Kalman Smother

Problem

Consider linear dynamic model:

$$
\begin{aligned}
x_{t+1} &= Ax_t + w_t, \quad w_t \sim N(0, \Gamma) \\
y_t &= Cx_t + v_t, \quad v_t \sim N(0, \Sigma) \\
\theta &= \{A = 1, C = 1, \Gamma = 1, \Sigma = 49\} \\
t &= \{t \in \mathbb{Z} | 0 \leq t \leq T\}
\end{aligned}
$$

Solve $\hat{\gamma}(x_t) = p(x_t | y_{1:T})$ by observation $y_{1:T} = (y_1, \ldots, y_T)$ and parameter θ. Since $\hat{\gamma}(x_t) = N\left(x_t; \hat{\mu_t}, \hat{V_t}\right)$, find $\hat{\mu_t}, \hat{V_t}$.

Algorithm

$\hat{\mu_t}, \hat{V_t}$ is found by the iteration $(t = T, T-1, T-2, \ldots)$:

$$
\begin{aligned}
J_t &= V_t A^T (P_t)^{-1} \\
\hat{\mu_t} &= \mu_t + J_t \left(\hat{\mu_{t+1}} - A\mu_t\right) \\
\hat{V_t} &= V_t + J_t \left(\hat{V_{t+1}} - P_t\right) J_t^T
\end{aligned}
$$

When there are not missing values in the observation series,

$$
\begin{aligned}
\hat{\mu_t} &= A\mu_{t-1} \\
P_{t-1} &= AV_{t-1}A^T + \Gamma \\
K_t &= P_{t-1}C^T \left(CP_{t-1}C^T + \Sigma\right)^{-1} \\
\mu_t &= \hat{\mu_t} + K_t \left(y_t - C\hat{\mu_t}\right) \\
V_t &= \left(I - K_t C\right) P_{t-1}
\end{aligned}
\tag{1}
$$

When there are missing values in the observation series,

$$
\begin{aligned}
P_{t-1} &= AV_{t-1}A^T + \Gamma \\
\mu_t &= A\mu_{t-1} \\
V_t &= AV_{t-1}A^T + \Gamma.
\end{aligned}
$$

Result