Lesson: Frameworks and goals of statistical modeling

Module: Introduction to statistical models

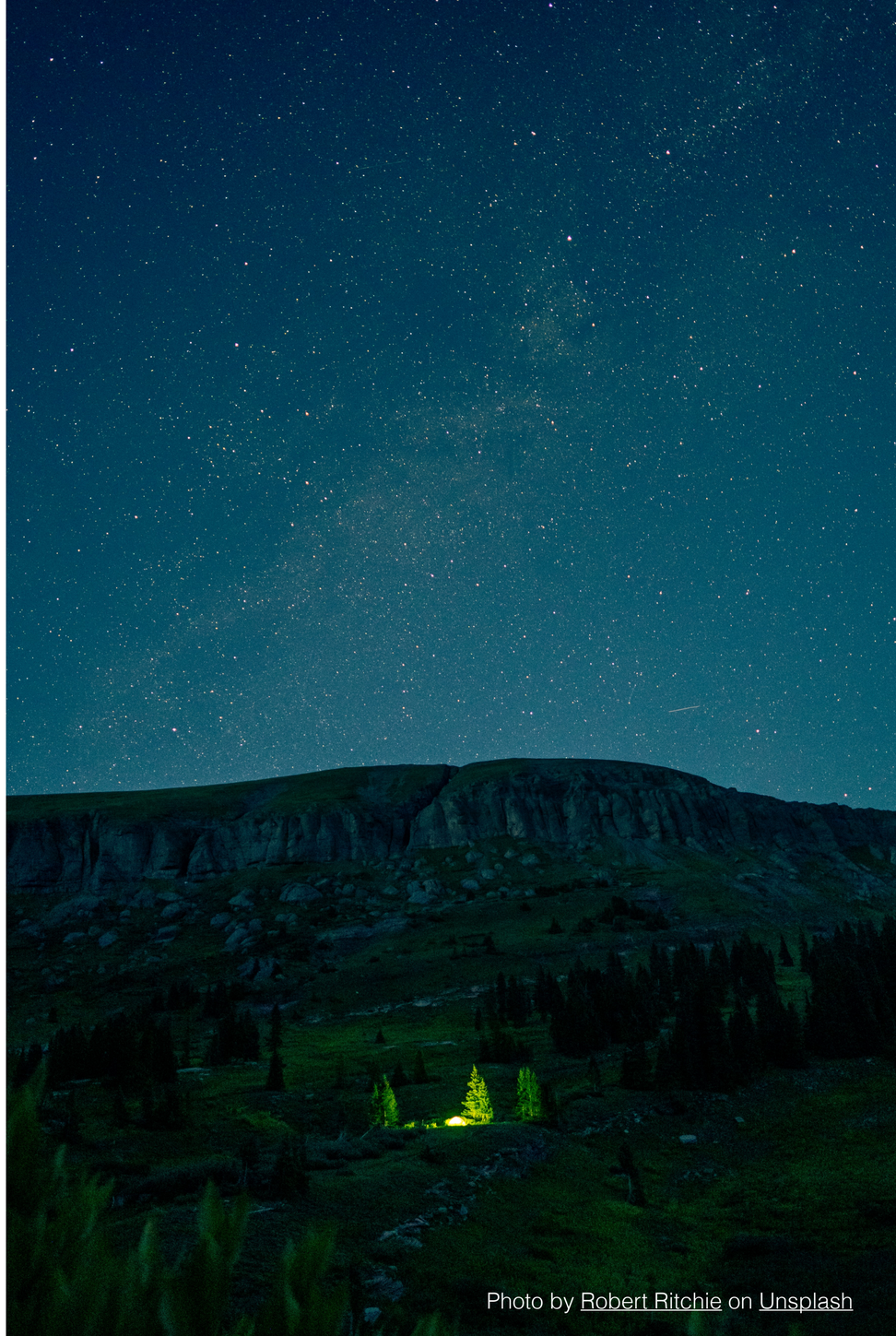
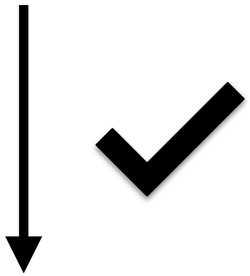


Photo by Robert Ritchie on Unsplash

switches + dials



Audio signal

Photo by Steve Harvey on Unsplash

$$Y = f(\mathbf{x}) + \varepsilon$$

output

Systematic component

random component

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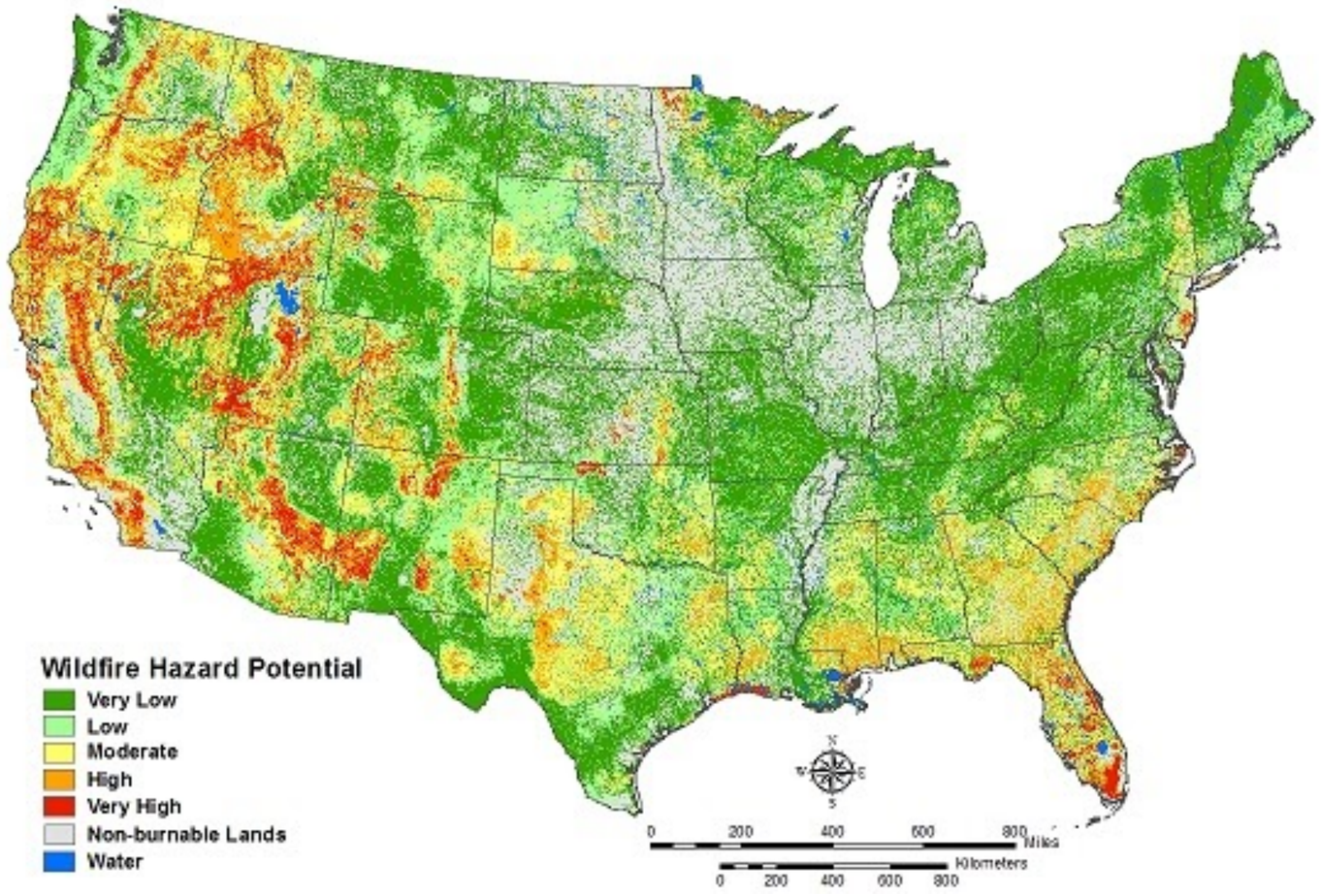

temperature, humidity + fuel density + elevation, lightening storms + number of nearby campsites + …

↓ ?

Likelihood of large wildfire

Image from firelab.org: https://bit.ly/37q9EGA

Definition: A *statistical unit* is one member of the set of entities being studied.

Definition: A *population* is a collection of units about which research questions are asked.

Definition: A *sample* is a subset of the population. Typically, samples should be *representative.*

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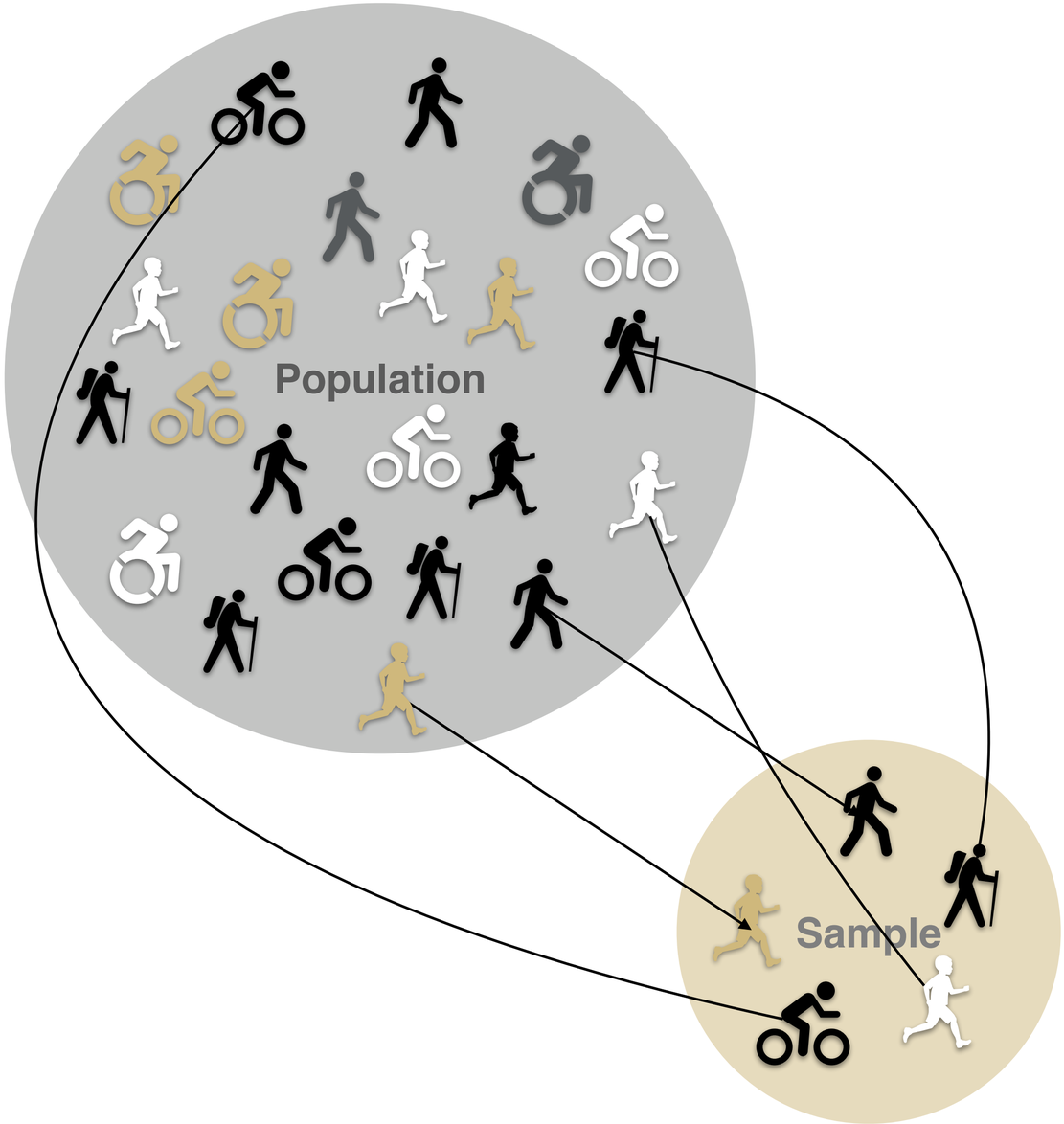


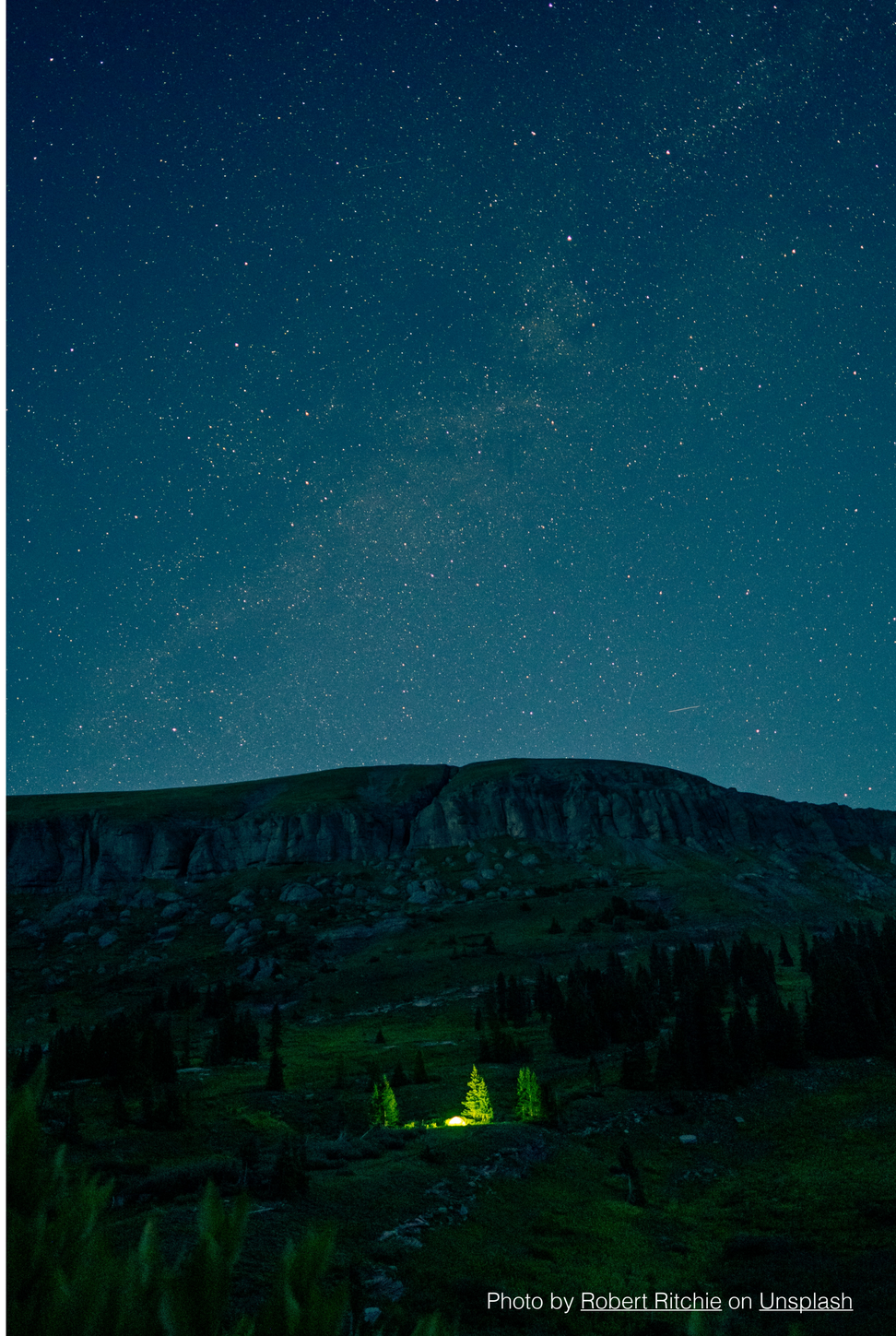
Photo by ismail mohamed - SoviLe on Unsplash

Definition: *Inferential statistics and data science* is the process of learning about relationships in a sample in a way that is reliable enough to *generalize* from the sample to a population of interest.

Photo by Danny Nee on Unsplash

Lesson: The assumption of *concept validity*

Module: Introduction to statistical models



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Is cycling any less safe in the winter than in the summer?

Photo by Max Adulyanukosol on Unsplash

Definition: To *operationalize* a concept means to derive a set of steps to measure the concept.

Definition: The *validity* of a dataset or measurement tool is the extent to which the dataset or measurement tool measures what it claims to measure.

Photo by Max Adulyanukosol on Unsplash

Study on cycling safety:

http://bit.ly/3oZNd0x

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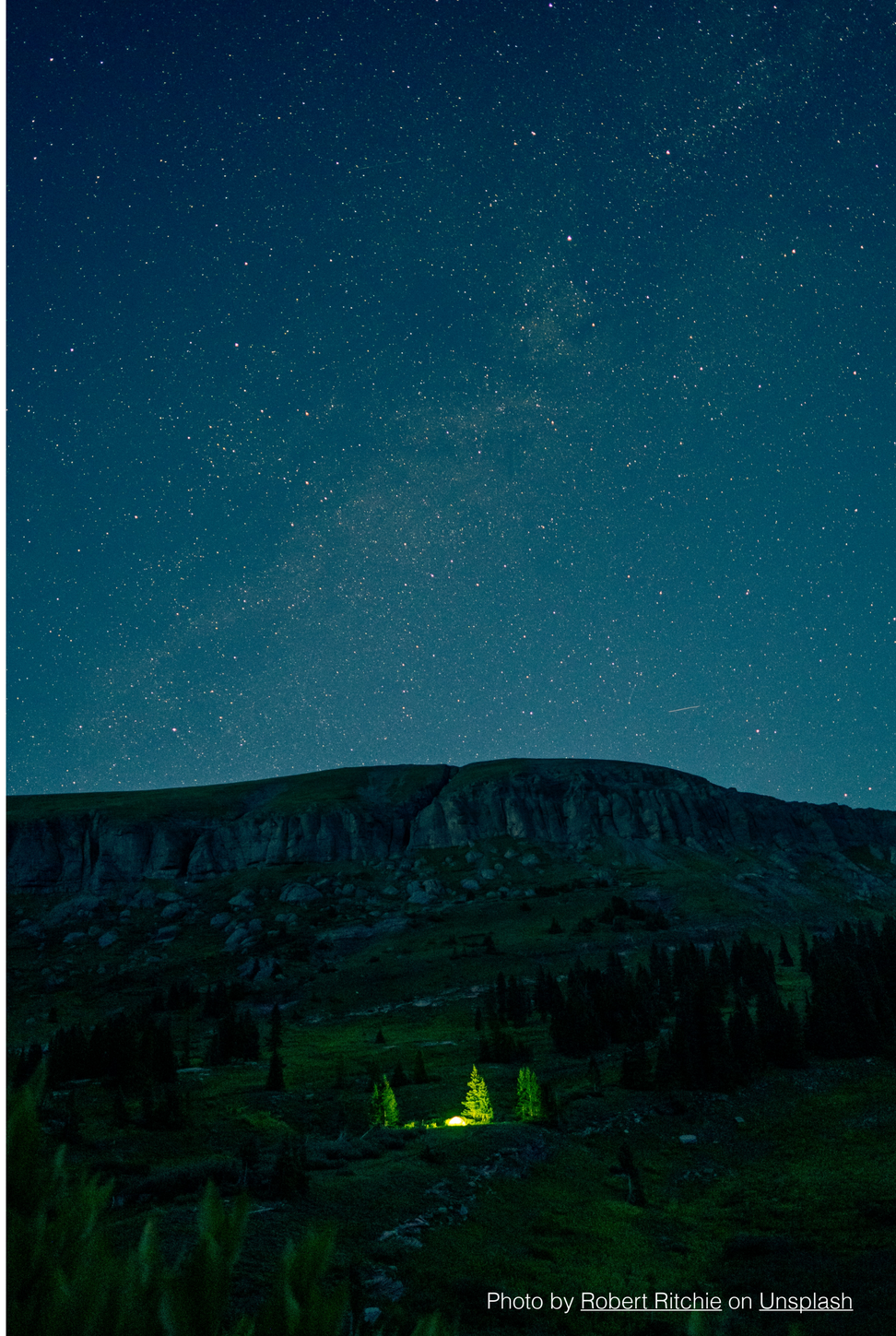
Lesson: The linear regression model

Module: Introduction to statistical models



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Motivation and Context

Basic statistical inference: estimate or test hypotheses about data generated from a distribution with a constant mean. For example:

$$Y_1, Y_2, \ldots, Y_n \overset{iid}{\sim} N(\mu, \sigma^2)$$

μ: est. the mean

But what if the mean depended on another variable? For example:

$$Y_1, \ldots, Y_n \overset{iid}{\sim} N\left(\mu(X_1, X_2, \ldots, X_p), \sigma^2\right)$$

$f(x) = y$

The Context of Linear Regression

Linear regression is used to explain or model the relationship between a single variable Y, and one or more variables $x_1, \ldots, x_p$.

Definition: Y is called the *response, outcome, output,* or *dependent* variable.

Definition: $x_1, \ldots, x_p$ are called *predictors*, *inputs*, *independent variables*, *explanatory variables,* or *features*. In some contexts, they are also called *covariates*.

We will assume, for now, that all variables are continuous (but will soon extend our methods to allow for discrete variables!).

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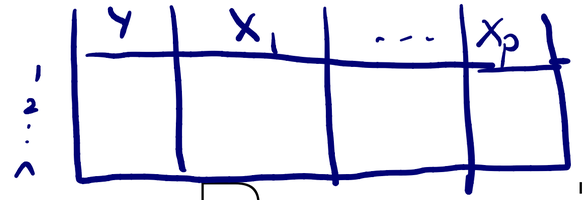


The Context of Linear Regression

The simplest relationship between these variables is a linear relationship: For $i = 1, \ldots, n$

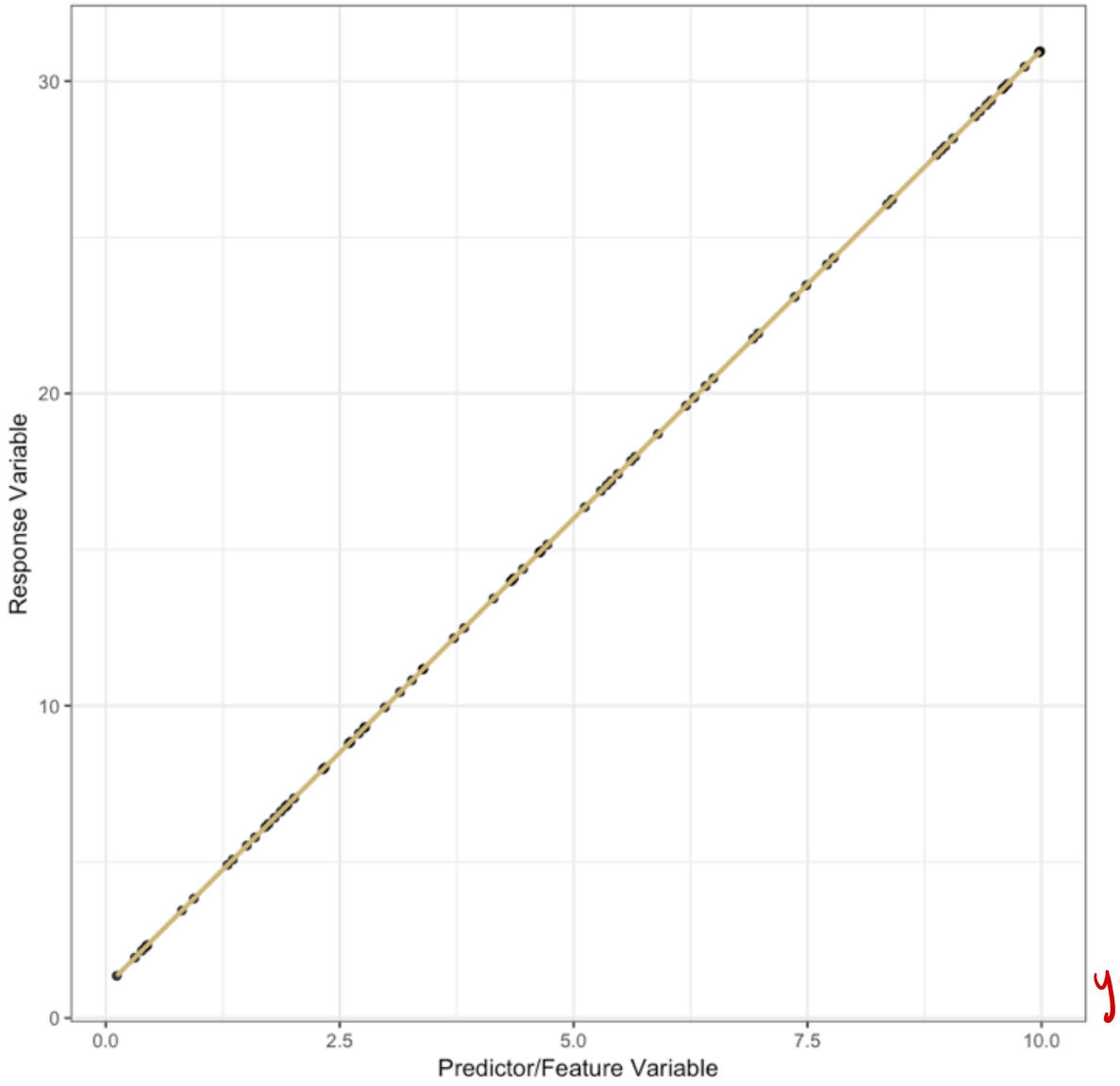
$$Y_i = \beta_0 + \beta_1 X_{i1} + \ldots + \beta_p X_{ip}$$

But note that, when we actually measure data, measurements aren't perfect—there is error. So, we might use the following to model our data:

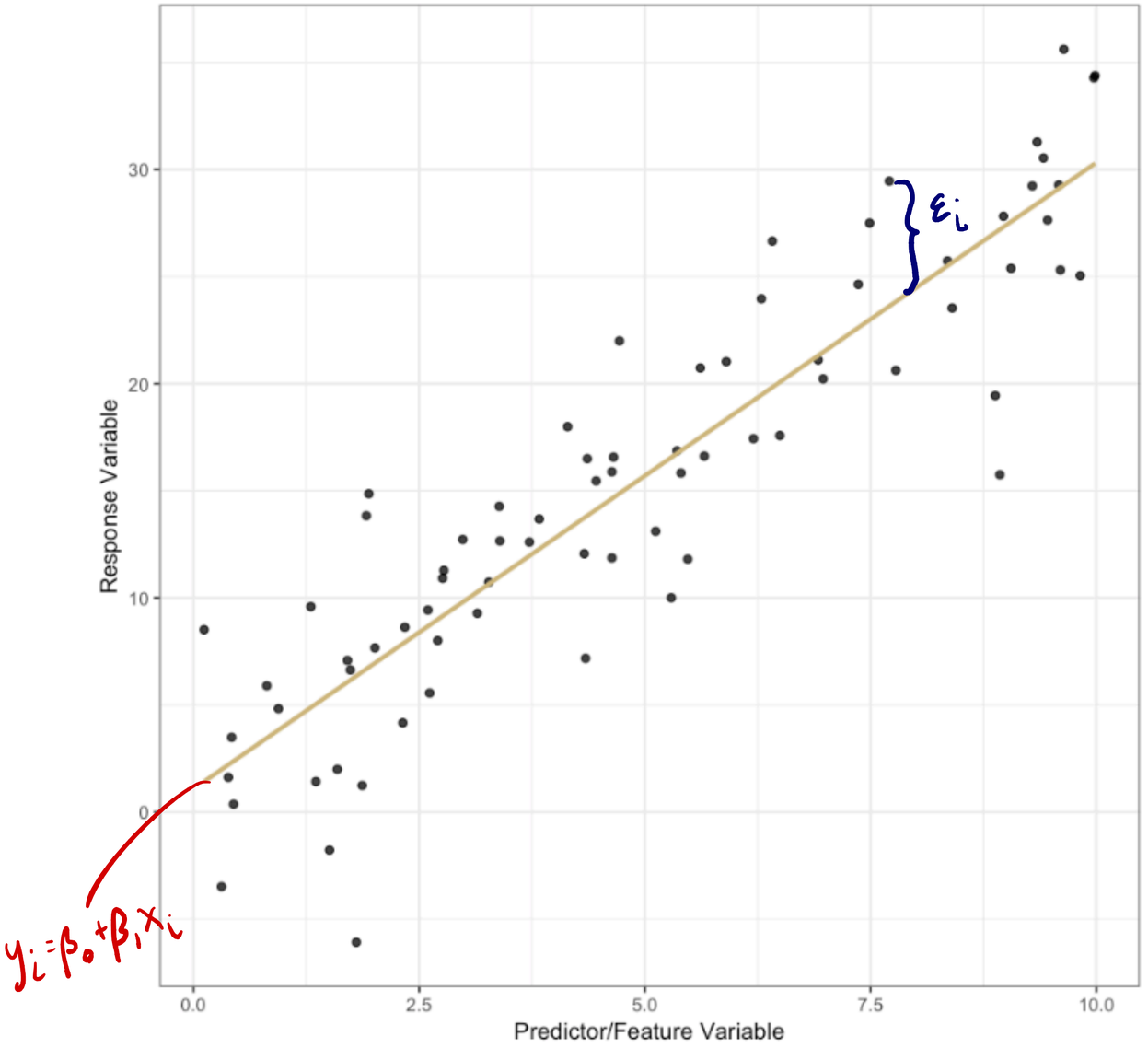
$$Y_i = \beta_0 + \beta_1 X_{i1} + \ldots + \beta_p X_{ip} + \varepsilon_i$$

ε_i: random variable

Exact Linear Relationship



Noisy Linear Relationship



The Context of Linear Regression

Regression analysis has two main objectives:

1. *Prediction:* Predict an unmeasured/unseen Y using observed $x_1, \ldots, x_p$.

2. *Explanation*: To assess the effect of, or explain the relationship between, Y and $x_1, \ldots, x_p$.

Can we infer causality?

Lesson: Matrix representation of the linear regression model

Module: Introduction to statistical models

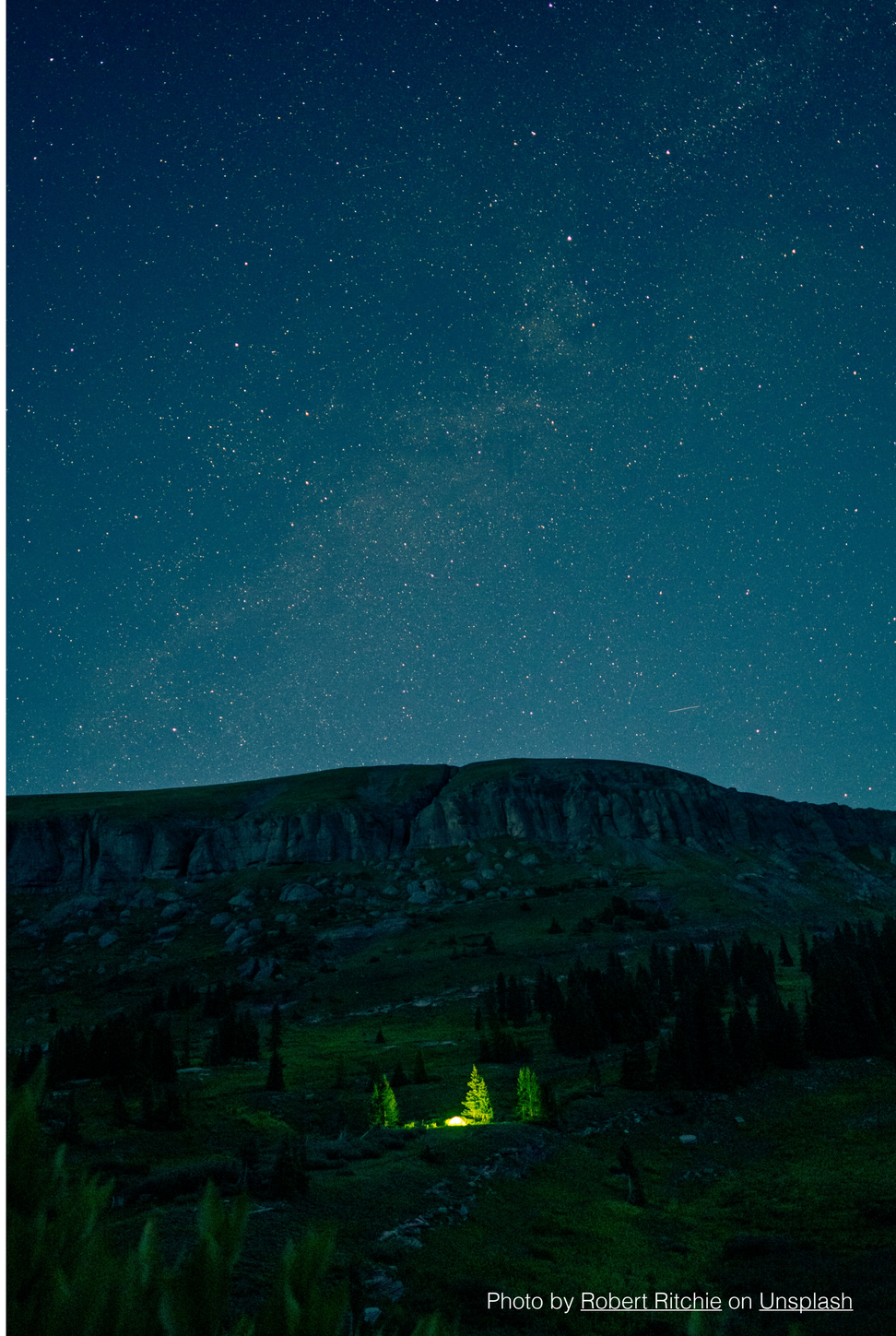


Photo by Robert Ritchie on Unsplash

$\begin{pmatrix} 1 & x_{11} & \cdots & x_{1p} \\ 1 & x_{21} & & x_{2p} \\ \vdots & \vdots & & \vdots \\ 1 & x_{n1} & & x_{np} \end{pmatrix}$

p = # of predictors

X r.v.
x realization

The Linear Regression Model

Let $\mathbf{Y} = (Y_1, \ldots, Y_n)^T$ be the response variable and $\mathbf{x_1} = \begin{pmatrix} x_{1,1} \\ x_{2,1} \\ \vdots \\ x_{n,1} \end{pmatrix}$, $\mathbf{x_2} = \begin{pmatrix} x_{1,2} \\ x_{2,2} \\ \vdots \\ x_{n,2} \end{pmatrix}$, $\ldots$, $\mathbf{x_p} = \begin{pmatrix} x_{1,p} \\ x_{2,p} \\ \vdots \\ x_{n,p} \end{pmatrix}$

$\underline{x}_1$

be predictors; we will collect the predictors in a matrix: $X = \begin{pmatrix} \mathbf{1} & \mathbf{x_1} & \mathbf{x_2} & \ldots & \mathbf{x_p} \end{pmatrix}$, where $\mathbf{1} = (1,1,...,1)^T$. Let $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_p)^T$ be a vector of parameters. Finally, let $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)^T$ be a vector of error terms. Then we can write our model as:

$n \times (p+1)$

$\underline{Y} = X\underline{\beta} + \underline{\varepsilon}$

$n \times 1$

$n \times 1$ $\quad$ $n \times (p+1)$ $\quad$ $(p+1) \times 1$

Matrix/vector representation

For $i = 1, \ldots, n$

$Y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip} + \varepsilon_i$

Single equation representation

The Linear Regression Model

Examples of linear models (i.e., models that linear methods can handle):

like least squares

1. $Y_i = \beta_0 + \beta_1 \underbrace{\log(x_i)}_{z_i} + \varepsilon_i$

2. $Y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \cdots + \beta_p x_i^p + \varepsilon_i$

Examples of nonlinear models:

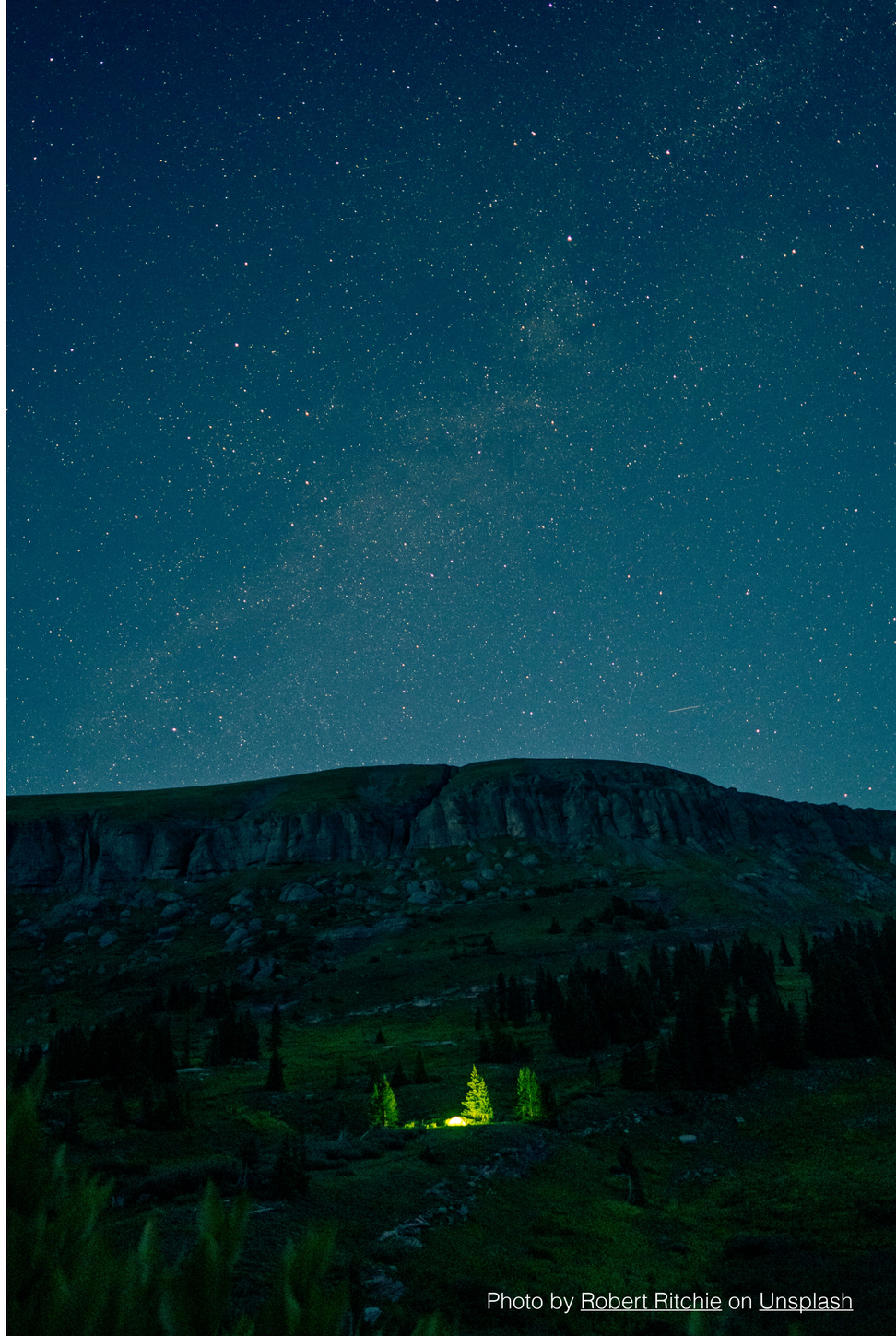
1. $Y_i = \beta_1 [\cos(x_i)]^{\beta_2} + \varepsilon_i$

2. $Y_i = \beta_0 + \beta_1 \sin(\beta_2 x_i) + \varepsilon_i$

Photo by Jason Leung on Unsplash

Lesson: Assumptions of the linear regression model

Module: Introduction to statistical models



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$\underline{Y} \sim N(X\beta, \sigma^2 I_n)$, $I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{n \times n}$

The Linear Regression Model

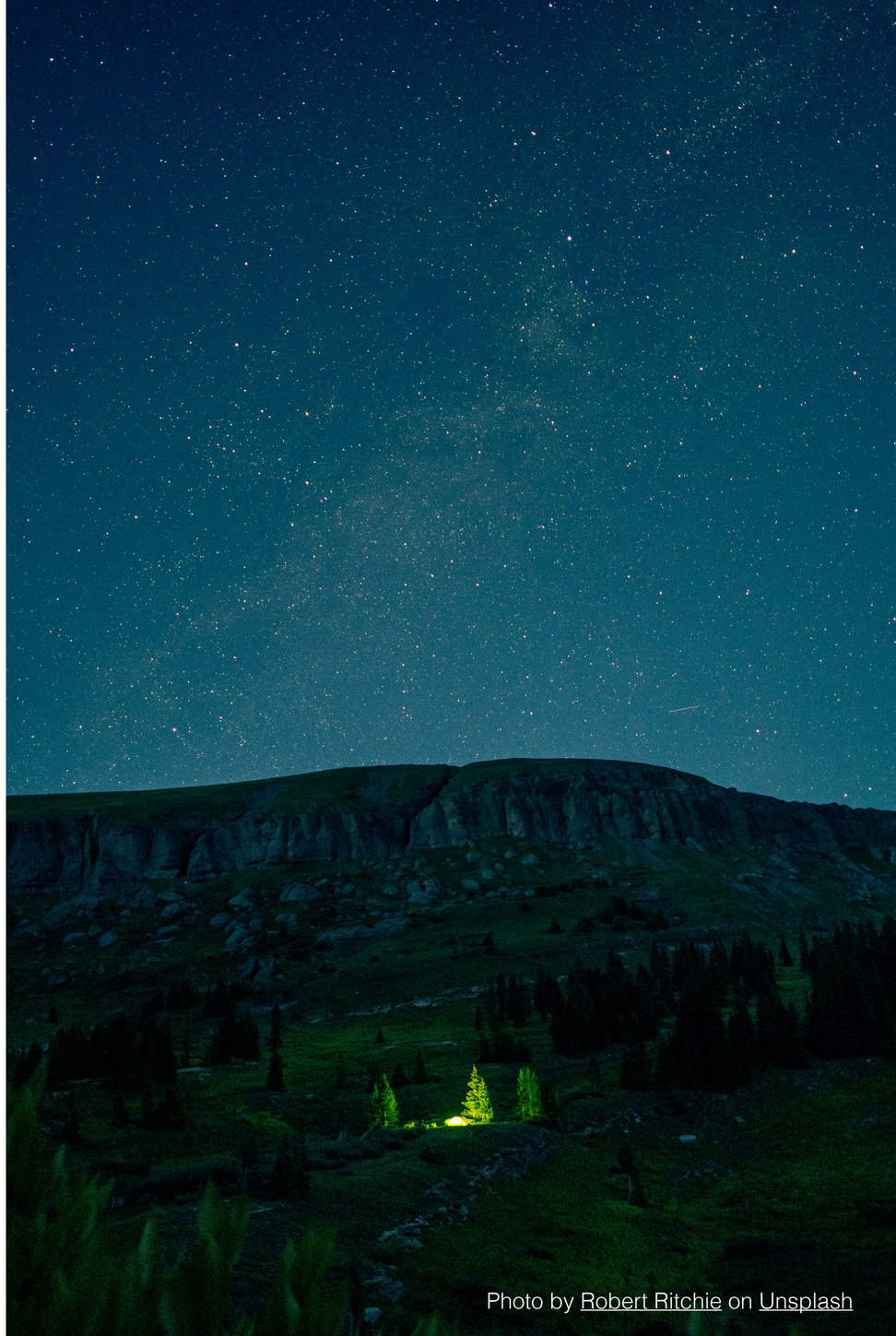
(Ind $\Rightarrow$ uncorrelated. uncorrelated $\not\Rightarrow$ Ind.)

<u>Definition/Assumptions</u> of the linear regression model:

1. *Linearity*: Linear relationship between $\underline{Y}$ and β
2. *Independence*: Y_i is independent from Y_j
   ($Cov(Y_i, Y_j) = 0 \quad i \neq j$)
3. *Homoskedasticity (constant variance)*:
   $Var(Y_i) = \sigma^2$ for all $i = 1, \ldots, n$
4. *Normality*:
   $Y_i \sim N(\beta_0 + \beta_1 X_{i1} + \cdots + \beta_p X_{ip}, \sigma^2)$

Lesson: Interpreting the linear regression model

Module: Introduction to statistical models



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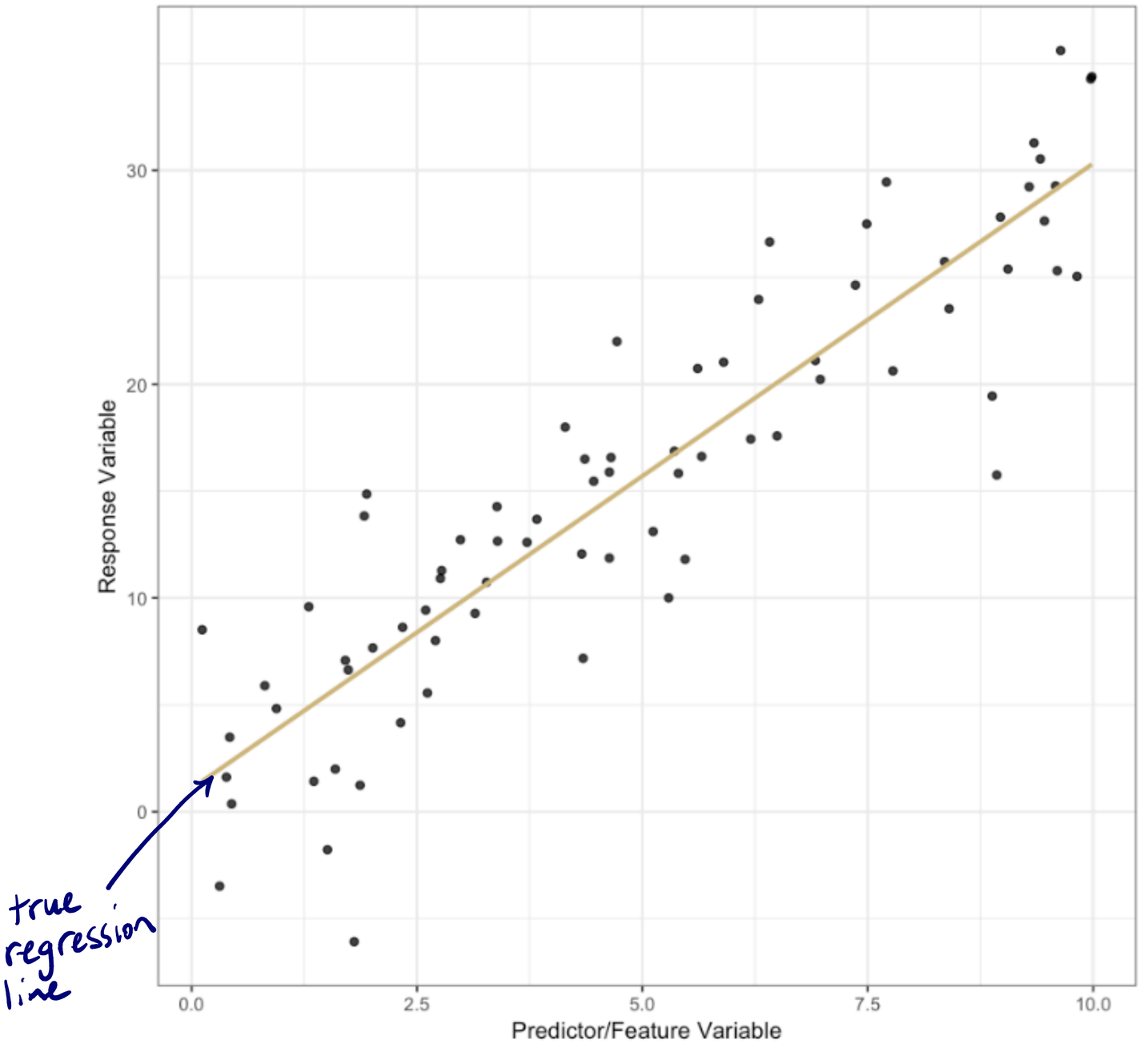

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Interpreting the Regression Parameters

Interpreting *simple* linear regression parameters:

1. β_0: the intercept of the true regression line
2. β_1: the slope of the true regression line.

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Interpreting the Regression Parameters

Interpreting *simple* linear regression parameters:

$Y = \beta_0 + \beta_1 X + \varepsilon$, $E(\varepsilon) = 0 \Rightarrow E(Y) = \beta_0 + \beta_1 X$

1. β_0 *is the average value of* Y *when x is zero.* Usually this is called the "baseline average".

2. β_1: *is the average change in* Y *associated with a 1-unit increase in the value of* x.

Interpreting the Regression Parameters

Interpreting *multiple* linear regression parameters:

1. β_0: the intercept of the true regression line. We interpret this as *the average value of* Y *when all of the* x*'s are zero.*

2. β_j: the slope of the true regression line, $j = 1,...,p$.

Interpreting the Regression Parameters

Interpreting *multiple* linear regression parameters:

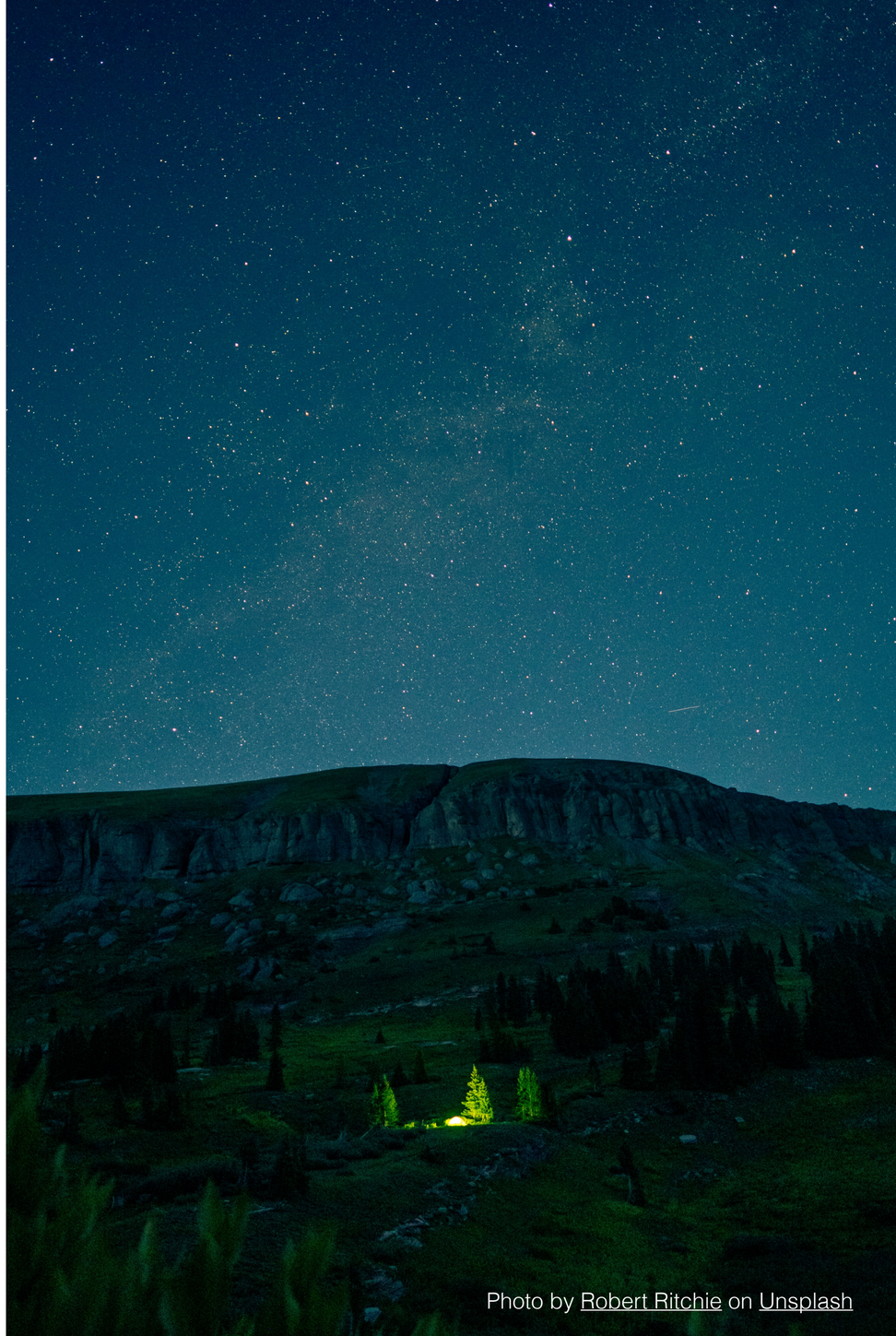
1. β_0 *is the average value of* Y *when all of the* x*'s are zero.*

2. β_j *is the average change in* Y *associated with a 1-unit increase in the value of* x_j *assuming all other predictors are held constant,* $j = 1,...,p$.

Thus, these "slope" parameters are called *partial* or *adjusted* regression parameters/coefficients.

Lesson: The appropriateness of linear regression

Module: Introduction to statistical models



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How do we know linear regression is appropriate?

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Does a theory or “law” tell us how the variables relate?

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Photo by James Allen on Unsplash

Does a theory or “law” tell us how the variables relate?

$$F = kx$$

Force

Spring Constant

Displacement

What past studies related to this research question have been conducted?

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Can we explore the data to learn whether there are linear relationships between variables?

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Definition: A *circular analysis* or *double dipping* is the process of exploring a dataset in an attempt to discover what relationships exist, and then test hypotheses related to that exploration on the same dataset.

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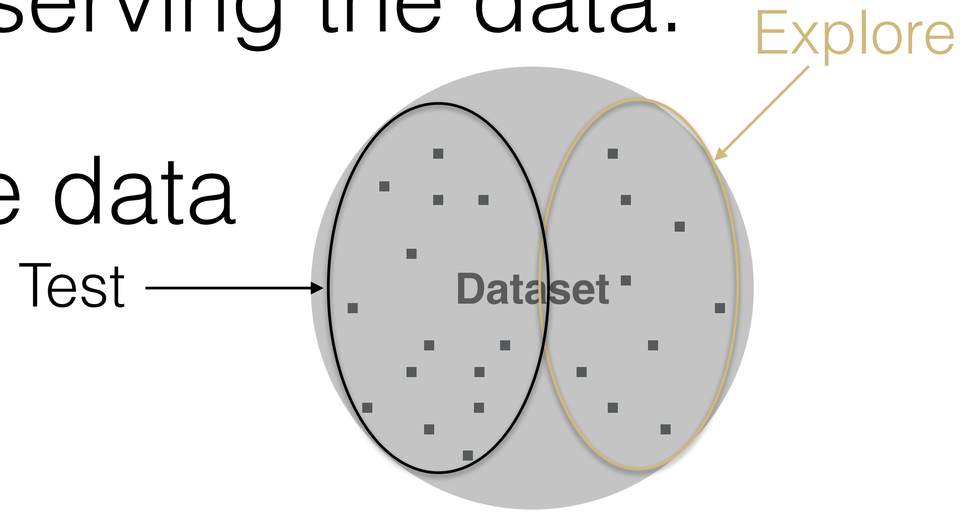
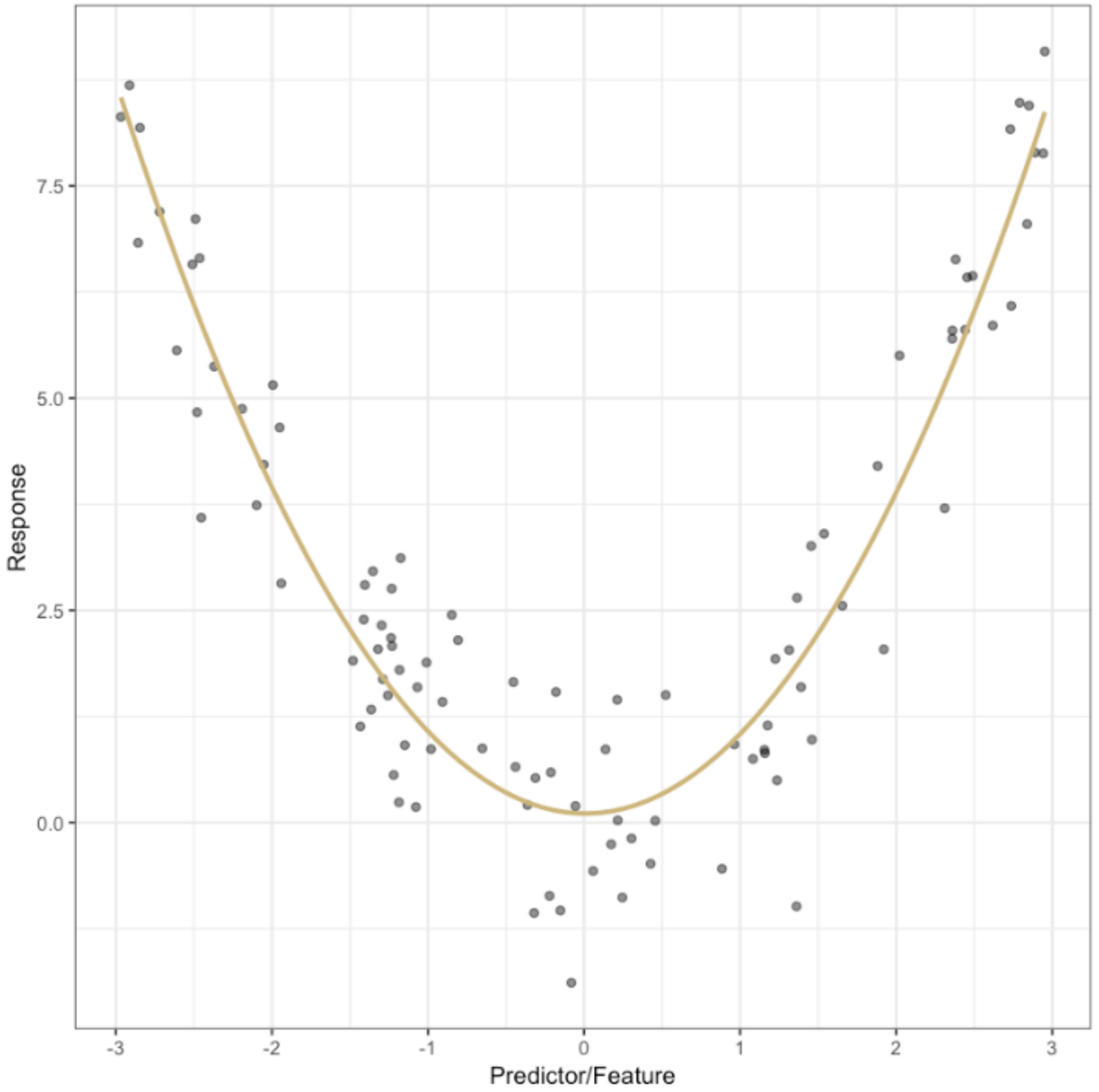
Ways to avoid circular analyses:

1. Design the analysis and pre-specify research hypotheses before observing the data.
2. Subset the data





Nonlinear relationships between the predictors and the response?