$$r_1 = a_1 x + b_1 + n_1 \quad , \quad r_2 = a_2 x + b_2 + n_2$$

$$\hat{x} = w_1 r_1 + w_2 r_2 - C$$

Determine the relationships between w_1, w_2, and C:

$$\langle \hat{X} \rangle = x \Rightarrow \langle w_1 r_1 + w_2 r_2 - C \rangle = x$$

$$\Rightarrow \langle w_1 (a_1 x + b_1 + n_1) + w_2 (a_2 x + b_2 + n_2) - c \rangle = x$$

$$\Rightarrow w_1 a_1 x + w b_1 + w \cancel{\langle n_1 \rangle}^{0} + w_2 a_2 x + w_2 b_2 + w_2 \cancel{\langle n_2 \rangle}^{0} - c = x$$

$$\Rightarrow w_1 a_1 x + w_1 b_1 + w_2 a_2 x + w_2 b_2 - C - x = 0$$

$$\Rightarrow \underbrace{(w_1 a_1 + w_2 a_2 - 1)}\, x + \underbrace{w_1 b_1 + w_2 b_2 - C} = 0$$

$$w_1 a_1 + w_2 a_2 = 1 \qquad\qquad C = w_1 b_1 + w_2 b_2$$

$$w_1 = \frac{1 - w_2 a_2}{a_1} \qquad\qquad w_1 = \frac{\frac{b_2 - C}{a_2}}{\frac{a_1 b_2}{a_2} - b_1}$$

or

$$w_2 = \frac{1 - w_1 a_1}{a_2} \qquad\qquad w_2 = \frac{\frac{b_1}{a_1} - C}{\left(\frac{b_1 a_2}{a_1} - b_2\right)}$$

So plug back to $\hat{x}$:

$$\hat{x} = \left(\frac{1-w_2 a_2}{a_1}\right) r_1 + \left(\frac{1-w_1 a_1}{a_2}\right) r_2 - \left(\frac{1-w_2 a_2}{a_1}\right) b_1 + \left(\frac{1-w_1 a_1}{a_2}\right) b_2$$

Now, min $\langle (\hat{x}-x)^2 \rangle$:

$$\left\langle \left(\frac{r_1 - w_2 a_2 \cdot r_1}{a_1} + \frac{r_2 \cdot r_2 w_1 a_1}{a_2} - \frac{b_1 - b_1 w_2 a_2}{a_1} + \frac{b_2 - w_1 a_1}{a_2} - x \right)^2 \right\rangle$$

$$\left\langle \left(\frac{a_1 x + b_1 + n_1 - w_2 a_2 (a_1 x + b_1 + n_1)}{a_1} \right. \right.$$

$$+ \frac{a_2 x + b_2 + n_2 - (a_2 x + b_2 + n_2) w_1 a_1}{a_2} - \frac{b_1 - b_1 w_2 a_2}{a_1}$$

$$\left. \left. + \frac{b_2 - w_1 a_1 b_2}{a_2} - x \right)^2 \right\rangle$$

$$\left\langle\left((1-w_2a_2)x+\left(\frac{1-w_2a_2}{a_1}\right)n_1+(1-a_1w_1)x+\left(\frac{1-w_1a_1}{a_2}\right)n_2-x\right)^2\right\rangle$$

$$\left\langle\left((1-w_2a_2-w_1a_1)x+\left(\frac{1-w_2a_2}{a_1}\right)n_1+\left(\frac{1-w_1a_1}{a_2}\right)n_2\right)^2\right\rangle$$

$$\Rightarrow\left\langle\left(\left(\frac{1-w_2a_2}{a_1}\right)n_1+\left(\frac{1-w_1a_1}{a_2}\right)n_2\right)^2\right\rangle$$

$$\Rightarrow\left\langle\left(w_1n_1+w_2n_2\right)^2\right\rangle$$

$$\Rightarrow\left\langle w_1^2n_1^2+2w_1n_1w_2n_2+w_2^2n_2^2\right\rangle$$

$$\Rightarrow w_1^2\langle n_1^2\rangle+w_2^2\langle n_2^2\rangle$$

$$\Rightarrow w_1^2\sigma^2+w_2^2\sigma^2$$

$$\Rightarrow \sigma^2\left(\frac{\frac{b_2}{a_2}-c}{\frac{a_1b_2}{a_2}-b_1}\right)+\sigma^2\left(\frac{\frac{b_1}{a_1}-c}{\left(\frac{b_1a_2}{a_1}-b_2\right)}\right)$$

So as $a_1\to\infty$ and $a_2\to\infty$, we result to,

$\sigma^2\left(\frac{c}{b_1}\right)+\sigma^2\left(\frac{c}{b_2}\right)$ which is the variance of the noise times the base line rate

1b) assume $x = 1$ and $x = -1$,

$$r_1 = \mathcal{H}(x) + n_1 \quad , \quad r_2 = \mathcal{H}(x) + n_2$$

$$\hat{x} = w_1 r_1 + w_2 r_2 - c$$

where, $\mathcal{H}(x) = \begin{cases} 1 & , x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$

For $x = 1$:

$$\langle \hat{x} - x \rangle = 0 \Rightarrow \langle \hat{x} - 1 \rangle = 0$$

$$\langle w_1 r_1 + w_2 r_2 - c - 1 \rangle = 0$$

$$\Rightarrow \langle w_1 (1 + n_1) + w_2 (1 + n_2) - c - 1 \rangle = 0$$

$$\Rightarrow w_1 + w_1 \cancelto{0}{\langle n_1 \rangle} + w_2 + w_2 \cancelto{0}{\langle n_2 \rangle} - c - 1 = 0$$

$$\Rightarrow w_1 + w_2 - c - 1 = 0 \Rightarrow w_1 + w_2 = c + 1$$

So,

$$w_1 = c + 1 - w_2 \quad , \quad w_2 = c + 1 - w_1$$

$$c = -1 + w_1 + w_2$$

$$\hat{x} = (c + 1 - w_2) r_1 + (c + 1 - w_1) r_2 - 1 + w_1 + w_2$$

For $x = -1$:

$$\langle \hat{x} + 1 \rangle = 0$$

$$\langle w_1 (\cancel{n_1}) + w_2 (\cancel{n_2}) - c + 1 \rangle = 0$$

0 0

$$c = 1$$

To find min $\langle (\hat{x} - x)^2 \rangle = 0$: for $x = -1$

$$\langle (w_1 n_1 + w_2 n_2)^2 \rangle = 0$$

$$\langle (w_1 n_1)^2 + 2 w_1 n_1 w_2 n_2 + (w_2 n_2)^2 \rangle = 0$$

$$(w_1^2 + w_2^2)\sigma^2 = 0$$

Now to find MSE for $-1 \leq x \leq 1$:

$x \geq 0$:

$$\langle (\hat{x} - x)^2 \rangle = 0$$

$$\langle (w_1 + w_1 n_1 + w_2 + w_2 n_2 - c - x)^2 \rangle = 0$$

$$\langle (w_1 n_1 + w_2 n_2)^2 \rangle = 0$$

$$\sigma^2 (w_1^2 + w_2^2) = 0$$

So when $x=-1$, we have $\sigma^2(w_1^2+w_2^2)$ and when $x=1$, we have $\sigma^2(w_1^2+w_2^2)$

Therefore when $x=0$ that is when our MSE will be the highest b/c at:

$$\langle (w_1 + w_1 n_1 + w_2 + w_2 n_2 - c - \overset{x}{\overset{\downarrow}{0}})^2 \rangle = 0$$

we do not subtract any value from our c.

Our estimation might be worse than part (a) b/c $x=0$ is at the discontinuity of the Heaviside function.