Homework1-1a-2c

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$1a)\qquad \tau_m u'(t) = -[u(t) - u_{rest}] + RI(t) \qquad$ where $u(0) = u_{rest}$

Let $u(t) - u_{rest} = y$ and \$y' = u(t): \$
$\tau_m y' = -y + RI$

$\tau_m y' + y = RI \Rightarrow y' + \frac{1}{1/\tau_m} y = \frac{RI}{\tau_m}$ where $\mu = e^{t/\tau_m}$

$\Rightarrow y' e^{\frac{t}{\tau_m}} + \frac{e^{\frac{t}{\tau_m}}}{\tau_m} y = \frac{RIe^{\frac{t}{\tau_m}}}{\tau_m} \Rightarrow \int \frac{d}{dt} ye^{\frac{t}{\tau_m}} = \int \frac{RIe^{\frac{t}{\tau_m}}}{\tau_m} \frac{d}{dt} \Rightarrow ye^{\frac{t}{\tau_m}} = RIe^{\frac{t}{\tau_m}} + C$

$\Rightarrow y = RI + Ce^{-t/\tau_m} \Rightarrow u = u_r est + RI + Ce^{-t/\tau_m}$

$\Rightarrow u(t) = u_{rest} + RI(1 - e^{-t/\tau_m})$

$1b)\qquad \lim_{t\to\infty} u(t) = u_{th}$

$\lim_{t\to\infty} u_{rest} + R\bar{I}(1 - e^{-t/\tau_m}) \Rightarrow u_{rest} + R\bar{I}(1 - 0)$
$\Rightarrow u_{rest} + R\bar{I} = u_{th}$

$\Rightarrow \bar{I} = \frac{u_{th} - u_{rest}}{r}$

$1c)\qquad u(t) = u_{rest} + RIe^{t/\tau_m} + C \qquad$ where $\qquad u(0) = u_{start}$

From similar work of 1a) we get:

$u(t) = u_{rest} + (u_{start} - u_{rest})e^{-t/\tau_m} + \frac{RA}{\tau_m}$

Solving for A:

$A = \frac{\tau_m(u_m - [u_{rest} + (u_{start} - u_{rest})e^{-t/\tau_m}]}{R}$

The minimum value of A depends mainly on R because R cannot equal 0. Additionally, it depends on what value u_{start} and u_{rest} are because the will decide how much amplitude you will need to create a spike. If u_{start} starts at a high voltage than you will need less amplitude. However, if u_{rest} starts at a very low point, you will need more amplitude to generate a spike.

$1d)\qquad u(jT^+) = -[u(jT^+) - u_{rest}]e^{-T/\tau_m} + \frac{RA}{\tau_m} + u_{rest} \qquad$ where $\qquad U = u(jT^+)$

$u = [u - u_{rest}]e^{-T/\tau_m} + \frac{RA}{\tau_m} + u_{rest}$

$\Rightarrow u(1 - e^{-T/\tau_m}) = u_{rest}(1 - e^{-T/\tau_m}) + \frac{RA}{\tau_m}$

$\Rightarrow u = u_{rest} + \frac{RA}{\tau_m(1 - e^{-T/\tau_m})}$

Solve for T:

$1 - e^{-T}\tau_m = \frac{(u - u_{rest})\tau_m}{RA} \Rightarrow \ln e^{-T/\tau_m} = \ln(1 - \frac{(u - u_{rest})\tau_m}{RA})$

$\Rightarrow T = \tau_m^2[\ln \frac{(u-u_{rest})}{RA}]$

$2a)$ $\quad \tau_x x' + x = \delta(t)$ $\quad$ where \$ x(0) = 0\$

Using laplace transforms:

$\tau_x \mathcal{L}[x'] + \mathcal{L}[x] = \mathcal{L}[\delta(t)]$

$\Rightarrow \mathcal{L}[x](s + \frac{1}{\tau_x}) = \frac{1}{\tau_x} \Rightarrow \mathcal{L}[x] = \frac{1}{\tau_x}\frac{1}{s+1/\tau_x} \Rightarrow \mathcal{L}^{-1}[x] = \frac{1}{\tau_x}\mathcal{L}^{-1}[\frac{1}{s+1/\tau_x}]$

$\Rightarrow x = \frac{e^{-t/\tau_x}}{\tau_x}$

Pluggin x into $\quad \tau_s I' = -I + I_0 x$ $\quad$ where $I(0) = 0$

$\tau_s I' = -I + \frac{I_0 e^{-t/\tau_x}}{\tau_x} \Rightarrow I' + \frac{I}{\tau_s} = \frac{I_0 e^{-t/\tau_m}}{\tau_s \tau_x}$ $\quad$ where $\quad \mu = e^{t/\tau_s}$

$I = I_0 \frac{e^{-t/\tau_x} - e^{-t/\tau_s}}{\tau_x - \tau_s}$

Plugging I into $\quad \tau_s u' = -u + RI$ $\quad$ where $u(0) = 0$

$\tau_m u' = -u + RI_0 \frac{e^{-t/\tau_x} - e^{-t/\tau_s}}{\tau_x - \tau_s} \Rightarrow u' + \frac{u}{\tau_m} = RI_0 \frac{e^{-t/\tau_x} - e^{-t/\tau_s}}{\tau_m(\tau_x - \tau_s)}$ $\quad$ where $\quad \mu = e^{t/\tau_m}$

$u = \frac{RI_0}{\tau_x - \tau_s}[\frac{\tau_x e^{-t/\tau_x}}{\tau_x - \tau_s} - \frac{\tau_s e^{-t/\tau_s}}{\tau_s - \tau_m} + \frac{\tau_s - \tau_x}{e^{t/\tau_m}}]$

From the equation above, the amplitude increases with R. Similarly with I_0. Since, they are the numerators the amplitude will increase.

As $\tau_s \to 0$, we obtain:

$$\lim_{\tau_s \to 0} u = RI_0(\frac{e^{-t/\tau_x}}{\tau_x - \tau_m} - \frac{1}{e^{t/\tau_m}})$$

and as $\tau_m \to 0$, we have:

$$\lim_{\tau_m \to 0} u = \frac{RI_0 e^{-t/\tau_x}}{\tau_x}$$

We are left with the Resistor and Input because τ_m and τ_s cancel the rest. τ_x is the only constant variable left.

$2b)$

$0 = -u_1 + I_1 + g(u_2 - u_1)$
$0 = -u_2 + g(u_1 - u_2)$

Looking for u_2 in terms of u_1:

$0 = gu_1 - gu_2 - u_2 \Rightarrow u_2 = \frac{gu_1}{g+1}$

Plug into first component:

$0 = -u_1 + I_1 + g(\frac{gu_1}{g+1} - u_1) \Rightarrow 0 = I + (\frac{g^2}{g+1} - g - 1)u_1$

$u_1 = -\frac{I}{\frac{g^2}{g+1} - g - 1}$

$R_{input}(g) = \frac{u_1}{I}$ plug in u_1

$R_{input}(g) = \frac{g+1}{2g+1}$

From the solution, it seems that as g gets large the denominator grows faster than the numerator therefore, reaching a limit closer to 1. Note that $g \neq \frac{1}{2}$ else that denominator is undefined.

$2c)$

$0 = -u_1 + I_1 + g(u_2 - u_1)$
$0 = -u_2 + g(u_1 - 2u_2 + u_3)$
$0 = -u_3 + g(u_2 - u_3)$

Start with u_3:

$0 = gu_2 - (g+1)u_3 \Rightarrow u_3 = \frac{-gu_2}{g+1}$

Plug into second compartment:

$0 = -u_2 + g(u_1 - 2u_2 + \frac{gu_2}{g+1}) \Rightarrow 0 = gu_1 - (\frac{g^2}{g+1} + 2g + 1)u_2$

$u_2 = u_1 \frac{g(g+1)}{3g^2+3g+1}$

Plug into first compartment:

$0 = -u_1 + I_1 + g(u_1 \frac{g(g+1)}{3g^2+3g+1} - u_1) \Rightarrow 0 = I + u_1 \frac{g^2(g+1)}{3g^2+3g+1} - gu_1 - u_1$

$u_1 = -\frac{I}{\frac{g^2(g+1)}{3g^2+3g+1} - g - 1}$

$R_{input}(g) = \frac{3g^2+3g+1}{(g+1)^2(2g+1)}$

As more compartments are added the denominator will begin to increase faster than the numerator. So, more compartments will have less of a significance to the input resistance.