Mechanical Control Systems (ME 4473)

Recitation - 11

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1. **Agenda:**
    - Revision
    - Problems(Root Locus)

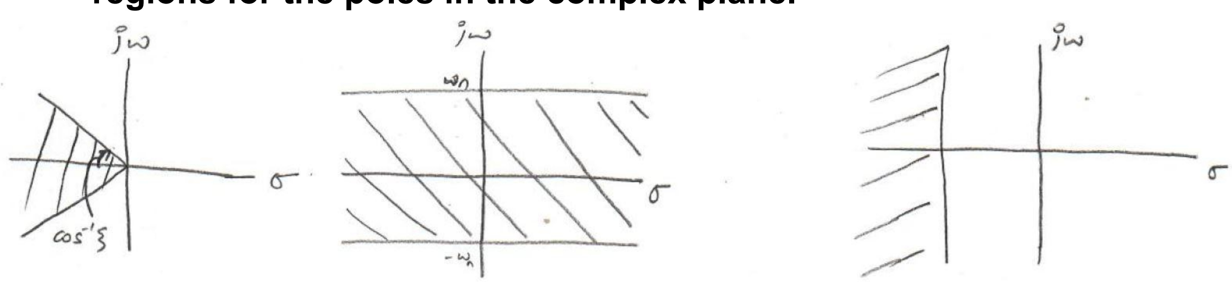
2. **Revision:**
    a. **Suppose you are a Controls Engineer assigned to work in designing an Automatic Controller for a plant. At what point in the design process would you go about evaluating the root locus?**

If the boss gives the system requirements in the form of damping ratio or natural frequency, the root locus analysis is crucial.

b. **Does the poles of a system change their location by themselves during a normal operation of the system?**

Unless there is a change in system parameters,(e.g. using a potentiometer to change the gain) the poles do not change their location by themselves on the complex plane.

c. **Your boss says that the natural frequency needs to be below w_n , damping ratio above ζ, and Time to decay the system modes to half T_d. For Each of these conditions, sketch the allowable regions for the poles in the complex plane.**



d. **What causes the poles to change their location?**

The unknown parameter 'K' can come up in the system transfer function as well as a controller parameter in the controller transfer function. Changing the value of the K essentially affects the actual location of poles in the transfer function. Root locus is basically the trajectory of the poles as we change the value of K.

e. **Why wouldn't a zero move?**

Zeros also move if the K parameter is embedded in the numerator polynomial of the loop transfer function.

f. **Fill in the blank:**

   i. **RL Magnitude Condition :** $|G_1(s)H_1(s)| = |1/K|$
   ii. **RL Phase Condition :** $\angle G_1(s)H_1(s) = (2K+1)\pi \quad ; K \geq 0$
   iii. **Angle of Departure:** The angle at which a locus leaves a complex pole in the s-plane.

g. **What are the steps to find the root locus of a control system?**

Step 1: Identify the poles of the loop transfer function
(These poles correspond to the points where K = 0)

Step 2: Identify the zeros of the loop-transfer function
(These zeros correspond to the points where K → ∞)

Step 3: Calculate the angles at which loci goes to inf

$\Theta_K = ((2K + 1)\ \pi)/(p - z)$; p,z => # of finite poles & zeros resp.
K = 0, 1, …, p-z-1

Step 4: Calculate the point along the real axis where the asymptotic line(s) intersects = center of gravity(c.g.)

c.g. = (∑poles - ∑zeros) / (p-z)
(Using this step & step 3, we can draw the line for asymptotes)

Step 5: On a given section(real axis), the root locus can be found in the section only if the total number of poles & zeros G(s)H(s) to the right of the section is odd (for K is positive)

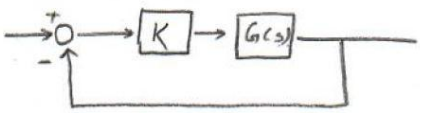
Step 6: Calculate the angle of departure from all complex poles and angles of arrival to all complex zeros

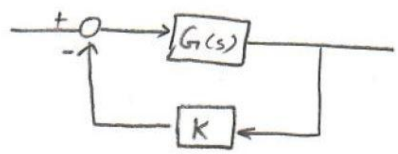
$$\angle G_1(s)H_1(s) = \sum_{i=1}^{m} \angle(s_1 + z_i) - \sum_{j=1}^{n} \angle(s_1 + p_j) = 2(K + 1)\pi$$

Step 7: Calculate the point where branches of the root locus crosses the imaginary axis (Routh Hurwitz for 1+GH = 0)
(Gives the max value of K)

Step 8: Calculate the break-away points along the real axis
$(d/ds)\,(G(s)H(s)) = 0$

h. **For the two systems provided below, would they have the same or different root locus?**





$$CLTF = \frac{K G(s)}{1 + K G(s)} \qquad CLTF = \frac{G(s)}{1 + K G(s)}$$

$$\boxed{1 + K G(s) = 0} \qquad \boxed{1 + k G(s) = 0}$$

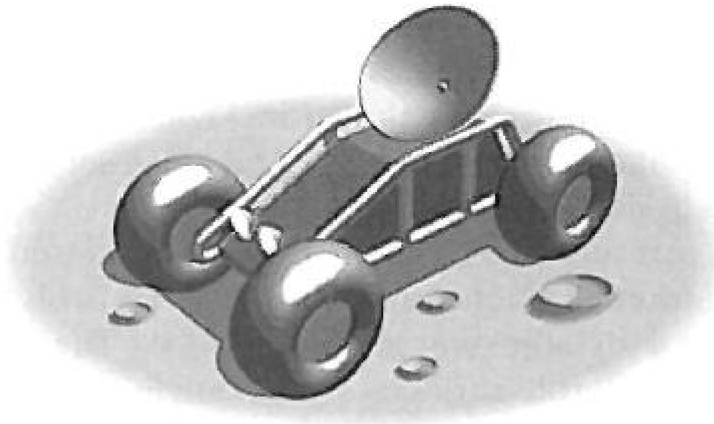
*Note:
- Both block diagram produce the same root locus
- Both are equivalent in terms of stability

i. **If the angle of departure of one of the poles of complex conjugate poles is a^0, what is the angle of arrival of another pole?**

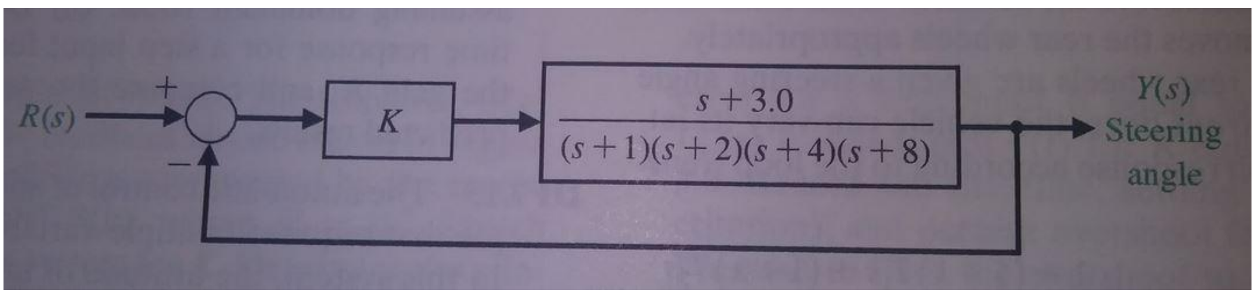
$$\Theta' = 180^0 - a^0$$

3. **Problems:**

D.P.7.12 A rover vehicle designed for use on other planets and moons is shown in the figure below:



The block diagram for steering control is shown below:



a. Sketch the root locus as K varies from 0 to 1000. Find the roots of K = 100, 300 and 600.

$$CLTF = \frac{\dfrac{k(s+3)}{(s+1)(s+2)(s+4)(s+8)}}{1 + \dfrac{k(s+3)}{(s+1)(s+2)(s+4)(s+8)}}$$

$$LTF = \frac{k(s+3)}{(s+1)(s+2)(s+4)(s+8)}$$

STEP 1:

$$(s+1)(s+2)(s+4)(s+8) = 0$$

$$\therefore \underline{\underline{s=-1 \; ; \; s=-2 \; ; s=-4 ; s=-8}}$$

STEP 2:

$$s+3=0$$

$$\therefore \underline{\underline{s=-3}}$$

STEP 3:

$$\theta_{k=0} = \frac{\pi}{3}$$

$$\theta_{k=1} = \pi$$

$$\theta_{k=2} = \frac{5}{3}\pi$$

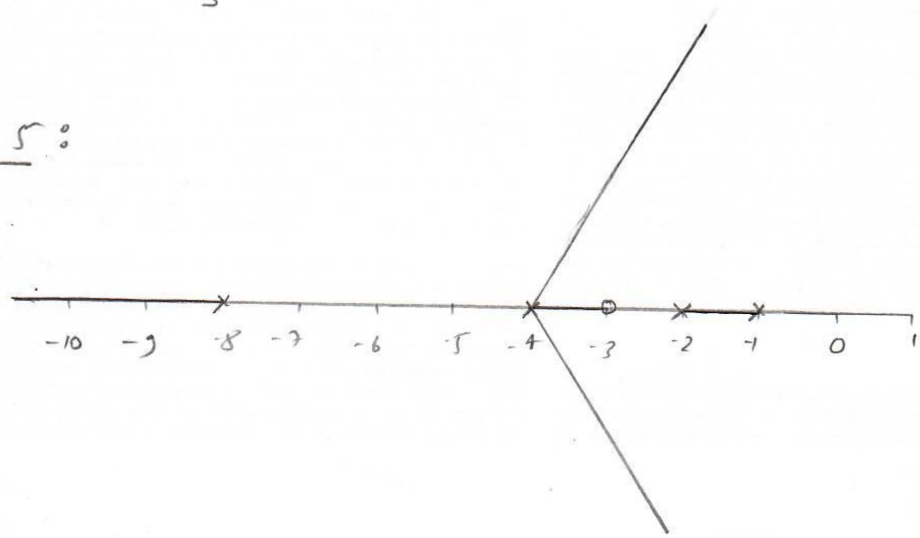
$$k = 4-1-1 = 2$$

$$P - Z = 4 - 1 = 3$$

STEP 4:

$$C\delta = \frac{(-1)+(-2)+(-4)+(-8)-(-3)}{3} = -4$$

STEP 5:



STEP 6: No Complex Conjugate Poles. Hence, N/A

STEP 7:

$$(s+1)(s+2)(s+4)(s+8) + K(s+3) = 0$$

$$(s^2+3s+2)(s^2+12s+32) + K(s+3) = 0$$

$$s^4 + 12s^3 + 32s^2 + 3s^3 + 36s^2 + 96s + 2s^2 + 24s + 64 + K(s+3) = 0$$

$$s^4 + 15s^3 + 70s^2 + 120s + 64 + Ks + 3K = 0$$

$$\therefore s^4 + 15s^3 + 70s^2 + (120+K)s + (64+3K) = 0$$

s^4:	1	70	$(64+3K)$
s^3:	15	$(120+K)$	0
s^2:	$\frac{1050-120-K}{15} = \frac{930-K}{15}$	$(64+3K)$	0
s^1:	$\dfrac{\frac{(930+K)(120+K)}{15} - 960 - 45K}{\frac{930-K}{15}} = \dfrac{(K+187.5-2.49i)(K+187.5+2.49i)}{930-K}$	0	
s^0:	$64+3K$		

$64+3K > 0$

$\underline{\underline{K > -21.3}}$

$930-K > 0$

$\underline{\underline{K < 930}}$

$(K+187.5-2.49i)(K+187.5+2.49i) >$

$\underline{\underline{\therefore K > -187.5 \pm 2.4}}$

STEP 8: Break-away points:
for characteristic eq^n $B(s) + K\,A(s) = 0$

$$\frac{d}{ds}\left(-\frac{B(s)}{A(s)}\right) = 0$$

$$\frac{d}{ds}\left(-\frac{(s+3)}{(s^4+15s^3+70s^2+120s+64)}\right) = 0$$

$$\frac{-(s^4+15s^3+70s^2+120s+64)\cdot 3 + (s+3)(4s^3+45s^2+140s+120)}{(s+1)^2(s+2)^2(s+4)^2(s+8)^2} = 0$$

$$\frac{-3s^4 - \cancel{45s^3} - 210s^2 - 360s - 192 + 4s^4 + \cancel{45s^3} + 140s^2 + 120s + 12s^3 + 135s^2 + 420s + 360}{(s+1)^2(s+2)^2(s+4)^2(s+8)^2} = 0$$

$$s^4 + 12s^3 + 65s^2 + 180s + 168 = 0$$

$$s_{1,2} = -2.8 \pm 3.65j$$

$$s_3 = -4.70$$

$$s_4 = -1.68$$

(Plot Root Locus in Matlab)
Correction: K = 10 is actually K = 100

$K = 100,\ 300,\ 600$

$$s^4 + 15s^3 + 70s^2 + (120 + k)s + (64 + 3k) = 0$$

$K = 10:$

$$s^4 + 15s^3 + 70s^2 + 220s + 364 = 0$$

$s_1 = -9.74$ $\quad s_4 = -3.10$

$s_{2,3} = -1.08 \pm 3.29j$

$K = 300:$

$$s^4 + 15s^3 + 70s^2 + 420s + 964 = 0$$

$s_1 = -11.45$

$s_{2,3} = -0.26 \pm 5.26j$

$s_4 = -3.03$

$K = 600:$

$$s^4 + 15s^3 + 70s^2 + 720s + 1864 = 0$$

$s_1 = -13.03$

$s_{2,3} = 0.52 \pm 6.87j$

$s_4 = -3.02$

b. Predict the overshoot, settling time(2% criterion), and steady state error for a step input, assuming dominant roots for K = 100.

(b) For $K = 100$,

If we only include the dominant roots, out tf becomes

$$TF = \frac{100(s+3)}{(s+1.08-3.29i)(s+1.08+3.29i)}$$

$$= \frac{100(s+3)}{(s+1.08)^2-(3.29i)^2}$$

$$= \frac{100(s+3)}{(s+1.08)^2+(3.29)^2}$$

$$= \frac{100(s+3)}{s^2+2.16s+12}$$

Characteristic Eqn is;

$$s^2+2.16s+12=0$$

Comparing with the characteristic eqn of 2nd order system,

$$s^2+2\zeta\omega_n s+\omega_n^2=0$$

$\omega_n^2 = 12$

$\therefore \omega_n = 3.46 \text{ rads}^{-1}$

$2\zeta\omega_n = 2.16$

$\zeta = \frac{2.16}{2\times 3.46} = 0.31$

$\therefore \zeta = 0.31$

$\therefore PO = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}}$

$\approx 35.7\%$

$T_s = \frac{4}{\zeta\omega_n}$

$\therefore T_s \approx 3.7s$

$$e_{ss} = \frac{1}{1+K_p}$$

$$= \frac{1}{1+\lim_{s\to 0} G(s)}$$

$$= \frac{1}{1+\lim_{s\to 0}\frac{1000(s+3)}{s^2+2.16s+12}}$$

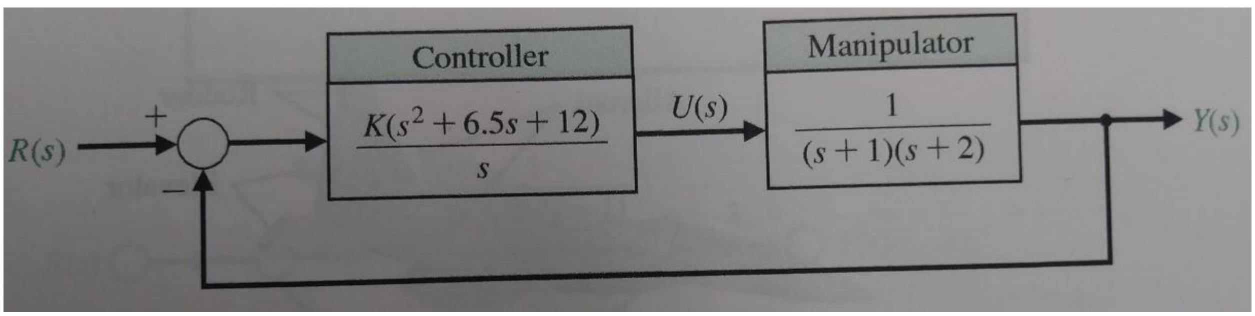
$$= \frac{1}{1+\frac{3000}{12}}$$

$\therefore e_{ss} = 0.00398$

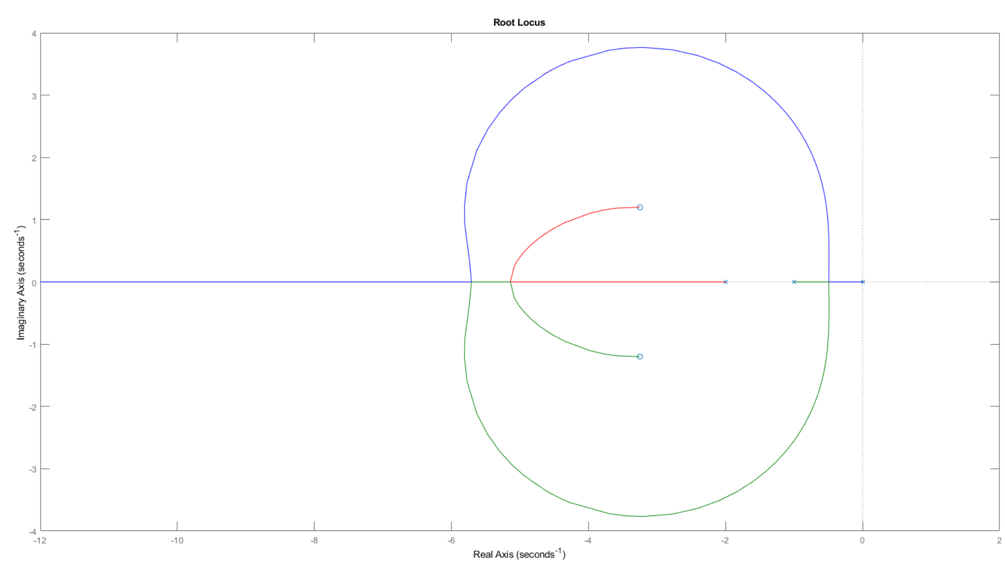
c. Determine the actual time response for a step input for 3 values of the gain K, and compare the actual results with the predicted results for K = 100

(In Matlab)

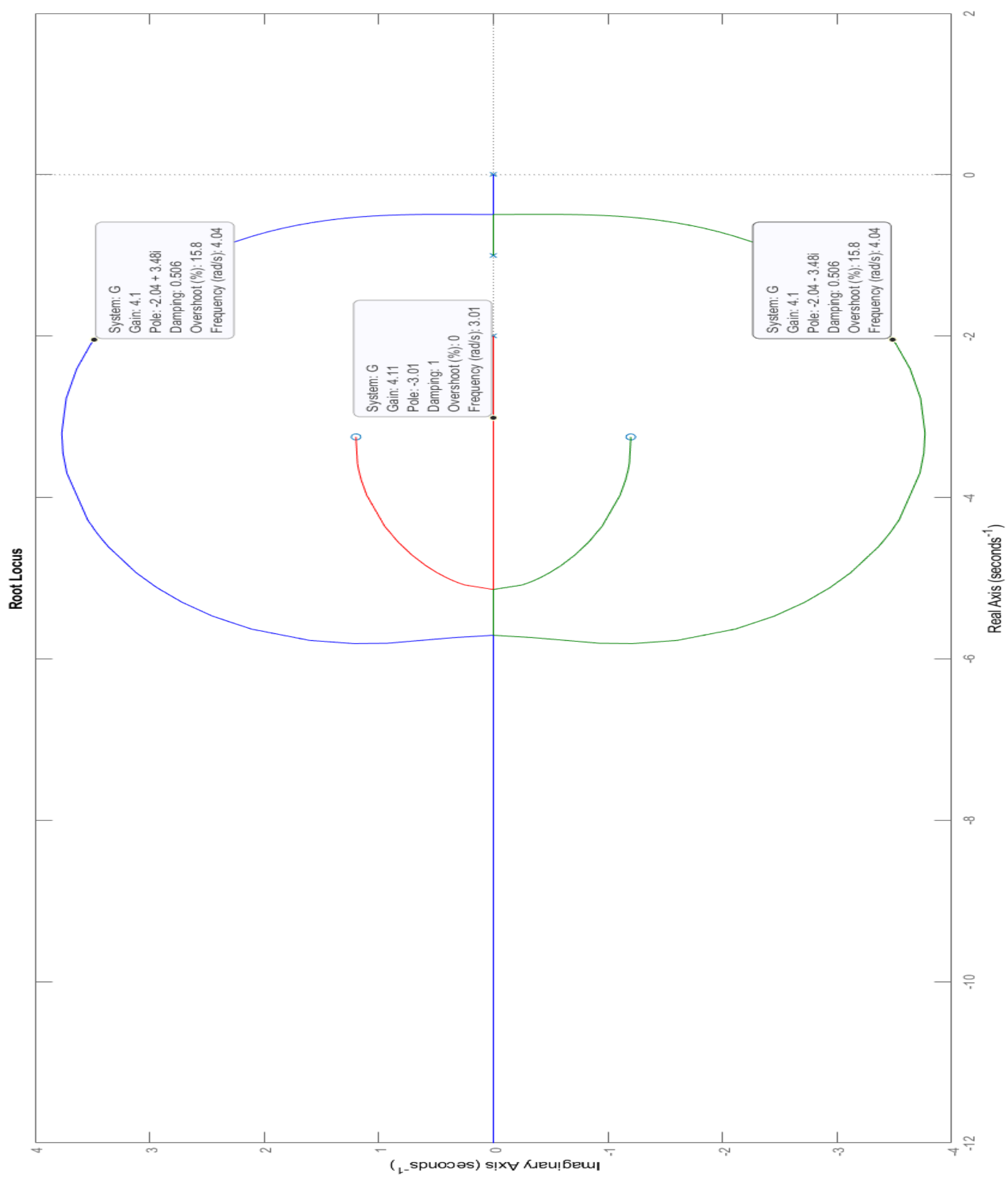
D.P.7.3 A rover vehicle has been designed for maneuvering at 0.25 mph over Martian Terrain. Because Mars is 189 million miles from Earth and it would take upto 40 mins each way to communicate with Earth[22,27], the rover must act independently and reliably. Resembling a cross between a small flatbed truck and jeep, the rover is constructed of three articulated sections, each with its own two independent, axle-bearing, one-meter conical wheels. A pair of sampling arms-one for chipping & drilling, the another for manipulating fine objects - extended from its front and end like pincers. The control of the arms can be represented by the system shown below.



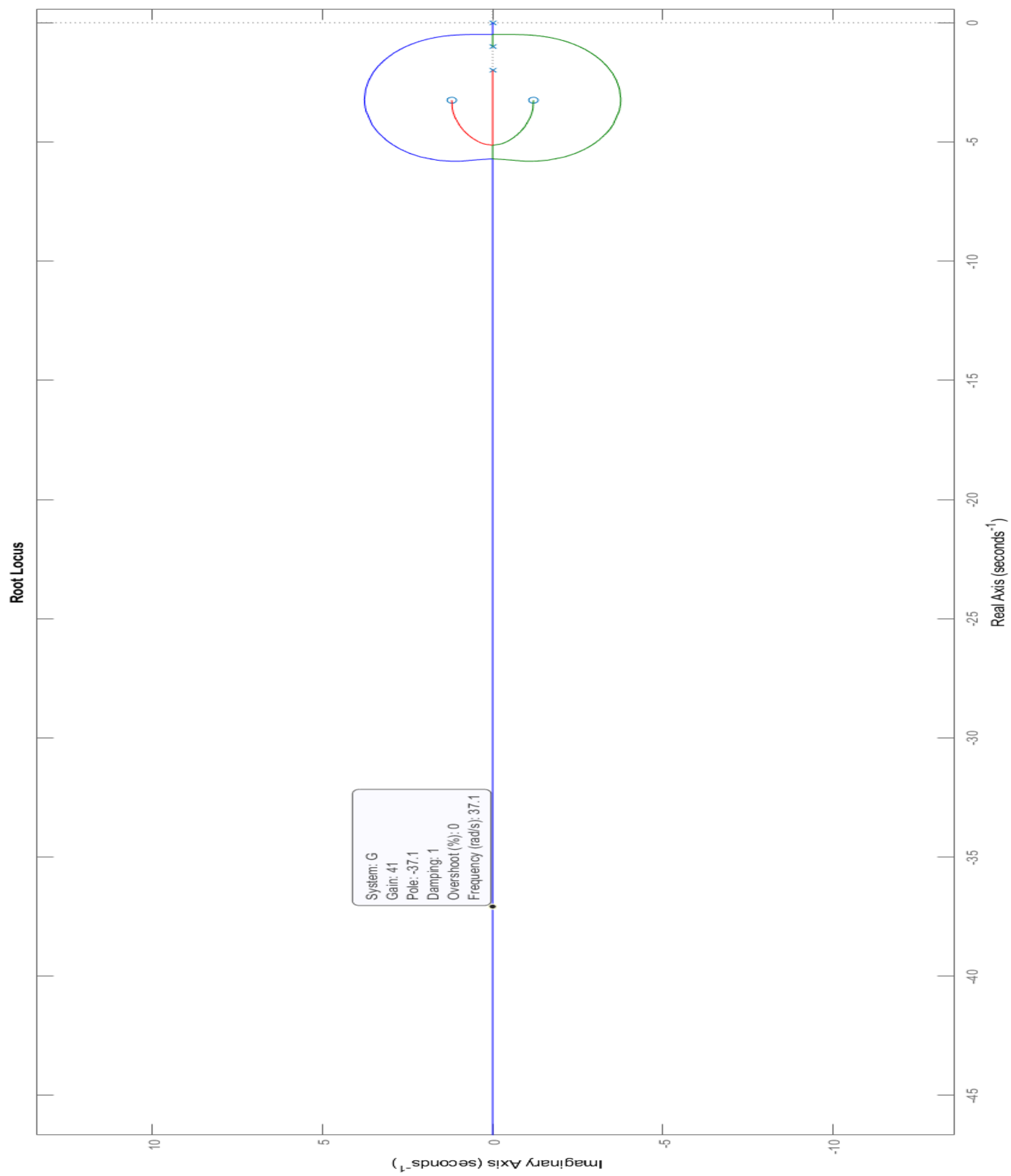
a. **Sketch the root locus for K & identify the K for K = 4.1 and 41.**



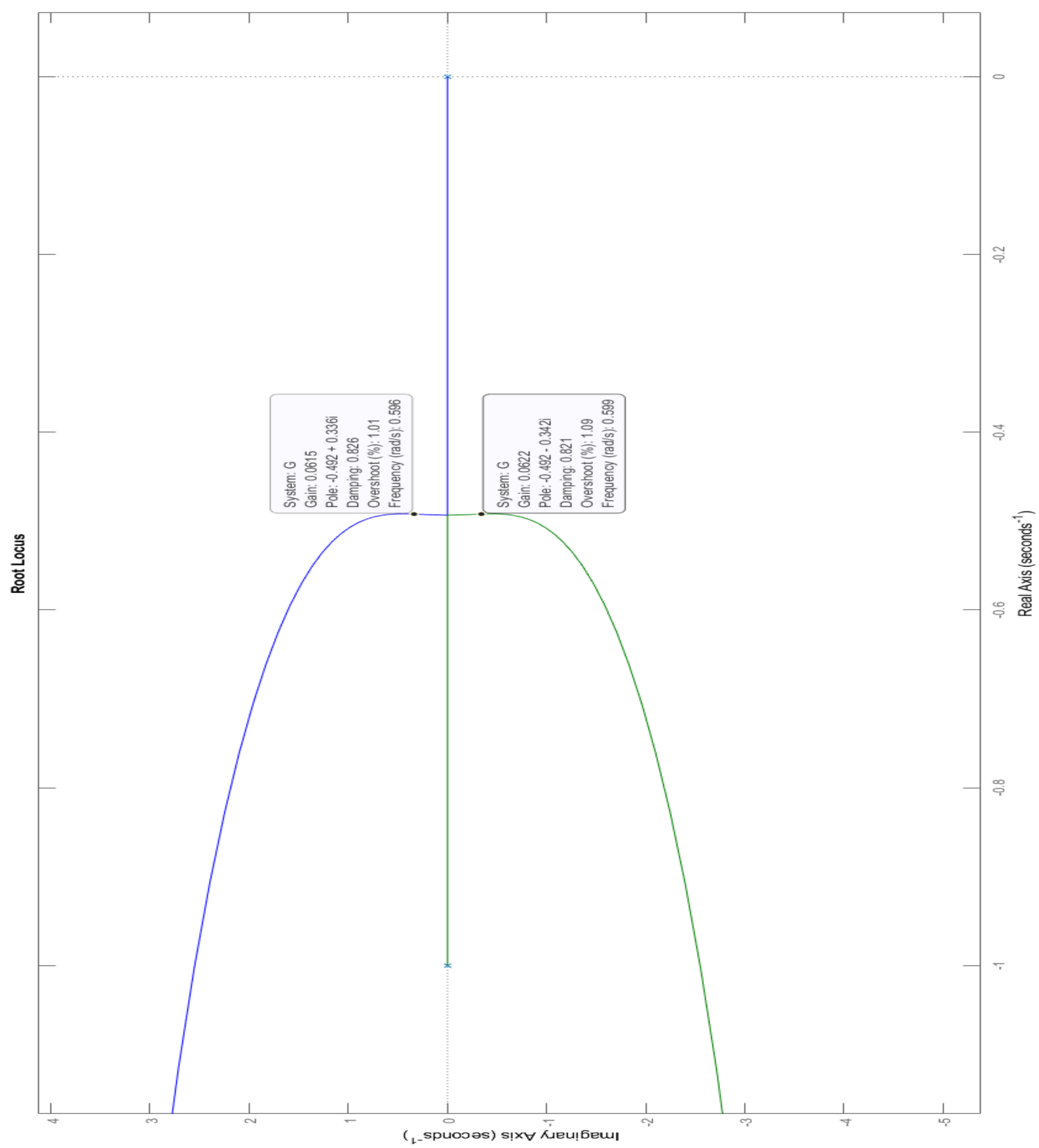
K = 4.1:


Root Locus
System: G
Gain: 4.1
Pole: -2.04 + 3.48i
Damping: 0.506
Overshoot (%): 15.8
Frequency (rad/s): 4.04
System: G
Gain: 4.11
Pole: -3.01
Damping: 1
Overshoot (%): 0
Frequency (rad/s): 3.01
System: G
Gain: 4.1
Pole: -2.04 - 3.48i
Damping: 0.506
Overshoot (%): 15.8
Frequency (rad/s): 4.04
Real Axis (seconds⁻¹)
Imaginary Axis (seconds⁻¹)

K = 41:


Root Locus
Real Axis (seconds^{-1})
Imaginary Axis (seconds^{-1})
System: G
Gain: 41
Pole: -37.1
Damping: 1
Overshoot (%): 0
Frequency (rad/s): 37.1

b. Determine the gain K that results in the percent overshoot to a step of P.O = 1%.



c. **Determine the gain that minimizes the settling time (with a 2% criterion) while maintaining a percent overshoot of P.O. ≤ 1%**

P.O. ≤ 1%

$$0.01 = e^{-\pi \zeta / \sqrt{1-\zeta^2}}$$

$$\ln(0.01) = \frac{-\pi \zeta}{\sqrt{1-\zeta^2}}$$

$$(\ln(0.01))^2 = \frac{-\pi^2 \zeta^2}{1-\zeta^2}$$

$$21.21 - 21.21\zeta^2 = -\pi^2 \zeta^2$$

$$21.21 = 21.21\zeta^2 - \pi^2 \zeta^2$$

$$\zeta^2 = \frac{21.21}{21.21 - \pi^2} = 1.87$$

$$\zeta = 1.37 \qquad \therefore \zeta \leq 1.37$$

$$T_s = \frac{4}{\zeta \omega_n}$$

$$T_s = \frac{4}{1.37\omega_n} = \frac{2.91}{\omega_n}$$

$$\omega_n = \frac{2.91}{T_s}$$

(For a really small T_s, ω_n is large)

(The natural frequency is increasing in the direction of increasing K in Rlocus plot)

Bibliography:

- **Dorf, Modern Control Systems, 13th ed**
- **https://www.youtube.com/watch?v=CRvVDoQJjYI, Root Locus Method**