Mechanical Control Systems (ME 4473)

Recitation - 5

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1. **Agenda:**
    - Revision(System Representation, Block Diagram, State Space)

2. **Things we learnt so far is representation of system using:**
    a. **Differential Equations**
        i. **Higher order coupled differential equations**
        ii. **First order system of differential equations**
    b. **Block Diagram representation**
    c. **Transfer Function representation**

Modeling using Differential Equations	
Pro	**Cons**
Precise	Tells no info about the system

Analogous Systems: The systems that can be modeled by the same differential equations

Modeling using Block Diagrams	
Pro	**Cons**
Easy to study the performance of individual subsystem and overall system	Does not provide information about the physical construction of the system & the source of energy

Modeling using Transfer Functions	
Pro	**Cons**
The whole system behavior is encapsulated in the form of poles and zeros	Can only be applied to LTI systems

3. **Write the matlab keywords to evaluate:**
    a. **Transfer function:** A = tf(num, den)
    b. **Step Response:** step(A)
    c. **Closed Loop Transfer function:** feedback(A,B)

Problem-1: For a given transfer function:

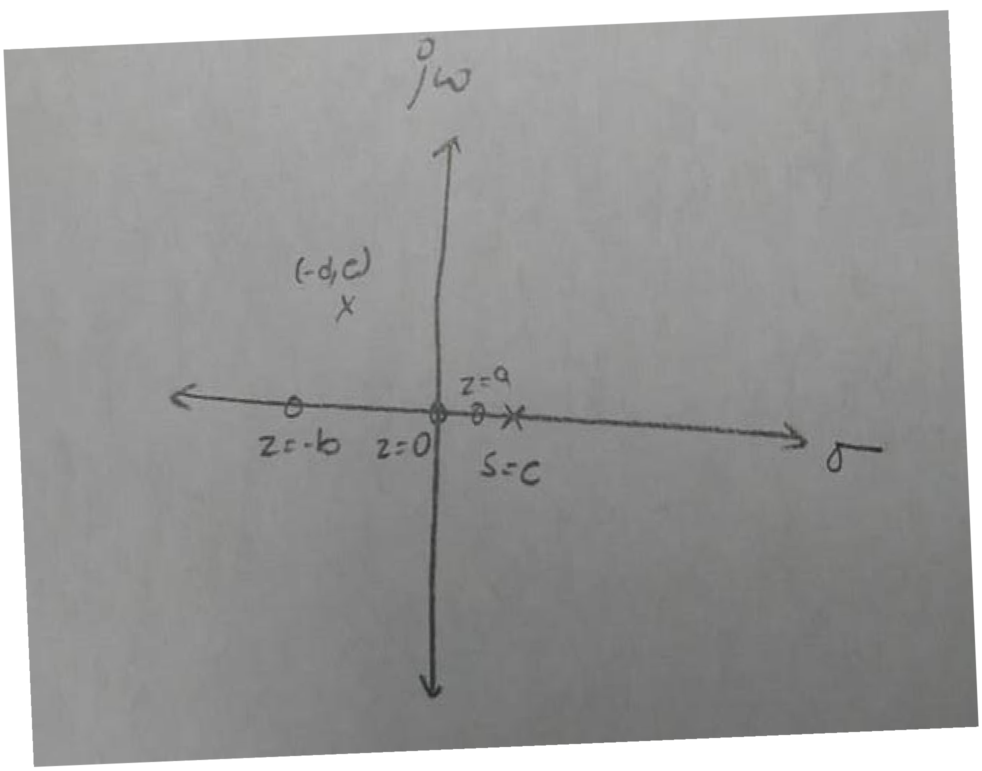
$$G(s) = \frac{s(s-a)(s+b)}{(s+c)(s+d-ej)}$$

where, a, b, c, d, e <0

a. What are the poles of the transfer function?
s = -c, s = -d+ej

b. What are the zeros of the transfer function?
z = 0, z = a, z = -b

c. Represent the poles and zeros in the σ vs. jw graph.

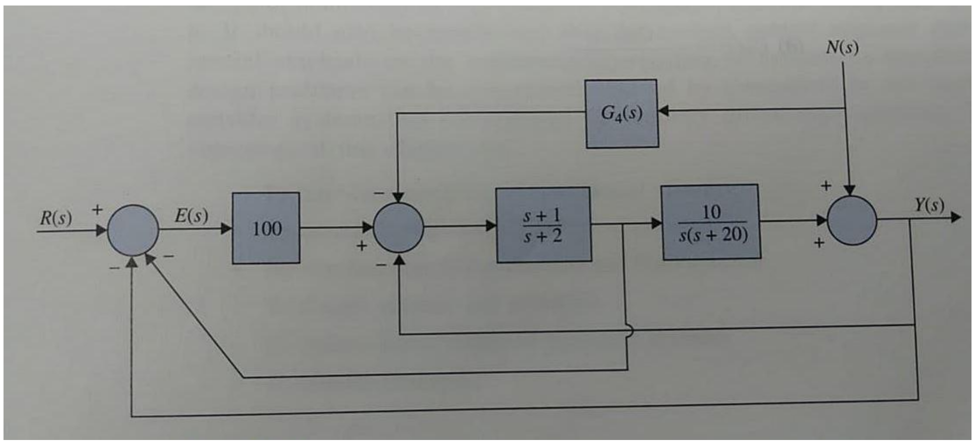


This plot has a logic error. Consult video for the accurate plot

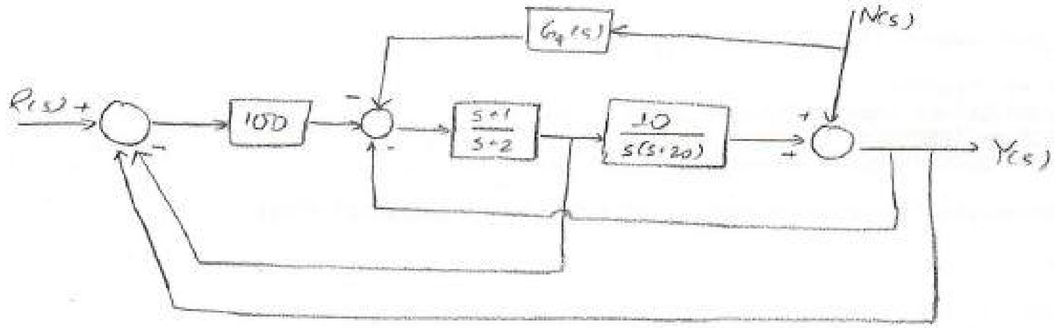
d. Is the system stable?

No poles lie in the right half part of the σ vs. jw graph

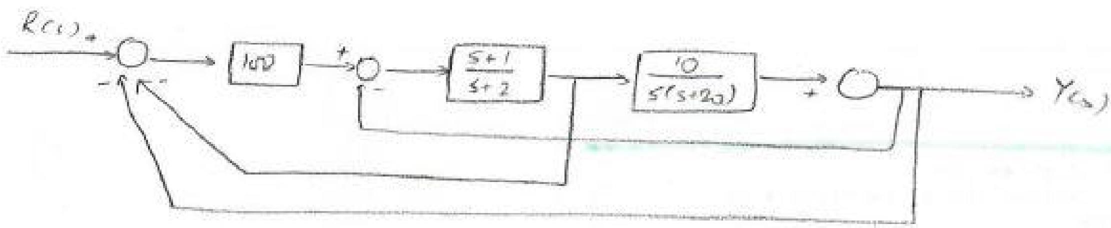
Problem 2: The block diagram below represents a Feedback system with noise. Transfer function G4(s) is for the reduction of effect of Noise.

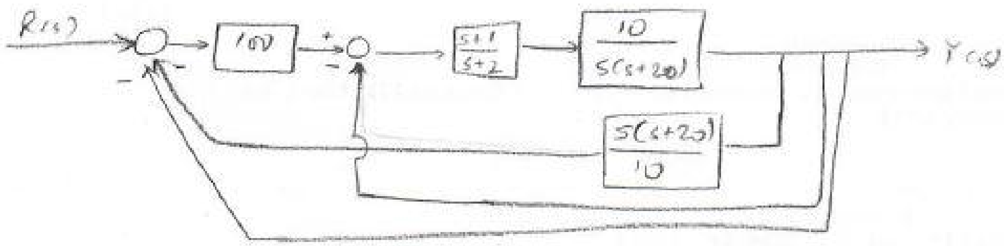


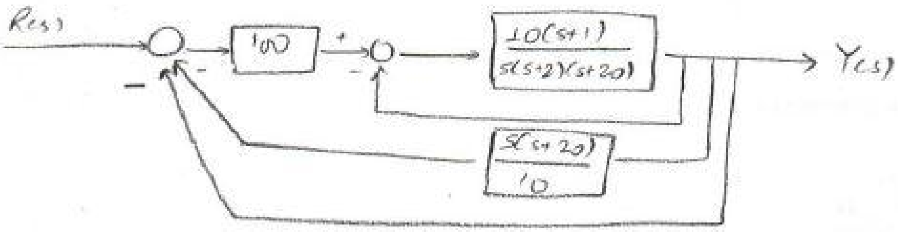
a. Evaluate the transfer function Y(s)/R(s) when N(s) = 0

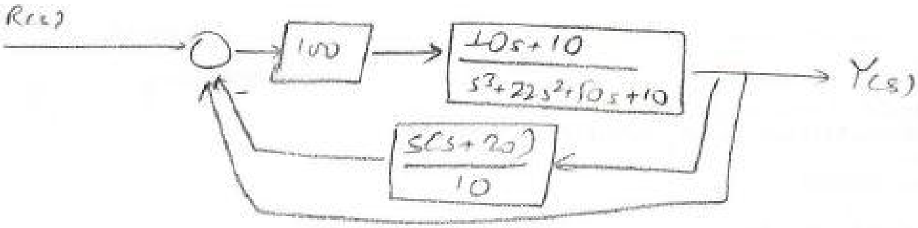


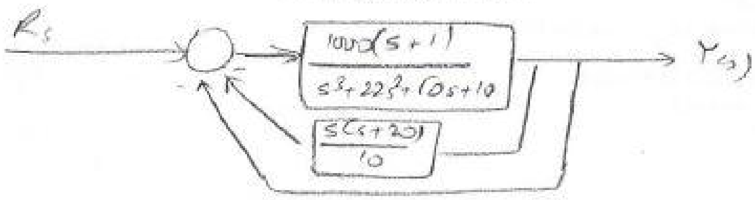
$$\left.\frac{Y(s)}{R(s)}\right|_{N=0} \qquad G_4(s)=0$$

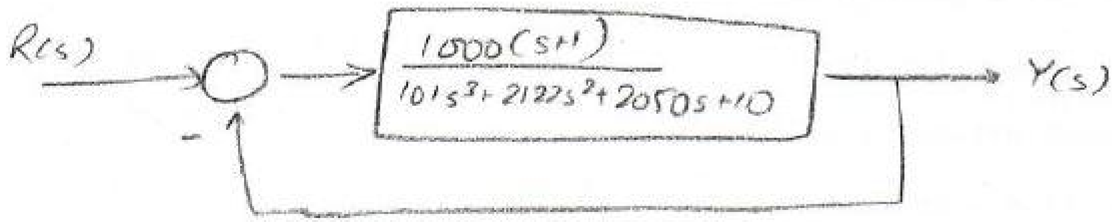










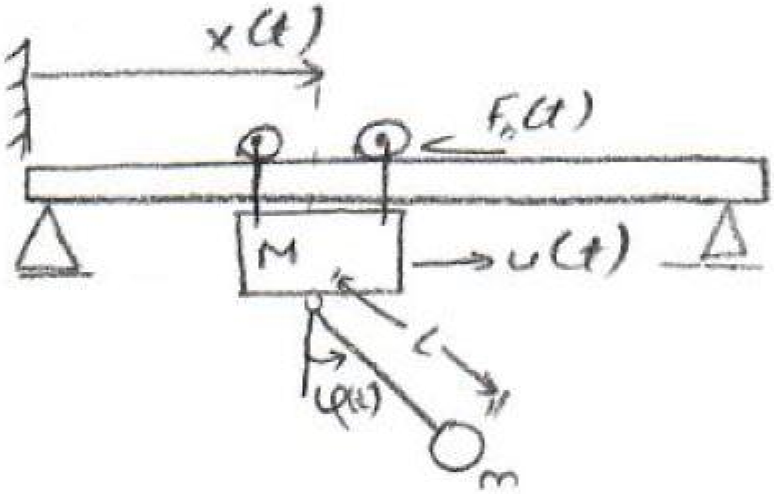


$$\frac{Y(s)}{R(s)} = \frac{\dfrac{1000(s+1)}{101s^3+2122s^2+2050s+10}}{1+\dfrac{1000(s+1)}{101s^3+2122s^2+2050s+10}}$$

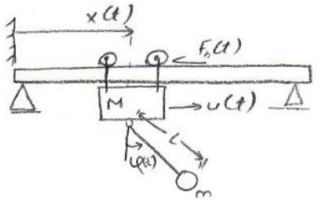
$$= \frac{1000(s+1)}{101s^3+2122s^2+2050s+10+1000s+1000}$$

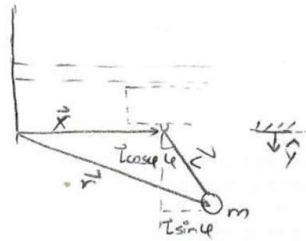
$$\boxed{\left.\frac{Y(s)}{R(s)}\right|_{N=0} = \frac{1000(s+1)}{101s^3+2122s^2+3050s+1010}}$$

Problem-3: Consider the hanging crane structure in the fig below. The mass of the cart is M, the mass of the payload is m, massless rigid connector has length 'L' and the friction is modeled as $F_b(t) = b\dot{x}$ where x(t) is the distance travelled by the cart.



a. **Write the equations of the motion describing the motion of the cart and the payload.**



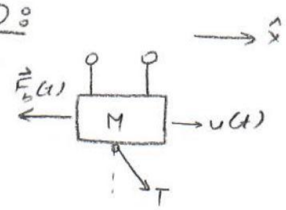


$$\vec{x}_p = \vec{x} + \vec{l}\sin\varphi$$
$$y_p = -l\cos\varphi$$

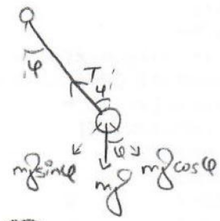
$$\dot{x}_p = \dot{x} + l\dot{\varphi}\cos\varphi$$
$$\dot{y}_p = l\dot{\varphi}\sin\varphi$$

$$\ddot{x}_p = \ddot{x} + l(\ddot{\varphi}\cos\varphi - (\dot{\varphi})^2\sin\varphi)$$
$$\ddot{y}_p = l(\ddot{\varphi}\sin\varphi + (\dot{\varphi})^2\cos\varphi)$$

FBO:



(A)



(B)

(A) $\sum F_x = m\ddot{x}$

$$\vec{u}(t) - \vec{T}\sin\varphi - \vec{F}_b(t) = M\ddot{\vec{x}}$$

$$\vec{T}\sin\varphi = \vec{u}(t) - M\ddot{\vec{x}} - \vec{F}_b(t) \quad \text{— (1)}$$

(B) $\sum F_{x_p} = m\ddot{x}_p$

$$-\vec{T}\sin\varphi = m(\ddot{\vec{x}} + l(\ddot{\varphi}\cos\varphi - (\dot{\varphi})^2\sin\varphi)) \quad \text{— (2)}$$

$\sum F_{y_p} = m\ddot{y}_p$

$$-\vec{T}\cos\varphi + mg = ml(\ddot{\varphi}\sin\varphi + (\dot{\varphi})^2\cos\varphi) \quad \text{— (3)}$$

Now, Eliminate $\vec{T}$ from eq^ns,

I. Substitute $\vec{T}\sin\varphi$ in ② from ①,

$$\vec{u}(t) - M\ddot{\vec{x}} - \vec{F}_b(t) = m\ddot{x} + mL\ddot{\varphi}\cos\varphi - mL(\dot{\varphi})^2\sin\varphi$$

$$(M+m)\ddot{x} + mL\ddot{\varphi}\cos\varphi - mL(\dot{\varphi})^2\sin\varphi + F_b(t) = u(t) \text{ — } ④$$

II Multiply ② by $\cos\varphi$ & ③ by $\sin\varphi$ & Add ②&③,

$$-\cancel{T\cos\varphi\sin\varphi} + mg\sin\varphi = mL\ddot{\varphi}\sin^2\varphi + \cancel{mL(\dot{\varphi})^2\cos\varphi\sin\varphi}$$

$$(+)\quad \cancel{T\sin\varphi\cos\varphi} \quad = m\ddot{x}\cos\varphi + mL\ddot{\varphi}\cos^2\varphi - \cancel{mL(\dot{\varphi})^2\sin\varphi\cos\varphi}$$

$$mg\sin\varphi = mL\ddot{\varphi}\sin^2\varphi + mL\ddot{\varphi}\cos^2\varphi + m\ddot{x}\cos\varphi$$

$$mg\sin\varphi = mL\ddot{\varphi} + m\ddot{x}\cos\varphi$$

$$\therefore\ mL\ddot{\varphi} + mg\sin\varphi + m\ddot{x}\cos\varphi = 0 \text{ — } ⑤$$

$$\boxed{\begin{aligned} &(M+m)\ddot{\vec{x}} + mL\ddot{\varphi}\cos\varphi - mL(\dot{\varphi})^2\sin\varphi + b\dot{\vec{x}}(t) = \vec{u}(t) \\ &mL\ddot{\varphi} + mg\sin\varphi + m\ddot{x}\cos\varphi = 0 \end{aligned}}$$

b. Linearize the system about small angles around stable equilibrium of the pendulum.

Linearize about $\varphi \approx 0$,

$\cos\varphi = 1$

$\sin\varphi = \varphi$

$\dot{\varphi} = 0$

$$\boxed{\begin{aligned} &(M+m)\ddot{\vec{x}} + mL\ddot{\varphi} + b\dot{\vec{x}} = \vec{u} \quad \text{— } ⑥ \\ &mL\ddot{\varphi} + mg\varphi + m\ddot{\vec{x}} = 0 \quad \text{— } ⑦ \end{aligned}}$$

c. **Decouple the linearized coupled differential equation.**

$$m\ddot{x} = -mL\ddot{\varphi} - mg\varphi$$

$$\ddot{x} = -L\ddot{\varphi} - g\varphi \quad \text{—} \quad (8)$$

<u>Substitute (8) in (6),</u>

$$(M+m)(-L\ddot{\varphi} - g\varphi) + mL\ddot{\varphi} + b\dot{x} = \vec{u}$$

$$-(M+m)L\ddot{\varphi} - (M+m)g\varphi + mL\ddot{\varphi} + b\dot{x} = \vec{u}$$

$$-ML\ddot{\varphi} - (M+m)g\varphi + b\dot{x} = \vec{u}$$

$$-ML\ddot{\varphi} = (M+m)g\varphi - b\dot{x} + \vec{u}$$

$$\ddot{\varphi} = \frac{(M+m)}{(-ML)}g\varphi + \frac{b}{ML}\dot{x} + \frac{1}{(-ML)}\vec{u} \quad \text{—} \quad (9)$$

<u>Substitute (9) in (8),</u>

$$\ddot{x} = -L\left(\frac{M+m}{-ML}g\right)\varphi - L\frac{b}{ML}\dot{x} - L\frac{1}{(-ML)}\vec{u}$$

$$\ddot{x} = \left(\frac{M+m}{M}g\right)\varphi - \frac{b}{M}\dot{x} + \frac{1}{M}\vec{u} \quad \text{———} \quad (10)$$

d. **Write the state space representation of the system.**

For eq^n ⑨ & ⑩, States are,

$$X_1 = \vec{x} \quad X_2 = \dot{\vec{x}} \quad X_3 = \vec{\varphi} \quad X_4 = \dot{\vec{\varphi}}$$

$$\dot{X}_1 = X_2$$

$$\dot{X}_2 = \ddot{\vec{x}} = \left(\frac{M+m}{M} g\right) X_3 - \frac{b}{M} X_2 + \frac{1}{M} \vec{u}$$

$$\dot{X}_3 = X_4$$

$$\dot{X}_4 = \ddot{\vec{\varphi}} = \left(\frac{(M+m)g}{-ML}\right) X_3 + \frac{b}{ML} X_2 + \frac{1}{(-ML)} \vec{u}$$

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{b}{M} & \frac{(M+m)g}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{b}{ML} & \frac{(M+m)g}{-ML} & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \left(\frac{1}{M}\right) \\ 0 \\ \left(\frac{1}{-ML}\right) \end{bmatrix} \vec{u}$$

Bibliography:

- Modern Control Systems, 13th ed
- Automatic Control Systems, 9th ed