a) $$\frac{dN}{dt} = -N^2 \quad , \quad N(0) = 1$$

$$\int -\frac{dN}{N^2} = \int dt \Rightarrow \frac{1}{N} = t + C$$

$$\Rightarrow N = \frac{1}{t+C} \quad \text{plug IVP,} \quad N = \frac{1}{t+1}$$

$$\frac{dR}{dt} = N(1-R) - \beta R \quad , \quad R(0) = 0$$

$$\frac{dR}{dt} = \frac{1}{t+1}(1-R) - \beta R$$

$$\Rightarrow R' + \left(\frac{1}{t+1} + \beta\right) R = \frac{1}{t+1}$$

$$\mu = e^{\int \frac{1}{t+1} + \beta \, dt}$$

$$\mu = e^{\ln(t+1) + \beta t}$$

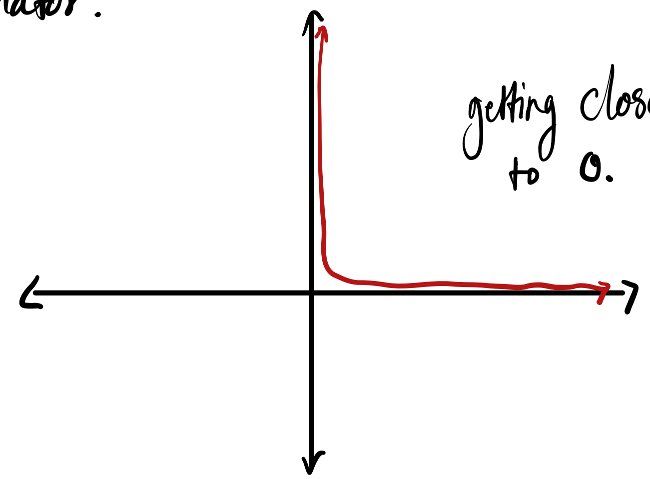
$$= (t+1)e^{\beta t}$$

$$\Rightarrow \int (t+1)e^{\beta t} R = \int e^{\beta t} dt$$

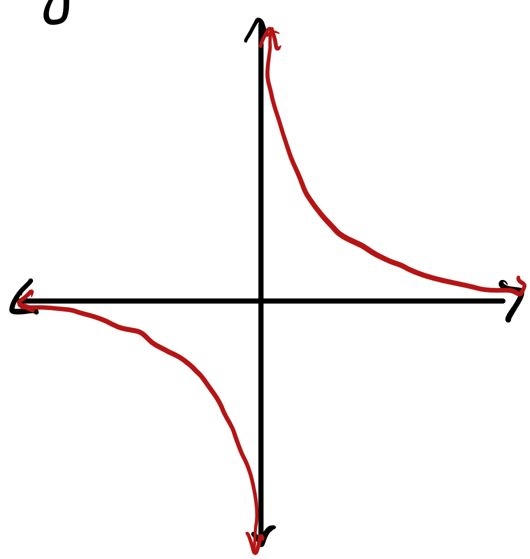
$$\Rightarrow (t+1)e^{\beta t} R = \frac{e^{\beta t}}{\beta} + C \Rightarrow R = \frac{1}{\beta(t+1)} + \frac{e^{-\beta t} C}{t+1}$$

$$\Rightarrow \text{Plug in IVP,} \quad \boxed{R = \frac{e^{-\beta t}}{\beta(t+1)} + \frac{1}{\beta(t+1)}}$$

For very large rates of β, the trajectory rate of $R(t)$ goes to 0 because of β bieng in the denominator.



For very small rates of β, the trajectory rates of $R(t)$ have a asymptotic trajectory. The smaller β gets, the further away from the center the asymptotic trajectories become.



b) $g_{syn}(t) = x(t)\,y(t)\,\bar{g}$

$$\frac{dx}{dt} = -\frac{x - x_F}{\tau_F} + f_F(1-x)\sum_{j=1}^{\infty}\delta(t-t_j), \quad x(0) = x_F$$

$$\frac{dy}{dt} = -\frac{y-1}{\tau_D} + f_D\, y\sum_{j=1}^{\infty}\delta(t-t_j), \quad y(0) = 1$$

Let, t_n spike periodically that is $t_1 = 100\,ms$, $t_2 = 200\,ms$, ...

$f_F = f_D = 1$ & $\tau_F = \tau_D = \tau = 100\,ms$ & $x_F = 0.5$

$$+\ x' = -\frac{x - 0.5}{\tau} + 1 - x$$

$$+\ y' = -\frac{y-1}{\tau} - y$$

$$+:\quad x' + x = -\frac{x - 0.5}{\tau} + 1$$

$$x' + \left(1 + \frac{1}{\tau}\right)x = +\frac{0.5}{\tau} + 1$$

$$\mu = e^{\int 1 + \frac{1}{\tau}\,dt} = e^{t + \frac{t}{\tau}}$$

$$\Rightarrow \int e^{t + \frac{t}{\tau}}\, x\, dt = \left(\frac{0.5}{\tau} + 1\right)\int e^{t + \frac{t}{\tau}}\,dt$$

$$\Rightarrow e^{t+\frac{t}{\tau}} x = \left(\frac{0.5}{\tau} + 1\right)\left[\frac{\tau e^{t+\frac{t}{\tau}}}{\tau+1} + c\right]$$

$$\Rightarrow e^{t+\frac{t}{\tau}} x = \frac{0.5 e^{t+\frac{t}{\tau}}}{\tau+1} + \frac{\tau e^{t+\frac{t}{\tau}}}{\tau+1} + C$$

$$\Rightarrow x = \frac{0.5}{\tau+1} + \frac{\tau}{\tau+1} + C e^{-t-\frac{t}{\tau}}, \quad x(0) = x_F = 0.5$$

$$\boxed{x = \frac{\tau+0.5}{\tau+1} + \left(0.5 - \frac{0.5-\tau}{\tau+1}\right) e^{-t-\frac{t}{\tau}}}$$

$$y' = \frac{-y-1}{\tau} - y$$

$$y' + \left(1 + \frac{1}{\tau}\right) y = +\frac{1}{\tau} \qquad \mu = e^{t+\frac{t}{\tau}}$$

$$\Rightarrow \int e^{t+\frac{t}{\tau}} y \, dt = \int + \frac{e^{t+\frac{t}{\tau}}}{\tau} dt$$

$$\Rightarrow e^{t+\frac{t}{\tau}} y = + \frac{e^{t+\frac{t}{\tau}}}{\tau+1} + C$$

$$\Rightarrow y = C e^{-t-\frac{t}{\tau}} + \frac{1}{\tau+1}, \quad y(0) = 1$$

$$\Rightarrow \boxed{y = \left(1 - \frac{1}{\tau+1}\right) e^{-t-\frac{t}{\tau}} + \frac{1}{\tau+1}}$$

Now $g_{syn} = x(t)\,y(t)\,\bar{g}$, when $t = 300$ and $t = 200$

$$g_{syn} = x(200^-)\,y(200^-)\,\bar{g}$$

$$+\quad x(200^-) = \frac{100.5}{101} + \left(0.5 + \frac{-100.5}{101}\right)e^{-202}$$

$$= .995$$

$$+\quad y(200) = \left(1 - \frac{1}{101}\right)e^{-202} + \frac{1}{101} = .009$$

$$\boxed{g_{syn} = (.995)(0.009)\,\bar{g} = 0.0089}$$

$$g_{syn} = x(300^-)\,y(300^-)\,\bar{g}$$

$$+\quad x(300^-) = \frac{100}{101} + \frac{0.5}{101} + \left(0.5 + \frac{-0.5}{101} - \frac{100}{101}\right)e^{-303}$$

$$= .995$$

$$+\quad y(300) = \left(1 - \frac{1}{101}\right)e^{-303} + \frac{1}{101}$$

$$= 0.009$$

$$\boxed{\begin{aligned} g_{syn} &= .995)(0.009)\,\bar{g} \\ &= 0.0089 \end{aligned}}$$

c) $x = \frac{\tau + X_F}{\tau+1} + \left(X_F + \frac{-X_F - \tau}{\tau+1} \right) e^{-t - \frac{t}{\tau}}$

$$y = \left(1 - \frac{1}{\tau+1}\right) e^{-t - \frac{t}{\tau}} + \frac{1}{\tau+1}$$

Long term pre-spike conductane :

$$x = \frac{\tau + X_F}{\tau+1} + \left(X_F - \frac{X_F - \tau}{\tau+1} \right) e^{-t - \frac{t}{\tau}}$$

From the above equation $1 < \tau < \infty$ so that our denominators do not evaluate ~~to~~ 0.
It varies with τ and X_F by τ making the conductance go to 0 or not and X_F as the strength of the spike.

To determine the maximum of $x(t)$.

$$0 = \frac{\tau + X_F}{\tau+1} + \left(X_F - \frac{X_F - \tau}{\tau+1} \right) e^{-t - \frac{t}{\tau}} \, dt$$

$$\Rightarrow 0 = \frac{\tau + X_F}{\tau+1} + \left(X_F - \frac{X_F - \tau}{\tau+1} \right)\left(-\frac{1}{\tau} - 1\right) e^{-t - \frac{t}{\tau}}$$

$$\Rightarrow 0 = \frac{\frac{\tau + x_F}{\tau+1}}{\left(x_F - \frac{x_F - \tau}{\tau+1}\right)\left(-\frac{1}{\tau} - 1\right)} + e^{-t - \frac{t}{\tau}}$$

$$e^{-t - \frac{t}{\tau}} = - \frac{\frac{\tau + x_F}{\tau+1}}{\left(x_F - \frac{x_F - \tau}{\tau+1}\right)\left(-\frac{1}{\tau} - 1\right)}$$

$$-t - \frac{t}{\tau} = \ln\left|\frac{\tau + x_F}{\tau+1}\right| - \ln\left|\left(x_F - \frac{x_F - \tau}{\tau+1}\right)\left(-\frac{1}{\tau} - 1\right)\right|$$

$$\boxed{t = \frac{\ln\left|\frac{\tau + x_F}{\tau+1}\right| - \ln\left|\left(x_F - \frac{x_F - \tau}{\tau+1}\right)\left(-\frac{1}{\tau} - 1\right)\right|}{\left(-1 - \frac{1}{\tau}\right)}}$$

This is where we will be obtaining our sustained spikes.