Mechanical Control Systems (ME 4473)

Recitation - 8

GTA: Shishir Khanal, khanshis@isu.edu

(Students are advised to have a calculator around while building Routh- Hurwitz table)

---

1. **Agenda:**
    - Revision
    - Problems(Routh Hurwitz Criterion)

2. **Questions:**

   Routh Hurwitz Criterion: The number of roots of q(s) with the positive real parts is equal to the number of changes in the sign of the first column of the Routh array.

   Necessary condition for stability: A system is stable iff all the elements in the first column of the Routh array are positive.

- **What is the meaning of:**
    - **Monic polynomial:** The polynomial that has the coefficient of the highest order term 1.
    - **Causal systems:** Systems that do not begin before t = 0. These systems are characterized by the transfer functions that have the highest order term of zeros less than or equal to the highest term of the poles. (i.e. number of poles more than or equal to that of zeros)

- **What are the steps to evaluate Routh-Hurwitz stability criterion?**

    1. Arrange the coefficients in two rows : Row n & Row n-1
    2. Evaluate Row n-2 by solving for b_i's (Look at Row n and Row n-1)
       $$b_i = (-)\,|A_1\ A_{i+1}|\ /\ A_1(1,1)$$
    3. Evaluate Row n-3 by solving for c_i's (Look at Row n-1 and Row n-2)
       $$c_i = (-)\,|B_1\ B_{i+1}|\ /\ B_1(1,1)$$
    4. If the elements in the 1st column are all positive, then all the roots are in the LHP. THe number of roots in the RHP equals the number of sigh changes in the first column.

Necessary condition: If there is a sign change in the first column, the system is essentially unstable

- **What are the special cases that arise in the rows and column elements during the evaluation of Routh Hurwitz criterion?**
    1. No zero in the first column
    2. Zero in the first column

Tactic: Put ε at indices where the values are 0

ε = is a really small positive number

3. Zeros in one row

Tactic: Switch the row with zeros with the coefficients of the first derivative of the auxiliary polynomial of the previous row

Q. How to construct an auxiliary polynomial?
For a row of interest, an auxiliary polynomial is constructed by starting at the power of s at the in the row of concern and using coefficients to form a polynomial that drops by the power of 2 till 0.

Routh array and stability:

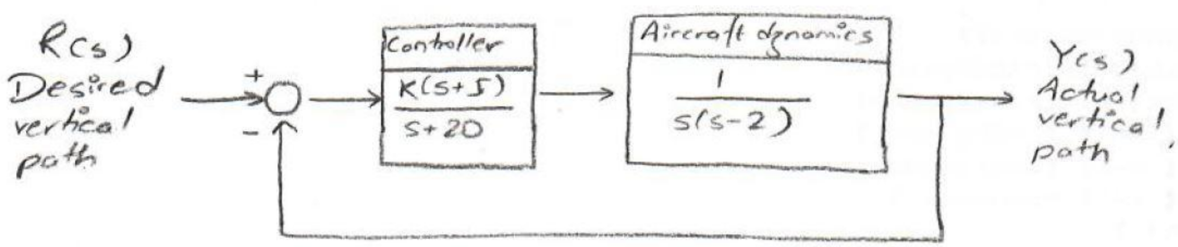
Roots on RHP = Total number of sign changes on first column
Roots on Imaginary Axis = Number of 0 rows
Roots on LHP = Remaining number of roots

Also,

Roots on RHP = Unstable poles
Roots on Imaginary Axis = Marginally Stable poles
Roots on LHP = Stable poles

3. **Problems:**

P6.19(Dorf) The goal of vertical takeoff and landing(VTOL) aircraft is to achieve operation from relatively small airports and yet operate as a normal aircraft as in a level flight. An aircraft taking off in a form similar to a rocket is inherently unstable. A control system using adjustable jets can control the vehicle, as shown in the picture below.



a. **Determine the range of the gain for which the system is stable.**

   1. **Find the equivalent closed loop transfer function**

$$G(s) = \frac{k(s+5)}{(s+20)} \times \frac{1}{s(s-2)}$$

$$= \frac{k(s+5)}{(s+20)(s^2-2s)} = \frac{k(s+5)}{s^3-2s^2+20s^2-40s} = \frac{k(s+5)}{s^3+18s^2-40s}$$

$$CLTF = \frac{\left(\frac{K(s+5)}{s^3+18s^2-40s}\right)}{\left(1+\frac{K(s+5)}{s^3+18s^2-40s}\right)} = \frac{k(s+5)}{s^3+18s^2-40s+Ks+5k} = \frac{k(s+5)}{s^3+18s^2+(K-40)s+5K}$$

2. **Write down the characteristic polynomial of the system from the equivalent closed loop transfer function.**

Characteristic Equation:

$$s^3 + 18s^2 + (K-40)s + 5K = 0$$

3. **Write down the Routh array table using the characteristic polynomial.**

Routh Array:

s^3	1	$(K-40)$
s^2	18	$5K$
s^1	$-\left(\frac{5K-18(K-40)}{18}\right) = \frac{13K-720}{18}$	0
s^0	$\frac{5K\left(\frac{13K-720}{18}\right)-0}{\left(\frac{13K-720}{18}\right)} = 5K$	

4. **Using the Routh-Hurwitz gain condition for stability, find the variable terms in the first column of the Routh array to find the range of K that will make the system stable.**

Stability:

$$\frac{13K-720}{18} > 0 \rightarrow K > \frac{720}{13}$$

$$\therefore K > 55.4$$

$$5K > 0$$

$$\therefore K > 0$$

b. **Determine the gain K for which the system is marginally stable and the roots of the characteristic equation for this value of K.**

1. **According to the Routh-Hurwitz stability condition, what kind of pattern in the Routh array table corresponds to the marginally stable poles?**
   All elements of a row are zero

2. **What could be the candidate value of the K that could bring such a pattern in the Routh array table?**
   K = 720/13

3. **Construct the Routh array for this particular value of K.**

(b) $K = \frac{720}{13}$

s^3	1	$\frac{200}{13}$	0
s^2	18	$\frac{3600}{13}$	0
(s^1	0	0)	← Corresponds to a pole at imaginary axis. Hence, Marginally stable
s^1	36	0	
s^0	276		

s

s^3

(Auxiliary polynomial: 18s^2 +(3600/13))

4. **How many poles are there in the RHP, Imaginary axis and RHP based on the Routh array?**
   RHP = No of sign changes in first column = 0
   Imaginary Poles = 2* Number of 0 element rows = 2
   LHP = Highest order of char. equation - RH poles - Imaginary poles
   LHP = 3-0-2 = 1

5. **Evaluate the roots of the characteristic polynomial using the candidate value of K.**

$$s^3 + 18s^2 + \frac{200}{13}s + \frac{3600}{13} = 0$$

$$s^3 + 18s^2 + 200s + 3600 = 0$$

$$s_1 = -18 \quad s_{2,3} = \pm 14.14j$$

(*Instructions: The characteristic equation will be provided during the Recitation session. There is no multiple choice quiz for this problem. Please solve this problem and submit your work as a pdf in moodle*)

Recitation Quiz: For the characteristic polynomial provided below,

I. **Build Routh Hurwitz array and provide the Sign change information**
II. **Enumerate how many poles lie on LHP, RHP and imaginary axis and provide your judgement if the system is stable or unstable.**
III. **Compute the roots using Matlab and validate your judgement.**

(For last part: only include the roots output from Matlab and few lines for the justification in your submission(No snapshots of Matlab code is required)**)**

a(s) = $s^5 + 3s^4 + 2s^3 + 6s^2 + 6s + 9$

Part-I:

s^5	1	2	6
s^4	3	6	9
s^3	$\frac{6-6}{3} = 0$	$\frac{18-9}{3} = 3$	0
s^3	ε	3	0
s^2	$\frac{6\varepsilon-9}{\varepsilon} \approx -\frac{9}{\varepsilon}$	$\frac{9\varepsilon-0}{\varepsilon} = 9$	0
s^1	$\frac{-\frac{27}{\varepsilon} - 9\varepsilon}{-9/\varepsilon} \approx 3$	0	
s^0	$\frac{27}{3} = 9$		

← Replace 0 by ε

$$C_1 = \frac{6\varepsilon - 9}{\varepsilon} = -\frac{9}{\varepsilon}$$

since, ε is really small, we can write

$$C_2 = \frac{-\frac{27}{\varepsilon} - 9\varepsilon}{-9/\varepsilon} = \frac{-27 - 9\varepsilon^2}{-9} \approx 3$$

Sign changes in first column:

s^5	+
s^4	+
s^3	+
s^2	−
s^1	+
s^0	+

Number of sign changes = 2

Part-II:

Number of poles in RHP = 2

Number of poles in Imaginary Axis = 0

Number of poles in LHP = 5 - 2 - 0 = 3

Part-III:

```
>> char = [1 3 2 6 6 9]

char =

     1     3     2     6     6     9

>> roots(char)

ans =

  -2.9043 + 0.0000i
   0.6567 + 1.2881i
   0.6567 - 1.2881i
  -0.7046 + 0.9929i
  -0.7046 - 0.9929i
```

Bibliography:

- **Dorf, Modern Control Systems, 13th ed**
- **Franklin, Feedback Control of Dynamic Systems, 5th ed**

(If you would like to know more about VTOL plane systems: https://www.youtube.com/watch?v=4GfqB7P6uAE)