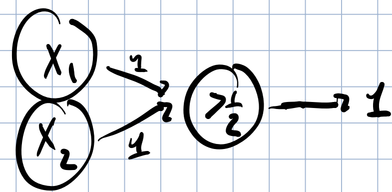
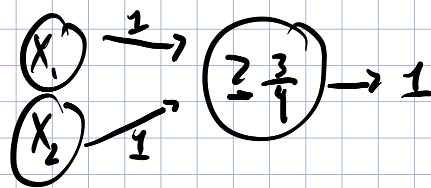
1a) $$y = H(w_1 x_1 + w_2 x_2 + \theta)$$

Let $\theta \neq 0$, $\theta > w_1 = w_2 > 0$,

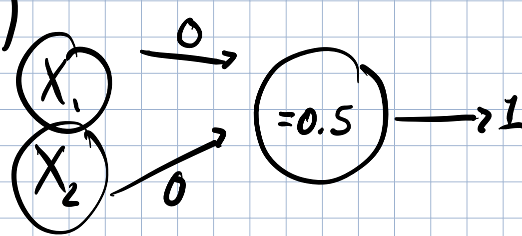
or gate: $y = H(w_1 x_1 + w_2 x_2 - \frac{1}{2})$



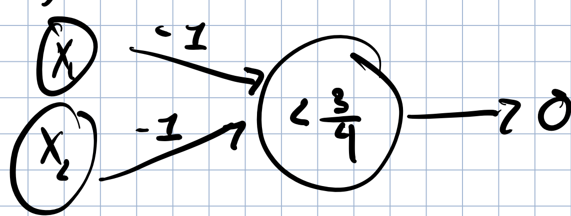
and gate: $y = H(w_1 x_1 + w_2 x_1 - \frac{3}{4})$



Nor gate: $y = H(0.5 - w_1 x_1 - w_2 x_2)$



Nand gate: $y = H(\frac{3}{4} - w_1 x_1 - w_2 x_2)$



We want to show that,

$$y = H(w_1 x_1 + w_2 x_2 + \theta) = 0$$

is not possible.

Xor gate

x_1	x_1	y	
0	0	0	✓
1	0	1	✓
0	1	1	✓
1	1	0	✗

Let $w_1, w_2 \geq 0$. Let $\theta = -\frac{1}{2}$: x_1 and $x_2 \in \{0,1\}$

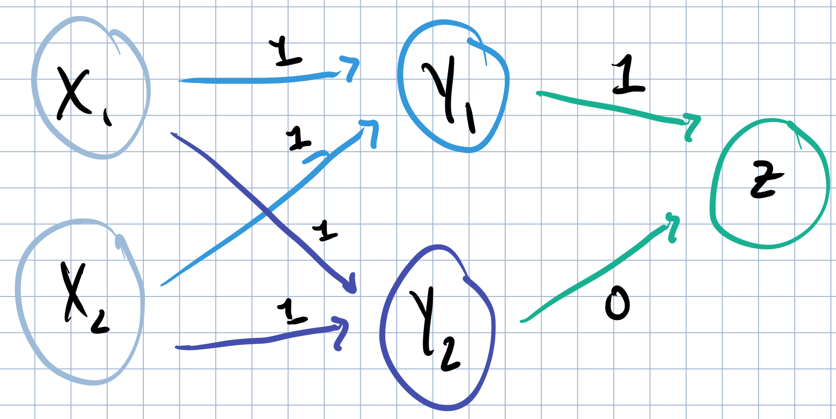
$$y = H(w_1 + w_2(0) - \tfrac{1}{2}) = 1$$
$$y = H(w_1(0) + w_2 - \tfrac{1}{2}) = 1$$
$$y = H(w_1(0) + w_2(0) - \tfrac{1}{2}) = 0$$
$$y = H(w_1 + w_2 - \tfrac{1}{2}) = 1 \quad \times$$

Similarly if $w_1, w_2 > 0$ and $\theta = 1$, x_1 and $x_2 \in \{0, \frac{1}{2}\}$

$$y = H(1 - w_1(0) - w_2 x_2) = 1$$
$$y = H(1 - w_1 x_1 - w_2(0)) = 1$$
$$y = H(1 - w_1 x_1 - w_2 x_2) = 0$$
$$y = H(1 - w_1(0) - w_2(0)) = 1$$

Therefore it is not possible with any combination or values of w_1, w_2, and θ.

1b)



$\theta = -\frac{1}{2}$, $w_{12} = w_{11} = 1$

$y_1 = H\left(x_1 + x_2 - \frac{1}{2} \right)$

$\theta = \frac{3}{2}$, $w_{21} = w_{22} = -1$

$y_2 = H\left(\frac{3}{2} - x_1 - x_2 \right)$

$\eta = -\frac{3}{2}$ $J_1 = J_2 = 1$

$z = H\left(y_1 + y_2 - \frac{3}{2} \right)$

XOR gate

x_1	x_2	z
0	0	0
1	0	1
0	1	1
1	1	0

1c)

$$y_1 = f(w_{11}x_1 + w_{12}x_2 + \theta_1)$$

$$y_2 = f(w_{21}x_1 + w_{22}x_2 + \theta_2)$$

$$z = f(J_1 y_1 + J_2 y_2 + \eta)$$

$w_{11} = w_{12} = w_{21} = w_{22} = J_1 = J_2 = 1$, $z = 0.05$, $r = 0.5$

$$J_k \to J_k - \frac{1}{2}\left(\frac{\partial E}{\partial J_k}\right) \qquad w_{jk} \to w_{jk} - \frac{1}{2}\left(\frac{\partial E}{\partial w_{jk}}\right)$$

$$E = (z - 0.05)^2$$

Initial Error:

$y_2 = y_1 = f(1) \approx 0.73$

$z = f(0.73 + 0.73 - 1) = 0.613$

$E = (0.613 - 0.05)^2 \approx 0.31$

$$J_1 \mapsto J_1 - \left(\frac{1}{2}\right)\frac{\partial E}{\partial J_1} :$$

$$\frac{\partial E}{\partial J_1} = (z - 0.05)^2 = 2(z - 0.05)\frac{\partial z}{J_1}$$

$$\frac{\partial z}{\partial J_1} = y_1 \sigma(J_1)(1 - \sigma(J_1))$$

Plug $J_1 = 1$:

$$\frac{\partial z}{\partial J_1} \simeq 0.196 \qquad \frac{\partial E}{\partial J_1} = 0.22$$

$$J_1 \mapsto J_1 - \frac{1}{2}\left(\frac{\partial E}{\partial J_1}\right) \simeq 0.919$$

Similarly for the case of J_2

$$J_2 \mapsto J_2 - \left(\frac{1}{2}\right)\left(\frac{\partial E}{\partial J_2}\right) \simeq 0.919$$

$$W_{11} = W_{11} - 0.5 \frac{\partial E}{\partial w_{11}}$$

$$\frac{\partial E}{w_{11}} = (z - 0.05)^2 = 2(z - 0.05)\frac{\partial z}{\partial w_{11}}$$

$$\frac{\partial z}{\partial w_{11}} = J_1 \cdot \sigma(y_1)(1 - \sigma(y_1)) \quad \frac{\partial y_1}{\partial w_{11}}$$

$$\frac{\partial y_1}{w_{11}} = w_{11}\left(\sigma(w_{11})(1 - \sigma(w_{11})\right)$$

$$\frac{\partial y_1}{\partial w_{11}} \approx 0.196 \quad \frac{\partial z}{\partial w_{11}} \approx 0.0431$$

$$\frac{\partial E}{w_{11}} \approx 0.0484$$

$$W_{11} \mapsto 1 - 0.5 \cdot \frac{\partial E}{\partial w_{11}} \approx 0.975$$

Similarly for the cases of W_{12}, W_{21}, W_{22}.

So our output after the weight update:

$$y_1 = f(w_{11}x_1 + w_{12}x_2 - 1) \approx 0.721$$

$$y_2 = f(w_{21}x_1 + w_{22}x_2 - 1) \approx 0.721$$

$$z = f(J_1 y_1 + J_2 y_2 - 1) \approx 0.580$$

$$E = (z - 0.05)^2 \approx 0.281$$

Yes, the error is reduced after the first step.
Yes, we can speed up the learning process by increasing r. However we do not want to increase r too much b/c we will overshoot and instead increase the error.