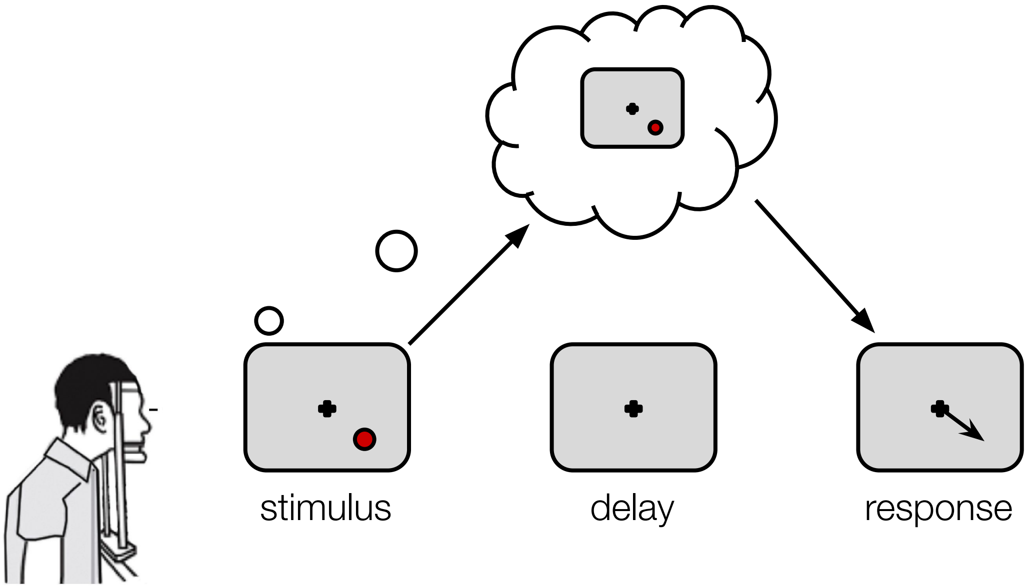
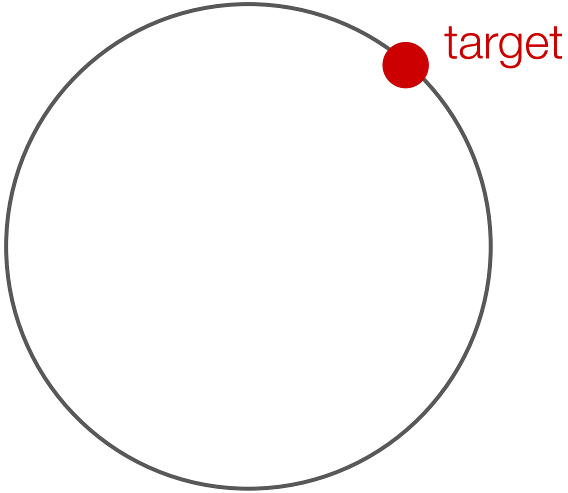
Mixture Models

Classic WM task



Response distributions



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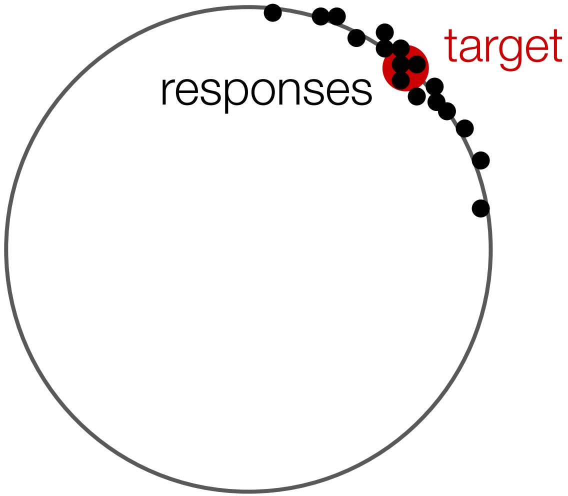
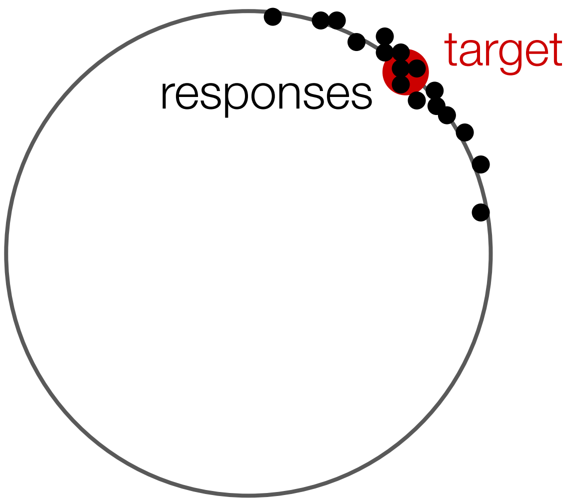


How precise is the working memory representation?

Response distributions



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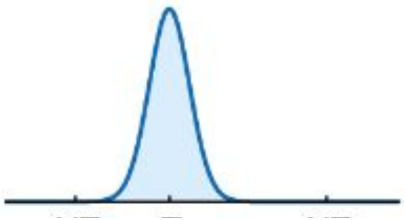

$$\theta = \{\mu, \sigma^2\}$$

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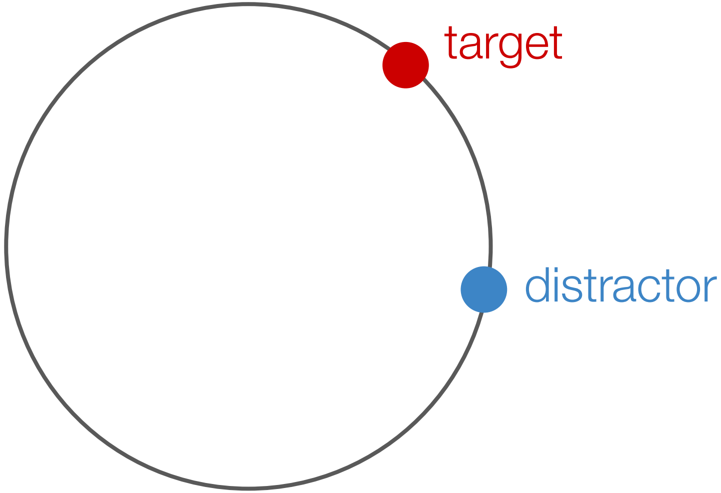

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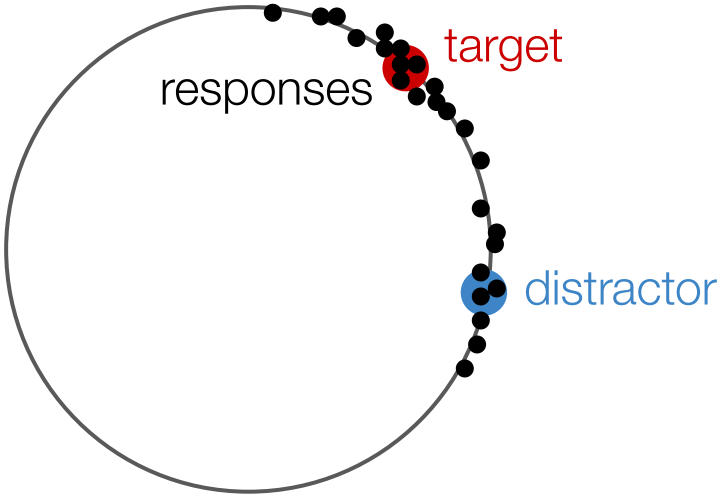

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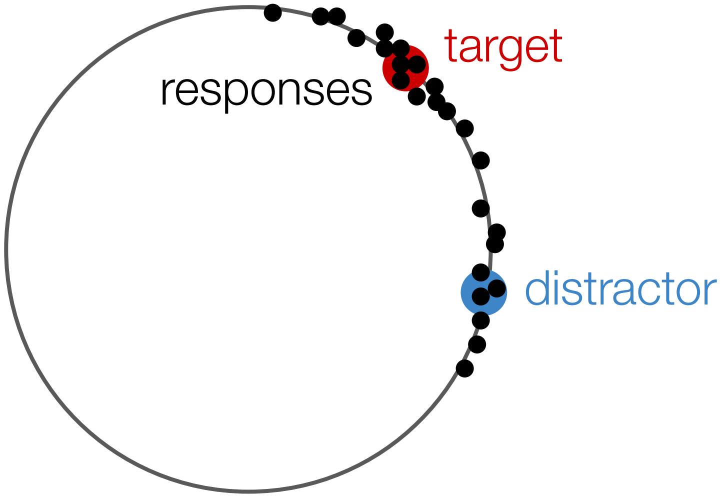

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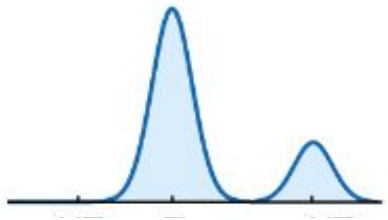

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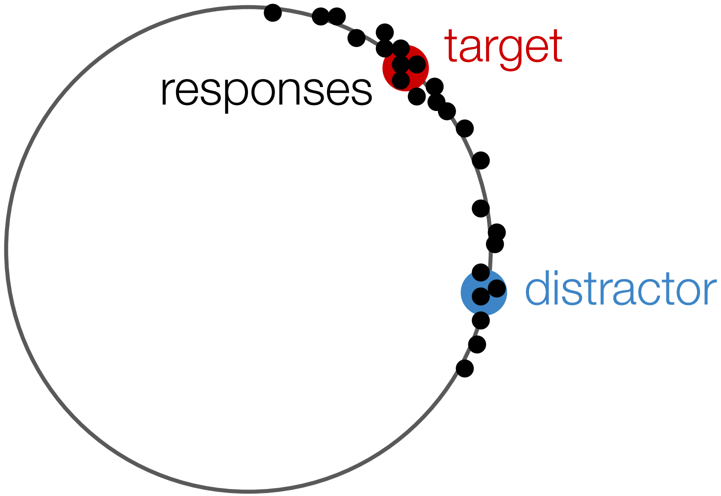

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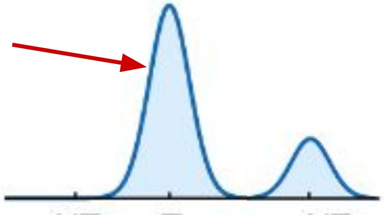

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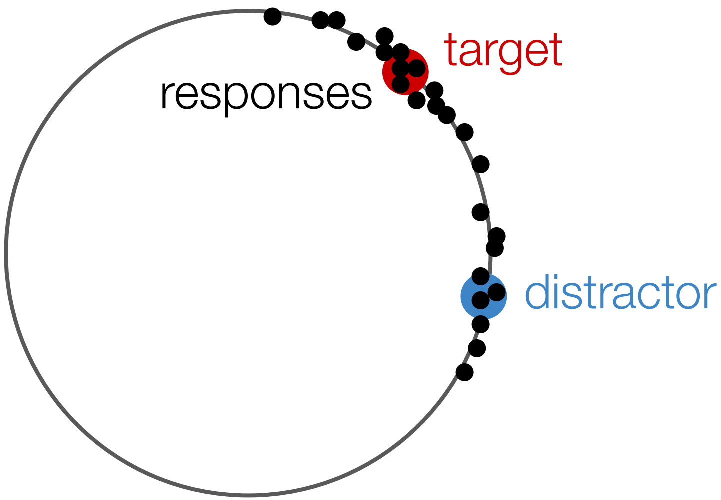

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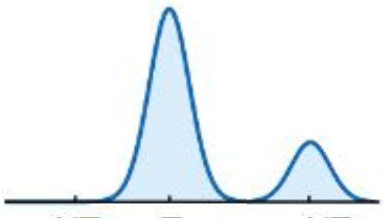

Which stimulus did the subject hold in mind in each trial?

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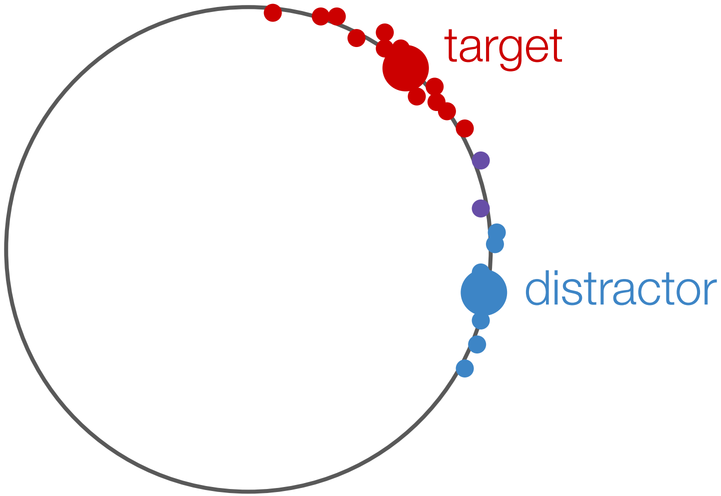

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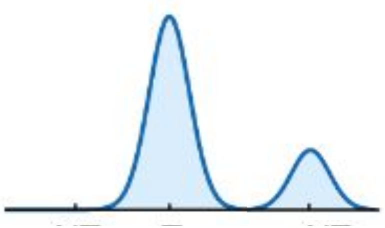

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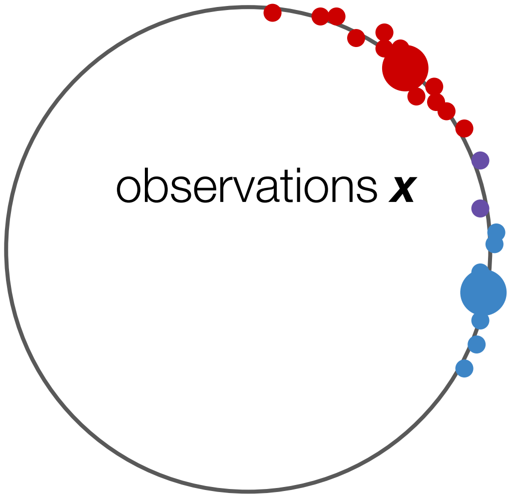
Mixture models

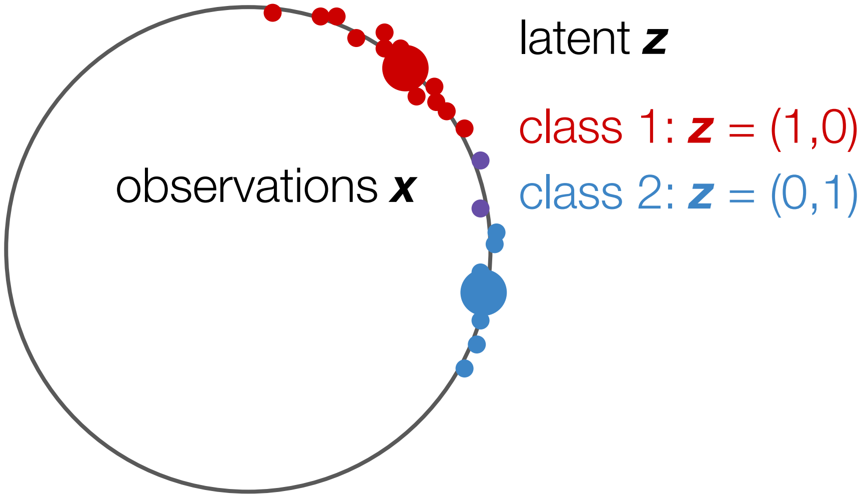


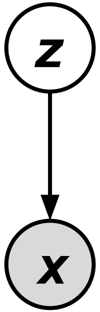
latent $\boldsymbol{z}$

class 1: $\boldsymbol{z} = (1,0)$

class 2: $\boldsymbol{z} = (0,1)$

Mixture models





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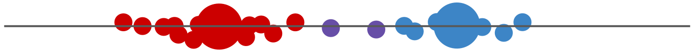

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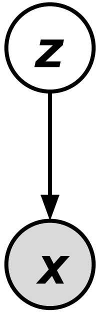

$$p(\boldsymbol{x}|\boldsymbol{z}_k = 1) = \mathcal{N}(\mu_k, \sigma_k^2)$$

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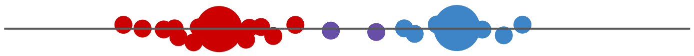

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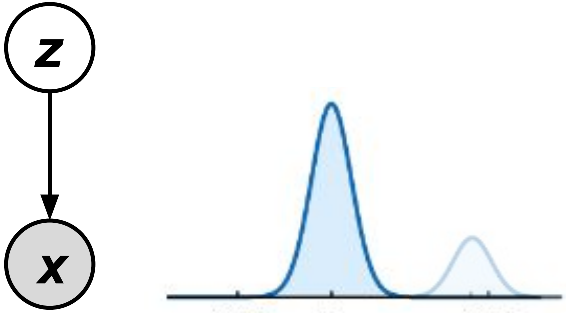

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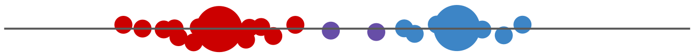

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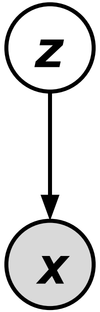
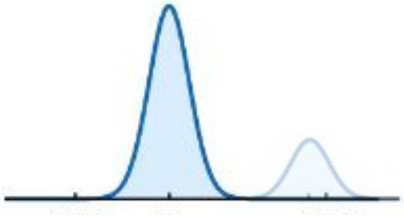


But: We don't know the value of ***z***!



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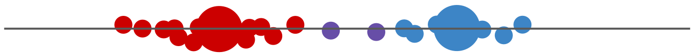

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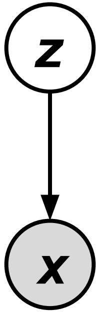

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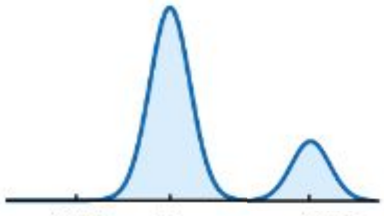

$$p(\boldsymbol{x}) = \sum_z p(\boldsymbol{x}, \boldsymbol{z}) = \sum_z p(\boldsymbol{x}|\boldsymbol{z})p(\boldsymbol{z})$$

marginalization

$$= \sum_k \pi_k \mathcal{N}(\mu_k, \sigma_k^2)$$

Expectation maximization (EM)

EM tackles two problems:

1) Determine which data point belongs to which class

2) Fit class-specific parameters

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   → Inference of the **posterior** over the latent indicator variable ***z***

$$
\begin{aligned}
p(\boldsymbol{z}|\boldsymbol{x}) &= \frac{p(\boldsymbol{z})p(\boldsymbol{x}|\boldsymbol{z})}{p(\boldsymbol{x})} \\
&= \frac{\pi_k \mathcal{N}(\mu_k, \sigma_k)}{\sum_k \pi_k \mathcal{N}(\mu_k, \sigma_k)}
\end{aligned}
$$

Expectation maximization (EM)

EM tackles two problems:

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   ML solution: Optimize $\ln p(\boldsymbol{x}) = \ln \sum_{z} p(\boldsymbol{x}, \boldsymbol{z}|\theta)$, but high computational cost

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   Solution: Optimize expected value $\mathbb{E}_{\mathbf{z}}\left[\ln p(\boldsymbol{x},\boldsymbol{z}|\theta')\right] = \sum_z p(\boldsymbol{z}|\boldsymbol{x},\theta) \ln p(\boldsymbol{x},\boldsymbol{z}|\theta')$

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   → Inference of the **posterior** $p(\boldsymbol{z}|\boldsymbol{x},\theta)$ over the latent indicator variable ***z***

2) Fit class-specific parameters
   → Find parameters that optimize expected complete-data log likelihood

$$\mathbb{E}_{\mathbf{z}}\left[\ln p(\boldsymbol{x},\boldsymbol{z}|\theta')\right] = \sum_{z} p(\boldsymbol{z}|\boldsymbol{x},\theta) \ln p(\boldsymbol{x},\boldsymbol{z}|\theta')$$

Iterate

Tutorial

- Simulate observations from a generative model

Data is two-dimensional!

$$\mu = (\mu_x, \mu_y)$$

$$\Sigma = \begin{pmatrix} \sigma_x^2 & \sigma_x\sigma_y \\ \sigma_y\sigma_x & \sigma_y^2 \end{pmatrix}$$

for each class

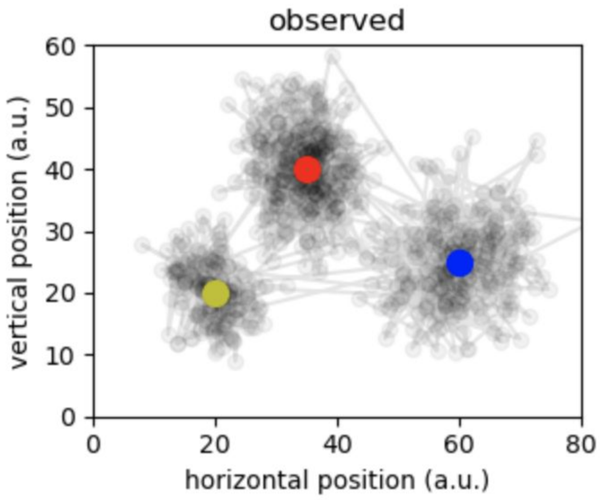
- Use EM to fit a Gaussian mixture model to simulated data

Mixture models

Hidden Markov Models

Hidden Markov Models

Same setup as before, but with sequential dependencies between data points



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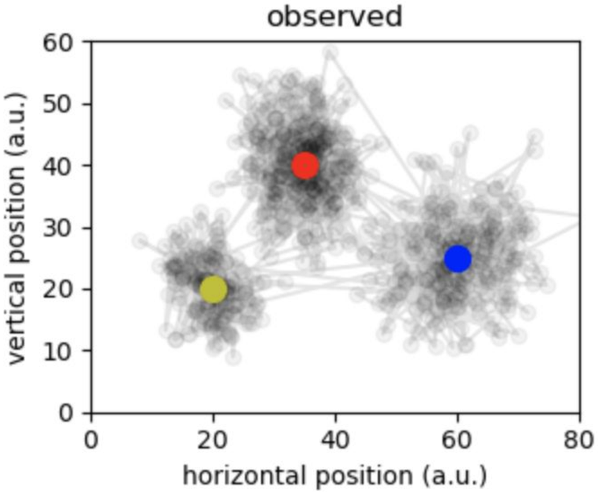
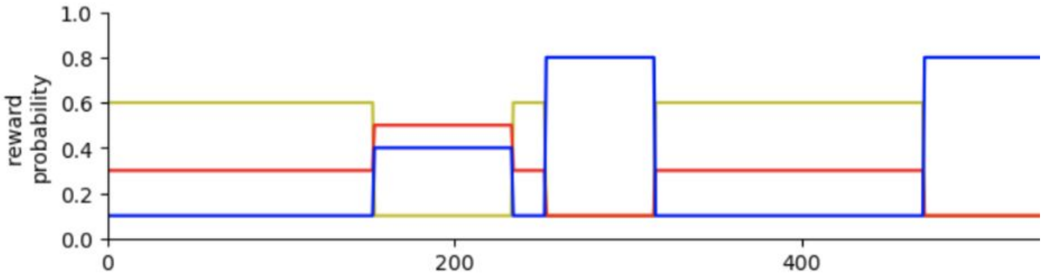


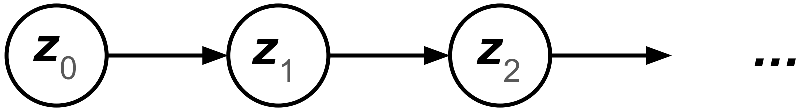


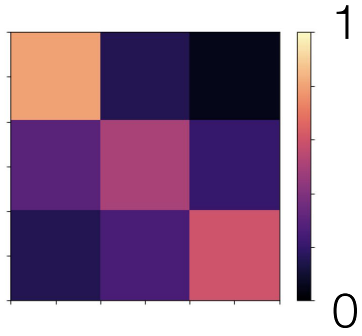
(e.g. because reward probabilities fluctuate in a block structure)

Hidden Markov Models

To capture sequential dependencies in choice behavior, we need an explicit model of transition structure of a latent variable, $p(\boldsymbol{z}_n | \boldsymbol{z}_{n-1})$

→ Transition matrix $\boldsymbol{A}$





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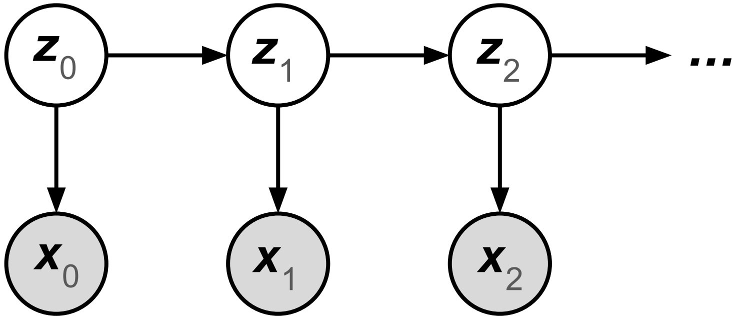
Saccades ***x*** are an "emission" of the latent choice variable



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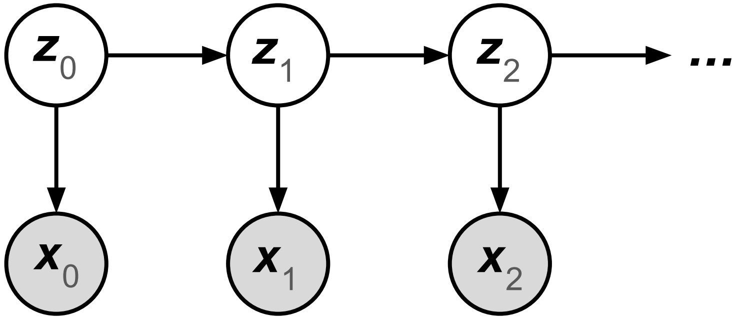

Joint distribution:

$$p(\boldsymbol{x}, \boldsymbol{z}|\theta)$$
$$= p(\boldsymbol{z}_0) \prod_{n=1}^{N} p(\boldsymbol{z}_n|\boldsymbol{z}_{n-1}) \prod_{n=0}^{N} p(\boldsymbol{x}_n|\boldsymbol{z}_n, \theta)$$

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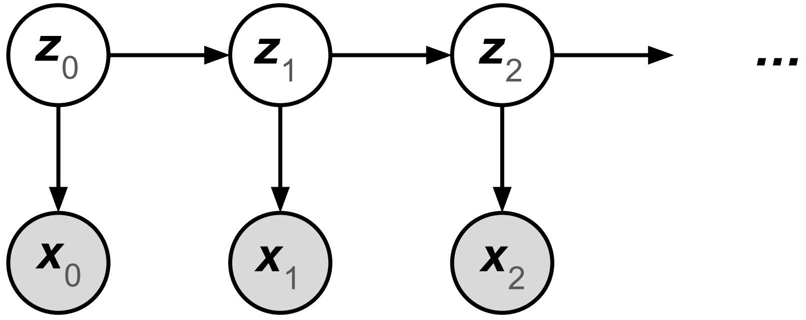
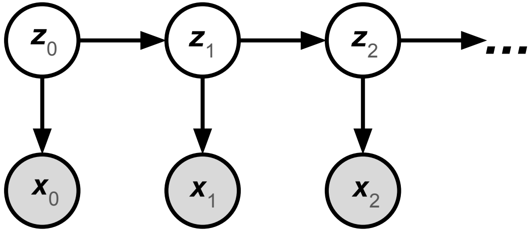


Joint distribution:

$$p(\boldsymbol{x}, \boldsymbol{z} | \theta) = \underbrace{p(\boldsymbol{z}_0)}_{\text{initial state } \pi} \underbrace{\prod_{n=1}^{N} p(\boldsymbol{z}_n | \boldsymbol{z}_{n-1})}_{\text{state transitions } A} \underbrace{\prod_{n=0}^{N} p(\boldsymbol{x}_n | \boldsymbol{z}_n, \theta)}_{\text{likelihoods}}$$

$$\mathcal{N}(\boldsymbol{x}_n; \mu_k, \Sigma_k)$$

EM for Hidden Markov Models



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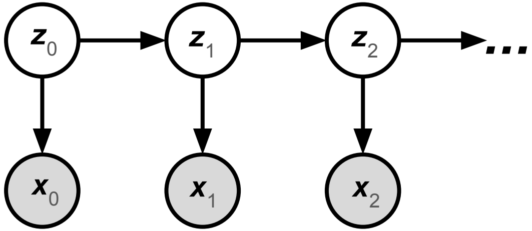

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We want to infer the latent states ***z***, and estimate parameters $\theta = \{\boldsymbol{\pi}, \boldsymbol{A}, \boldsymbol{\mu}, \boldsymbol{\Sigma}\}$

E-Step: infer posteriors and calculate initial state and state transition probabilities $\boldsymbol{\pi}, \boldsymbol{A}$

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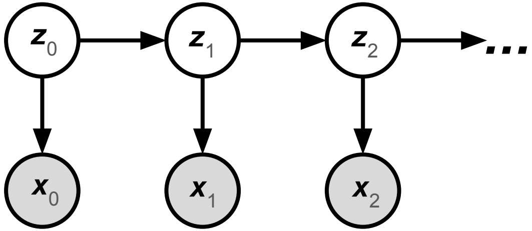

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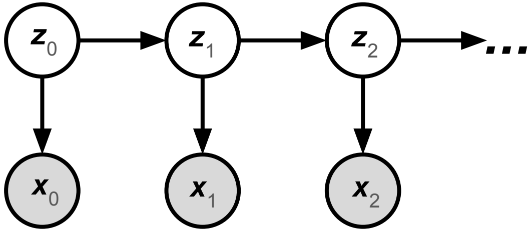

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$\rightarrow$ **easy!**

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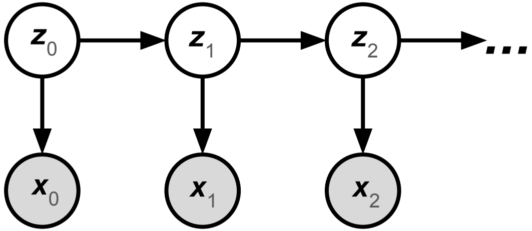

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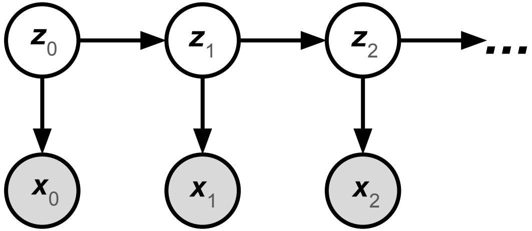

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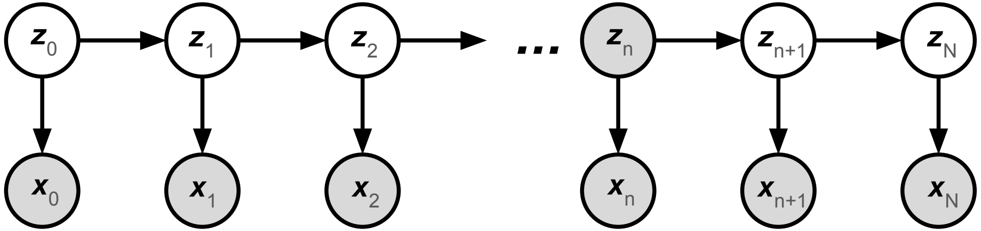
E-Step: infer posteriors
→ In HMMs, there are two posteriors over $\boldsymbol{z}$:

$p(\boldsymbol{z}_n|\boldsymbol{x}, \theta)$	probability of each latent state value
$p(\boldsymbol{z}_n, \boldsymbol{z}_{n-1}|\boldsymbol{x}, \theta)$	probability of observing a pair of subsequent states

EM for Hidden Markov Models

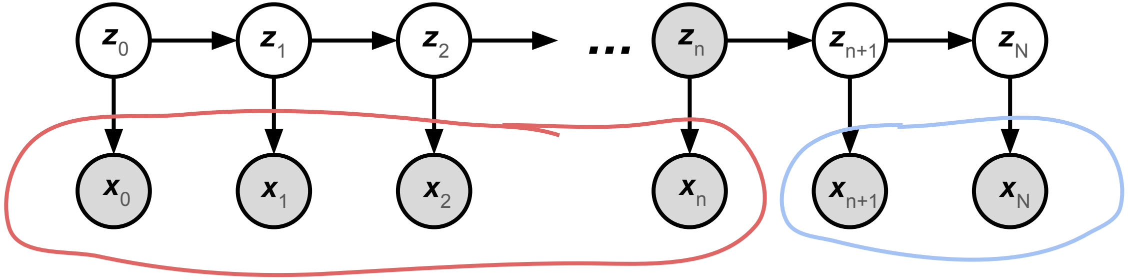


E-Step: infer posteriors

For this, again we use Bayes theorem $p(\boldsymbol{z}_n|\boldsymbol{x}_{0:N},\theta) = \frac{p(\boldsymbol{x}_{0:N}|\boldsymbol{z}_n,\theta)p(\boldsymbol{z}_n)}{p(\boldsymbol{x}_{0:N})}$

(and equivalent for $p(\boldsymbol{z}_n, \boldsymbol{z}_{n-1}|\boldsymbol{x},\theta)$)

EM for Hidden Markov Models



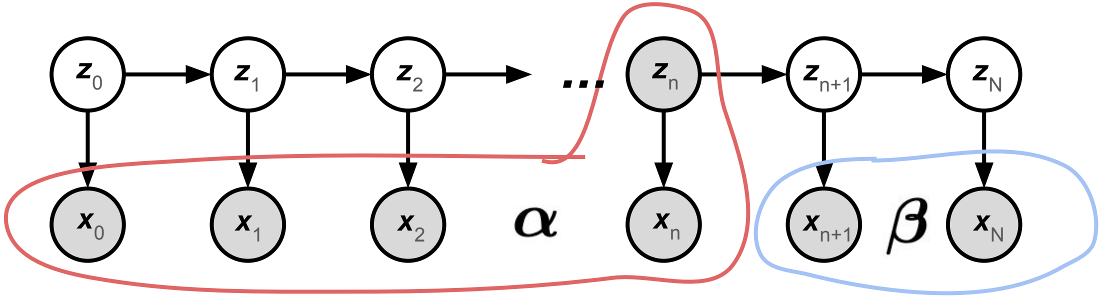
For a given state $\boldsymbol{z}_n$ in trial n, we can split the likelihood in two terms:

$p(\boldsymbol{x}_{0:n}|\boldsymbol{z}_n, \theta)$, $p(\boldsymbol{x}_{n+1:N}|\boldsymbol{z}_n, \theta)$

E-Step: infer posteriors

For this, again we use Bayes theorem $\quad p(\boldsymbol{z}_n|\boldsymbol{x}_{0:N}, \theta) = \dfrac{p(\boldsymbol{x}_{0:N}|\boldsymbol{z}_n, \theta)p(\boldsymbol{z}_n)}{p(\boldsymbol{x}_{0:N})}$

EM for Hidden Markov Models



We include $p(\boldsymbol{z}_n)$:

$$\underbrace{p(\boldsymbol{x}_{0:n}, \boldsymbol{z}_n|\theta)}_{\boldsymbol{\alpha}}, \quad \underbrace{p(\boldsymbol{x}_{n+1:N}|\boldsymbol{z}_n, \theta)}_{\boldsymbol{\beta}}$$

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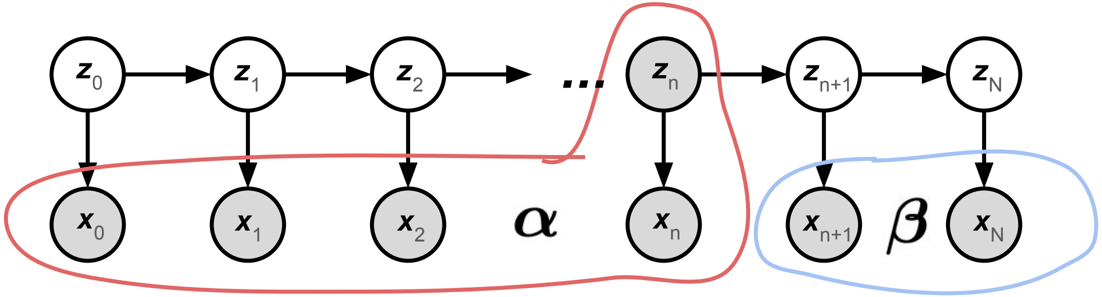
EM for Hidden Markov Models



We include $p(\boldsymbol{z}_n)$:

$p(\boldsymbol{x}_{0:n}, \boldsymbol{z}_n|\theta)$, $p(\boldsymbol{x}_{n+1:N}|\boldsymbol{z}_n, \theta)$

$\boldsymbol{\alpha}$ $\boldsymbol{\beta}$

E-Step: infer posteriors

For this, again we use Bayes theorem $$p(\boldsymbol{z}_n|\boldsymbol{x}_{0:N}, \theta) = \frac{p(\boldsymbol{x}_{0:N}|\boldsymbol{z}_n, \theta)p(\boldsymbol{z}_n)}{p(\boldsymbol{x}_{0:N})}$$

Then the posterior becomes $$p(\boldsymbol{z}_n|\boldsymbol{x}_{0:N}, \theta) = \frac{\boldsymbol{\alpha}(\boldsymbol{z}_n)\boldsymbol{\beta}(\boldsymbol{z}_n)}{p(\boldsymbol{x}_{0:N})}$$

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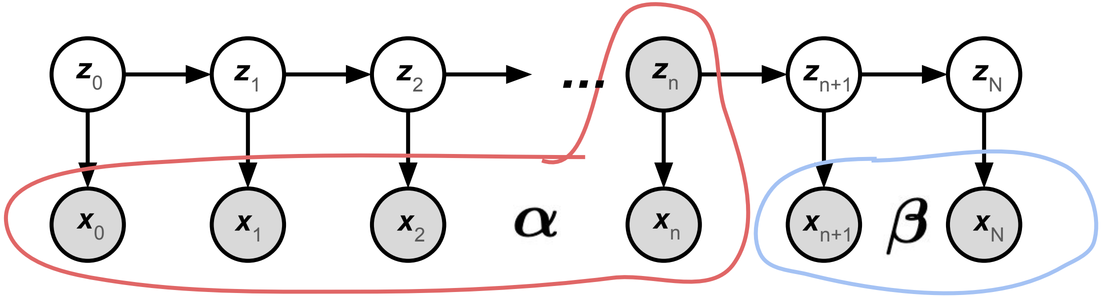

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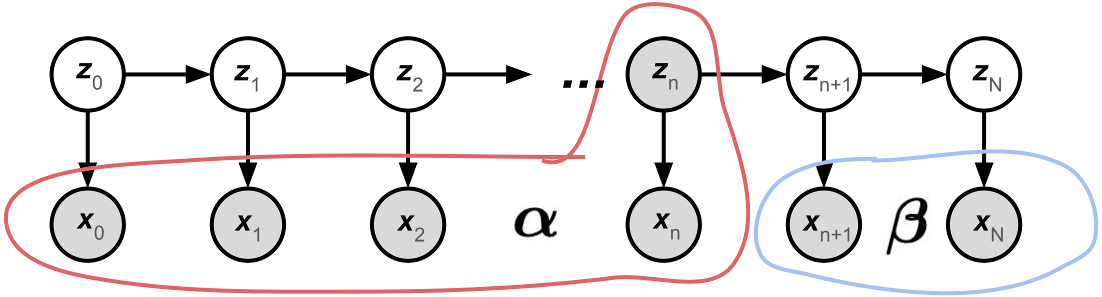

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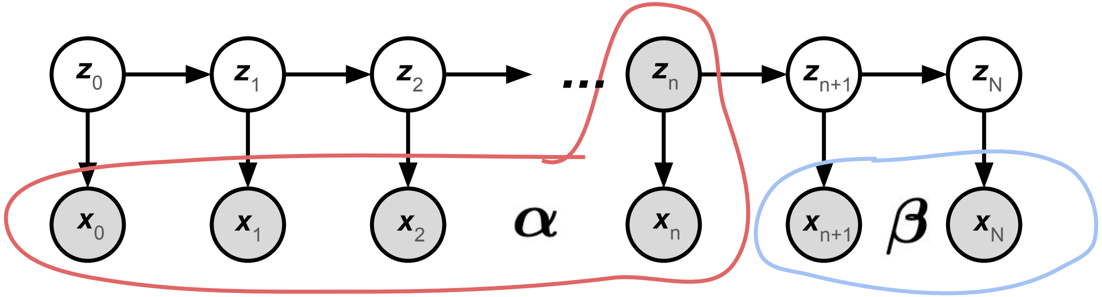

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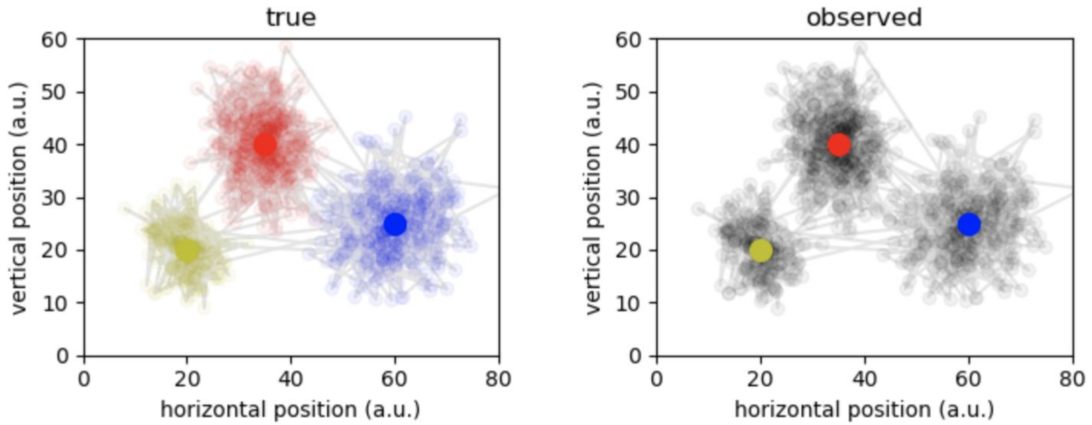
3. M-Step is straightforward

Tutorial

- Simulate observations from a generative model



- Use EM with forward-backward algorithm to fit an HMM to simulated data

Closing remarks

Mixture Models

More flexible models are possible! e.g. when target positions change from trial to trial

$$x^{(1)} = x_{saccade} - x_{target} \qquad x^{(2)} = x_{saccade} - x_{distractor}$$

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Mixtures of different distributions (e.g. Gaussian, uniform, Student t… → last exercise)

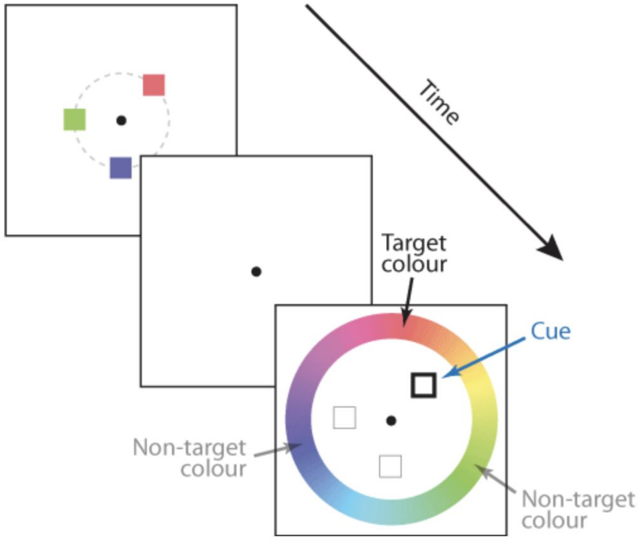
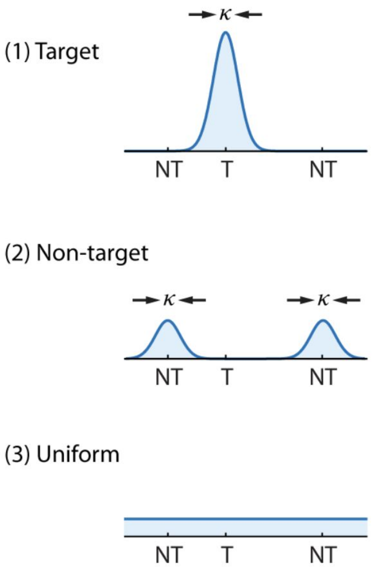


Figure 1 | The colour report task.



Bays, Catalao & Husain, *J Vis* (2009); Schneegans & Bays, *Cortex* (2016)

Remarks on emission models

Gaussian emission models: $p(\boldsymbol{x}|z_k = 1, \theta_k) = \mathcal{N}(\boldsymbol{x}|\mu_k, \Sigma_k)$

Other (including more complex) emission models are possible:

- Student t (see tutorial), Poisson, etc… (continuous observations ***x***)
- Categorical emissions (discrete observations ***x***)

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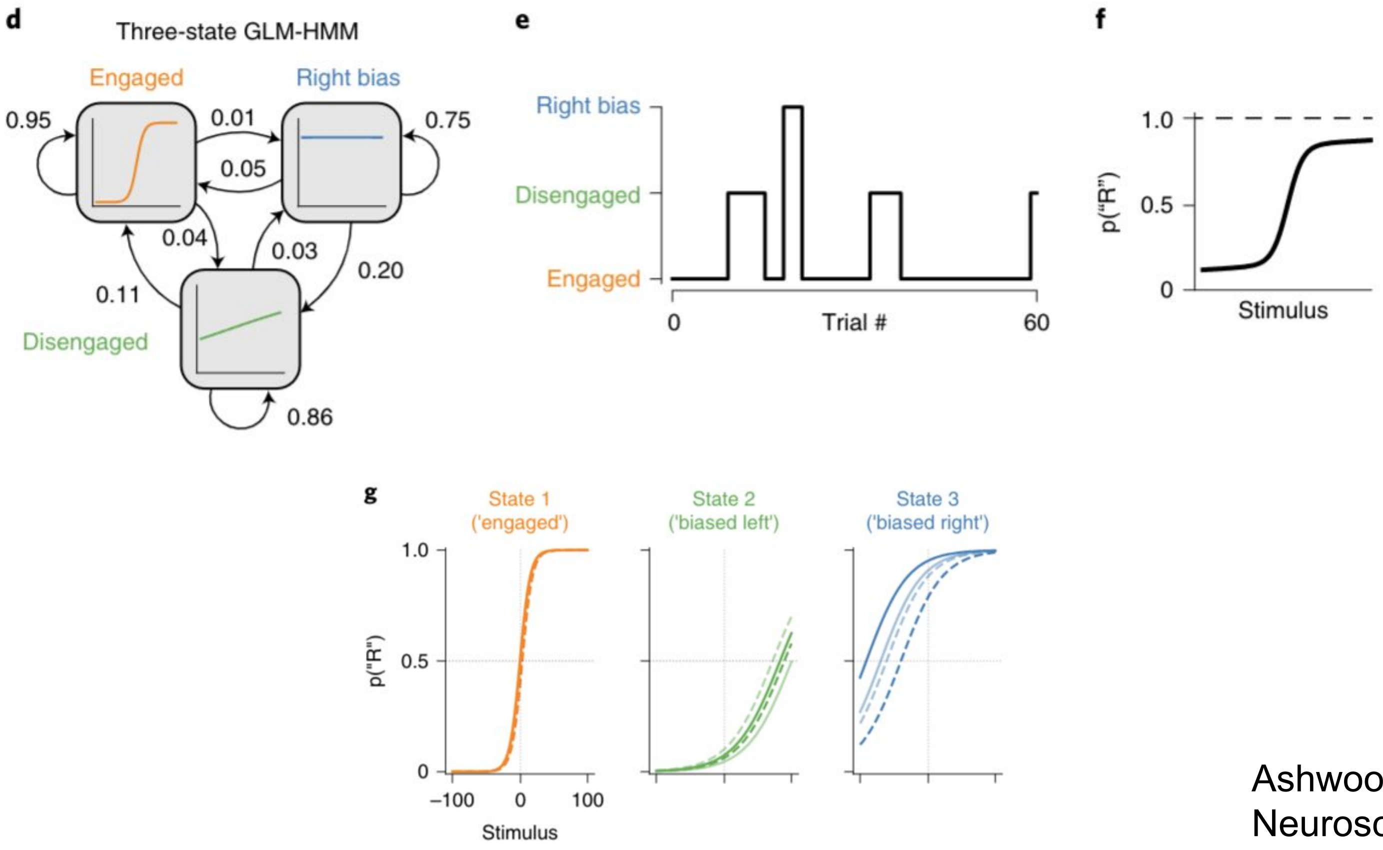
- Switching factor analysis (if you include an extra set of continuous latents that depend on the state)

Examples of HMMs in neuroscience



Ashwood et al., Nat Neurosci, 2022

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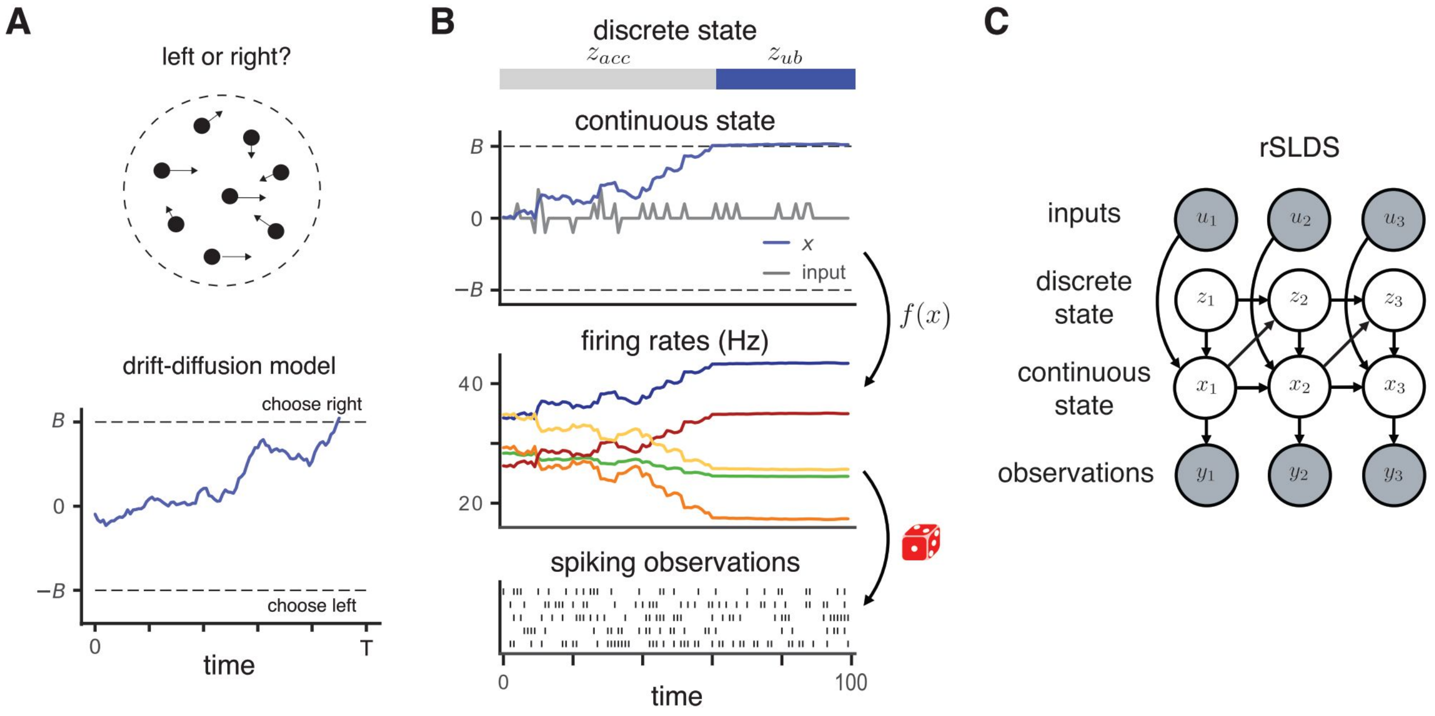

Zoltowski et al., ICML, 2020

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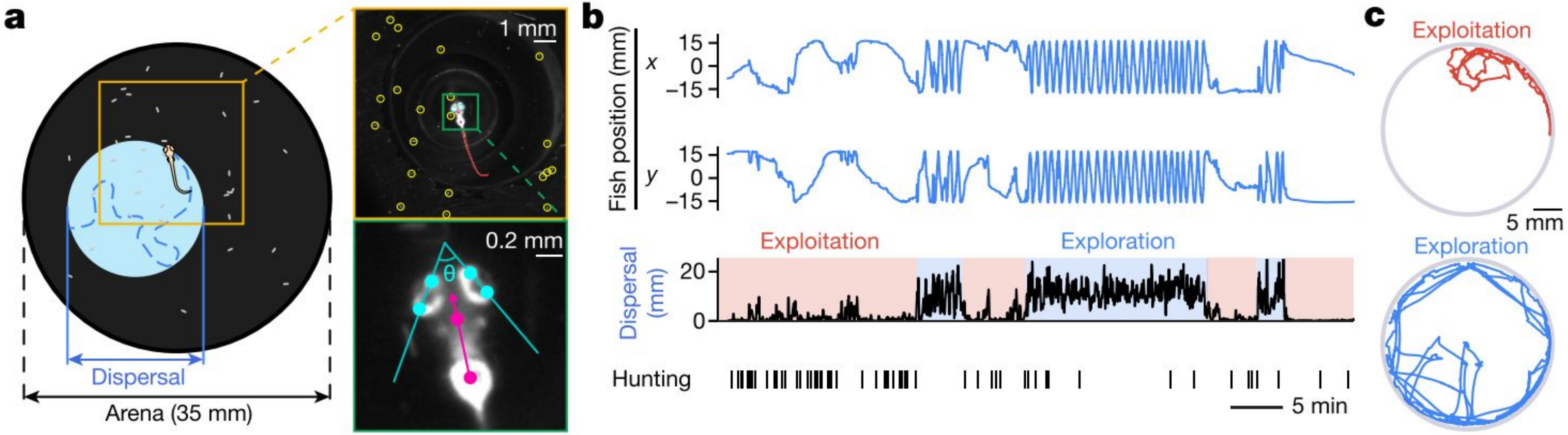

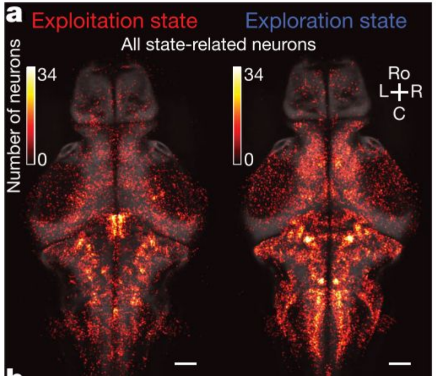

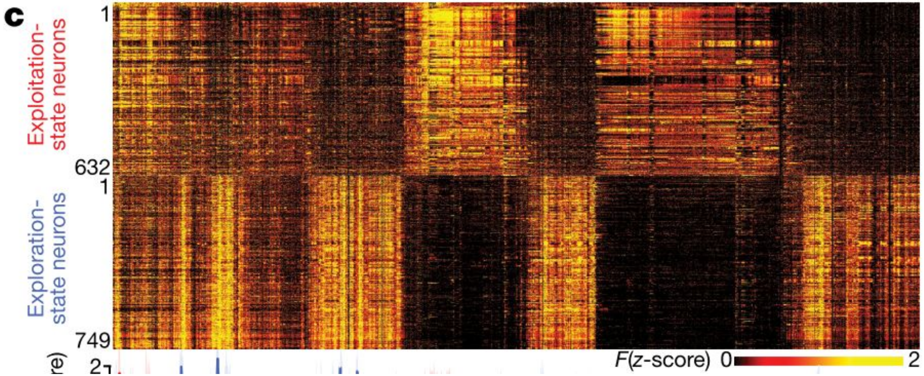

Marques et al., Nature, 2019

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Wiltschko et al., Neuron, 2015