CPTEC's Dynamical Core Documentation

This document presents the new version of the spectral dynamical core of the Global Circulation Atmospheric Model of CPTEC (the BAM model, Figueroa et.al (2016)), incorporating the changes made to use hybrid vertical coordinates (the vertical discretization used with a sigma vertical coordinate was described in a previous document). This model, based on the hydrostatic Primitive Equations, is a three-time-level semi-implicit scheme with both Eulerian and Semi-Lagrangian options, employing a reduced Gaussian grid (as proposed in Courtier and Naughton (1994)). Transport of moist variables, as well as tracers transport, is done entirely in a semi-Lagrangian fashion, with the variables represented only on the model grid. The model is fully parallelized, employing MPI for message passing and OPENMP for shared memory within a cluster, and runs on several thousands of processors of CPTEC's Cray XE6. The MPI parallelization of the model was done in similar lines as in Barros et al. (1995).

1 Primitive Equations on hybrid vertical coordinates

The equations written on latitude (θ) / longitude (λ) horizontal coordinates and on hybrid (η) vertical coordinates are as following. The momentum equations in advection formulation:

$$\frac{\partial U}{\partial t}+\frac{1}{a\cos^2\theta}(U\frac{\partial U}{\partial \lambda}+V\cos\theta\frac{\partial U}{\partial \theta})+\dot{\eta}\frac{\partial U}{\partial \eta}-fV+\frac{1}{a}(\frac{\partial \Phi}{\partial \lambda}+R_dT_v\frac{\partial \ln p}{\partial \lambda})=F_u \quad (1)$$

$$\frac{\partial V}{\partial t}+\frac{1}{a\cos^2\theta}(U\frac{\partial V}{\partial \lambda}+V\cos\theta\frac{\partial V}{\partial \theta})+\dot{\eta}\frac{\partial V}{\partial \eta}+fU+\frac{\cos\theta}{a}(\frac{\partial \Phi}{\partial \theta}+R_dT_v\frac{\partial \ln p}{\partial \theta})+\frac{\sin\theta}{a\cos^2\theta}(U^2+V^2)=F_v \quad (2)$$

Thermodynamic equation:

$$\frac{\partial T}{\partial t} + \frac{1}{a\cos^2\theta}(U\frac{\partial T}{\partial \lambda} + V\cos\theta\frac{\partial T}{\partial \theta}) + \dot{\eta}\frac{\partial T}{\partial \eta} - \frac{\kappa T_v \omega}{(1+(\delta-1)q)p} = F_T \tag{3}$$

Moisture equation:

$$\frac{\partial q}{\partial t} + \frac{1}{a\cos^2\theta}(U\frac{\partial q}{\partial \lambda} + V\cos\theta\frac{\partial q}{\partial \theta}) + \dot{\eta}\frac{\partial q}{\partial \eta} = F_q \tag{4}$$

Continuity equation (integrated form):

$$\frac{\partial \ln p_s}{\partial t} + \frac{1}{p_s}\int_0^1 \nabla\cdot(\mathrm{v}_H\frac{\partial p}{\partial \eta})\,d\eta = 0 \tag{5}$$

Hydrostatic equation (diagnostic):

$$\frac{\partial \phi}{\partial \eta} + \frac{R_d T_v}{p}\frac{\partial p}{\partial \eta} = 0 \tag{6}$$

Vertical velocity equations (diagnostic - comes from continuity equation):

$$\dot{\eta}\frac{\partial p}{\partial \eta} + \frac{\partial p}{\partial t} + \int_0^\eta \nabla\cdot(\mathrm{v}_H\frac{\partial p}{\partial \eta})\,d\eta = 0 \tag{7}$$

$$\omega = -\int_0^\eta \nabla\cdot(\mathrm{v}_H\frac{\partial p}{\partial \eta})\,d\eta + \mathrm{v}_H\cdot\nabla p \tag{8}$$

Model Variables:
The horizontal velocity field is $\mathrm{v}_H = (u, v)$, $U = u\cos\theta$, $V = v\cos\theta$, T_v is the virtual temperature and T is temperature, p_s is the surface pressure and q is the specific humidity. D will represent the horizontal Divergence,

ξ the relative vorticity and ϕ the Geopotential height. The hybrid vertical coordinate η is defined according to Simmons and Burridge (1981) with $\eta = \eta(p, p_s)$ varying monotonically with pressure at each grid point from 0 at the top of the atmosphere to 1 at the ground surface. The model layers are defined from the pressure values at the layer interfaces, given by $p_{k+1/2} = a_{k+1/2} + b_{k+1/2} p_s$, for $k = 0$ (at the top) to $k = N_l$ at the bottom layer, with two sets of constants $a_{k+1/2}$ and $b_{k+1/2}$. The η values are derived accordingly as $\eta_{k+1/2} = a_{k+1/2}/p_0 + b_{k+1/2}$ at the layer interfaces, with a constant value for p_0. At the model levels, when necessary, we define η_k as the average of the values at the neighbouring level interfaces. The constants a and b are chosen such that the model levels vary from terrain following (close to the ground) to constant pressure surfaces (close to the top of the atmosphere), with $a_{1/2} = b_{1/2} = a_{N_{l+1/2}} = 0$ and $b_{N_{l+1/2}} = 1$. The η-coordinate vertical velocity is $\dot{\eta}$, while $\omega = dp/dt$ is the p-coordinate vertical velocity. R_d and R_v are gas constants (for dry air and water vapor, respectively), C_{pd} and C_{pv} are specific heats at constant pressure (also for dry air and water vapor), with $\kappa = R_d/C_{pd}$. The coriolis term is given by $f = 2\Omega \sin\theta$ and F_U, F_V, F_T and F_q are forcing terms due to the physical parameterization processes.

2 Prognostic Variables and Spectral discretization

The prognostic variables

- D - Divergence field
- ξ - Vorticity field
- The velocities U and V will be derived from ξ and D.
- T_v - Virtual temperature
- q - specific humidity
- $\ln p_s$ - log of surface pressure (this is a two-dimensional field).

will be stored in spectral space (with the exception of q that normally will be only a grid-point variable). At each level k, we store the coefficients F_n^m of an expansion in spherical harmonics with triangular truncation of the field F, given by $F(\lambda, \theta) = \sum_{m=-M}^{M} \sum_{n=|m|}^{M} F_n^m P_n^m(\sin\theta) e^{im\lambda}$.

3 The vertical discretization

The vertical discretization follows closely Ritchie et al. (1995). The atmosphere is divided into N_l layers, which are defined by the pressure values at their interfaces (as described before), whose values are given by

$$p_{k+1/2} = a_{k+1/2} + b_{k+1/2} p_s \quad , k = 0, ..., N_l. \tag{9}$$

All the prognostic variables are evaluated at the middle of a vertical layer, corresponding to a pressure value p_k. The vertical velocities will be initially computed at the level interfaces. We first describe how the vertical integrals will be evaluated. We approximate

$$\begin{aligned} \int_0^{\eta_{k+1/2}} \nabla \cdot (\mathrm{v}_H \frac{\partial p}{\partial \eta})\ d\eta &\approx \sum_{j=1}^{k} \nabla \cdot (\mathrm{v}_j \Delta p_j) \\ &= \sum_{j=1}^{k} (D_j \Delta p_j + \mathrm{v}_j \cdot \nabla \Delta p_j) \\ &= \sum_{j=1}^{k} (D_j \Delta p_j + \mathrm{v}_j \cdot \nabla p_s \Delta b_j) \end{aligned} \tag{10}$$

where the subscripts denote the vertical layer, $\Delta p_j = p_{j+1/2} - p_{j-1/2}$, $\Delta b_j = b_{j+1/2} - b_{j-1/2}$ and D_j is the horizontal divergence at layer j, given by:

$$D_j = \frac{1}{a \cos^2 \theta} (\frac{\partial U_j}{\partial \lambda} + \cos \theta \frac{\partial V_j}{\partial \theta}) \ . \tag{11}$$

These integral approximations will be used in the discretization of (6), (7) and (8). The vertical advection of a variable x, in the Eulerian formulation, will be approximated by:

$$\left(\dot{\eta} \frac{\partial x}{\partial \eta} \right)_k = \frac{1}{2 \Delta p_k} \left[\left(\dot{\eta} \frac{\partial p}{\partial \eta} \right)_{k+\frac{1}{2}} (x_{k+1} - x_k) + \left(\dot{\eta} \frac{\partial p}{\partial \eta} \right)_{k-\frac{1}{2}} (x_k - x_{k-1}) \right] \ , \tag{12}$$

with $\dot{\eta} \frac{\partial p}{\partial \eta}$ derived from (7) and (10) as

$$\left(\dot{\eta} \frac{\partial p}{\partial \eta} \right)_{k+1/2} = -p_s b_{k+1/2} \frac{\partial}{\partial t} (\ln p_s) - \sum_{j=1}^{k} (D_j \Delta p_j + p_s (\mathrm{v}_j \cdot \nabla \ln p_s) \Delta b_j) \ , \tag{13}$$

where $\frac{\partial}{\partial t}(\ln p_s)$ is discretized from (5) and (10)

$$\frac{\partial}{\partial t}(\ln p_s) = -\sum_{j=1}^{N_l}(\frac{1}{p_s}D_j\Delta p_j + (\mathrm{v}_j \cdot \nabla \ln p_s)\Delta b_j) \ . \tag{14}$$

The Geopotential is obtained from the hydrostatic equation (6), as

$$\phi_{k+1/2} - \phi_{k-1/2} = -R_d(T_v)_k \left(\ln p_{k+1/2} - \ln p_{k-1/2}\right) \tag{15}$$

from which we get

$$\phi_{k-1/2} = \phi_s + \sum_{j=k}^{N_l} R_d(T_v)_j \ln\left(\frac{p_{j+1/2}}{p_{j-1/2}}\right) \ . \tag{16}$$

In order to get the Geopotential (and its gradient) at layer k we use

$$\phi_k = \phi_{k+1/2} + \alpha_k R_d(T_v)_k \ , \tag{17}$$

with (as proposed in Simmons and Burridge (1981))

$$\alpha_k = 1 - \frac{p_{k-1/2}}{\Delta p_k}\ln\left(\frac{p_{k+1/2}}{p_{k-1/2}}\right), \text{ for } k > 1 \text{ and } \alpha_1 = \ln 2 \ . \tag{18}$$

4 The semi-implicit Eulerian formulation

Let W be the vector of the prognostic variables for the primitive equations and let us write the system of equations as:

$$\frac{\partial W}{\partial t} + L(W) + N(W) = 0 \tag{19}$$

where the differential operator of the system is separated into an operator N including the non-linearities of the system and a linear operator L. In the Eulerian option of the model, system (19) is discretized in time as:

$$(I + \Delta t L)(W(t+\Delta t)) = (I - \Delta t L)(W(t-\Delta t)) - 2\Delta t N(W(t)) \tag{20}$$

The non-linear operator $N(W)$ is evaluated with the help of Spectral transforms. It includes the horizontal advection terms $\vec{V}.\nabla F$ (where F is U, V, T or q). The products are computed in grid-point space and the derivatives are obtained through spectral transforms. The solution of the system is entirely done spectrally.

We now describe in more detail, how the splitting into the operators L and N is done in each prognostic equation and how they are discretized.

5 The momentum equations

The horizontal and vertical advection (discretized as in (12)), the Coriolis terms and the metric term are all treated explicitly and are therefore part of the operator N for these two equations. The pressure gradient term and its discretization need to be detailed, before we can say which part is treated explicitly or implicitly. This term at level k is given by (using (17)):

$$\nabla\phi_k + R_d(T_v)_k\nabla\ln p_k = \nabla\phi_s + R_d\alpha_k\nabla(T_v)_k + R_d(T_v)_k\nabla\alpha_k + \sum_{j=k+1}^{N_l} R_d\left((T_v)_j\nabla\ln\left(\frac{p_{j+1/2}}{p_{j-1/2}}\right) + \ln\left(\frac{p_{j+1/2}}{p_{j-1/2}}\right)\nabla(T_v)_j\right) + R_d(T_v)_k\nabla\ln p_k \ . \tag{21}$$

In order to conserve angular momentum (cf. Simmons and Burridge(1981)), the following discretization is employed for the last term of (21)

$$\begin{aligned} R_d(T_v)_k\nabla\ln p_k &= \frac{R_d(T_v)_k}{\Delta p_k}\left(\alpha_k\nabla(\Delta p_k) + \ln\left(\frac{p_{k+1/2}}{p_{k-1/2}}\right)\nabla p_{k-1/2}\right) \\ &= R_d(T_v)_k\left(\alpha_k\Delta b_k + b_{k-1/2}\ln\frac{p_{k+1/2}}{p_{k-1/2}}\right)\frac{\nabla p_s}{\Delta p_k} \end{aligned} \tag{22}$$

We now derive some of the expressions involved in (21).

$$\begin{aligned} \nabla\alpha_k &= -\nabla\left(\frac{p_{k-1/2}}{\Delta p_k}\ln\frac{p_{k+1/2}}{p_{k-1/2}}\right) \\ &= -\frac{p_{k-1/2}}{\Delta p_k}\nabla\left(\ln\frac{p_{k+1/2}}{p_{k-1/2}}\right) - \ln\frac{p_{k+1/2}}{p_{k-1/2}}\left(\frac{\nabla p_{k-1/2}}{\Delta p_k} - \frac{p_{k-1/2}\nabla(\Delta p_k)}{\Delta^2 p_k}\right) \\ &= -\frac{p_{k-1/2}}{\Delta p_k}\left(\frac{b_{k+1/2}}{p_{k+1/2}} - \frac{b_{k-1/2}}{p_{k-1/2}}\right)\nabla p_s - \ln\frac{p_{k+1/2}}{p_{k-1/2}}\left(\frac{b_{k-1/2}\nabla p_s}{\Delta p_k} - \frac{p_{k-1/2}\Delta b_k\nabla p_s}{\Delta^2 p_k}\right) \\ &= \left(-b_{k+1/2}\frac{p_{k-1/2}}{p_{k+1/2}} + b_{k-1/2}\left(1 - \ln\frac{p_{k+1/2}}{p_{k-1/2}}\right) + \frac{p_{k-1/2}\Delta b_k}{\Delta p_k}\ln\frac{p_{k+1/2}}{p_{k-1/2}}\right)\frac{\nabla p_s}{\Delta p_k} \end{aligned}$$

and therefore

$$\nabla\alpha_k = \left(-b_{k+1/2}\frac{p_{k-1/2}}{p_{k+1/2}} + b_{k-1/2}\left(1 - \ln\frac{p_{k+1/2}}{p_{k-1/2}}\right) + (1-\alpha_k)\Delta b_k\right)\frac{\nabla p_s}{\Delta p_k} \ . \tag{23}$$

Now, combining (22) and (23) we get

$$
\begin{aligned}
R_d(T_v)_k(\nabla \ln p_k + \nabla \alpha_k) &= \frac{R_d(T_v)_k}{\Delta p_k}\left(b_{k-1/2} + \Delta b_k - b_{k+1/2}\frac{p_{k-1/2}}{p_{k+1/2}}\right) p_s \nabla \ln p_s \\
&= \frac{R_d(T_v)_k}{\Delta p_k}\left(b_{k+1/2}\left(1 - \frac{p_{k-1/2}}{p_{k+1/2}}\right)\right) p_s \nabla \ln p_s \\
&= R_d(T_v)_k\left(\frac{b_{k+1/2}}{p_{k+1/2}}\right) p_s \nabla \ln p_s
\end{aligned} \tag{24}
$$

We also need an expression for the following term

$$
\begin{aligned}
\nabla \ln \frac{p_{j+1/2}}{p_{j-1/2}} &= \nabla \ln p_{j+1/2} - \nabla \ln p_{j-1/2} = \frac{\nabla p_{j+1/2}}{p_{j+1/2}} - \frac{\nabla p_{j-1/2}}{p_{j-1/2}} \\
&= \left(\frac{b_{j+1/2}}{p_{j+1/2}} - \frac{b_{j-1/2}}{p_{j-1/2}}\right) \nabla p_s = -\frac{c_j}{p_{j+1/2}\ p_{j-1/2}} p_s \nabla \ln p_s
\end{aligned} \tag{25}
$$

$$\text{where } c_j = a_{j+1/2}\ b_{j-1/2} - a_{j-1/2}\ b_{j+1/2}$$

We can now describe how the pressure gradient term is split between the linear part L (to be treated implicitly) and the operator N with non-linearities which will be treated explicitly. The operator L incorporates some linearizations of terms of the pressure gradient. We will have (at level k):

$$
L = \alpha_k^r R_d \nabla (T_v)_k + \sum_{j=k+1}^{N_l} R_d \ln\left(\frac{p_{j+1/2}^r}{p_{j-1/2}^r}\right) \nabla (T_v)_j + R_d T^r \nabla \ln p_s \tag{26}
$$

$$\text{where } p_{j+1/2}^r = a_{j+1/2} + b_{j+1/2}\ p_s^r, \quad \alpha_k^r = 1 - \frac{p_{k-1/2}^r}{\Delta p_k^r} \ln\left(\frac{p_{k+1/2}^r}{p_{k-1/2}^r}\right)$$

and $p_s^r = 800\text{hPa}$ and $T^r = 300\text{K}$ are constant reference values.

By defining $\pi_k^r = \ln \frac{p_{k+1/2}^r}{p_{k-1/2}^r}$ and the matrix

$$
A = R_d \begin{bmatrix} \alpha_1^r & \pi_2^r & .. & .. & \pi_{N_l}^r \\ 0 & \alpha_2^r & \pi_3^r & .. & \pi_{N_l}^r \\ 0 & 0 & \alpha_3^r & .. & .. \\ .. & .. & .. & .. & \pi_{N_l}^r \\ 0 & .. & .. & 0 & \alpha_{N_l}^r \end{bmatrix} \quad ,
$$

the discretized momentum equations will have the form:

$$U^{n+1} + \frac{\Delta t}{a}(A\frac{\partial T_v}{\partial \lambda} + R_d T^r \frac{\partial \ln p_s}{\partial \lambda})^{n+1} = U^{n-1} + 2\Delta_t T_U \tag{27}$$

and

$$V^{n+1} + \frac{\Delta t \cos\theta}{a}(A\frac{\partial T_v}{\partial \theta} + R_d T^r \frac{\partial \ln p_s}{\partial \theta})^{n+1} = V^{n-1} + 2\Delta_t T_V \tag{28}$$

where

$$\begin{aligned} T_U &= F_u^n - Ad(U^n) + fV^n - \frac{1}{a}(\frac{\partial \Phi}{\partial \lambda} + R_d T_v \frac{\partial \ln p}{\partial \lambda})^n \\ &+ \frac{1}{a}(A\frac{\partial T_v}{\partial \lambda} + R_d T^r \frac{\partial \ln p_s}{\partial \lambda})^n - \frac{1}{2a}(A\frac{\partial T_v}{\partial \lambda} + R_d T^r \frac{\partial \ln p_s}{\partial \lambda})^{n-1} \end{aligned}$$

and

$$\begin{aligned} T_V &= F_v^n - Ad(V^n) - fU^n - \frac{\cos\theta}{a}(\frac{\partial \Phi}{\partial \theta} + R_d T_v \frac{\partial \ln p}{\partial \theta})^n \\ &+ \frac{\cos\theta}{a}(A\frac{\partial T_v}{\partial \theta} + R_d T^r \frac{\partial \ln p_s}{\partial \theta})^n \\ &- \frac{\sin\theta}{a\cos^2\theta}(U^2 + V^2)^n - \frac{\cos\theta}{2a}(A\frac{\partial T_v}{\partial \theta} + R_d T^r \frac{\partial \ln p_s}{\partial \theta})^{n-1} \end{aligned}$$

with

$$Ad(F) = \frac{1}{a\cos^2\theta}(F\frac{\partial F}{\partial \lambda} + V\cos\theta\frac{\partial F}{\partial \theta}) + (\dot{\eta}\frac{\partial F}{\partial \eta})$$

representing the advection of a field F.

The pressure gradient terms are discretized according to (21), (24) and (25).

6 The thermodynamic equation:

The treatment of the thermodynamic equation is described here in detail. First, we point out that the prognostic variable in the model is T_v (the virtual temperature) and not the temperature itself. So, equation (3) is written as:

$$\frac{\partial T_v}{\partial t} + Ad(T_v) - (1+\epsilon q)\frac{\kappa T_v \omega}{(1+(\delta-1)q)p} = \tilde{F_T} = (1+\epsilon q)F_T + \epsilon T\frac{dq}{dt} \tag{29}$$

where $\epsilon = \frac{R_v}{R_d} - 1$, $T_v = (1+\epsilon q)T$ and $\frac{dT_v}{dt} = (1+\epsilon q)\frac{dT}{dt} + \epsilon T \frac{dq}{dt}$. The last term in equation (29) is computed only in the physics (as in Sela (2009)) and is incorporated to the forcing term F_T. The energy conversion term will be discretized at level k as

$$(1+\epsilon q)\frac{\kappa T_v \omega}{(1+(\delta-1)q)p} = \kappa (T_v)_k \frac{1+\epsilon q_k}{1+(\delta-1)q_k}\left(\frac{\omega}{p}\right)_k \tag{30}$$

where the last term is discretized, based on the equation (8) for ω, as in Simmons and Burridge (1981), employing (10) and (22):

$$\begin{aligned}
\left(\frac{\omega}{p}\right)_k &= -\frac{1}{\Delta p_k}\left(\ln\frac{p_{k+1/2}}{p_{k-1/2}}\sum_{j=1}^{k-1}\nabla\cdot(\mathrm{v}_j\Delta p_j) + \alpha_k \nabla\cdot(\mathrm{v}_k\Delta p_k)\right) \\
&\quad + \mathrm{v}_k\cdot\left(\frac{\nabla p}{p}\right)_k \\
&= -\frac{1}{\Delta p_k}\left(\ln\frac{p_{k+1/2}}{p_{k-1/2}}\sum_{j=1}^{k-1}(D_j\Delta p_j + p_s(\mathrm{v}_j\cdot\nabla\ln p_s)\Delta b_j)\right) \\
&\quad - \frac{\alpha_k}{\Delta p_k}\Big(D_k\Delta p_k + p_s(\mathrm{v}_k\cdot\nabla\ln p_s)\Delta b_k)\Big) \\
&\quad + \frac{p_s}{\Delta p_k}\left(\Delta b_k + \frac{c_k}{\Delta p_k}\ln\frac{p_{k+1/2}}{p_{k-1/2}}\right)(\mathrm{v}_k\cdot\nabla\ln p_s)
\end{aligned} \tag{31}$$

The linearized part, to be treated implicitly, is given (at level k) by

$$L = \kappa T^r\left(\alpha_k^r D_k + \frac{1}{\Delta p_k^r}\ln\frac{p_{k+1/2}^r}{p_{k-1/2}^r}\sum_{j=1}^{k-1}(D_j\Delta p_j^r)\right) \tag{32}$$

with the reference values defined as for the momentum equations. With the introduction of the matrix

$$B = \kappa T^r \begin{bmatrix} \alpha_1^r & 0 & .. & .. & 0 \\ \frac{\Delta p_1^r}{\Delta p_2^r}\pi_2^r & \alpha_2^r & 0 & .. & 0 \\ \frac{\Delta p_1^r}{\Delta p_3^r}\pi_3^r & \frac{\Delta p_2^r}{\Delta p_3^r}\pi_3^r & \alpha_3^r & 0 & 0 \\ .. & .. & .. & .. & .. \\ \frac{\Delta p_1^r}{\Delta p_{N_l}^r}\pi_{N_l}^r & .. & .. & \frac{\Delta p_{N_l-1}^r}{\Delta p_{N_l}^r}\pi_{N_l}^r & \alpha_{N_l}^r \end{bmatrix},$$

the discretized thermodynamic equation takes the form

$$T_v^{n+1} + \Delta t B D^{n+1} = T_v^{n-1} + 2\Delta t T_T \tag{33}$$

with

$$\begin{aligned} T_T &= \tilde{F_T} - Ad(T_v^n) + (1+\epsilon q^n)\frac{\kappa T_v^n}{(1+(\delta-1)q^n)}\left(\frac{\omega}{p}\right)^n \\ &\quad +BD^n - \frac{1}{2}BD^{n-1} \end{aligned} \tag{34}$$

and $(\frac{\omega}{p})^n$ discretized as in (31).

7 Surface Pressure equation:

The continuity equation (5) is discretized with help of (10). The linearized term, to be treated implicitly is

$$L = \frac{1}{p_s^r}\sum_{j=1}^{N_l} D_j \Delta p_j^r = \delta \cdot D \tag{35}$$

$$\text{with} \quad \delta = \frac{1}{p_s^r}(\Delta p_1^r, ..., \Delta p_{N_l}^r) \ .$$

The discrete equation takes the form

$$\ln p_s^{n+1} + \Delta t \delta \cdot D^{n+1} = \ln p_s^{n-1} + 2\Delta t T_{p_s} \tag{36}$$

with

$$T_{p_s} = -\sum_{j=1}^{N_l}(\frac{1}{p_s^n} D_j^n \Delta p_j^n + (\mathrm{v}_j^n \cdot \nabla \ln p_s^n)\Delta b_j) + \delta \cdot D^n - \frac{1}{2}\delta \cdot D^{n-1} \tag{37}$$

8 Specific humidity equation:

In the current version of the model, the specific humidity q will be preferably treated in a Lagrangian way (even in the Eulerian version of the model), being handled only on the model grid. This discretization will be described together with the description of the transport of tracers and other moist variables. However, we keep an option to handle q as a spectral variable in a Eulerian way. In this case the discretization of this equation is done as

$$q^{n+1} = q^{n-1} + 2\Delta t T_q \tag{38}$$

where

$$T_q = F_q^n - Ad(q^n)$$

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9 Semi-implicit Semi-Lagrangian discretization

In the semi-Lagrangian version we write the momentum, thermodynamic and specific humidity equations as

$$\frac{dW}{dt} = \frac{\partial W}{\partial t} + A(W) = -L(W) - \tilde{N}(W) \tag{39}$$

where we have isolated the advection terms in $A(W)$, L is again the (same) linear part to be treated implicitly and $\tilde{N}$ contains the remaining non-linear contributions. The semi-Lagrangian discretization of (39) will have the form:

$$(I + \Delta t L)W^{n+1} = (I - \Delta t L)W_*^{n-1} - \Delta t(\tilde{N}W^n + \tilde{N}W_*^n) \tag{40}$$

where $*$ denotes the departure point (at time $t_{n-1} = t_n - \Delta t$) of a particle arriving at a Gaussian grid-point at time t_{n+1}. The non-linear contributions to the tendencies are evaluated at time t_n in the middle of the Lagrangian trajectory, approximated as the average in space between the arrival and departure point. The linearized terms L are treated implicitly, with the values at the departure points at instant t_{n-1} obtained through interpolation of their values at gaussian grid points. The Lagrangian trajectories are computed following Ritchie(1987,1991), as done in the previous sigma coordinate version of the model, but now the vertical velocity is $\dot{\eta}$, given by

$$\dot{\eta}_k = \frac{1}{2}\left[\left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{k-1/2} + \left(\dot{\eta}\frac{\partial p}{\partial \eta}\right)_{k+1/2}\right]\left(\frac{\partial p}{\partial \eta}\right)_k^{-1} , \tag{41}$$

using (13) and the approximation (as in Ritchie et al (1995))

$$\left(\frac{\partial p}{\partial \eta}\right)_k = \frac{\Delta p_k}{\Delta \eta_k} = p_s\frac{\Delta a_k/p_s + \Delta b_k}{\Delta a_k/p_0 + \Delta b_k} . \tag{42}$$

The momentum equations are treated in vector form, with the metric terms being absorbed in the Lagrangian trajectories. In this formulation, the prognostic equations will have the form:

$$
\begin{aligned}
\mathbf{V}^{n+1} + \Delta t(A\nabla T_v + R_d T^r \nabla \ln p_s)^{n+1} &= \mathbf{V}_*^{n-1} - \Delta t(A\nabla T_v + R_d T^r \nabla \ln p_s)_*^{n-1} \\
&\quad + \Delta t((\tilde{N}_{\mathbf{V}}^n) + (\tilde{N}_{\mathbf{V}}^n)_*) \qquad (43)
\end{aligned}
$$

where

$$\tilde{N}_{\mathbf{V}}^n = F_{\mathbf{V}}^n - f\mathbf{k} \times \mathbf{V}^n - (\nabla\Phi + R_d T_v \nabla \ln p)^n \ ,$$

with the pressure gradient terms discretized as in the Eulerian scheme. Since the equation is in vector form, the changes in the local orientations of the coordinates along the trajectory must be taken into account when the two components of the equation are written. For this, we follow Ritchie (1988).

The thermodynamic equation assumes the form (40) with L as in (32) and $\tilde{N}$ given by

$$\tilde{N} = \tilde{F_T} + (1 + \epsilon q)\frac{\kappa T_v}{(1 + (\delta - 1)q)}\left(\frac{\omega}{p}\right) + BD \qquad (44)$$

with $(\frac{\omega}{p})$ discretized as in (31).

In the specific humidity equation the term $\tilde{N}$ is formed only by the forcing term F_q.

For the Lagrangian treatment of the continuity equation we introduce a new variable

$$l' = \ln p_s + \frac{\phi_s}{R_d T^r} \qquad (45)$$

which is much smoother than $\ln p_s$ and contributes to reduce noise generated by orographic forcing, as suggested by Ritchie and Tanguay (1996). Moreover, as proposed in Barros and Garcia (2007) and also adopted in the sigma coordinate version of the present model, the integrated form of the continuity equation leads to a natural interpretation that $\ln p_s$ is advected by a mean wind, resulting from the vertical integration of the wind field (compare equation (14)). We therefore define the mean wind

$$\bar{\mathbf{V}} = \sum_{j=1}^{N_l} \mathrm{v}_j \Delta b_j \qquad (46)$$

and discretize the continuity equation as

$$(l')^{n+1} + \Delta t \delta \cdot D^{n+1} = (l')^{n-1}_{*_2} - \Delta t \delta \cdot D^{n-1}_{*_2} - \Delta t((\tilde{N}_{p_s})^n + (\tilde{N}_{p_s})^n_{*_2}) \quad (47)$$

where the subscript $*_2$ denotes the departure point of the two dimentional trajectory (given by the mean wind $\bar{\mathbf{V}}$) arriving at a grid-point and with

$$(\tilde{N}_{p_s})^n = \sum_{j=1}^{N_l} \left(\frac{1}{p_s^n} D_j^n \Delta p_j^n + \frac{(\mathrm{v}_j^n \cdot \nabla \phi_s)}{R_d T^r} \Delta b_j \right) - \delta \cdot D^n \ . \quad (48)$$

10 Time-step Evolution

At the beginning of a time-step the variables at time t are stored in spectral space. At this point we also assume that we have all variables and their necessary derivatives at time $t - \Delta t$ stored in grid-point-space. The time-step will then begin by the evaluation of the necessary variables at time t on the grid through spectral to grid transforms. Then, on the grid, the right-hand-side of the discretized equations will be computed. Also in grid-point-space we will carry out the time filtering process. Finished the grid-point computations (including the dry physics) we will transform the right-hand-side of the equations to spectral space. In spectral space, we will solve the system of equations and obtain the new variables at time $t + \Delta t$. This solution process will also entail the semi-implicit computations. After getting the updated values of the Temperature, humidity and surface pressure, the humidy physical processes are carried out and the values of humidity and temperature are adjusted and (normally only the temperature, since humidity is only a grid-point variable in our standard choice) again transformed to spectral space, where the horizontal diffusion is then applied to the prognostic variables. In case of necessity, we apply at the end of the time-step a selective enhanced diffusion, depending on the maximum wind speeds at each layer. In the following sections we provide some detail of the several parts which compose the time-step.

11 Spectral-to-grid transforms

The first step is to derive the coefficients of some fields which need to be transformed. This is the case for the velocities U and V, as well as for meridional components of the gradients of T_v and $\ln p_s$ (and of q if the humidity is treated in an Eulerian way). Coefficients of U and V are obtained from those of divergence and vorticity through the relations:

$$U_n^m = \frac{1}{a}\left(im\chi_n^m + (n-1)\varepsilon_n^m \psi_{n-1}^m - (n+2)\varepsilon_{n+1}^m \psi_{n+1}^m\right) \tag{49}$$

$$V_n^m = \frac{1}{a}\left(im\psi_n^m - (n-1)\varepsilon_n^m \chi_{n-1}^m + (n+2)\varepsilon_{n+1}^m \chi_{n+1}^m\right) \tag{50}$$

where $\chi_n^m = -(a^2/n(n+1))D_n^m$, $\psi_n^m = -(a^2/(n(n+1))\xi_n^m$ and $\chi_0^0 = \psi_0^0 = 0$ (ε_n^m is defined at the end of this document, in the section about spherical harmonics). The coefficients for the meridional component of the gradient of any field F are given by:

$$\left(\cos\theta \frac{\partial F}{\partial \theta}\right)_n^m = -(n-1)\varepsilon_n^m F_{n-1}^m + (n+2)\varepsilon_{n+1}^m F_{n+1}^m \tag{51}$$

After this we are ready to perform the Legendre transforms:

$$F^m(\theta) = \sum_{n=|m|}^{M} F_n^m P_n^m(\sin\theta) \tag{52}$$

for the following fields: U, V, D, ξ, T, $\ln ps$, $\cos\theta\partial T/\partial\theta$, and $\cos\theta\partial\ln ps/\partial\theta$ (in a total of 6 three-dimensional fields and 2 two-dimensional fields transformed). These are followed by Fourier transforms ($\sum_{m=-M}^{M} F^m(\theta)e^{im\lambda}$) in order to get the following fields in grid-point-space: U, V, D, ξ, T, $\ln ps$, $\cos\theta\partial T/\partial\theta$, $\cos\theta\partial\ln ps/\partial\theta$, $\partial U/\partial\lambda$, $\partial V/\partial\lambda$, $\partial T/\partial\lambda$ and $\partial\ln ps/\partial\lambda$. We point out, that for computing the values of a field F and of its derivative $\partial F/\partial\lambda$, we only need to perform one Legendre transform (getting $F^m(\theta)$) followed by two Fourier transforms (one for the field itself and one for the λ-derivative - after multiplying $F^m(\theta)$ by im). In the total we have Fourier transforms of 9 three-dimensional and of 3 two-dimensional fields. (In the semi-Lagrangian scheme we don't need to transform ξ, $\partial U/\partial\lambda$ and $\partial V/\partial\lambda$. This amounts to saving another 3-D Legendre transform and 3 3-D Fourier transforms.) If the humidity field q is kept as a spectral variable treated in a Eulerian way, extra transforms are needed.

In the Eulerian scheme it is still necessary to complete the wind gradients. We only compute the zonal derivatives of the winds through spectral transforms. In order to get the meridional derivatives, instead of performing two more Legendre transforms, followed by the corresponding Fourier transforms, the following relations are used on the computational grid:

$$\cos\theta\frac{\partial U}{\partial\theta} = \frac{\partial V}{\partial\lambda} - a\cos^2\theta\xi \tag{53}$$

$$\cos\theta\frac{\partial V}{\partial\theta} = -\frac{\partial U}{\partial\lambda} + a\cos^2\theta D \tag{54}$$

12 Transforms from grid-point to spectral space

After completing the evaluation of the tendencies (these shall also include the contributions from the physics parametrization processes) the implicit system has still to be solved. For this we will first complete the right-hand-sides of each equation. (For a prognostic variable F we build the right-hand-side as $R_F = F^{n-1} + 2\Delta t T_F$. Notice that in the semi-Lagrangian scheme the field value at time t_{n-1} has to be interpolated to the departure points locations. We will then rather add it to the tendency contributions at this time-level before interpolating.) Then, the RHS's of the equations for T and $\ln ps$ will be transformed to spectral space as they are. The momentum equations will be changed into equations for Vorticity and Divergence (by taking curl and divergence of the equations).

Before describing how this is done, we first remember how a grid-point field $R(\lambda,\theta)$ is transformed to spectral space (or in other words, how we obtain the spectral coefficients R_n^m of R from its grid-point values. Knowing the function R, its spherical harmonics coefficients are given by:

$$R_n^m = \frac{1}{2\pi}\int_{-1}^{1}\int_{-\pi}^{\pi} R(\lambda,\mu)P_n^m(\mu)e^{-im\lambda}d\lambda d\mu \tag{55}$$

where $\mu = \sin(\theta)$. This integral is evaluated in two steps, the first one consists of a Fourier transform (this is the integral with respect to λ through a multiple trapezoidal rule):

$$R_n(\mu_j) = \frac{1}{N_\lambda}\sum_{l=1}^{N_\lambda} R(\lambda_l,\mu_j)e^{-im\lambda_l} \tag{56}$$

where N_λ is usually taken as approximately $3M$ (where M is the spectral truncation). The second step is a Legendre transform, actually consisting of a gaussian integration for the integral with respect to μ, given by:

$$R_n^m = \sum_{j=1}^{N_\theta} \omega_j R_n(\mu_j) P_n^m(\mu_j) \tag{57}$$

where the ω_j's are the Gaussian weights from the integration formula and N_θ is the number of latitudes ($N_\theta = N_\lambda/2$).

We now proceed to the transformation of the momentum equations:

$$U^{n+1} + \frac{\Delta t}{a}\frac{\partial}{\partial \lambda}\left(AT_v + R_d T^r \ln p_s\right)^{n+1} = R_U \tag{58}$$

$$V^{n+1} + \frac{\Delta t \cos\theta}{a}\frac{\partial}{\partial \theta}\left(AT_v + R_d T^r \ln p_s\right)^{n+1} = R_V \tag{59}$$

into the equations for vorticity and divergence:

$$\xi^{n+1} = \frac{1}{a\cos^2\theta}\left(\frac{\partial R_V}{\partial \lambda} - \cos\theta\frac{\partial R_U}{\partial \theta}\right) = R_\xi \tag{60}$$

$$D^{n+1} + \Delta t \nabla^2\left(AT_v + R_d T^r \ln p_s\right)^{n+1} = \frac{1}{a\cos^2\theta}\left(\frac{\partial R_U}{\partial \lambda} + \cos\theta\frac{\partial R_V}{\partial \theta}\right) = R_D \tag{61}$$

The transfomed right-hand-sides R_ξ and R_D are given by the expressions:

$$R_{\xi_n}^{\ m} = \frac{1}{a}\int_{-\pi/2}^{\pi/2}\left(im\tilde{V}^m(\theta)P_n^m(\sin\theta) + \tilde{U}^m(\theta)H_n^m(\sin\theta)\right)\cos\theta d\theta \tag{62}$$

$$R_{D_n}^{\ m} = \frac{1}{a}\int_{-\pi/2}^{\pi/2}\left(im\tilde{U}^m(\theta)P_n^m(\sin\theta) - \tilde{V}^m(\theta)H_n^m(\sin\theta)\right)\cos\theta d\theta \tag{63}$$

which are obtained by the transforms (55), with application of integration by parts. In these formulas we have that

$$\tilde{U}^m(\theta) = \frac{1}{2\pi\cos^2\theta}\int_{-\pi}^{\pi} R_U(\theta,\lambda)e^{-im\lambda}d\lambda \tag{64}$$

$$\tilde{V}^m(\theta) = \frac{1}{2\pi\cos^2\theta}\int_{-\pi}^{\pi} R_V(\theta,\lambda)e^{-im\lambda}d\lambda \tag{65}$$

(which are computed through FFT's). We first compute (using two Legendre transforms)

$$\tilde{U}_n^m = \int_{-\pi/2}^{\pi/2} \tilde{U}^m(\theta) P_n^m(\sin\theta)\cos\theta d\theta \tag{66}$$

and

$$\tilde{V}_n^m = \int_{-\pi/2}^{\pi/2} \tilde{V}^m(\theta) P_n^m(\sin\theta)\cos\theta d\theta \tag{67}$$

Then we use that

$$H_n^m(\sin\theta) = -n\varepsilon_{n+1}^m P_{n+1}^m(\sin\theta) + (n+1)\varepsilon_n^m P_{n-1}^m(\sin\theta) \tag{68}$$

in order to get

$$R_{\xi_n}^{\ m} = \frac{1}{a}\left(im\tilde{V}_n^m - n\varepsilon_{n+1}^m \tilde{U}_{n+1}^m + (n+1)\varepsilon_n^m \tilde{U}_{n-1}^m\right) \tag{69}$$

and

$$R_{D_n}^{\ m} = \frac{1}{a}\left(im\tilde{U}_n^m + n\varepsilon_{n+1}^m \tilde{V}_{n+1}^m - (n+1)\varepsilon_n^m \tilde{V}_{n-1}^m\right) \tag{70}$$

for $|m| \leq n \leq M$.

Summarizing, we transform the RHS's R_U, R_V, R_T and R_{ps} from grid-point space to spectral space, amounting to Fourier and Legendre transforms of 3 three-dimensional fields and 1 two-dimensional (assuming that the humidity q remains as a grid-point variable only). After this we already have the new value of vorticity through equation (60) and of the specific humidity through equation (4). The remaining prognostic variables will be determined in the semi-implicit computations.

13 Semi-implicit computations

It remains to solve the system which couples the values of T_v, D and $\ln ps$ for the time-step $t + \Delta t$ in the different vertical layers. This system has the form:

$$\begin{aligned} D + \Delta t \nabla^2 (AT_v + R_d T^r \ln p_s) &= R_D \\ T + \Delta t BD &= R_T \\ \ln ps + \Delta t \delta \cdot D &= R_{ps} \end{aligned} \tag{71}$$

We will solve the system in spectral space, it is coupled only in the vertical, being independent for each wave-number (n, m). It assumes the following form:

$$\begin{aligned} \tilde{D}_n^m + \bar{A}\tilde{T_v}_n^m + \tilde{\beta} \ln {p_s}_n^m &= \tilde{R_D}_n^m \\ \tilde{T_v}_n^m + \bar{B}\tilde{D}_n^m &= \tilde{R_T}_n^m \\ \ln {p_s}_n^m + \tilde{\delta}.\tilde{D}_n^m &= {R_{ps}}_n^m \end{aligned} \tag{72}$$

where $\bar{A} = -(\Delta t n(n+1)/a^2)A$, $\bar{B} = \Delta t B$, $\tilde{\beta} = -(\Delta t n(n+1)/a^2)R_d T^r$, $\tilde{\delta} = \Delta t \delta$ and the $\tilde{}$'s denote vertical vectors.

The solution of this system for each pair (n, m) is obtained by first substituting the values of T_v and $\ln p_s$ from the second and third equation into the first, to get:

$$\tilde{D}_n^m + \bar{A}(\tilde{R_T}_n^m - \bar{B}\tilde{D}_n^m) + \tilde{\beta}({R_{ps}}_n^m - \tilde{\delta}.\tilde{D}_n^m) = \tilde{R_D}_n^m$$

which leads to:

$$\tilde{D}_n^m = (I - \bar{A}\bar{B} - \tilde{\beta}.\tilde{\delta}^T)^{-1}(\tilde{R_D}_n^m - \tilde{\beta}{R_{ps}}_n^m - \bar{A}\tilde{R_T}_n^m)$$

The matrix $(I - \bar{A}\bar{B} - \tilde{\beta}.\tilde{\delta}^T)$ is explicitly computed and inverted. Once $\tilde{D}_n^m$ has been computed, one obtains $\tilde{T_v}_n^m = \tilde{R_T}_n^m - \bar{B}\tilde{D}_n^m$ and $\ln {p_s}_n^m = {R_{ps}}_n^m - \tilde{\delta}.\tilde{D}_n^m$.

14 Horizontal Diffusion

An implicit method is employed for the horizontal diffusion, in which the diffusion equation

$$\frac{\partial F}{\partial t} = T_F - c(-1)^p \nabla^{2p} F \tag{73}$$

is discretized spectrally by a fractional step method as:

$$\tilde{F}_n^m(t + \Delta t) = \bar{F}_n^m(t + \Delta t) - 2c\Delta t \left(\frac{n(n+1)}{a^2}\right)^p \tilde{F}_n^m(t + \Delta t) \tag{74}$$

where $\bar{F}_n^m$ is a temporary value for $\tilde{F}_n^m$ at the new time-step, obtained after the semi-implicit computations (the implicit horizontal diffusion will be applied at the end of the time-step). For the virtual temperature and the humidity field no horizontal diffusion is being applied.

15 Enhanced Diffusion

An extra diffusion term is applied at a given level k when the maximum wind speed at this level $V_{max}(k)$, computed during the grid-point computations, exceeds a critical value V_{crit} (taken as $85m/s$). We define a factor $\beta = V_{crit} n_0 \Delta t_0 / \Delta t$, with $\Delta t_0 = 900s$, $n_0 = 63$ (Eulerian time-step for a reference resolution n_0) and Δt the time-step of the model in use. We obtain a critical wave-number $n_{crit} = \beta / V_{max}$. The spectral coefficients of the prognostic variables at level k will be selectively dumped, but only for wave-numbers above the critical value. The dumping factor for each $n > n_{crit}$ is defined as $cdump_n = (1 + \alpha \Delta t V_{max}(n - n_{crit})/a)^{-1}$, with $\alpha = 2.5$, and each spectral coefficient at this wave-number multiplied by $cdump$. (These values are chosen such that an advection equation with velocity V_{max} would be integrated stably with an Eulerian explicit scheme)

16 Time-Filter

A Robert-Asselin time filter is applied (for a prognostic variable F) in the form:

$$\bar{F}(t) = F(t) + \gamma \left(\bar{F}(t - \Delta t) - 2F(t) + F(t + \Delta t) \right) \tag{75}$$

where the overbar denotes filtered quantities. This filter will be applied in grid-point space in two steps:

$$\tilde{F}(t) = F(t) + \gamma \left(\bar{F}(t - \Delta t) - 2F(t) \right)$$

at the end of a time-step and

$$\bar{F}(t - \Delta t) = \tilde{F}(t - \Delta t) + \gamma F(t)$$

when $F(t)$ becomes available in grid-point space. $\tilde{F}$ will be stored until the next step, when it is used in the computation of $\bar{F}$. These filtered values at time $t - \Delta t$ (and unfiltered at time t) will be used in the computations of the right-hand-sides of the equations.

17 Spherical Harmonics

For completeness we include here a general description about expansions in Spherical Harmonics and its basic properties. Spherical harmonics are defined on the sphere, for each (integer) zonal wavenumber m and total wavenumber n ($n \geq |m|$), as:

$$Y_n^m(\lambda, \theta) = e^{im\lambda} P_n^m(sin\theta) \tag{76}$$

where P_n^m is an associated Legendre Polynomial of order m and degree n. They are given by Rodrigues formula as:

$$P_n^m(\mu) = \sqrt{\frac{(2n+1)(n-m)!}{2(n+m)!}} \frac{(1-\mu^2)^{|m|/2}}{2^n n!} \frac{d^{n+|m|}(1-\mu^2)^n}{d\mu^{n+|m|}} \tag{77}$$

The Legendre polynomials in the above formula are normalized such that, for each order m, they build an orthonormal family with respect to the inner product $< f, g >= \int_{-1}^{1} f(\mu) g(\mu) d\mu$. On the other hand, the spherical harmonics constitute a complete orthonormal family with respect to integration over the sphere (taking $\mu = sin\theta$):

$$\frac{1}{2\pi} \int_{-1}^{1} \int_{-\pi}^{\pi} Y_n^m \bar{Y}_{n'}^{m'} d\lambda d\mu = \delta_{n,n'} \delta_{m,m'} \tag{78}$$

where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise.

Another important property of the spherical harmonics is that they are eingefunctions of the Laplace operator on the two-dimensional sphere of radius a. We have:

$$\nabla^2 Y_n^m = \frac{-n(n+1)}{a^2} Y_n^m \tag{79}$$

This property is used throughout the model. Furthermore we have that $\partial Y_n^m / \partial \lambda = im Y_n^m$, meaning that the spherical harmonics are also eigenfunctions of the zonal differentiation operator. The same is not true for differentiation in the meridional direction. In this case we will make use of the relation

$$H_n^m(\mu) = (\mu^2 - 1) \frac{dP_n^m(\mu)}{d\mu} = n \varepsilon_{n+1}^m P_{n+1}^m(\mu) - (n+1) \varepsilon_n^m P_{n-1}^m(\mu) \tag{80}$$

where

$$\varepsilon_n^m = \sqrt{\frac{n^2 - m^2}{4n^2 - 1}}$$

In the global spectral model we develop the meteorological fields in spherical harmonic expansions, for a given truncation M. Our choice is to use a triangular truncation, which provides an isotropic resolution over the globe (in other words, the resolution of a triangular truncation would be invariant under any rotation of the coordinate system). Any field F will therefore have a truncated expansion like

$$F(\lambda, \theta) = \sum_{m=-M}^{M} \sum_{n=|m|}^{M} F_n^m Y_n^m(\lambda, \theta) = \sum_{m=-M}^{M} \sum_{n=|m|}^{M} F_n^m e^{im\lambda} P_n^m(\sin\theta) \tag{81}$$

Given a field F on the sphere, its spectral coeffcients F_n^m can be obtained (using (78)) as

$$F_n^m = \frac{1}{2\pi} \int_{-1}^{1} \int_{-\pi}^{\pi} F(\lambda, \mu) \bar{Y}_n^m(\lambda, \mu) d\lambda d\mu \tag{82}$$

where $\mu = \sin\theta$.

This computation is done numerically in two steps (for a field F known on a spherical grid of N_λ by N_θ points). The integral with respect to λ is computed as

$$F_m(\mu_j) = \frac{1}{N_\lambda} \sum_{l=1}^{N_\lambda} F(\lambda_l, \mu_j) e^{-im\lambda_l} \tag{83}$$

which is a discrete Fourier transform. It is followed by a numerical integration with respect to μ, done as

$$F_n^m = \sum_{j=1}^{N_\theta} \omega_j F_m(\mu_j) P_n^m(\mu_j) \tag{84}$$

which is a Gaussian integration formula with weights ω_j. In order to have optimal precision in the numerical evaluation of the integral with respect to μ, the latitudes on the grid are chosen such that $\mu_j = \sin\theta_j$ are the zeros of a Legendre Polynomial of order zero and degree N_θ (number of latitudes on the grid). Longitudes are uniformly spaced. N_θ and N_λ are chosen such that the spectral coefficients of quadratic non-linear products are obtained with no errors (alias free). For this, N_θ has to be at least $(3M+1)/2$ and $N_\lambda = 2N_\theta$.

In the evaluation of the Legendre transform (84) one should use the fact that $P_n^m(-\mu) = (-1)^{n+|m|} P_n^m(\mu)$ in order to save computations (see Temperton (1991)).

The computation of the grid values of a function given by its spectral coefficients is also done in two steps, an inverse Legendre transform

$$F^m(\theta) = \sum_{n=|m|}^{M} F_n^m P_n^m(\sin\theta) \tag{85}$$

followed by an inverse Fourier transform

$$F(\lambda,\theta) = \sum_{m=-M}^{M} F^m(\theta) e^{im\lambda} \tag{86}$$

In the inverse transforms it is used that $P_n^m(\mu) = P_n^{-m}(\mu)$ and that $\bar{F}_n^m = F_n^{-m}$ (for a real field) in order to save computations.

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