x) = f(x)$$

$$\downarrow$$

$$y_1(x)\frac{du(x)}{dx} + \underbrace{u(x)\left[\frac{dy_1(x)}{dx} + P(x)y_1(x)\right]}_{\text{equal to zero}} = f(x)$$

$$\downarrow$$

$$y_1(x)\frac{du(x)}{dx} = f(x)$$

$$\downarrow$$

$$du(x) = \frac{f(x)}{y_1(x)}dx \longrightarrow u(x) = \int \frac{f(x)}{y_1(x)}dx \longrightarrow \boxed{y_p(x) = y_1(x)\int \frac{f(x)}{y_1(x)}dx}$$

$$\boxed{y_c = ce^{-\int P(x)dx}}$$

$$\boxed{y_p(x) = e^{-\int P(x)dx}\int [e^{\int P(x)dx} f(x)]dx}$$

solution of the linear 1st order DE:

$$\boxed{y(x) = c\,e^{-\int P(x)dx} + e^{-\int P(x)dx}\int [e^{\int P(x)dx} f(x)]dx}$$

where c is any constant

$e^{\int P(x)dx}$: integrating factor

Solution

(Step 1) Obtain the **standard form** and find $P(x)$

(Step 2) Calculate $e^{\int P(x)dx}$

(Step 3a) The standard form of the linear 1st order DE can be rewritten as:

$$\boxed{\frac{d}{dx}\left[e^{\int P(x)dx}y\right]=e^{\int P(x)dx}f(x)}$$

remember it

(Step 3b) Integrate both sides of the above equation

$$e^{\int P(x)dx}y=\int e^{\int P(x)dx}f(x)dx+c,$$

$$\boxed{y=e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)dx+ce^{-\int P(x)dx}}$$

or remember it, skip Step 3a

Example 2

$$\boxed{\frac{dy}{dx} - 3y = 6}$$

$\downarrow$

Step 1 $P(x) = -3$

$\downarrow$

Step 2 $e^{\int P(x)dx} = e^{-3x}$

$\downarrow$

Step 3 $\frac{d}{dx}\left[e^{-3x}y\right] = 6e^{-3x}$

$\downarrow$

Step 4 $e^{-3x}y = -2e^{-3x} + c$

$\downarrow$

$$\boxed{y = -2 + ce^{3x}}$$

We can also skip step 3 and use the formula below directly:

$$y = e^{-\int P(x)dx}\int e^{\int P(x)dx} f(x)dx + ce^{-\int P(x)dx}$$

Example 3

$$\boxed{x\frac{dy}{dx}-4y=x^6e^x}$$

↓

Step 1 $\dfrac{dy}{dx}-4\dfrac{y}{x}=x^5e^x,\quad P(x)=-\dfrac{4}{x}$

↓

Step 2 $y_c=ce^{-\int P(x)dx}=ce^{4\ln|x|}=c|x|^4$. Consider $x>0$, $y_c=cx^4$

↓

Step 3 $Let\ y_p=u(x)y_c, \dfrac{dy_p}{dx}=u'(x)x^4+4u(x)x^3$

↓

Step 4 $u'(x)x^4+4u(x)x^3-4x^{-1}u(x)\ x^4=\ x^5e^x \rightarrow u'(x)=xe^x$

↓

Step 5 $u(x)=xe^x-e^xdx=xe^x-e^x=(\mathrm{x}-1)e^x,\quad y_p=(\mathrm{x}-1)e^x\cdot x^4$

↓

Step 6 $\boxed{y=(x^5-x^4)e^x+cx^4}$

Example 4

$$\boxed{\left(x^2-9\right)\frac{dy}{dx}+xy=0}$$

$$\downarrow$$

$$\frac{dy}{dx}+\frac{x}{x^2-9}y=0$$

$$\downarrow$$

$$P(x)=\frac{x}{x^2-9}$$

$$\downarrow$$

$$y_c=ce^{-\int P(x)dx}=ce^{-\frac{1}{2}\left|x^2-9\right|}=c\left(\left|x^2-9\right|\right)^{-\frac{1}{2}}$$

$$\downarrow$$

$$\boxed{y_c=\frac{c}{\sqrt{\left|x^2-9\right|}}}$$

Multiple solutions

$$\frac{dy}{dt} = ky$$

Separable? Exact? Linear?

Exercise

1. $(xy^2 - x)dx + (x^2y + y)dy = 0$, with $\mathrm{y(x = 0)} = \sqrt{2}$
2. $(y - x + 1)dx - (y - x + 5)dy = 0$, with $\mathrm{y}(0) = 4$

Exercise

3. $(2\cos y + 4x^2)dx = x \sin y\, dy$

4. $(2xy + 3y)dx + (4y^3 + x^2 + 3x + 4)dy = 0, y(0) = 1$

Exercise

5. $\dfrac{dy}{dx} + (\tan x)y = \sin 2x\,,\, y(0) = 1$

6. $\dfrac{dy}{dx} = \dfrac{y}{2x + y^3 e^y}$

Exercise

7. $\frac{dy}{dx} = \frac{y^2 + 2y}{y^4 + 2xy + 4x}$

8. $(4x + 3y^2)dx + 2xydy = 0$