a) Show nullclines:

$$\frac{du}{dt} = u\left(u-\frac{1}{2}\right)(1-u) - w + I$$

$$\frac{dw}{du} = \varepsilon(u-w)$$

Set $u' = 0$ & $w' = 0$

$$u' = 0 : \quad 0 = u\left(u-\frac{1}{2}\right)(1-u) - w + I$$

$$w' = 0 : \quad 0 = \varepsilon(u-w)$$

We obtain,

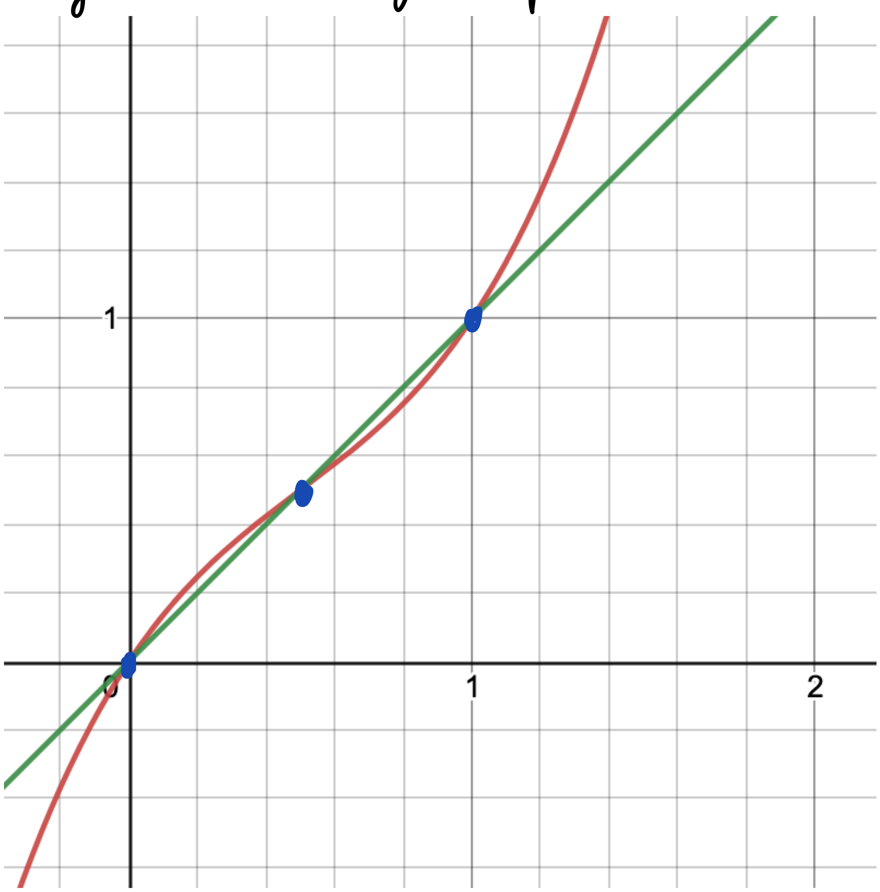
$$I = u - u\left(u-\frac{1}{2}\right)(1-u)$$

$$w = u$$

Jacobian associated with its linear stability:

$$J(\bar{u},\bar{w}) = J\begin{pmatrix} F_u & F_w \\ G_u & G_w \end{pmatrix} = \begin{pmatrix} -3u^2+3u-\frac{1}{2} & -1 \\ \varepsilon & -\varepsilon \end{pmatrix}$$

The reason why we have 3 fixed pts is because 'I' is a cubic function and 'w' is a linear nullcline. They can only be up to 3 fixed pts because the first fixed pt is for u_{rest}, the second fixed pt is when we reach threshold u_{th}, and the last fixed pt we reach is an equilibrium pt that can change depending on voltage input.



b) $I = 0$ and show $\bar{u} = \bar{w} = 0$

$$0 = u - u\left(u - \tfrac{1}{2}\right)(1 - u)$$

$$w = u$$

$$0 = u + \left(-u^2 + \tfrac{u}{2}\right)(1 - u)$$

$$= u - u^2 + u^3 + \frac{u}{2} - \frac{u^2}{2}$$

$$= u\left(1 - u + u^2 + \tfrac{1}{2} - \tfrac{u}{2}\right)$$

$$\underline{0 = u = w}$$

So,

$$\bar{u} = 0 = \bar{w}$$

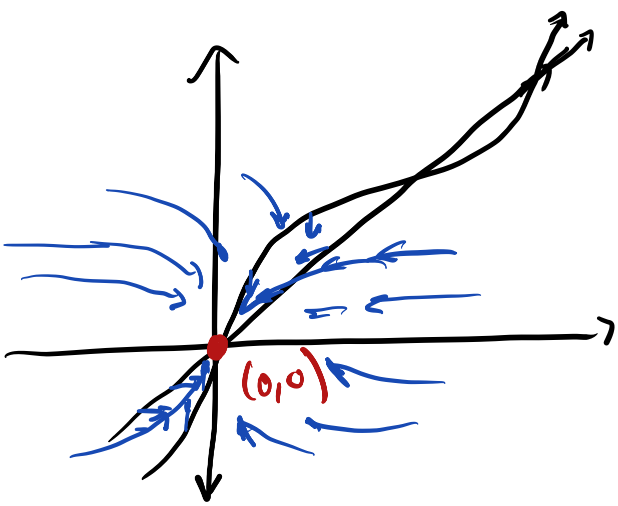
$$J(0,0) = \begin{pmatrix} -\frac{1}{2} & -1 \\ \varepsilon & -\varepsilon \end{pmatrix} = \begin{vmatrix} \lambda + \frac{1}{2} & -1 \\ \varepsilon & \lambda + \varepsilon \end{vmatrix}$$

$$\Rightarrow \boxed{\lambda = \frac{-\frac{1}{2} - \varepsilon}{2} \pm \frac{\sqrt{\left(-\frac{1}{2} - \varepsilon\right)^2 - \varepsilon}}{2}}$$

Note that $\varepsilon > 0$,

Then $-\frac{1}{2} - \varepsilon < 0$, will have $\lambda_\pm$ with '$-$' real part, because for $\varepsilon > 0$ $\sqrt{\left(\frac{1}{2}-\varepsilon\right)^2 - \varepsilon} < \left(\frac{1}{2} - \varepsilon\right)$ if its real.

With a stable node at $(0,0)$



Yes, when $\varepsilon < 0$ will make the node unstable and create repeatitive spikes.

c) $$J\left(\frac{1}{2}, \frac{1}{2}\right) = \begin{pmatrix} \frac{1}{4} & -1 \\ \varepsilon & -\varepsilon \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4} & -1 \\ \varepsilon & -\varepsilon \end{pmatrix} = \begin{vmatrix} \lambda - \frac{1}{4} & -1 \\ \varepsilon & \lambda + \varepsilon \end{vmatrix}$$

$$\Rightarrow \lambda^2 + \left(\varepsilon + \frac{1}{4}\right)\lambda + \frac{5\varepsilon}{4}$$

$$\boxed{\pm\lambda = \frac{\frac{1}{4} - \varepsilon}{2} \pm \frac{\sqrt{\left(\frac{1}{4} - \varepsilon\right)^2 - \varepsilon}}{2}}$$

When $\varepsilon = 1$, we have a stable node since $\frac{1/4 - \varepsilon}{2} < 0$. However when $\varepsilon < \frac{1}{4}$ the real part becomes '+' and our node becomes unstable and our node begins to sustain oscillation.