Lecture 12: Introduction to Vector and Fiber Bundles

definitions, constructions, parallelizability and the Hopf Bundle

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1. What is a Vector Bundle?

Motivation: Let M be a smooth n-manifold, with atlas $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in J}$. For a point $p \in M$, take $(U_\alpha, \varphi_\alpha) \in \mathcal{U}$ to be a chart around p. We see that TU_α and $U_\alpha \times \mathbb{R}^n$ are diffeomorphic! This might not be true globally but we are sure that this holds locally; we are interested what these local homeomorphisms tell us about the global geometry of M. This motivates our definition of a vector bundle:

Definition 1. A **vector bundle of rank** k is defined as $\xi = (B, E, p)$ where B and E are topological spaces and $p : E \to B$ is an onto continuous function; such that the following conditions hold for every $b \in B$:

i) $p^{-1}(b)$ is a k-dimensional vector space.
ii) An open neighbourhood U of b and a homeomorphism $\varphi : p^{-1}(U) \to U \times \mathbb{R}^k$ exist such that $\varphi|_{p^{-1}(b)} : p^{-1}(b) \to \{b\} \times \mathbb{R}^k$ is a linear isomorphism and such that $\pi_1 \circ \varphi = p$, where π_1 is projection on the first coordinate.

We call B the **base space**, E the **total space**, $p^{-1}(b)$ the **fiber** over b and the homeomorphisms φ **local trivializations**. If B and E are manifolds, p is smooth and the local trivializations are diffeomorphisms, we call ξ a **smooth vector bundle**. If $k = 1$, we call ξ a **line bundle**.

In a sense, the vector spaces situated above each point of B are "continuously changing" as b changes continuously on B. Note that set-theoretically, $E = B \times \mathbb{R}^k$ but this might not hold on a topological or differential scale.

Example 1. On the smooth n-manifold M, let p send each tangent vector $(x, v) \in M \times \mathbb{R}^n$ to the point x. In this case, (M, TM, p) is a smooth vector bundle of rank n.

Example 2. Take B to be any topological space or manifold, let $E = B \times \mathbb{R}^k$ and define $p(x, v) = x$ (i.e. projection on the first coordinate). In this case, (B, E, p) is a vector bundle which we call the **trivial vector bundle** of rank k on B.

Example 3. We can use usual definitions from linear algebra to define new vector bundles from old ones. To name a few, let $\xi_i = (B, E_i, p_i)$ be vector bundles of rank k and l. We can define:

i) the **dual bundle** (B, E^*, p^*) where ${p^*}^{-1}(b)$ is the dual vector space to $p^{-1}(b)$.
ii) the **Whitney-sum bundle** $(B, E_1 \oplus E_2, p_1 \oplus p_2)$ of rank $k + l$, where $(p_1 \oplus p_2)^{-1}(b) = p_1^{-1}(b) \oplus p_2^{-1}(b)$.
iii) the **tensor bundle** $(B, E_1 \otimes E_2, p_1 \otimes p_2)$ of rank kl, where $(p_1 \otimes p_2)^{-1}(b) = p_1^{-1}(b) \otimes p_2^{-1}(b)$.
iv) the **exterior algebra bundle** $(B, \bigwedge^m E, \bigwedge^m p)$ of rank $\binom{k}{m}$ where each fiber is $\bigwedge^m p^{-1}(b)$.

For example, the dual bundle of TM for a smooth bundle M is the **cotangent bundle** T^*M.

2. Morphisms and Sections of Vector Bundles

We shall now define morphisms between vector bundles and afterwards, sections on vector bundles; these algebraic notions are crucial in working with vector bundles.

Definition 2. For $i \in \{1, 2\}$, let $\xi_i = (B, E_i, p_i)$ be vector bundles. We call a continuous map $f : E_1 \to E_2$ a **bundle morphism** whenever the restriction of f to the fiber $p_1^{-1}(b)$ is a linear isomorphism onto the fiber $p_2^{-1}(b)$, and the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

If f is a homeomorphism, the two bundles are called **isomorphic** and f is called a **bundle isomorphism** (We replace "continuous" and "homeomorphism" with "smooth" and "diffeomorphism" if we are talking about smooth bundles).

Lemma 1. *Let $\xi_1 = (B, E_1, p_1)$ and $\xi_2 = (B, E_2, p_2)$ be two vector bundles and let $f : E_1 \to E_2$ be a continuous map such that $p_2 \circ f = p_1$. In this case, f will be an isomorphism of bundles if and only if the restriction of f to each fiber is a linear isomorphism.*

Proof. If f is an isomorphism then the lemma is trivial; so we only prove the converse. We therefore assume that f is a bundle morphism such that the restriction of f to fibers is a linear isomorphism. We will proceed locally; for two local trivializations (U, φ) for ξ_1 and (V, ψ) for ξ_2 we will show that we can define a continuous inverse for $g := \psi^{-1} \circ f \circ \varphi : U \times \mathbb{R}^k \to V \times \mathbb{R}^k$. How can we do this? We can see that $g(b, v) = (b, A(b)(v))$ where $A : B \to M_k(\mathbb{R})$ is a continuous matrix-valued map. The condition of f (and locally, g) restricing to linear isomorphisms on fibers says that $A(b)$ is an invertible matrix for each $b \in B$; by Cramer's rule from linear algebra, we can therefore see that the map $A^{-1} : B \to GL_n(\mathbb{R})$ defined as $A^{-1}(b) := (A(b))^{-1}$ is a continuous inverse for g. Now we have locally found inverses for our local trivializations; glueing these together we reach a global continuous inverse for f, which proves f is a homeomorphism and therefore a bundle isomorphism. $\square$

Definition 3. Let $\xi = (B, E, p)$ be a vector bundle. A **section** of this bundle is a continuous map $\sigma : B \to E$ such that $p \circ \sigma = id_B$. If ξ is a smooth bundle, we say σ is a **smooth section** if σ is also a smooth map.

Example 4. For a smooth n-manifold M, the (smooth) sections of the tangent bundle (M, TM, p) are exactly the (smooth) vector fields on M. Also, differential m-forms are (smooth) sections of the bundle $\bigwedge^m(T^*M)$!

Example 5. We define a specific section on (B, E, p) as follows: for every $b \in B$ let $\sigma(b) := 0_{p^{-1}(b)}$; we call this section the **zero section**. Note that this section is a homeomorphism of B onto $\sigma(B)$.

3. Parallelizability

Definition 4. The vector bundle $\xi = (B, E, p)$ of rank k is called **trivial** whenever it is isomorphic to the trivial bundle $(B, B \times \mathbb{R}^k, \pi_1)$ on B. We call a smooth n-manifold M **parallelizable** whenever the bundle (M, TM, p) is trivial.

Example 6. We define a non-trivial line bundle on $\mathbb{S}^1$ called the **Möbius bundle**: Take $\mathbb{S}^1 \times \mathbb{R}$ as the total space, and identify the points $(0, t)$ and $(1, -t)$. The quotient space resulting from this identification is a vector bundle of rank 1 on $\mathbb{S}^1$. Why is this not trivial? Denote the Möbius bundle by E, and take the zero section σ and define:

$$\Gamma = \left\{(b, \sigma(b))\middle| b \in \mathbb{S}^1\right\}$$

It can be seen that $E - \Gamma$ is connected, while $(\mathbb{S}^1 \times \mathbb{R}) - \Gamma$ is not; therefore these two total spaces cannot be homeomorphic and therefore E cannot be trivial.

Now we provide a lemma which gives us an equivalent condition to triviality using global sections:

Proposition 1. *A vector bundle $\xi = (B, E, p)$ of rank k is trivial if and only if one can find k sections $\sigma_1, ..., \sigma_k$ such that at each base point $b \in B$, the vectors $\sigma_1(b), ..., \sigma_k(b)$ are linearly independant.*

Example 7. We shall use the proposition to show that $\mathbb{S}^1$ is parallelizable. The tangent space $T_x\mathbb{S}^n$ is exactly the orthogonal complement of span(x) in $\mathbb{R}^{n+1}$. Using this idea, we can provide a nowhere-zero section on $\mathbb{S}^1$: take $\sigma(z) = (z, iz)$ for each $z \in \mathbb{S}^1$, viewed as a unit complex number. We therefore readily see that $T\mathbb{S}^1$ is trivial and therefore $\mathbb{S}^1$ is parallelizable. The same can be done for $\mathbb{S}^3$ using quaternions and for $\mathbb{S}^7$ using octonions! It has been shown that these are the only parallelizable spheres; the proof is quite cumbersome and requires heavy machinery from algebraic topology!

4. Fiber Bundles and the Hopf Bundle

In order to define a fiber bundle of fiber space F, we simply replace $\mathbb{R}^k$ with F:

Definition 5. A **fiber bundle** is defined as $\xi = (B, E, p, F)$ where B, E and F are topological spaces and $p : E \to B$ is an onto continuous function; such that for every $b \in B$, an open neighbourhood U of b and a homeomorphism $\varphi : p^{-1}(U) \to U \times F$ exist such that $\varphi|_{p^{-1}(b)} : p^{-1}(b) \to \{b\} \times F$ is a homeomorphism and such that $\pi_1 \circ \varphi = p$, where π_1 is projection on the first coordinate. The terminology is the same as vector bundles; we call F the **fiber space**. If all the spaces are smooth manifolds, then the same comment about smooth bundles also applies here. The concept of sections, of bundle morphisms and of trivial bundles also readily generalize to fiber bundles.

One might inquire why this generalization of vector bundles into fiber bundles is important at all. In order to partially answer this, we construct a non-trivial fiber bundle (which is also not a vector bundle) that is an important fiber bundle in many different parts of geometry, topology and physics!

Example 8. We construct a fiber bundle on the sphere $\mathbb{S}^2$ called the **Hopf bundle** which is $H = (\mathbb{S}^2, \mathbb{S}^3, h, \mathbb{S}^1)$; we let $\mathbb{S}^3$ be pairs of complex numbers (z_0, z_1) such that $|z_0|^2 + |z_1|^2 = 1$, and we similarly let $\mathbb{S}^2$ be pairs of complex and real numbers (w, x) such that $|w|^2 + x^2 = 1$. Using these, we define h as:

$$h(z_0, z_1) = (2z_0\bar{z}_1, |z_0|^2 - |z_1|^2)$$

One can easily check that h is a continuous map from $\mathbb{S}^3$ to $\mathbb{S}^2$. Now, let $Z = (z_0, z_1)$ and $W = (w_0, w_1)$ be two points on $\mathbb{S}^3$ that map to the same point under h. It is easy to check that in this case, a unit complex number λ must exist such that $Z = \lambda W$; therefore, the fiber situated on every point in $\mathbb{S}^2$ is exactly a unit circle ($\mathbb{S}^1$) and therefore we have constructed a fiber bundle on $\mathbb{S}^2$ with a total space of $\mathbb{S}^3$ and a fiber space of $\mathbb{S}^1$. Topologically, one can see that $\mathbb{S}^3$ and $\mathbb{S}^2 \times \mathbb{S}^1$ are not homeomorphic; therefore this bundle is not trivial.

Exercises

1. Show that any Lie group G is parallelizable. Using this, give a proof for why $\mathbb{S}^3$ is parallelizable.
2. Let $\xi = (B, E, p)$ be a vector bundle of rank k and let $f : B' \to B$ be a continuous map. We define:

$$E' = \left\{(b', e) \in B' \times E \,\middle|\, f(b') = p(e)\right\}, \qquad p'(b', e) = b'$$

   Show that $f^*\xi = (B', E', p')$ is a vector bundle of rank k; we call this the **pullback bundle** of ξ relative to f.
3. Complete the proof described in Example 8; explicitly, show that for any point $(w, x) \in \mathbb{S}^2$, the preimage $h^{-1}(w, x)$ is homeomorphic to $\mathbb{S}^1$.