10-2 Calculus with Parametric Curves

師大工教一

Tangent Lines and Areas

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Parametric Formula for $\dfrac{dy}{dx}$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \cdot \frac{dx}{dt} \Rightarrow \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

If all three derivatives exist and $\dfrac{dx}{dt} \neq 0$, then $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

Parametric Formula for $\frac{d^2y}{dx^2}$

If the equations $x = f(t), y = g(t)$ define y as a twice-differentiable function of x, then at any point where $\frac{dx}{dt} \neq 0$ and $y' = \frac{dy}{dx}$, $\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$.

Ex1(p607) Find the tangent line to the curve $x = \sec t, \quad y = \tan t, \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$, at the point $\left(\sqrt{2}, 1\right)$, where $t = \frac{\pi}{4}$.

$$\frac{dy}{dx} = \frac{\sec^{\cancel{2}} t}{\cancel{\sec t}\,\tan t} = \frac{\sec t}{\tan t}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = \frac{\sec\frac{\pi}{4}}{\tan\frac{\pi}{4}} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

Eq. of tangent line: $y - 1 = \sqrt{2}(x - \sqrt{2})$

Ex2(p607) Find $\frac{d^2y}{dx^2}$ as a function of t if $x = t - t^2$ and $y = t - t^3$.

$$\frac{dy}{dx} = \frac{1-3t^2}{1-2t}$$

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{1-3t^2}{1-2t}\right)'}{1-2t} = \frac{\frac{(-6t)(1-2t) - (1-3t^2)(-2)}{(1-2t)^2}}{1-2t}$$

$$= \frac{\frac{-6t+12t^2+2-6t^2}{1-4t+4t^2}}{1-2t} = \frac{\frac{6t^2-6t+2}{4t^2-4t+1}}{1-2t} = \frac{2(3t^2-3t+1)}{(1-2t)^3}$$

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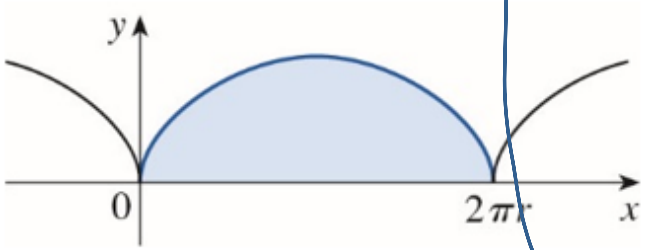
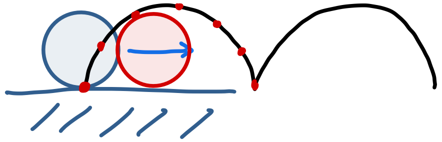
跟 t 產生的方向有關

Area

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)f'(t)\,dt \quad \left[\text{or } \int_\beta^\alpha g(t)f'(t)\,dt\right]$$

Ex Find the area under one arch of the $\boxed{\text{cycloid}}$ $x = r(\theta - \sin\theta), y = r(1 - \cos\theta)$.





$f(\theta) = x = r(\theta - \sin\theta)$

$g(\theta) = y = r(1 - \cos\theta)$

$f'(\theta) = r(1 - \cos\theta)$

$$\int_0^{2\pi} r(1-\cos\theta)(r(1-\cos\theta))\,d\theta$$

$$= r^2\int_0^{2\pi} (1-\cos\theta)^2\,d\theta = r^2\int_0^{2\pi} 1 - 2\cos\theta + \cos^2\theta\,d\theta = r^2\left(\theta - 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big|_0^{2\pi}$$

$$= r^2(3\pi - 0)$$

$$= 3\pi r^2$$

Definition If a curve C is defined parametrically by

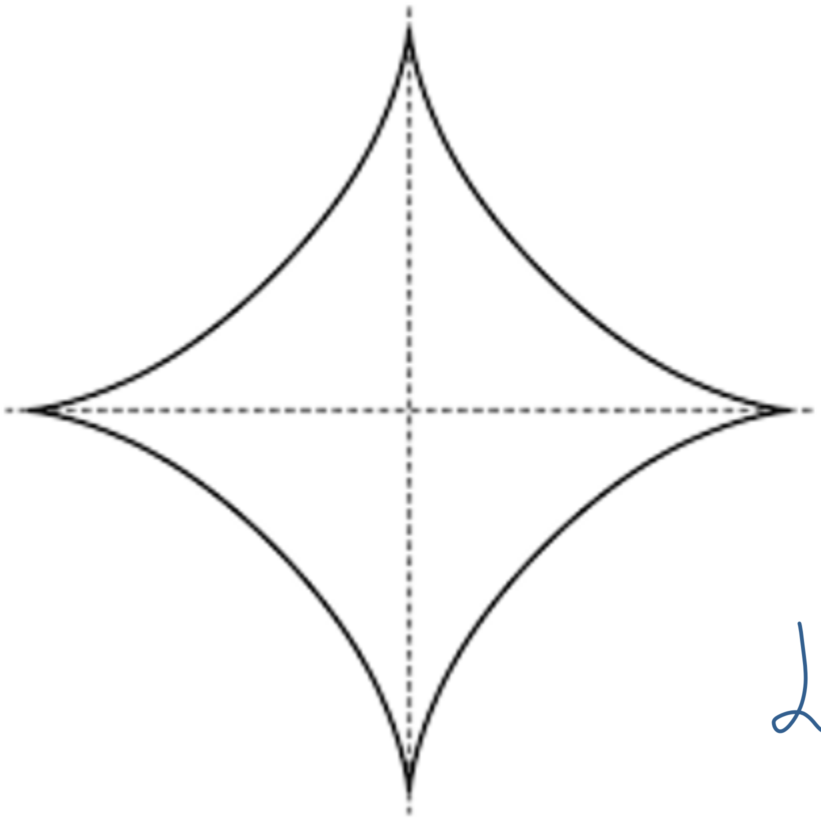
$x = f(t), y = g(t), a \le t \le b$, where f' and g' are continuous and not

simultaneously zero on $[a,b]$, and if C is traversed exactly once as t

increases from $t = a$ to $t = b$, then **the length of** C is the definite integral

$$L = \int_a^b \sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2}\, dt\,.$$

Ex5(p610) Find the length of the astroid 星狀線 $x=\cos^3 t,\quad y=\sin^3 t,\quad 0\le t\le 2\pi$.



$$L=\int_a^b \sqrt{(f'(t))^2+(g'(t))^2}\,dt$$

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=\cos^2 t+\sin^2 t=1$$

$f(t)=\cos^3 t$, $g(t)=\sin^3 t$

$f'(t)=-3\cos^2 t\sin t$, $g'(t)=3\sin^2 t\cos t$

$$L=4\int_0^{\frac{\pi}{2}}\sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t}\,dt$$

$$=4\int_0^{\frac{\pi}{2}}3\sqrt{\cos^2 t\sin^2 t(\cos^2+\sin^2 t)}\,dt$$

$$=4\int_0^{\frac{\pi}{2}}3\sqrt{\cos^2 t\sin^2 t}\,dt=12\int_0^{\frac{\pi}{2}}\cos t\sin t\,dt=12\left(\frac{\sin^2 t}{2}\Big|_0^{\frac{\pi}{2}}\right)$$

$$=6$$

Area of Surface of Revolution for Parametrized Curve

If a smooth curve $x = f(t), y = g(t), a \le t \le b$, is traversed exactly once as t increases from a to b, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1.Revolution about the $x-$axis $(y \ge 0)$:

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

2.Revolution about the $y-$axis $(x \ge 0)$:

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

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4. Given a parametric curve $x = \sqrt{3}t^2$, $y = 3t - \frac{1}{3}t^3$, $-3 \le t \le 3$.

$(2\sqrt{3}t)^2 \quad (3 - t^2)^2$

$12t^2 \quad 9 - 6t^2 + t^4$

(1) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(2) Find the area enclosed by the curve.

(3) Find the area of the surface generated by revolving the curve about the x-axis.

(1) $$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3 - t^2}{2\sqrt{3}t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\left(\frac{3-t^2}{2\sqrt{3}t}\right)'}{2\sqrt{3}t} = \frac{\frac{-2t(2\sqrt{3}t) - (3-t^2)(2\sqrt{3})}{(2\sqrt{3}t)^2}}{2\sqrt{3}t} = \frac{-2\sqrt{3}t^2 - 6\sqrt{3}}{(2\sqrt{3}t)^3}$$

(2) $$A = \int_{-3}^{3} \left(3t - \frac{1}{3}t^3\right)(2\sqrt{3}t)\,dt = \int_{-3}^{3} \left(6\sqrt{3}t^2 - \frac{2\sqrt{3}}{3}t^4\right)dt$$

$$= 2\sqrt{3}t^3 - \frac{2\sqrt{3}}{15}t^5 \Big|_{-3}^{3} = 54\sqrt{3} - \frac{162\sqrt{3}}{5} + 54\sqrt{3} - \frac{162\sqrt{3}}{5}$$

$$= \frac{216\sqrt{3}}{5} \quad \#$$

$$S = \int_0^3 2\pi\left(3t - \frac{1}{3}t^3\right)\sqrt{12t^2 + 9 - 6t^2 + t^4}\,dt$$

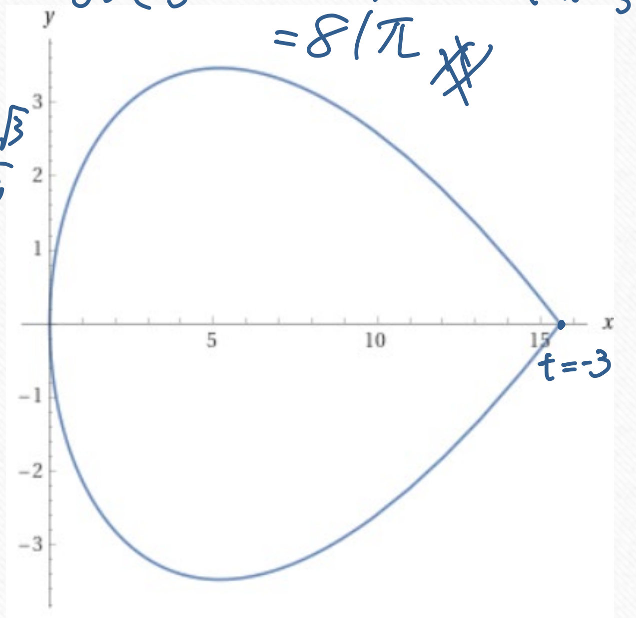
$$= 2\pi \int_0^3 \left(3t - \frac{1}{3}t^3\right)\sqrt{t^4 + 6t^2 + 9}\,dt$$

$$= 2\pi \int_0^3 \left(3t - \frac{1}{3}t^3\right)(t^2 + 3)\,dt$$

$$= 2\pi \int_0^3 3t^3 + 9t - \frac{t^5}{3} - t^3\,dt$$

$$= 2\pi \int_0^3 \left(-\frac{t^5}{3} + 2t^2 + 9t\right)dt = 2\pi\left(-\frac{t^6}{18} + \frac{2t^3}{3} + \frac{9}{2}t^2 \Big|_0^3\right)$$

$$= 81\pi \quad \#$$



HW10-2

- **HW:11,14,15,25,28,30**

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4. Given a parametric curve

$$x(t) = t - \sin t, \quad y(t) = 1 - \cos t, \quad 0 < t < \pi.$$

(a) (8 points) Find the tangent line of the curve at the point where $t = \frac{\pi}{4}$.

(b) (8 points) Show that the curve is concave downward.