Finding Polynomial Integrals in Simple Dynamical Systems

Sharif University of Technology

Bamdad Torabi

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1 Introduction

Integrability is a vital property one can discuss when examining dynamical systems. We include the definition here in order to set a standard:

Defintion 1. *Let X be a topological space and let $f : X \to X$ be a map which defines a (discrete-time) dynamical system on X. We call a function $T : X \to \mathbb{R}$ (or $\mathbb{C}$) an **integral** for f whenever for all $x \in X$ we have $T(f(x)) = T(x)$. We call f an **integrable** dynamical system whenever a continuous and non-constant integral T exists.*

During the autumn of 2024, I took a course titled "Introduction to Dynamical Systems". A question was posed as an exercise which caught my eye. The question was originally from Professor Oliver Knill's lecture notes from the course under the same name in Harvard University in 2005 [3]. The question goes as follows:

Question 2. [3, page 138, question 2.5]
a. Prove that the cat map $f(x, y) = (2x + y, x + y)$ on the torus is not integrable.
b. Prove that the cat map as defined on the plane is integrable.

A cute argument using continuity and the stable and unstable manifolds of the hyperbolic point at the origin were sufficient to prove the first part, and for the second part one can easily verify that the quadratic $T(x, y) = x^2 - xy - y^2$ is an integral for the cat map on the plane. This question raised two big questions for me regarding the integrability of dynamical systems:

Question 3. Given a topological space X and a dynamical system $f : X \to X$ how can one classify the set of all integrals for this dynamical system? Also, when can we expect to have polynomials (or one step further, analytic power series) as our integrals?

These questions opened up a rabbit hole which has led me to writing this article. After discussing the history of integrability in mathematical literature, we shall set up the necessary notation and definitions and then we will prove some propositions regarding polynomial integrals for some simple class of dynamical systems. Using these, we shall reduce the problem of finding polynomial integrals to a simpler problem. Afterwards we will construct an algorithm based on those results to find all polynomial integrals in simple cases, and at the end we shall work on a few examples with the new algorithm and classify polynomial integrals in these systems.

2 History

Integrals in Mathematical Physics

Dynamical systems have historically been applied under the guise of mathematical physics. The first person to use their knowledge of mathematics to examine an ever-challenging physical problem, in celestial mechanics, was **Johannes Kepler** (1571-1630). After trying to decipher the proportions of the solar system by (creatively) invoking their "heavenly nature" by means of the Platonic solids, he then tried using his knowledge of conic sections to describe the motions of the planets. The result is now widely known as **"Kepler's laws of planetary motion"**. in which he describes the orbits, velocities and periods of the planets' orbits with an ellipse with one focus on the sun. His results are still widely regarded with high praise for their accuracy for their time. A few decades after this, it was **Isaac Newton** (1642-1727) who took inspiration from Kepler and deduced his **Universal Law of Gravitation** from his work. Here we can say, many argue, that mathematical physics was born. [6, slides 6 - 14]

It is actually quite a curious thing that an orbit of a planet should even be well-behaved; every planet's orbit is bounded and we expect them to be closed. Two and a half centuries after Kepler, it was **Joseph Bertrand** (1822-1900) who in 1873 stated that this "miraculous" occurence was an exception to the rule, not vice-versa:

Theorem 4. *(Bertrand, 1873) If we have a central force system (i.e. one where the magnitude of the force only depends on the distance r to a single point) in which all bounded orbits are closed, then the force is either proportional to r^{-2} or is proportional to r.*

This showed that the laws described before him (Newton's law of gravitation, Coulomb's law of electrostatic charge, Hooke's law for springs), which all fell into either of the two cases, were the only ways in which we could have a closed orbit for an object moving in this manner. He, alongside **Jean-Gaston Darboux** (1842-1917), also proved the same was true if we simply assume all our orbits to be conic sections. [6, slides 15 - 19]

The next big achievement in this direction came to us courtesy of **William Rowan Hamilton** (1805-1865). He used the **Euler-Lagrange equations** that described classical mechanics and showed that they could be rewritten as what is now known as a **Hamiltonian system**: if $\vec{x} = (x_1, ..., x_n)$ is the vector of the positions of n particles and $\vec{p} = (p_1, ..., p_n)$ is the vector of their momenta, then a function $H : \mathbb{R}^{2n} \to \mathbb{R}$ called the **Hamiltonian** exists such that:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} \qquad \frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}$$

In modern dynamical system terms, this means that H is an integral for the physical system, since the defining equations show that the trajectories are perpendicular to the gradient of H and therefore the value of H remains unchanged. Using this we can restrict the "movement" of the flow generated by the laws of motion to a submanifold of $\mathbb{R}^{2n}$. Finding other integrals in the system helps knock down the dimension of the solution space and therefore is also crucial to solving the problems. [6, slide 21]

This, and other attempts to deal with "conserved" quantities (such as energy and momentum) in closed systems were some of the first attempts to find integrals for systems which govern their evolution and movement. Later on, it was discovered that a rich geometric intuition lies behind this formulation: integrals for a Hamiltonian system exists in an infinite-dimensional Lie algebra called the **Poisson algebra** and can nowadays be formulated using **symplectic geometry** [6, slides 22 - 27]. This idea can be described more generally on differentiable manifolds: if M is an m-dimensional $\mathcal{C}^r$ manifold and $X_1, ..., X_k$ are vector fields on M such that at each point $x \in M$ we have that $\{X_1(x), ..., X_k(x)\}$ is linearly independant, we call the subspaces spanned at each point by these vector fields a **regular distribution of rank** k on M. If we define the Lie bracket of two vector fields X and Y to be $[X, Y]f = X \cdot Y \cdot f - Y \cdot X \cdot f$, the following theorem holds:

Theorem 5. *(Frobenius' integrability theorem) A regular distribution Δ on a manifold M is* ***completely integrable****, meaning that a submanifold N of M exists such that at each $x \in N$ we have $T_xN = \Delta_x$, if and only if it is closed under the Lie bracket, meaning that for any two vector fields X, Y in the vector space of vector fields in Δ, $[X, Y] \in \Delta$.*

The theorem is proven by "straightening out" the vector fields in the $M = \mathbb{R}^n$ case by using the **Picard-Lindelöf theorem** from ODE theory, and then piecing them together in the general case. This theorem essentially gives a unified description of how to attain integrals (submanifolds of the original manifold which are tangent to specific vector fields), given a certain condition on the laws of motion (the vector fields). For more details see any introductory textbook on differential manifolds; for example, [4].

Much more can be said of integrals in physics; we will suffice at one more important example. In 1918, **Emmy Noether** (1882-1935) published a theorem which relates well-known integrals and conserved quantities to continuous symmetries of the system; the integrals are described using calculus of variations and the Euler-Lagrange equations, while the symmetries are described through actions of Lie groups. The theorem is as follows:

Theorem 6. *(Noether, 1918) For every symmetry in a physical system, a "conservation law" exists; meaning that a quantity exists that remains constant under the laws of the system.*

This profound theorem in mathematical physics established a duality between the daily integrals we encouter in physics (the conservation laws of energy, momentum, charge, etc) and the symmetries they exhibit [6, slides 28 - 34].

Now that we mentioned the duality between symmetries (i.e. groups acting on our space) and integrals (i.e. conserved quantities, or submanifolds to move on), we can discuss the other, more modern view in which conserved quantities and integrals are examined.

Invariant Theory

One can in general mention many similar problems to finding integrals in dynamical systems in algebraic and differential geometry; to name one, the **Jacobi inversion problem** comes from **Carl Gustav Jacob Jacobi** (1804-1851) trying to identify the geodesics on an ellipsoid. This lead to him using, in his own words, "a remarkable substitution" to find an explicit equation describing the geodesic in question [6, slides 57 - 63]. Here however we shall be specifically mentioning **invariant theory** and how it is related to the problem addressed in the rest of the article.

In 1845 **Arthur Cayley** (1821-1895) wrote the first article in invariant theory (although he himself credits the creation of the fields to questions posed in an 1841 paper by **George Boole** (1815-1864)). The field concern this basic question: let G be a group and let V be a finite-dimensional vector space over a field K. By considering a **representation** of G over V, i.e. a group homomorphism $\rho : G \to GL(V)$, one can see that this induces an action of G into $K[V]$ (the space of all polynomial functions on V with coefficients in K). A natural question to ask is how we can determine which polynomials are fixed under G; if we denote this algebra by $K[V]^G$, we are interested in discovering what this algebra is and how it behaves. [2]

This is a different approach to what we have seen up until now in mathematical physics: it isn't a dynamical system as we have no notion of a time component. However, the same question still applies: when is a polynomial fixed under some sort of action from a group? This is the question I engaged with from the context of polynomial integrals of simple dynamical systems, and it lies at the heart of invariant theory.

Many people have worked in invariant theory: **Felix Klein** (1849-1925) (who computed the invariant rings of finite group actions of $\mathbb{C}^2$), **Issai Schur** (1875-1941), **Hermann Weyl** (1885-1955), **Élie Cartan** (1869-1951), ... to name a few. This culminated in two great results: the first is the so called "symbolic method" which was pioneered by Cayley and others to manipulate algebraic expressions with no regard to their real meaning which proved to be a great success; the second great result comes from **David Hilbert** (1862-1943) who by proving his famous Basis Theorem showed that for a large class of polynomials, their invariant rings were finitely generated. This paper of Hilbert is sometimes acclaimed as the "first paper in modern algebra". [2]

Alas, because of Hilbert's momumental success in the field, no new "magnificient" problem arose and the subject started to die out. Many attempts were made to revive the topic but none were as influential as **David Mumford** (born 1937) who, in his 1965 book titled **"Geometric Invariant Theory"**, developed the theory by addressing the more general situation of group actions on schemes. Central to this theory is the concept of **"moduli spaces"**; it is defined as the quotient X/G where X is a scheme parametrizing a set of objects and G is a group acting on this scheme. In other words, we identify all points where the scheme is fixed under the action of G and therefore are essentially looking at the "orbit space" of this action, to put it in a more dynamical system context. [2][1]

As it shall be evident, the problem and solution I encountered and which I have written about can be traced back to the beginning ideas of invariant theory and is quite elementary, to be frank. However, reading through the plethora of references about invariant theory has made me believe that much can be done in this path to generalize and expand what I have done in this article, and I am eager to get the chance to work on this problem in the future, hopefully with an expanded and more sophisticated toolbox.

For more on integrals in a historic view, see [6] and [8]. For more on invariant theory see [2], [7] and [1].

3 Notation and Definitions

We first begin by setting up some useful notation:

Defintion 7. *Let X be a topological space and $f : X \to X$ a dynamical system on X. We can define the **integral algebra of** f as follows:*

$$\mathcal{I}_0(f, X) = \left\{ T \in \mathcal{C}(X, \mathbb{R}) \middle| \forall x \in X \quad T(f(x)) = T(x) \right\}$$

We will omit X whenever it is clear from context.

We now justify our use of the word "algebra" in our definition:

Defintion 8. *Let $\mathbb{F}$ be a field, and let $(A, +, \times)$ be a ring with unity. If A is also a vector space on $\mathbb{F}$ such that the multipication in the ring and the scalar multiplication commute (so $r(a \times b) = (ra) \times b = a \times (rb)$) then A is called an $\mathbb{F}$**-algebra**. If $B \subseteq A$ is another $\mathbb{F}$-algebra such that $1_B = 1_A$ then we call B a **subalgebra** of A.*

For more information about $\mathbb{F}$-algebras refer to any book on abstract algebra such as [5].

Proposition 9. *If f is a dynamical system on the topological space X, then the integral algebra $\mathcal{I}_0(f)$ is an $\mathbb{R}$-algebra; more specifically, it is a subalgebra of $\mathcal{C}(X, \mathbb{R})$. Also, all constant functions lie in $\mathcal{I}_0(f)$.*

Proof. Let $S, T \in \mathcal{I}_0(f)$ and $r \in \mathbb{R}$. For any $x \in X$ we compute:

$$\begin{aligned}(S+T)(f(x)) &= S(f(x)) + T(f(x)) = S(x) + T(x) = (S+T)(x) && \Rightarrow S+T \in \mathcal{I}_0(f)\\ (rT)(f(x)) &= r \cdot T(f(x)) = r \cdot T(x) = (rT)(x) && \Rightarrow rT \in \mathcal{I}_0(f)\\ (ST)(f(x)) &= \Big(S(f(x))\Big)\Big(T(f(x))\Big) = \Big(S(x)\Big)\Big(T(x)\Big) = (ST)(x) && \Rightarrow ST \in \mathcal{I}_0(f)\end{aligned}$$

So by definition $\mathcal{I}_0(f)$ is an $\mathbb{R}$-algebra. Now we notice that the constant function $f_1 : X \to \mathbb{R}$ where $f(x) = 1$ is in $\mathcal{I}_0(f)$ and is also the multiplicative identity in $\mathcal{C}(X, \mathbb{R})$. Therefore, since $\mathcal{C}(X, \mathbb{R})$ is an $\mathbb{R}$-algebra itself (by a similar argument to $\mathcal{I}_0(f)$), we can conclude that $\mathcal{I}_0(f)$ is a subalgebra of $\mathcal{C}(X, \mathbb{R})$. Finally, by defining $f_r = r \cdot f_1 \in \mathcal{C}(X, \mathbb{R})$ for all $r \in \mathbb{R}$ we see that $f_r \in \mathcal{I}_0(f)$ (by the defintion of an $\mathbb{R}$-algebra) and therefore the $\mathbb{R}$-algebra of constant functions is a subalgebra of $\mathcal{I}_0(f)$. ∎

Remark 10. The defintion above lets us talk about integrability of a dynamical system in a simpler way: if f is a dynamical system on a topological space X, we always have constant functions in our integral algebra and therefore $\mathbb{R} \hookrightarrow \mathcal{I}_0(f, X)$. Therefore, f is integrable if and only if $\mathcal{I}_0(f, X) \not\simeq \mathbb{R}$. Also the definitions above and the proposition also work for $\mathbb{C}$, or any other topological field.

From this point on we shall introduce our main cause: let $A \in M_n(\mathbb{R})$ be an $n \times n$ matrix, and let $f(x) = Ax$ be a dynamical system on $\mathbb{R}^n$. We wish to be able to find all integrals for f which are polynomials in $x_1, ..., x_n$ (the coordinates of our space). Thus we define:

Defintion 11. *Let $X = \mathbb{R}^n$, and let $f : X \to X$ be a dynamical system on X. We define the* ***polynomial integral algebra of*** f *as follows:*

$$I_p(f) = \Big\{T \in \mathbb{R}[x_1, ..., x_n] \Big| \forall x \in X \quad T(f(x_1, ..., x_n)) = T(x_1, ..., x_n)\Big\}$$

Proposition 12. *If f is a dynamical system on the topological space X, then the polynomial integral algebra $\mathcal{I}_p(f)$ is an $\mathbb{R}$-algebra; more specifically, it is a subalgebra of $\mathcal{I}_0(f)$. Also, all constant functions lie in $\mathcal{I}_p(f)$.*

Proof. The proof is entirely the same as Proposition 9. ∎

The following notations shall also prove useful in the next section when refering to the polynomial integral algebra of a linear function, to monomials in a multivariate polynomial, and to diagonal matrices (respectively):

Notation 13. In the definition of the polynomial integral algebra (Definition 11), if f is a linear function, i.e. we can write $f(x_1, ..., x_n) = A[x_1, ..., x_n]^T$ where A is an $n \times n$ real matrix, then by abuse of notation we write $\mathcal{I}_0(A)$ instead of $\mathcal{I}_0(f)$, and we similarly define $\mathcal{I}_p(A)$.

Notation 14. Let $I = (a_1, ..., a_n)$ be a vector where for all i, a_i is a non-negative integer. In this case if we let $x = (x_1, ..., x_n)$ then the expression x^I is understood to refer to the monomial ${x_1}^{a_1} ... {x_n}^{a_n}$. Also, we define ΣI as $\Sigma I := a_1 + ... + a_n$.

Notation 15. If $\lambda_1, ..., \lambda_n \in \mathbb{F}$ are values in a field $\mathbb{F}$ (not necessarily distinct) then the $n \times n$ matrix $D := \mathrm{diag}(\lambda_1, ..., \lambda_n)$ is defined as the diagonal matrix with entries $\lambda_1, ..., \lambda_n$ (in order) on the main diagonal.

4 Classifying polynomial integrals

In this section we shall be focusing on linear functions on $\mathbb{R}^n$; in other words, our space is $X = \mathbb{R}^n$ for some natural number n and $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $f(x) = Ax$ where $A \in M_n(\mathbb{R})$. Also, note that everything being done can also be done for $\mathbb{C}$; this will prove to be useful since eigenvalues of real matrices might not necessarily be real and might possibly be complex. First we give show that integrals behave well under equivalent dynamical systems:

Proposition 16. *Let X and Y be two topological spaces and let $f : X \to X$ be a dynamical system on X with an integral $T \in \mathcal{I}_0(f, X)$. If $h : Y \to X$ is a homeomorphism and we define $g : Y \to Y$ as $g(y) := (h^{-1} \circ f \circ h)(y)$, then the function $S : Y \to \mathbb{R}$ as defined as $S(y) = (T \circ h)(y)$ is an integral for g; in other words, $S \in \mathcal{I}_0(g, Y)$.*

Proof. Firstly, S is continuous since T and h are continuous. Secondly we can compute:

$$S(g(y)) = (T \circ h) \circ (h^{-1} \circ f \circ h)(y) = (T \circ f \circ h)(y) \overset{T \in \mathcal{I}_0(f,X)}{=} (T \circ h)(y) = S(y)$$

Therefore by definition, $S \in \mathcal{I}_0(g, Y)$ and S is an integral for g. ■

This helps us in working with linear maps: if A is a diagonalizable $n \times n$ matrix with eigenvalues $\lambda_1, ..., \lambda_n$ (not necessarily distinct), we know that we can find an invertible matrix $P \in M_n(\mathbb{R})$ such that $A = P^{-1}DP$ where $D = \text{diag}(\lambda_1, ..., \lambda_n)$; therefore classifying any sort of integrals for D will swiftly allow us to classify the same sort of integrals for A. Now we prove our main theorem, which classifies all polynomial integrals for a diagonal matrix D:

Theorem 17. *Let $\lambda_1, ...\lambda_n$ be real or complex numbers (not neccesarily distinct) and let $D \in M_n(\mathbb{C})$ so that $D = \text{diag}(\lambda_1, ..., \lambda_n)$.*
a. The monomial $p(x) = {x_1}^{a_1}...{x_n}^{a_n}$ is an integral for D if and only if $p(\lambda_1, ..., \lambda_n) = 1$.
b. Let $P(x) = \sum_{\Sigma I \leq k} a_I x^I$ be a polynomial of degree k. Then P is an integral for D if and only if every non-zero monomial of P such as $q(x) = a_I x^I$ is also a polynomial integral of D.

Proof. For part a, p is a polynomial and is therefore always continuous. By naming $\vec{\lambda} = (\lambda_1, ..., \lambda_n)$, we directly compute:

$$p \in \mathcal{I}_0(D, \mathbb{R}^n) \Leftrightarrow \forall x \; p(x) = p(Dx) \Leftrightarrow {x_1}^{a_1}...{x_n}^{a_n} = (\lambda_1 x_1)^{a_1}...(\lambda_n x_n)^{a_n} = p(\lambda)p(x)$$
$$\Leftrightarrow p(x)(p(\lambda) - 1) = 0 \overset{p(x) \not\equiv 0}{\Longleftrightarrow} p(\lambda) = 1$$

Therefore $p(x)$ is an integral for $x \mapsto Dx$ if and only if $p(\lambda) = 1$. For part b, we compute P on x and Dx again:

$$P(x) = P(Dx) \Leftrightarrow \sum_{\Sigma I \leq k} a_I x^I = \sum_{\Sigma I \leq k} a_I \lambda^I x^I \Leftrightarrow \sum_{\Sigma I \leq k} a_I (\lambda^I - 1) x^I = 0$$

The condition on the right is an equality of polynomials and therefore every coefficient on either side must be equal. In other words, this is equivalent to $a_I(\lambda^I - 1) = 0$. If we only consider non-zero terms of $P(x)$, that means that we only consider the indices I where $a_I \neq 0$. Therefore this is equivalent to $\lambda^I = 1$ for any index I where $a_I \neq 0$. However by part a we know that this is equivalent to the monomial x^I being an integral for D whenever $a_I \neq 0$.

If, on the other hand, all the aforementioned monomials are integrals, then since $\mathcal{I}_0(D)$ is an $\mathbb{R}$-algebra, every linear combination of elements is also in $\mathcal{I}_0(D)$; and since P is a linear combination of these monomials, therefore $P \in \mathcal{I}_0(D)$. ■

Therefore this gives us an algorithm to classify all polynomial integrals of $f(x) = Ax$ for a diagonalizable matrix A: first we diagonalize A into $A = P^{-1}DP$ for some diagonal matrix D and some invertible matrix P. Then, since the entries on the main diagonal of D are exactly the eigenvalues of A, we solve the problem in Part a of Theorem 17 of finding all non-negative integers $a_1, ..., a_n$ such that $\prod_{i=1}^{n} {\lambda_i}^{a_i} = 1$. After we have found all such monomials, by Part b of Theorem 17 we know that $\mathcal{I}_0(D)$ is exactly the $\mathbb{R}$-algebra generated by these monomials. Finally, by using Proposition 16 we transform this generating set of monomials into a generating set of polynomials for $\mathcal{I}_0(A)$.

5 Examples

Example 18. Recall Arnold's cat map on the plane: $f(x, y) = (2x + y, x + y)$. We intend to find all polynomial integrals of f. We go through our algorithm: first we need to diagonalize the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow \det(\lambda I_2 - A) = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 1 \end{vmatrix} = \lambda^2 - 3\lambda + 1 = 0 \Rightarrow \lambda_{1,2} = \frac{3 \pm \sqrt{5}}{2}$$

If we let $\varphi = \dfrac{1 + \sqrt{5}}{2}$ be the golden ratio, then we see that $\lambda_1 = 1 + \varphi$ and $\lambda_2 = 2 - \varphi$. We now need to find non-negative integers a and b such that $\lambda_1^a \lambda_2^b = 1$. However we can notice that $\lambda_2 = \dfrac{1}{\lambda_1}$ (by looking at the coefficients of the characteristic polynomial of A). So, we have $\lambda_1^{a-b} = 1$ and therefore $a - b = 0 \Rightarrow a = b$. Therefore any polynomial integral for $D = \operatorname{diag}(\lambda_1, \lambda_2)$ is generated by monomials of the form $(xy)^a$ for some non-negative integer a.

At last, in order to recover the polynomial integrals of A, we need to compute the change of basis matrix P. For that we shall find the eigenvectors:

$$\begin{cases} \begin{bmatrix} 2x + y \\ x + y \end{bmatrix} = \lambda_1 \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow 2x + y = \lambda_1 x \Rightarrow y = (\lambda_1 - 2)x = (\varphi - 1)x \Rightarrow (x, y) = ((\varphi - 1)t, t) \\ \begin{bmatrix} 2x + y \\ x + y \end{bmatrix} = \lambda_2 \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow y = (\lambda_2 - 2)x = -\varphi x \Rightarrow (x, y) = (-\varphi t, t) \end{cases}$$

So by setting $t = \varphi$ in the first eigenvector formula and $t = 1$ in the second (by using the property that $\varphi(\varphi - 1) = 1$) we see that $P = \begin{pmatrix} 1 & -\varphi \\ \varphi & 1 \end{pmatrix}$ is the desired change of basis matrix. So now finally we compute the polynomial integrals of A, using Proposition 16:

$$\begin{aligned} Q(x, y) = xy \Rightarrow Q'(x, y) &= Q(P \begin{bmatrix} x \\ y \end{bmatrix}) = Q(x + \varphi y, -\varphi x + y) = (x + \varphi y)(-\varphi x + y) \\ &= -\varphi x^2 + (1 - \varphi^2)xy + \varphi y \overset{\varphi^2 - \varphi - 1 = 0}{=} -\varphi x^2 + \varphi xy + \varphi y^2 = -\varphi(x^2 - xy - y^2) \end{aligned}$$

Since polynomial integrals also form an $\mathbb{R}$-algebra, we can multiply by $\dfrac{-1}{\varphi}$ to get that $x^2 - xy - y^2$ is a generator of the $\mathbb{R}$-algebra of polynomial integrals of A. Therefore, we can write that $\mathcal{I}_p(A) = \mathbb{R}[x^2 - xy - y^2]$. ∎

Example 19. Let $\theta \in [0, 2\pi)$ and consider the rotation matrix by θ radians in the plane: $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Multiplication by this matrix is a well-known dynamical system; especially when we restrict it to the unit circle $\mathbb{S}^1$. We will use the same algorithm as above (albeit, with complex numbers instead of real numbers) to find all the integrals of this dynamical system. First we take care of two edge cases: if θ is equal to 0 or π, we have $\sin\theta = 0$ and therefore we shall have $R_\theta = \pm I_2$ where I_2 is the 2×2 identity matrix. If $\theta = 0$, $R_\theta = I_2$ and therefore the system is trivial and therefore every polynomial is an integral. If $\theta = \pi$ and $R_\theta = -I_2$ then by each application of our map, we oscillate between (x, y) and $(-x, -y)$. Therefore, only monomials with an even total degree will be integrals for this system; in other words, our polynomial integrals are all the even polynomials in x and y.

So now we can assume $\theta \notin \{0, \pi\}$ and therefore $\sin\theta \neq 0$. Now, we diagonalize the matrix:

$$\det(\lambda I_2 - R_\theta) = \begin{vmatrix} \lambda - \cos\theta & \sin\theta \\ -\sin\theta & \lambda - \cos\theta \end{vmatrix} = \lambda^2 - 2\cos\theta\lambda + 1 = 0$$
$$\Rightarrow \lambda_{1,2} = \frac{2\cos\theta \pm 2i\sin\theta}{2} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$$

Now to finish the linear algebra we shall also compute the change of basis matrix:

$$\begin{cases} R_\theta \begin{bmatrix} x \\ y \end{bmatrix} = e^{i\theta} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \cos\theta x - \sin\theta y = (\cos\theta + i\sin\theta)x \xRightarrow{\sin\theta\neq 0} y = -ix \Rightarrow (x, y) = (t, -it) \\ R_\theta \begin{bmatrix} x \\ y \end{bmatrix} = e^{i\theta} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \cos\theta x - \sin\theta y = (\cos\theta - i\sin\theta)x \xRightarrow{\sin\theta\neq 0} y = ix \Rightarrow (x, y) = (t, it) \end{cases}$$

By choosing $t = i$ in the first eigenvector and $t = 1$ in the second, we reach the matrix $P = \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$.

Now we are looking for non-negative integers a and b where $(e^{i\theta})^a (e^{-i\theta})^b = 1$ or equivalently, where $\theta(a - b) = 2\pi k$ for some integer k. Now we tackle two different cases: first assume that $\frac{\theta}{2\pi} \notin \mathbb{Q}$. In that case, we have $a - b = k\frac{2\pi}{\theta}$ and we notice that the left side is rational while the right side can only by rational if $k = 0$. So $a = b$ and therefore, the only monomial integral here is xy. We find all our integrals for R_θ now by a change of variable:

$$Q(x, y) = xy \Rightarrow Q'(x, y) = Q(P \begin{bmatrix} x \\ y \end{bmatrix}) = Q(ix + y, x + iy) = (ix + y)(x + iy) = i(x^2 + y^2)$$

Again because out integrals form a $\mathbb{C}$-algebra we see that $x^2 + y^2$ is the single generator of the polynomial integrals. Note that even when we restrict ourselves to real polynomials, this single quadratic still generates all our polynomial integrals.

So now we need to tackle the other case: when $\frac{\theta}{2\pi} \in \mathbb{Q} \Leftrightarrow \theta = \frac{2\pi r}{s}$ where we can assume r and s are integers with $\gcd(r, s) = 1$ and $s > 0$. So we can conclude that $a - b = \frac{ks}{r}$ and therefore if we take $t \in \mathbb{Z}$ such that $k = rt$, then $a - b = st$. Now if $t = 0$, $a = b$ and therefore our monomial integral is $(xy)^a$ for some natural number a, just like the irrational case. If $t > 0$, we can write $x^a y^b = (xy)^b x^{st}$; and if $t < 0$ we can write $x^a y^b = (xy)^a y^{s(-t)}$. In all three cases, we see that xy, x^s and y^s generate all monomials and therefore generate all integral polynomials of this dynamical system.

So all that is left is to change back to our original system by changing our basis using the matrix P. For xy, we know that this yields the polynomial $x^2 + y^2$. For the other two, we get

$(ix+y)^s$ and $(x+iy)^s$ respectively. We can see that in the first example by factoring i out, we reach $i^s(x-iy)^s$ and since i^s is a constant, it can be removed as mentioned before. Now, we can notice that if we name $(x+iy)^s = A_s + B_s i$ for some $A_s, B_s \in \mathbb{R}[x,y]$, then we shall have that $(x-iy)^s = A_s - B_s i$ and therefore since our polynomial integrals form a $\mathbb{C}$-algebra, therefore A_s and B_s are themselves polynomial integrals for the rotation matrix. So in the end we see that our polynomial integral algebra for R_θ lies in $\mathbb{R}[x^2+y^2, A_s(x,y), B_s(x,y)]$. ■

Remark 20. Note the fact that regardless of whether $\dfrac{\theta}{2\pi}$ is rational or not, x^2+y^2 is always a polynomial integral for the rotation matrix. This was evident even before we started our calculations, since rotation matrices leave the distance between a point and the origin intact, and x^2+y^2 is precisely the square of this distance.

Remark 21. The real polynomials A_s and B_s defined in Example 19 are related to families of polynomials which play an important role in functional analysis; they are known as the **Chebyshev polynomials of the first and second kind**. They are defined for every non-negative integer n as follows:

$$T_n(\cos\theta) = \cos(n\theta), \qquad U_n(\cos\theta)\sin\theta = \sin((n+1)\theta)$$

for every θ. By naming $x = R\cos\alpha$ and $y = R\sin\alpha$, we see that:

$$(x+iy)^s = R^s(\cos(s\alpha) + i\sin(s\alpha)) \Rightarrow \begin{cases} A_s(R\cos\alpha, R\sin\alpha) = R^s T_s(\cos\alpha) \\ B_s(R\cos\alpha, R\sin\alpha) = R^s \sin\alpha(U_{s-1}(\cos\alpha)) \end{cases}$$

By noting that $R = \sqrt{x^2+y^2}$ we can conclude that for $\theta = \dfrac{2r\pi}{s}$, we have the following characterization:

$$\mathcal{I}_p(R_{\frac{2r\pi}{s}}) = \mathbb{R}\Big[x^2+y^2, T_s(\frac{x}{\sqrt{x^2+y^2}}), \frac{y}{\sqrt{x^2+y^2}}U_{s-1}(\frac{x}{\sqrt{x^2+y^2}})\Big]$$

6 Further Remarks and Questions

While reading about integrability and invariant theory for this article and while thinking on these results, I found myself asking many questions about how one may continue this line of reasoning to achieve more results. The following remarks and questions have been posed so that one may continue thinking about this problem in different ways:

Question 22. What happens if the matrix A is not diagonalizable? It may be possible to continue this train of thought using the Jordan normal form of A; however the calculations seemed too tedious to carry out. Maybe using other decompositions could be the key to cracking the general case of the linear case.

Question 23. Can we extend these ideas to other well-behaved functions? For example if $f : \mathbb{R}^n \to \mathbb{R}^n$ has a hyperbolic fixed point at $x_0 \in \mathbb{R}^n$, then by the Hartman-Grobman Theorem we know that a neighbourhood $U \subseteq \mathbb{R}^n$ containing x_0 and a homeomorphism $h : U \to \mathbb{R}^n$ exist such that $h(x_0) = 0$ and such that the dynamical system $h \circ f \circ h^{-1}$ is equal to the linear map $x \mapsto Ax$ where A is the Jacobian matrix of f at the point x_0. In the case where A is diagonalizable, what can we deduce about the integrals of f in this neighbourhood? If we find an integral in $\mathcal{I}_0(A)$ or $\mathcal{I}_p(A)$ and change it into an integral for f on U, can we possibly extend it to a global one? What else can hyperbolic dynamics do in tandem with our results to understand $\mathcal{I}_0(f)$?

Question 24. Remark 20 raises the question whether we can necessarily use any geometric intuition to facilitate finding integrals. Also, Example 19 showed us that a link might be found between our integrals and some groups which may act upon our space; this is similar to ideas introduced in invariant theory and also is an important talking point of the celebrated theorem by Emmy Noether which relates integrals in classical mechanics (conserved quantities) to symmetries of that action. Can we somehow relate the geometry of the space our system acts on, or the symmetries of it, to the way our dynamical system behaves?

Question 25. Are the properties of the Chebyshev polynomial families (as discussed in Remark 21) reflected, in any capacity, in the dynamical systems in question? If so, how can using other orthogonal families of polynomials (such as Legendre polynomials) give us insight into other dynamical systems and their integrals?

Question 26. What can be done for non-linear dynamical systems? For example for Arnold's cat map, the function $f(x, y) = \sin\left(\frac{\pi}{\ln(1+\varphi)} \ln(\frac{\varphi x + y}{x - \varphi y})\right)$ is surprisingly an integral! It can be checked, however, that f cannot be expressed as a function of xy; therefore, our characterization for $\mathcal{I}_p$ does not hold up for $\mathcal{I}_0$. How can we deduce the nature of the integral algebra from the orbits of the system and the geometry of the space?

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