$$u_1' = -u_1 - w u_2 + 1$$
$$u_2' = -u_2 - w u_1 + 1$$

Find nullclines: $u_1' = 0$, $u_2' = 0$

$$0 = -u_1 - w u_2 + 1 \Rightarrow u_1 = 1 - w u_2$$

$$0 = -u_2 - w u_1 + 1 \Rightarrow u_2 = 1 - w u_1$$

$$\Rightarrow u_1 = \frac{1-w}{(1-w^2)}$$

$$u_2 = \frac{1-w}{(1-w^2)}$$

So when $w = 1$, we have an infinite amount of fixed pts. therefore $w_c = 1$

So fixed pts: $\left(\frac{1-w}{1-w^2} , \frac{1-w}{1-w^2} \right)$

$$J(u_1, u_2) = \begin{pmatrix} -1 & -w \\ -w & -1 \end{pmatrix}$$

$$\det(\lambda I - J) = \begin{pmatrix} \lambda + 1 & w \\ w & \lambda + 1 \end{pmatrix} = 0$$

$$\Rightarrow \lambda^2 + 2\lambda + 1 - w^2 = 0$$

$$\pm\lambda = -1 \pm \frac{\sqrt{4 - 4(1-w^2)}}{2}$$

2

$\lambda = -1 \pm \omega$, so for $\omega = 1$

$$\boxed{\lambda = 0, -2}$$

So we knew $\omega > 0$ therefore λ is stable when $\lambda = -\omega - 1$. When $\lambda = 0$, we have no direction. What that means neurobiological is that $\omega_c = 1$ is when there is a deadlock and neuron u_1 and u_2 will compete when either u_1 or u_2 is greater.

2b)

$$u_1' = -u_1 - \omega u_2 + 1 + I_1(t)$$
$$u_2' = -u_2 - \omega u_1 + 1 + I_2(t)$$

Solving for this linear system:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \underbrace{\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}}_{A} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 1 + I_0 - I_0 H(t-1) \\ 1 \end{pmatrix}}_{\vec{f}}$$

For $\lambda = -2$:

$$\lambda I - A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\Rightarrow -v_1 + v_2 \Rightarrow v_1 = v_2 \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda = 0$:

$$\lambda I - A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\Rightarrow v_1 + v_2 = 0 \Rightarrow v_1 = -v_2 \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So now we look for $A = U \Lambda U^{-1}$:

$$U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$ Eigenvector matrix

$$\Lambda = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$$ Diagnol Matrix of our eigenvalues

$$U^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$ Inverse Matrix

So,

$$\vec{u}' = U \Lambda U^{-1} \vec{u} + \vec{f}$$

and $\vec{v} = U^{-1} \vec{u} \rightarrow \vec{u} = U \vec{v}$

$$\Rightarrow (U\vec{v})' = U \Lambda \vec{v} + \vec{f}$$

$$\vec{v}' = \Lambda \vec{v} + U^{-1} \vec{f}$$

$$\Rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}' = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 + I_0 + I_0 H(t-1) \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} -2v_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 + I_0 + I_0 H(t-1) \\ 1 \end{pmatrix}$$

$$v_1' = -2v_1 + \frac{1}{2}\left[1 + I_0 + I_0 H(t-1)\right] + \frac{1}{2}$$

$$v_2' = \frac{1}{2}\left[1 + I_0 + I_0 H(t-1)\right] - 1/2$$

For $t < 1$:

$$v_1' = -2v_1 + 1 + \frac{I_0}{2}$$

$$v_2' = \frac{I_0}{2}$$

Solving v_1:

$$v_1' + 2v_1 = 1 + \frac{I_0}{2} \qquad \mu = e^{2t}$$

$$\int e^{2t} v_1 \frac{d}{dt} = \int \left[1 + \frac{I_0}{2}\right] e^{2t} \, dt$$

$$e^{2t} v_1 = \left[1 + \frac{I_0}{2}\right] \frac{e^{2t}}{2} + C$$

$$v_1 = \frac{1}{2}\left[1 + \frac{I_0}{2}\right] + Ce^{-2t}$$

Now v_2:

$$v_2' = \frac{I_0}{2} \Rightarrow v_2 = \frac{I_0 t}{2} + C_2$$

and we know $\vec{u} = U\vec{v}$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 + v_2 \\ v_1 - v_2 \end{pmatrix}$$

$$u_1 = Ce^{-2t} + \frac{1}{2} + \frac{I_0}{4} + \frac{I_0 t}{2} + C_2$$

$$u_2 = Ce^{-2t} + \frac{I_0}{4} + \frac{1}{2} - \frac{I_0 t}{2} + C_2$$

$u_1(0) = u_2(0) = 1/2$:

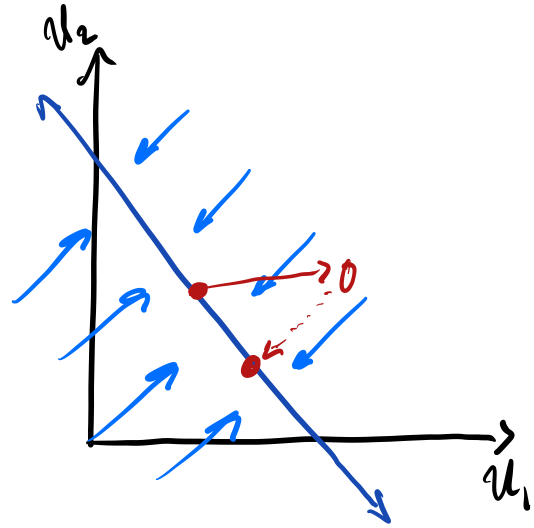
$$u_1 = \left(-\frac{I_0}{4}\right)e^{-2t} + \frac{1}{2} + \frac{I_0}{4} + \frac{I_0 t}{2}$$

$$u_2 = \left(-\frac{I_0}{4}\right)e^{-2t} + \frac{I_0}{4} + \frac{1}{2} - \frac{I_0 t}{2}$$

Now for $t > 1$, just plug $I_0 = 0$:

$$u_2 = u_1 = \frac{1}{2}$$

As the $\lim_{t\to\infty} u_1$ and $\lim_{t\to\infty} u_2$ we are only left with the constant $1/2$. Therefore u_1 and u_2 only depend on I_0 when $0<t<1$, after that u_1 and u_2 do not depend on I_0.



2c)

$$u_1' = -u_1 + \frac{1}{1+e^{-r(1-2u_2)}}$$

$$u_2' = -u_2 + \frac{1}{1+e^{-r(1-2u_1)}}$$

First lets show the nullclines:

$u_1'=0$, $u_2'=0$ so,

$$u_1 = \frac{1}{1+e^{-r(1-2u_2)}}$$

$$u_2 = \frac{1}{1+e^{-r(1-2u_1)}}$$

So when $\bar{u}_1 = \bar{u}_2 = \frac{1}{2}$:

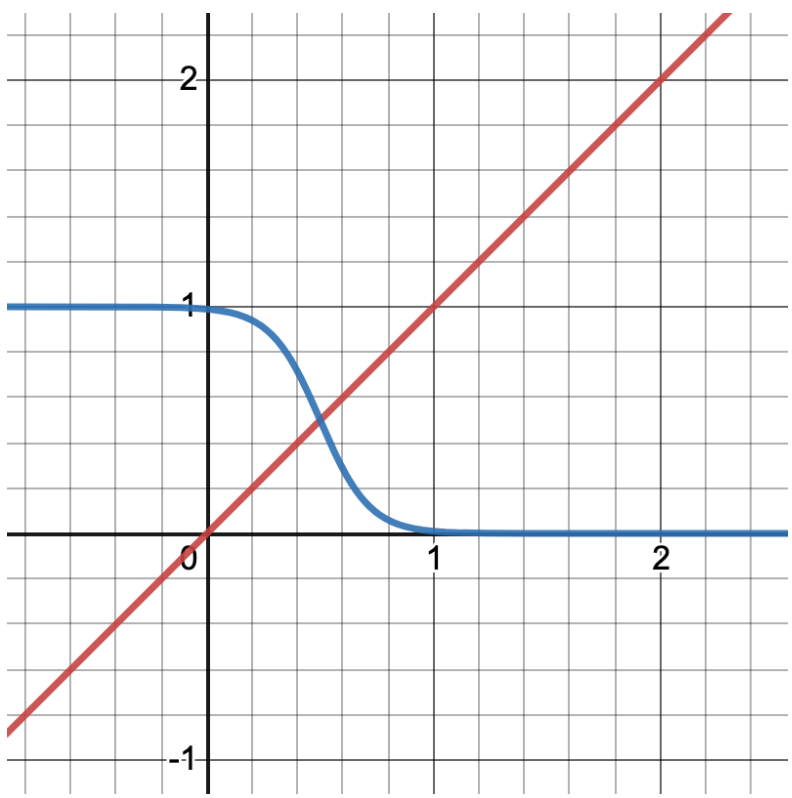
$$\bar{u}_1 = \frac{1}{2} = \frac{1}{1+e^{-r(1-1)}} = \frac{1}{2}$$

$$\bar{u}_2 = \frac{1}{2} = \frac{1}{1+e^{r(1-1)}} = \frac{1}{2}$$

Now $f(u) = u$ and $g(u) = \frac{1}{1+e^{-r(1-2u)}}$:

$$u = \frac{1}{1+e^{-r(1-2u)}}$$

If we plot this we get:



Now show symmetric fixed pt at $\bar{u}_1 = \bar{u}_2 = 1/2$.

$$J(u_1, u_2) = \begin{pmatrix} -1 & -\frac{2re^{-r(1-2u_2)}}{(1+e^{-r(1-2u_2)})^2} \\ -\frac{2re^{-r(1-2u_1)}}{(1+e^{-r(1-2u_1)})^2} & -1 \end{pmatrix}$$

$$J(1/2, 1/2) = \begin{pmatrix} -1 & -\frac{r}{2} \\ -\frac{r}{2} & -1 \end{pmatrix}$$

$$\Rightarrow \det(\lambda I - J) = \begin{pmatrix} \lambda+1 & \frac{r}{2} \\ \frac{r}{2} & \lambda+1 \end{pmatrix}$$

$$\Rightarrow \lambda^2 + 2\lambda + 1 - \frac{r^2}{4}$$

$$\Rightarrow \lambda_\pm = \frac{-1 \pm \sqrt{4 - 4(1 - \frac{r^2}{4})}}{2}$$

$$\Rightarrow \quad \lambda = -1 \pm \frac{r}{2} \Rightarrow \lambda = -1 + \frac{r}{2}, \; -1 - \frac{r}{2}$$

So for $+\lambda$, when $r < 2$ it will be stable.

$-\lambda$, it will always be stable. Hence, when $r \geq 2$ that is when our fixed pt is unstable and the network becomes competitive.