Stoer-Wagner Algorithm

A Minimum Cut Algorithm for Undirected Graphs

BigNews

CS214: Algorithms and Complexity
Shanghai Jiao Tong University

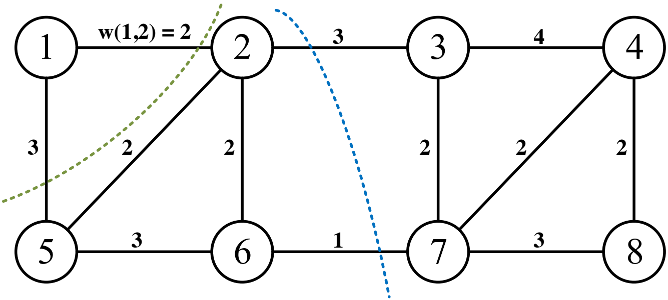
2016.12.06

Outline

Global Minimum Cut

Definition

Given an undirected graph $G(V, E)$, a global min-cut is a partition of V into two subsets (S, T) such that the sum of weights of edges between S and T is minimized.



Outline

The Algorithm
Global Minimum Cut
Stoer-Wagner Algorithm

Correctness
Lemma & Theorem
Proof

Running Time
Proof

Stoer-Wagner Algorithm (1)

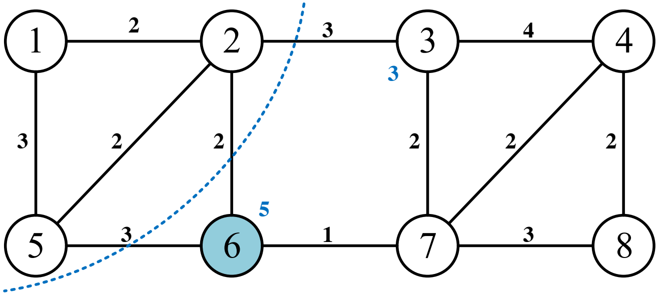
MinimumCutPhase(G, w, a)

$A \leftarrow \{a\}$
while $(A \neq V)$ **do**
    add to A the most tightly connected vertex
end
store the *cut-of-the-phase*
shrink G by merging the two vertices added last

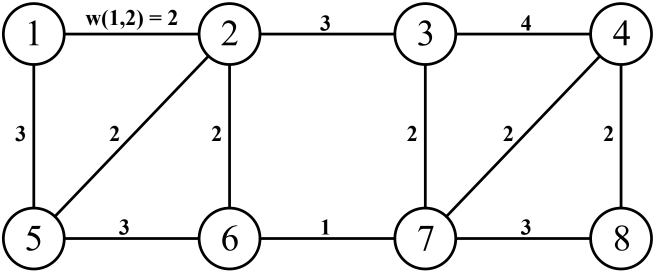
Most Tightly Connected

Definition

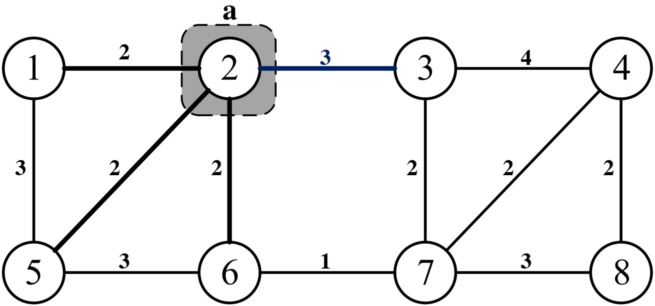
A vertex t are called **most ptightly connected** to vertex set A if sum of the weights of the edges connected between t and A is the highest among other vertices not belonging to A.



A graph $G = (V, E)$ with edge-weights

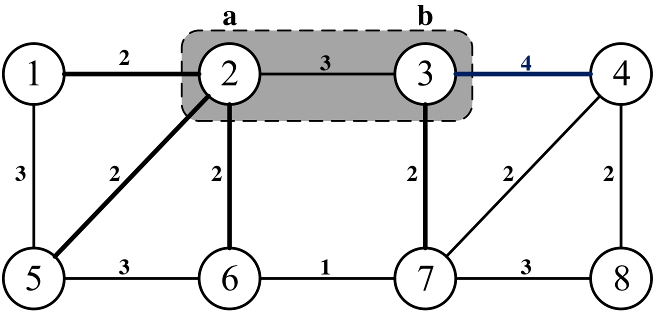


Start with vertex a = 2



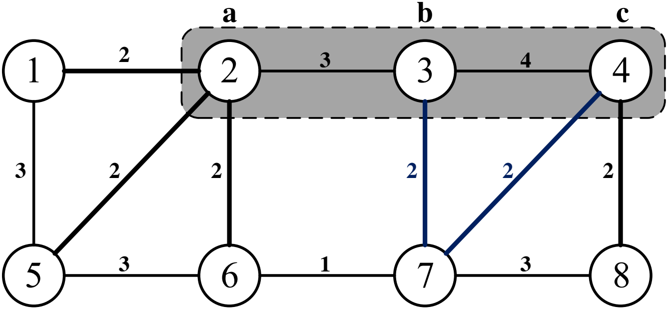
$A = \{2\}$

Add vertex b = 3



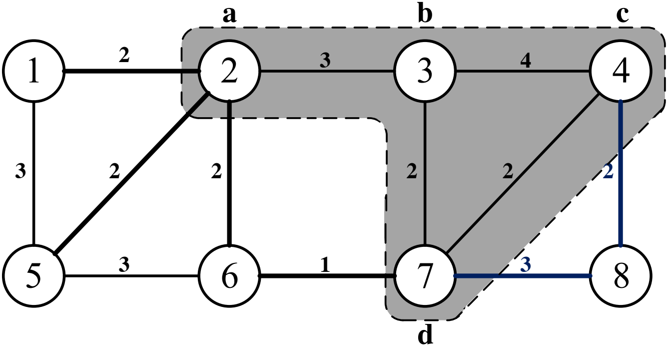
$A = \{2, 3\}$

Add vertex c = 4



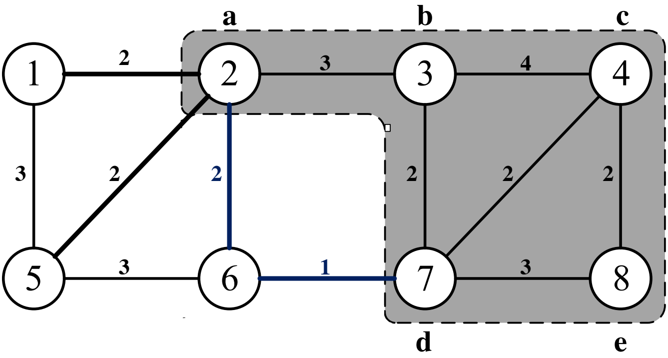
$A = \{2, 3, 4\}$

Add vertex d = 7



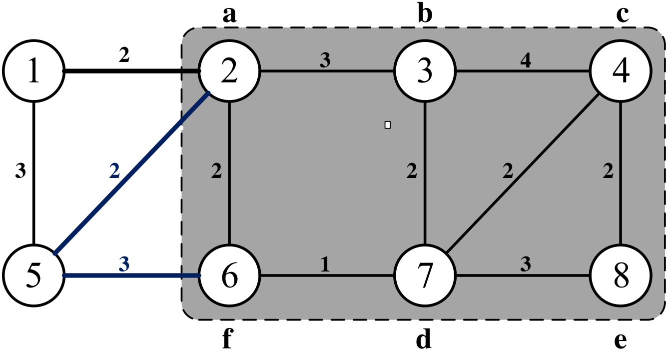
$A = \{2, 3, 4, 7\}$

Add vertex e = 8



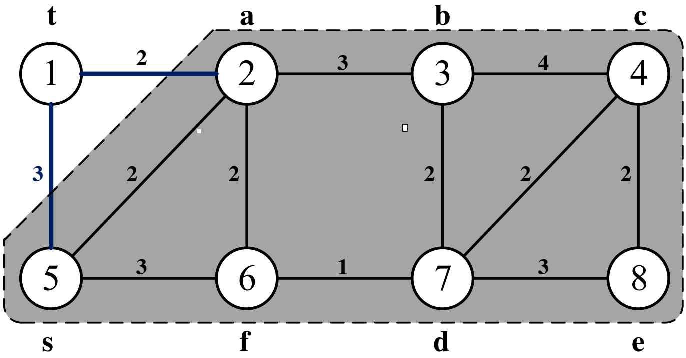
$A = \{2, 3, 4, 7, 8\}$

Add vertex f = 6



$A = \{2, 3, 4, 7, 8, 6\}$

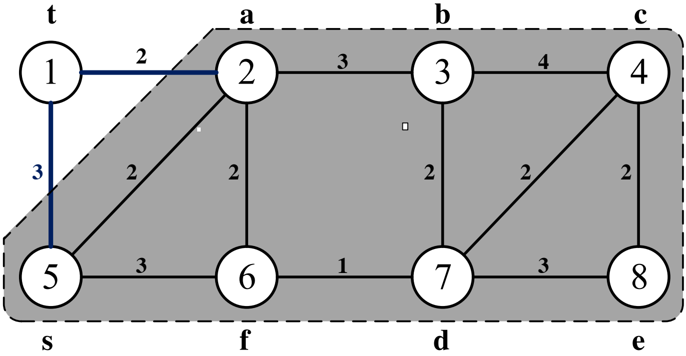
Add vertex s = 5 and vertex t = 1



$A = \{2, 3, 4, 7, 8, 6, 5\}$
$A = \{2, 3, 4, 7, 8, 6, 5, 1\}$
s and t are the last two vertices (in order) added to A, and we get a cut $C(A\text{-}t, t)$, which is so-called *cut-of-the-phase*.

Lemma

Each ***cut-of-the-phase*** *is a* ***minimum*** **s-t** ***cut*** *in the current graph, where* **s** *and* **t** *are the two vertices added last in the phase.*



Implications

What if the global min cut of G separates s and t?

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Then min s-t cut is also a global min cut of G.

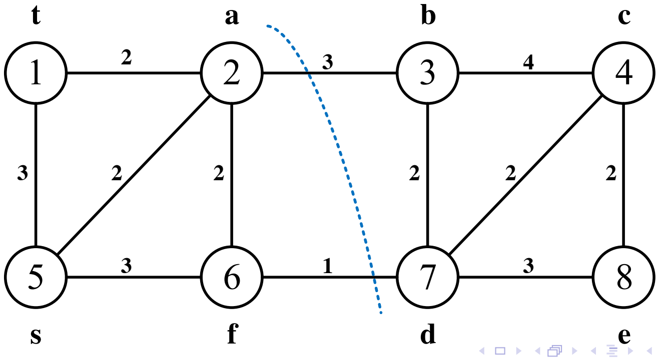
What if min cut of G does not separate s and t?

Then s and t are in the same partition of the global min cut, and we can **merge** them without changing the global min cut.

Merge

Definition

The two vertices are replaced by a new vertex and any edges from the two vertices to a remaining vertex are replaced by an edge weighted by the sum of the weights of the previous two edges, while edges joining the merged nodes are removed.



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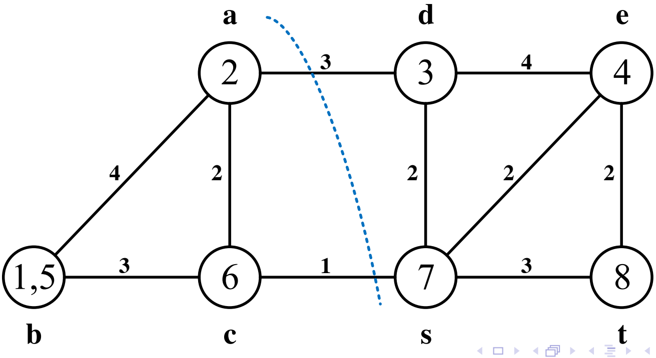
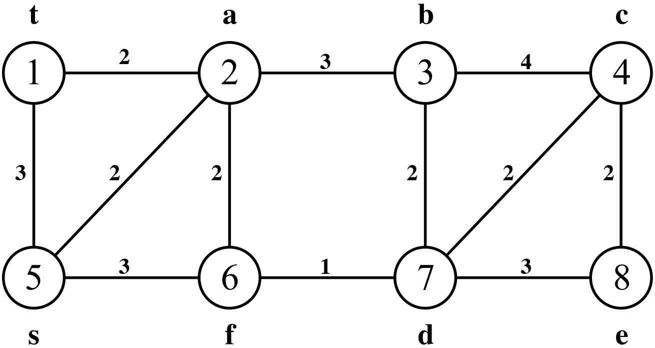


Stoer-Wagner Algorithm (2)

```
MinimumCut(G, w, a)
while (| V | > 1) do
    MinimumCutPhase(G, w, a)
    if the cut-of-the-phase is lighter than the current minimum cut
    then
        store the cut-of-the-phase as the current minimum cut
    end
end
```

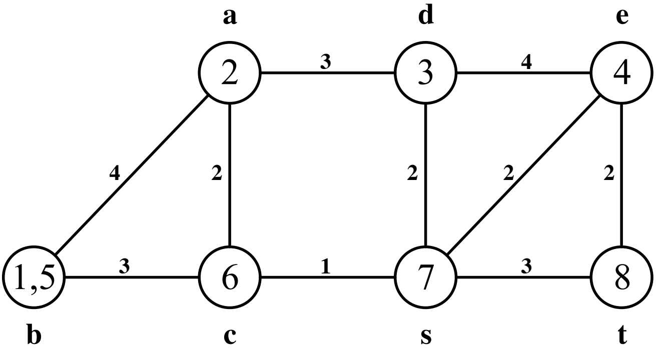
After the 1^{st} MinimumCutPhase(G, ω, a), $a = 2$



vertex ordering: a, b, c, d, e, f, s, t

cut-of-the-phase: $\{1\}, \{2, 3, 4, 5, 6, 7, 8\}$ $\quad \omega = 5$

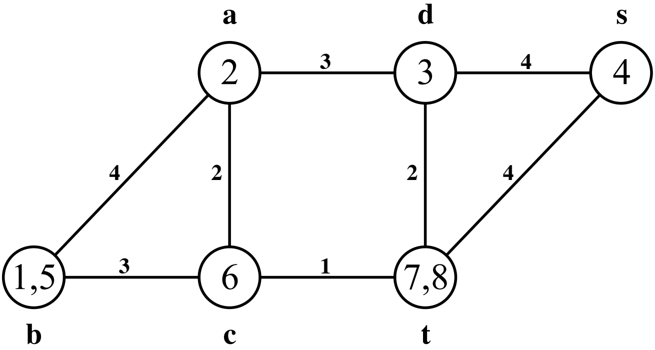
After the 2^{nd} MinimumCutPhase(G, ω, a), $a = 2$



vertex ordering: a, b, c, d, e, s, t

cut-of-the-phase: $\{8\}, \{1, 2, 3, 4, 5, 6, 7\}$ $\quad \omega = 5$

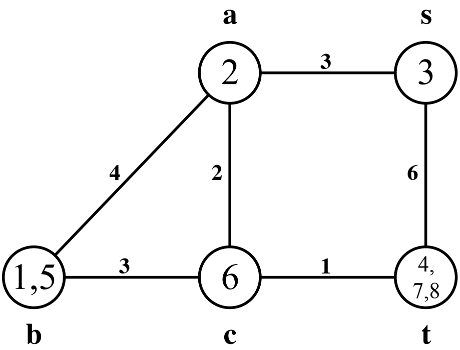
After the 3^{rd} MinimumCutPhase(G, ω, a), $a = 2$



vertex ordering: a, b, c, d, s, t

cut-of-the-phase: $\{7, 8\}, \{1, 2, 3, 4, 5, 6\}$ $\quad \omega = 7$

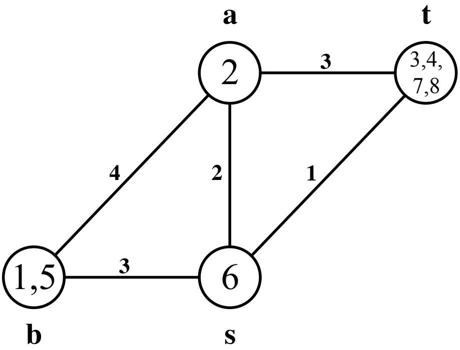
After the 4^{th} MinimumCutPhase(G, ω, a), $a = 2$



vertex ordering: a, b, c, s, t

cut-of-the-phase: $\{4, 7, 8\}, \{1, 2, 3, 5, 6\}$ $\quad \omega = 7$

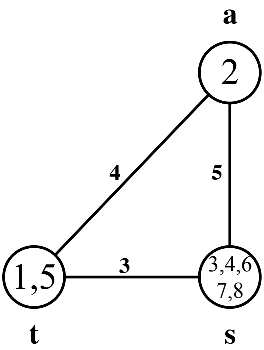
After the 5^{th} MinimumCutPhase(G, ω, a), $a = 2$



vertex ordering: a, b, s, t

cut-of-the-phase: $\{3, 4, 7, 8\}, \{1, 2, 5, 6\}$ $\quad \omega = 4$

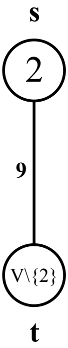
After the 6^{th} MinimumCutPhase(G, ω, a), $a = 2$



vertex ordering: a, s, t

cut-of-the-phase: $\{1, 5\}, \{2, 3, 4, 6, 7, 8\}$ $\omega = 7$

After the 7^{th} MinimumCutPhase(G, ω, a), $a = 2$



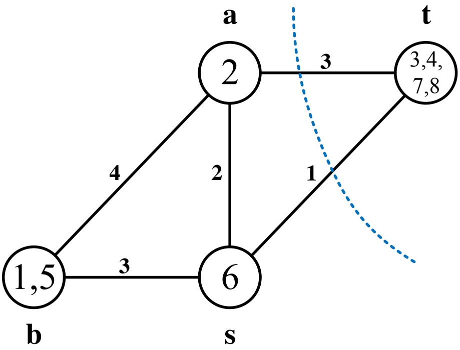
vertex ordering: s, t

cut-of-the-phase: $\{2\}, \{1, 3, 4, 5, 6, 7, 8\}$ $\quad \omega = 9$

Cut-of-the-phase

cut-of-the-phase	ω
$\{1\};\{2,3,4,5,6,7,8\}$	5
$\{8\};\{1,2,3,4,5,6,7\}$	5
$\{7,8\};\{1,2,3,4,5,6\}$	7
$\{4,7,8\};\{1,2,3,5,6\}$	7
$\{\mathbf{3},\mathbf{4},\mathbf{7},\mathbf{8}\};\{\mathbf{1},\mathbf{2},\mathbf{5},\mathbf{6}\}$	**4**
$\{1,5\};\{2,3,4,6,7,8\}$	7
$\{2\};\{1,3,4,5,6,7,8\}$	9

The Minimum Cut of the Graph G



vertex ordering: a, b, s, t

cut-of-the-phase: $\{3, 4, 7, 8\}, \{1, 2, 5, 6\}$ $\quad \omega = 4$

Outline

Lemma

*Each **cut-of-the-phase** is a **minimum*** **s-t** ***cut*** *in the current graph, where* **s** *and* **t** *are the two vertices added last in the phase.*

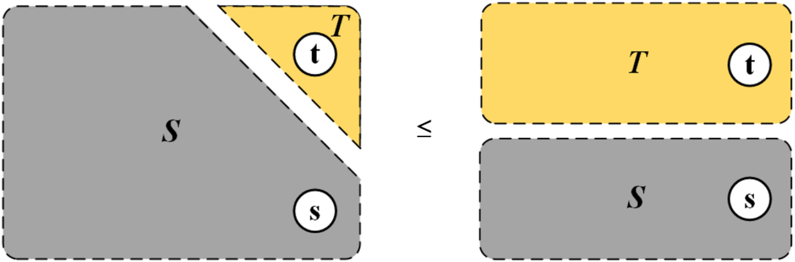
Theorem

*The **lightest** of these **cuts-of-the-phase** is the **minimum cut** of G.*

Assuming that the lemma holds, the theorem can be proved by a simple case distinction. Thus our proof is focused on the claimed property of the *cut-of-the-phase*.

Outline

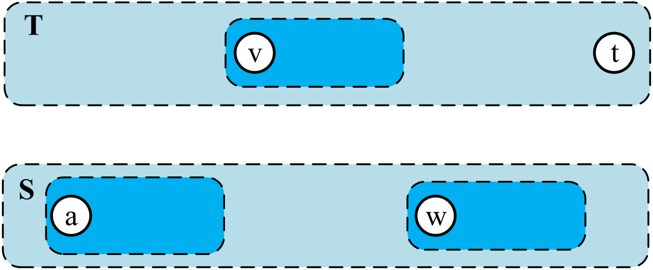
Proof



$\omega(A_t, t) \leq \omega(C_t)$

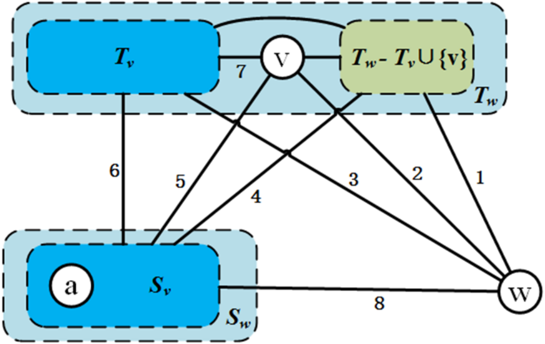
Definition

We call a vertex v *active* when v and the vertex added just before v are in different parts of the cut.



active vertices: $a, \ldots, v, w, \ldots, t$

Proof

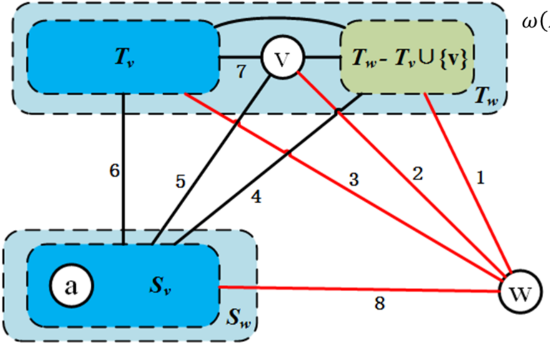


$$A_v = S_v \cup T_v$$

$$A_w = S_w \cup T_w$$

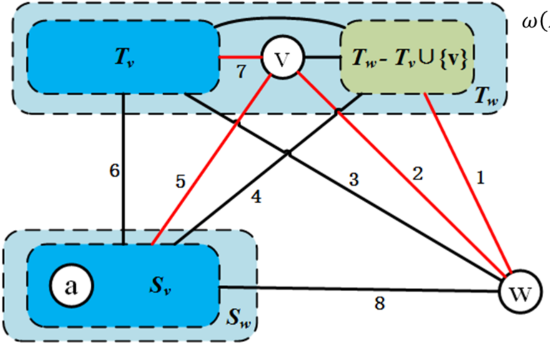
$$S_w = S_v$$

Proof



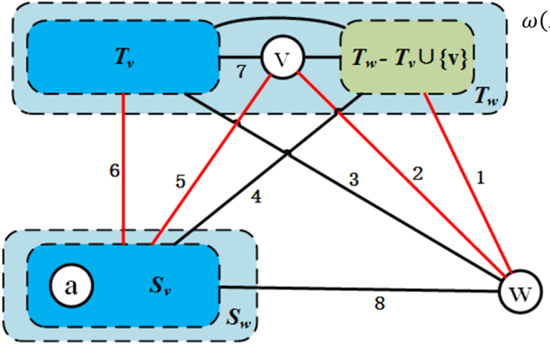
$$\omega(A_w, w) = (1+2) + (3+8)$$
$$= \omega(A_w \backslash A_v, w) + \omega(A_v, w)$$

Proof



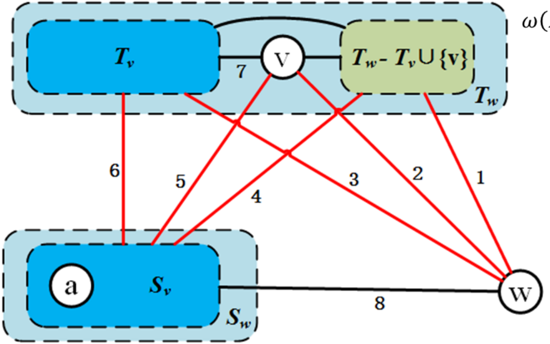
$$\begin{aligned}\omega(A_w, w) &= (1+2)+(3+8)\\ &= \omega(A_w \backslash A_v, w) + \omega(A_v, w)\\ &\leq \omega(A_w \backslash A_v, w) + \omega(A_v, v)\\ &= (1+2)+(5+7)\end{aligned}$$

Proof



$$\begin{aligned}
\omega(A_w, w) &= (1+2) + (3+8) \\
&= \omega(A_w \backslash A_v, w) + \omega(A_v, w) \\
&\leq \omega(A_w \backslash A_v, w) + \omega(A_v, v) \\
&= (1+2) + (5+7) \\
&\leq \omega(A_w \backslash A_v, w) + \omega(C_v) \\
&= (1+2) + (5+6)
\end{aligned}$$

Proof



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$$= \omega(A_w \backslash A_v, w) + \omega(A_v, w)$$
$$\leq \omega(A_w \backslash A_v, w) + \omega(A_v, v)$$
$$= (1+2) + (5+7)$$
$$\leq \omega(A_w \backslash A_v, w) + \omega(C_v)$$
$$= (1+2) + (5+6)$$
$$\leq \omega(C_w)$$
$$= (1+2+3+4+5+6)$$

Outline

MinimumCut Algorithm

MinimumCut(G, w, a)

while $(|V| > 1)$ **do**

  MinimumCutPhase(G, w, a)

  if the *cut-of-the-phase* is lighter than the current minimum cut **then**

    store the *cut-of-the-phase* as the current minimum cut

  end

end

MinimumCutPhase Algorithm

MinimumCutPhase(G, w, a)

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while $(A \neq V)$ **do**
  add to A the most tightly connected vertex
end
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MinimumCutPhase Algorithm

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Proposition

The running time for a single phase is $O(|E| + |V| \log |V|)$.

Proof.

1. All vertices that are not in A reside in a priority queue.

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3. $|V|$ *ExtractMax* $|E|$ *IncreaseKey*

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2. Whenever a vertex v is added to A,
   the priority queue does *ExtractMax* and *IncreaseKey* operations.
3. $|V|$ *ExtractMax* $|E|$ *IncreaseKey*
4. Fibonacci heaps
   ExtractMax $O(\log |V|)$
   IncreaseKey $O(1)$
   Time for one phase is $O(|E| + |V| \log |V|)$.

□

Proposition

The overall running time of Stoer-Wagner algorithm for minimum cut in undirected graphs is $O(|V||E| + |V|^2 \log |V|)$.

Proof.

1. $|V| - 1$ phases
2. Time for single phase $O(|E| + |V| \log |V|)$
3. Overall running time $O(|V||E| + |V|^2 \log |V|)$

$\square$

Thank you!

Q & A