a) $$\frac{du}{dt} = -u + \mathcal{H}(u - \tfrac{1}{2}) - w$$

$$\frac{dw}{dt} = \varepsilon(2u - w)$$

where $\mathcal{H}(u - 1/2) = \begin{cases} 0, & u < 1/2 \\ 1, & u \geq 1/2 \end{cases}$

Set $u' = 0$ & $w' = 0$

$$0 = -u + \mathcal{H}(u - 1/2) - w$$
$$0 = \varepsilon(2u - w)$$

We obtain,

$$w = \mathcal{H}(u - 1/2) - u$$
$$w = 2u$$

Solving for fix points:

$$2u = \mathcal{H}(u - 1/2) - u \qquad w = 2u$$

Jacobian associated with its linear stability

$$J(\bar{u}, \bar{w}) = J\begin{pmatrix} F_u & F_w \\ G_u & G_w \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathcal{H}(u - \frac{1}{2})}{\partial u} - 1 & -1 \\ 2\varepsilon & -\varepsilon \end{pmatrix}$$

When $u < \frac{1}{2}$:

$$2u = 0 - u \qquad \bar{w} = 0$$

$$\bar{u} = 0$$

$$J(0,0) = \begin{pmatrix} -1 & -1 \\ 2\varepsilon & -\varepsilon \end{pmatrix} = \begin{vmatrix} \lambda+1 & -1 \\ 2\varepsilon & \lambda+\varepsilon \end{vmatrix}$$

$$= (\lambda+1)(\lambda+\varepsilon) + 2\varepsilon$$

$$= \lambda^2 + \varepsilon\lambda + \lambda + \varepsilon + 2\varepsilon$$

$$= \lambda^2 + \lambda(1+\varepsilon) + 3\varepsilon$$

$$\lambda = \frac{-(1+\varepsilon)}{2} \pm \frac{\sqrt{(1+\varepsilon)^2 - 12\varepsilon}}{2}$$

When $u \geq 1/2$, we have a contradictory u.

$2u = 1 - u \Rightarrow u = \frac{a}{3}$ when we said $u \geq \frac{1}{2}$, not possible

b) Assume $u(0) = 3/4$ and $w(0) = 0$.

$$\frac{du}{dt} = -u + H(u - 1/2) \qquad \frac{dw}{dt} = \varepsilon(2u - w)$$

Since $u(0) > \frac{1}{2}$,

$\frac{du}{dt} = \frac{1}{4}$ and $\frac{1}{4}$ acts much faster than ε.

So,

$$u = \overbrace{H\left(u - \tfrac{1}{2}\right)}^{1} - \cancel{w}^{\,0}$$

$$w' = \varepsilon(2(1-w) - w)$$

We see that $u \to 1$ b/c $\frac{1}{4}$ acts much faster than ε and our heaviside function will set $u = 1 - w$, where $w(0) = 0$. Since w relies on $\varepsilon \ll 1$, $u \to 1$.

Once u reaches 1, 'w' begins to climb slowly b/c of ε.

Our slow equation becomes:

$$w' = \varepsilon\left[2(1-w)-w\right)$$

$$w' + 3\varepsilon w = \varepsilon 2 \quad , \quad \mu = e^{3\varepsilon t}$$

$$\int e^{3\varepsilon t} w \frac{d}{dt} = \int \varepsilon 2 e^{3\varepsilon t} dt$$

$$e^{3\varepsilon t} w = \frac{\varepsilon 2}{\varepsilon 3} e^{3\varepsilon t} + C$$

$$w = \frac{2}{3} + Ce^{-3\varepsilon t} \quad , \quad w(0) = 0$$

$$0 = \frac{2}{3} + C \Rightarrow C = -\frac{2}{3}$$

So the slow equation for w:

$$w = \frac{2}{3}\left(1 - e^{-3\varepsilon t}\right)$$

Solve for $w(T) = 1/2$

$$\frac{1}{2} = \frac{2}{3}\left(1 - e^{-3\varepsilon T}\right)$$

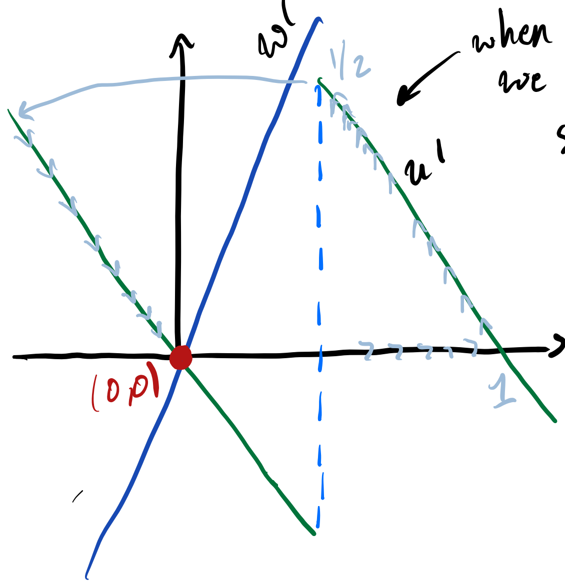
$$\frac{3}{4} = 1 - e^{-3\varepsilon T} \Rightarrow e^{-3\varepsilon T} = \frac{1}{4}$$

$$\Rightarrow \quad -3\varepsilon T = \ln\left(\tfrac{1}{4}\right)$$

$$T = \ln\left(\tfrac{1}{4}\right) \cdot \frac{1}{-3\varepsilon}$$

$$T = \frac{\ln(4)}{3\varepsilon}$$

When ε is decreased, T reaches its ending of the spike slowly. This makes sense b/c at $w(T) = 1/2$ we shoot back to our u' and the neuron slowly settles at its fixed pt.



when ε is decreased we reach ending of spike slowly.