APPM 4370/5370, Computational Neuroscience: Homework 5

Due Wed. Nov. 3, 2021
by 11:30am (by email)

PAGES 1-2: THREE MATH EXERCISES AND NO COLLABORATION PROBLEM. PAGE 3: PYTHON EXERCISE.

Math exercises. Taken from Gerstner et al, Chapter 19

1. **Normalization of firing rate.** Consider a learning rule for a feedforward network with two presynaptic and one postsynaptic neuron:

$$w_1' = \gamma(v - v_\theta)u_1, \qquad w_2' = \gamma(v - v_\theta)u_2,$$

where synaptic weights only change if a presynaptic neuron is active ($u_j > 0$) and $v_\theta > 0$. The direction of the change is determined by the activity of the postsynaptic neuron v. The postsynaptic firing rate is given by $v = f(w_1u_1 + w_2u_2)$ where f is monotone increasing and positive. Assume presynaptic firing rates u_j are constant.

(a) Show the system is in equilibrium when $v = v_\theta$.

(b) Determine the stability of the fixed point. Consider the cases $\gamma > 0$ and $\gamma < 0$. For which case is the fixed point unstable? Why might this be a problem in a real network of neurons?

(c) Is this rule Hebbian, anti-Hebbian, or non-Hebbian? Consider both cases $\gamma > 0$ and $\gamma < 0$.

2. **Receptive field development with the BCM rule.** Two presynaptic neurons with firing rates u_1 and u_2 connect to the same postsynaptic neuron which fires at a rate $v = w_1u_1 + w_2u_2$. Synaptic weights change according to the BCM rule

$$w_1' = \eta v(v - v_\theta)u_1, \qquad w_2' = \eta v(v - v_\theta)u_2 \tag{1}$$

with a hard lower bound $0 \leq w_{ij}$, $\eta > 0$, and $v_\theta = 2$ Hz.

(a) Initially $w_1(0) = w_2(0) = 1$. There are two possible input patterns presented: for $\mu = 1$, a $u_1 = 3$ Hz and $u_2 = 0$ Hz; and for $\mu = 2$, $u_2 = 1$ Hz and $u_1 = 0$ Hz. Inputs alternate between both inputs several times, back and forth. Each pattern presentation lasts for time $T_s > 0$. How do the weights w_1 and w_2 evolve? Explain why the postsynaptic neuron becomes specialized to one input. You do not need to solve the resulting differential equations explicitly, but use them to justify your answer.

(b) Now consider the situation in which for input pattern $\mu = 2$, $u_1 = 0$ Hz and $u_2 = 2.5$ Hz. In this case, how do the weights w_1 and w_2 evolve?

(c) Lastly, consider a time dependent threshold v_θ which is a low pass filtered version of v^2, so

$$v_\theta' = v^2 - v_\theta.$$

For the case $\eta = 1$, $u_1 > 0$, and $u_2 = 0$, find the fixed points and stability of the above equation coupled to the w_1 equation in Eq. (1). There should be two possible fixed points: one with a silent synapse ($w_1 = 0$) and the other with a nonzero synapse ($w_1 > 0$). Over what range of u_1 is the nonzero synapse equilibrium stable? Interpret your findings biologically.

3. **Principal component analysis with Oja's learning rule.** Consider a discrete time, vector version of Oja's learning rule:

$$\mathbf{w}_{n+1} = \mathbf{w}_n + \gamma v_n(\mathbf{u}_n - \mathbf{w}_n v_n), \tag{2}$$

where $\mathbf{w}_n$ is the vector of weights after trial n, γ is the learning rate, $\mathbf{u}_n$ is the presynaptic input on trial n which is randomly drawn from a distribution with correlation matrix $C = \langle \mathbf{u}_n \mathbf{u}_n^T \rangle$, and the postsynaptic response is $v_n = \mathbf{u}_n^T \mathbf{w}_n$. In the following you will show the long term equilibrium $\lim_{n\to\infty} \mathbf{w}_n \approx \bar{\mathbf{w}}$ approximates the first principal component of the correlation matrix C.

(a) Assume Eq. (2) eventually reaches some approximate equilibrium so $\langle \mathbf{w}_{n+1} - \mathbf{w}_n \rangle \approx 0$ and $\mathbf{w}_n \approx \bar{\mathbf{w}}$ for n large. Show that $C\bar{w} \approx \mu \bar{\mathbf{w}}$ where $\mu = \langle v_n^2 \rangle$, and conclude $\bar{\mathbf{w}}$ is roughly an eigenvector of C with eigenvalue μ.

(b) Now, prove $||\bar{\mathbf{w}}||^2 = 1$ by multiplying the equation from (a) by $\bar{\mathbf{w}}^T$ and simplifying and using the formula $\mu = \langle v_n^2 \rangle$. You can switch your $\approx$ from before to $=$ now for simplicity.

(c) Now to prove $\bar{\mathbf{w}}$ is the principal component, you will perform a linear stability analysis. To start, plug $\bar{\mathbf{w}}_n = \bar{\mathbf{w}} + \boldsymbol{\epsilon}_n$ into the following version of the update Eq. (2), assuming $\gamma = 1$:

$$\bar{\mathbf{w}}_{n+1} - \bar{\mathbf{w}}_n = \Delta \bar{\mathbf{w}}_n = C\bar{\mathbf{w}}_n - (\bar{\mathbf{w}}_n^T C \bar{\mathbf{w}}_n)\bar{\mathbf{w}}_n,$$

isolate terms of order $||\boldsymbol{\epsilon}_n||$ (ignore larger terms (of order 1) and those of order $||\boldsymbol{\epsilon}_n||^2$ and higher) to obtain the relationship

$$\Delta \bar{\mathbf{w}}_n = (C - \mu I)\boldsymbol{\epsilon}_n - 2\mu(\boldsymbol{\epsilon}_n^T \bar{\mathbf{w}})\bar{\mathbf{w}}.$$

(d) Now, left multiply the expression in (c) by $\mathbf{e}_j^T$ (the jth eigenvector of C where $j > 1$) and use orthogonality of eigenvectors from different eigenvalues ($\mathbf{e}_j^T \mathbf{w} = 0$ if $j > 1$) to show $\mathbf{e}_j^T \Delta \bar{\mathbf{w}}_n = (\lambda_j - \mu)\mathbf{e}_j^T \boldsymbol{\epsilon}_n$. This implies stable convergence to $\bar{\mathbf{w}}$ only if $\lambda_j < \mu$ (μ is the maximal eigenvalue).

4. ***NO COLLABORATION PROBLEM!*** **E/I network oscillations.** Here you will work out conditions for oscillations in both linear and nonlinear e/i population rate models.

(a) Consider the following linear excitatory/inhibitory population model:

$$v_e' = w_{ee} v_e - w_{ei} v_i, \qquad v_i' = v_e - v_i.$$

Derive conditions involving $w_{ee} > 0$ and $w_{ei} > 0$ that ensure stability of the steady state.

(b) Now consider the following nonlinear excitatory network:

$$v_1' = -v_1 + \frac{v_2^2}{\kappa + v_2^2}, \qquad v_2' = -v_2 + \frac{v_1^2}{\kappa + v_1^2}.$$

where $\kappa > 0$ is the firing threshold of neurons in the network. Find a critical threshold κ_c, so for $\kappa < \kappa_c$ there are three symmetric fixed points, where $v = \bar{v}_1 = \bar{v}_2$. Then, determine all three of these fixed points' stability when $\kappa = 3/16$. Can this network maintain some memory of initial conditions?

python exercise. You will write your own code for the following problem.

5. **Testing BCM rule predictions.** Here you will validate your theory from Ex. 2.

   (a) Using the parameters and equations from Ex. 2a, write a python code that simulates the system of differential equations for the BCM rule when $\eta = 1$ and $T_s = 1$, repeating the pattern presentations (in alternation) 10 times each.
   Use forward Euler method to solve the differential equations and use the 'min' function after each Euler step to cap the values of w_1 and w_2 at 10 (e.g., $w_1 = \min(w_1, 10)$). This will prevent your program from crashing due to the weights blowing up.
   Confirm a match with your previous predictions.

   (b) Now, using parameters from Ex. 2b, write a code again for when $\eta = \Delta t = 1$. Confirm a match with your previous predictions.

   (c) Now to validate predictions from Ex. 2c. Choose a u_1 value from each different qualitative stability range and simulate the BCM rule with the dynamic threshold (as given in Ex. 2c). You may need to run simulations for a long time in order to see the trajectories converge to the predicted fixed points. Also, make sure you initialize $w_1(0) > 0$ and $v_\theta > 0$, so that you don't get stuck at the silent synapse equilibrium. What happens when the non-silent equilibrium is unstable? Is there a limit cycle (oscillations)? If so, how does u_1 affect the frequency and amplitude of the oscillations?