Mechanical Control Systems (ME 4473)

Recitation - 6

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(Special thanks to Dr Schoen for providing problems for this worksheet)

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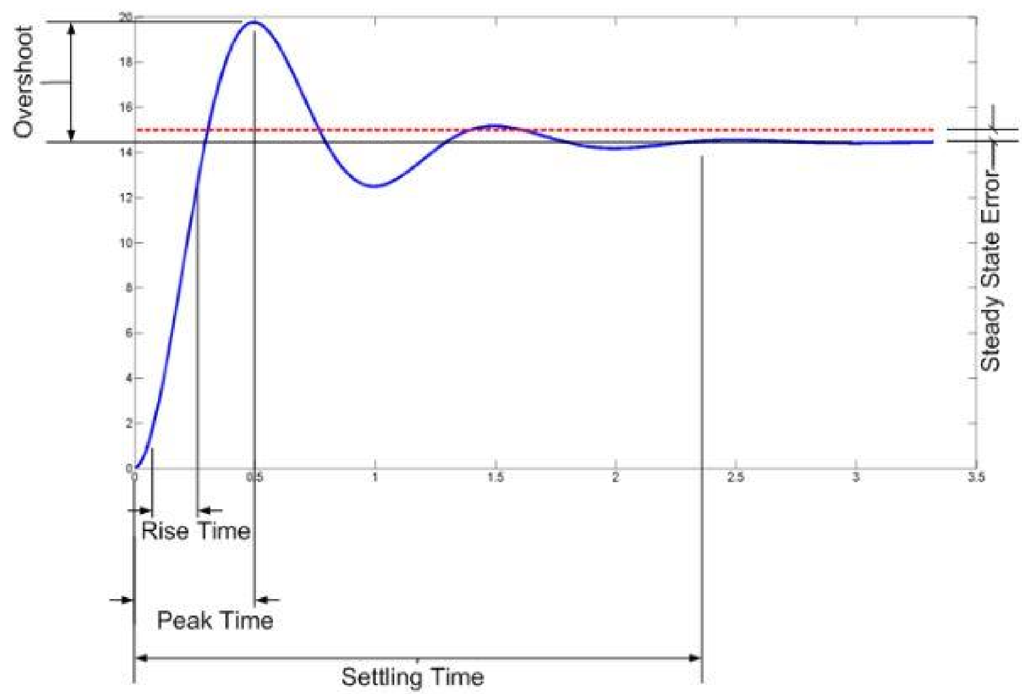
1. Agenda:

- Revision
- Problems(Overshoot, Gain, Response)

2. Questions:

- **Given the definitions, label the graph of the step response:**

  Rise Time - the time taken for the output to go from 10% to 90% of the final value.
  Peak Time - the time taken for the output to reach its maximum value.•
  Overshoot - (max value−final value)/final value×100.
  Settling Time - The time taken for the signal to be bounded to within a tolerance of x% of the steady state value.
  Steady State Error- The difference between the input step value (dashed line) and the final value



- **For a step response of a 2nd order system, what are the equations to evaluate:**
  - **Overshoot**

$$\boxed{P.O. = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}}}$$

- Settling Time(response 2% of final value)

$$e^{-\zeta\omega_n T_s} < 0.02$$

$$\boxed{T_s = \frac{4}{\zeta\omega_n}}$$

- **Peak Time**

$$\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}$$

- **Transfer function of a 2nd Order system:**

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- **Final Value Theorem**

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

- **Rise Time**

$$\boxed{T_{r1} = \frac{2.16\zeta + 0.60}{\omega_n},}$$

- **Fill in the blanks:**
  - A zero in the LHP increases the overshoot if the zero is within a factor of **4** of the real part of the complex pole
  - A zero in the RHP **decreases** the overshoot and may cause an initial undershoot (= response starts out in the wrong direction)
  - This information is important because designing a dynamic controller for a system is equivalent to adding poles to the system.

3. Problems:

A. **Consider a unit feedback system as shown in Figure 1. Specify the gain and pole location of the compensator so that the overall closed-loop response to a unit step input has an overshoot of no more than 25%, and a 1% settling time of no more than 0.1 seconds. Compute first by hand and show all steps, then use Matlab™ to verify your results.**

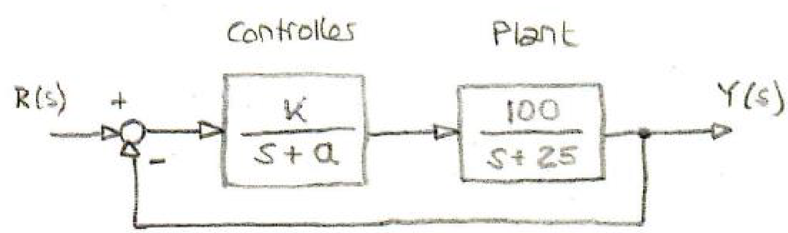


Figure 1: Setup for Problem No. A

I. What is the closed loop Transfer function of this system?

I. $TF = \dfrac{(100K)/(s+a)(s+25)}{1 + \dfrac{(100k)}{(s+a)(s+25)}}$

$TF_2 = \dfrac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$

$= \dfrac{100k}{(s+a)(s+25) + 100K}$

$\therefore TF = \dfrac{100K}{s^2 + (25+a)s + (100K + 25a)}$

II. **Using the equations of the overshoot & settling time, Find the unknown parameters for the general equation for the transfer function for this system.**

II. $P.O = 25 = 100\, e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$

$\ln\left(\frac{1}{4}\right) = \dfrac{-\pi\xi}{\sqrt{1-\xi^2}}$

$1.92 = \dfrac{\pi^2\xi^2}{1-\xi^2}$

$1.92 - 1.92\xi^2 = \pi^2\xi^2$

$\therefore \xi = \pm 0.404$

$\boxed{\therefore |\xi| \geq 0.404}$

For settling time of 1%

$e^{-\xi\omega_n T_s} \leq 0.01$

$-\xi\omega_n T_s \approx -4.61$

$\omega_n \approx \dfrac{4.61}{0.404 \times 0.1}$

$\boxed{\therefore \omega_n \approx 114.4 \text{ rad s}^{-1}}$

III. **Compare the transfer function of this system with the general transfer function to find the values of the variables.**

III.

$$\frac{100K}{s^2+(25+a)s+(100K+25a)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$2\zeta\omega_n = 25+a \qquad \omega_n^2 = (100K+25a) = (100K+25\times 67.4) = (114.4\,rads^{-1})^2$$

$$a = 2\zeta\omega_n - 25$$

$$= 2\times 0.404\times 114.4\,rads^{-1} - 25$$

$$\boxed{\therefore a = 67.4\,rads^{-1}} \qquad \boxed{\therefore K = 114}$$

IV. Verify the results using Simulink

@Simulink

B. Consider an open-loop transfer function of a unit feedback system as

$$G(s) = \frac{K}{s(s+2)}$$

Suppose the desired system response to a step input is specified as a peak time of 1 second and overshoot of 5%.

I. Determine whether both specifications can be met simultaneously by selecting the right value of K.

A. Closed Loop Transfer Function

A.

$$TF = \frac{G(s)}{1+G(s)} = \frac{\frac{K}{s(s+2)}}{1+\frac{K}{s(s+2)}} = \frac{K}{s(s+2)+K}$$

$$\boxed{\therefore TF = \frac{K}{s^2+2s+K}}$$

B. Equate the coefficients of like polynomial terms

B.

$$\frac{K}{s^2+2s+K} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$2\zeta\omega_n \cancel{s} = 2 \qquad \omega_n^2 = K$$

$$\zeta\omega_n = 1 \qquad \boxed{\therefore \omega_n = \sqrt{K}}$$

$$\boxed{\therefore \zeta = \frac{1}{\sqrt{K}}}$$

C. Evaluate the parameters from Overshoot and peak time equations

C.

$$0.05 = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$\ln(0.05) = \frac{-\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$(\ln(0.05))^2 = \frac{+\pi^2\zeta^2}{1-\zeta^2}$$

$$8.97 - 8.97\zeta^2 = \pi^2\zeta^2$$

$$\cancel{\text{So}}\ \zeta^2 = \frac{8.97}{8.97+\pi^2}$$

$$\boxed{\therefore \zeta = 0.69}$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 1\,sec$$

$$\frac{\pi^2}{\omega_n^2(1-\zeta^2)} = 1\,sec$$

$$\cancel{\omega_n = \sqrt{\pi}}$$

$$\omega_n = \frac{\pi}{\sqrt{1-\zeta^2}}$$

$$\omega_n = \frac{\pi}{\sqrt{1-0.69^2}}$$

$$\boxed{\therefore \omega_n = 4.34}$$

D. Result:

D. $\xi \cdot \omega_n = 0.69 \times 4.34 = 2.99 \neq 1$

Hence, desired input specifications can't be achieved by varying K.

II. **Now let's design for some compromise to obtain a set of parameters for gain, overshoot and peak time that is near the desired set of parameters.**

(Now we need to adjust our overshoot and peak times such that we get an achievable parameters with the K around these values)

A. **Introduce a relaxation factor for the overshoot and for the actual location of closed loop roots and evaluate the parameters**

A.

$$OS = r \times 0.05 = e^{\frac{-\pi \xi}{\sqrt{1-\xi^2}}} \qquad t_p = r \times 1\,sec = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\xi^2}}$$

$$\ln(r \times 0.05) = \frac{-\pi \frac{1}{\sqrt{K}}}{\sqrt{1-\frac{1}{K}}} \qquad r = \frac{\pi}{\sqrt{K-1}} \quad \text{— (ii)}$$

B. **Substitute the conditions from the closed loop transfer function of the system to find K & r**

B.

$$\ln(r \times 0.05) = \frac{-\pi}{\sqrt{K-1}} \quad \text{— (i)}$$

(i) & (ii),

$$\ln(r \times 0.05) = -r$$

$$\boxed{\therefore r = 2.21}$$

(ii),

$$r^2 = \frac{\pi^2}{K-1}$$

$$K = \frac{\pi^2}{r^2} + 1 = \frac{\pi^2}{(2.21)^2} + 1$$

$$\boxed{\therefore K = 3.02}$$

III. **Use Matlab to validate part a.**

@Matlab™

C. The equations of motion for the DC motor shown in the Figure below are given as:

$$J_m\ddot{\theta}_m + \left(b + \frac{K_t K_e}{R_a}\right)\dot{\theta}_m = \frac{K_t}{R_a} v_a.$$

$$J_m = 0.01 \text{ kg.m}^2,$$

$$b = 0.001 \text{ N.m.sec},$$

$$K_e = 0.02 \text{ V.sec},$$

$$K_t = 0.02 \text{ N.m/A},$$

$$R_a = 10\ \Omega$$

(a) Find the transfer function between the applied voltage v_a and the motor speed $\dot{\theta}_m$.

(b) What is the steady-state speed of the motor after a voltage $v_a = 10$ V has been applied?

(c) Find the transfer function between the applied voltage v_a and the shaft angle θ_m.

(d) Suppose feedback is added to the system in part (c) so that it becomes a position servo device such that the applied voltage is given by

$$v_a = K(\theta_r - \theta_m),$$

where K is the feedback gain. Find the transfer function between θ_r and θ_m.

(e) What is the maximum value of K that can be used if an overshoot $M_p < 20\%$ is desired?

(f) What values of K will provide a rise time of less than 4 sec? (Ignore the M_p constraint.)

(g) Use MATLAB to plot the step response of the position servo system for values of the gain K = 0.5, 1, and 2. Find the overshoot and rise time for each of the three step responses by examining your plots. Are the plots consistent with your calculations in parts (e) and (f)?

a) Transfer Function(Voltage and motor speed)

(a)

$$J_m \ddot{\theta}_m + \left(b + \frac{K_t K_e}{R_a}\right)\dot{\theta}_m = \frac{K_t}{R_a} V_a$$

$$J_m \Theta_m(s)\, s^2 + \left(b + \frac{K_t K_e}{R_a}\right) \Theta_m(s) . s = \frac{K_t}{R_a} V_a(s)$$

$$\frac{s\,\Theta_m(s)}{V_a(s)} = \frac{\frac{K_t}{R_a J_m}}{s + \frac{b}{J_m} + \frac{K_t K_e}{R_a J_m}}$$

$$\boxed{\frac{\dot{\Theta}_m(s)}{V_a(s)} = \frac{0.2}{s+0.104}}$$

b) Steady-state speed

(b) FVT,

$$\lim_{t\to\infty} \dot{\theta}_m(t) = \lim_{s\to 0} s\,\dot{\Theta}_m(s) \qquad V_a(s) = \frac{10}{s}$$

$$= \lim_{s\to 0} \frac{s \cdot 0.2 \cdot V_a(s)}{s + 0.104}$$

$$= \lim_{s\to 0} \frac{\cancel{s} \cdot 0.2 \cdot 10}{\cancel{s}(s+0.104)}$$

$$= \frac{(0.2)(10)}{0.104}$$

$$\boxed{\therefore \lim_{t\to\infty} \dot{\theta}_m(t) = 19.23}$$

c) Transfer Function(Voltage and shaft angle)

c) $$\frac{\dot{\Theta}_m(s)}{V_a(s)} = \frac{0.2}{s+0.104}$$

$$\frac{s\Theta_m(s)}{V_a(s)} = \frac{0.2}{s+0.104}$$

$$\therefore \boxed{\frac{\Theta_m(s)}{V_a(s)} = \frac{0.2}{s(s+0.104)}}$$

d) Transfer function (Θ_r and Θ_m)

(d) $$\theta_m(s) = \frac{0.2}{s(s+0.104)} K(\theta_r - \theta_m)$$

$$= \frac{0.2K\theta_r - 0.2K\theta_m}{s^2+0.104s}$$

$$\theta_m(s)\left(1 + \frac{0.2K}{s^2+0.104s}\right) = \frac{0.2K\,\theta_r(s)}{s^2+0.104s}$$

$$\theta_m(s)\left(\frac{(s^2+0.104s+0.2K)}{\cancel{s^2+0.104s}}\right) = \frac{0.2K\,\theta_r(s)}{\cancel{(s^2+0.104s)}}$$

$$\boxed{\therefore \frac{\theta_m(s)}{\theta_r(s)} = \frac{0.2K}{s^2+0.104s+0.2K}}$$

e) Value of K for Overshoot < 20%

(e)

$$20 = 100\, e^{\frac{-\pi \zeta}{\sqrt{1-\zeta^2}}}$$

$$\ln\left(\frac{1}{5}\right) = \frac{-\pi \zeta}{\sqrt{1-\zeta^2}}$$

$$1.6 = \frac{\pi \zeta}{\sqrt{1-\zeta^2}}$$

$$2.6 - 2.6\zeta^2 = \pi^2 \zeta^2$$

$$\zeta^2 = 0.21$$

$$\boxed{\therefore \zeta = 0.46}$$

$$T.F = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$2\zeta\omega_n = 0.104 \qquad \omega_n^2 = 0.2K$$

$$2 \times 0.46 \times \sqrt{0.2K} = 0.104$$

$$\sqrt{0.2K} = 0.113$$

$$K = 0.064$$

$$\boxed{K < 0.064}$$

f) Value of K for rise time less than 4 sec

(f) $t_r < 4\,sec$

Eqn 5.17,

$$T_r = \frac{2.16\zeta + 0.60}{\omega_n}$$

$$4_s = \frac{2.16 \times 0.404 + 0.60}{\omega_n}$$

$$\omega_n = 0.368$$

$$\omega_n^2 = 0.136$$

$$0.2K = 0.136$$

$$\boxed{\therefore K = 0.68}$$

$$\therefore k > 0.68$$

g) Step response plot
@Matlab