Thought-Augmented Policy Optimization: Bridging External Guidance and Internal Capabilities

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Abstract

Reinforcement learning (RL) has emerged as an effective method for training reasoning models. However, existing RL approaches typically bias the model's output distribution toward reward-maximizing paths without introducing external knowledge. This limits their exploration capacity and results in a narrower reasoning capability boundary compared to base models. To address this limitation, we propose TAPO (Thought-Augmented Policy Optimization), a novel framework that augments RL by incorporating external high-level guidance ("thought patterns"). By adaptively integrating structured thoughts during training, TAPO effectively balances model-internal exploration and external guidance exploitation. Extensive experiments show that our approach significantly outperforms GRPO by 99% on AIME, 41% on AMC, and 17% on Minerva Math. Notably, these high-level thought patterns, abstracted from only 500 prior samples, generalize effectively across various tasks and models. This highlights TAPO's potential for broader applications across multiple tasks and domains. Our further analysis reveals that introducing external guidance produces powerful reasoning models with superior explainability of inference behavior and enhanced output readability.

1 Introduction

Reinforcement learning (RL) has demonstrated remarkable success in enhancing the reasoning capabilities of large language models (LLMs), as exemplified by OpenAI-o1 [1], DeepSeek-R1 [2], and Kimi-1.5 [3]. In contrast to traditional approaches that rely on human-curated annotations [4, 5], contemporary RL training paradigms [2, 6] directly optimize base language models using simple, automatically computable reward function. This approach enables models to develop sophisticated Chain-of-Thought (CoT) [7] capabilities, and autonomously incentivize advanced reasoning behaviors, including problem decomposition, self-reflection, and iterative refinement [8, 9].

Recent RL research primarily focuses on enhancing training stability, efficiency and performance through two key aspects: (1) addressing inherent limitations of RL algorithm [10–12], such as length bias and KL divergence constraints; and (2) improving data organization and reducing data dependency [13–15], such as enabling no-supervision training. Despite these advancements, existing GRPO-based methods typically bias the model's self-generated output distribution toward reward-maximizing trajectories without incorporating external knowledge. This inherent limitation constrains exploration capacity and results in narrower reasoning capabilities compared to base models [9, 8].

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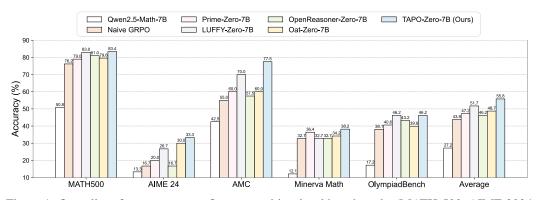


Figure 1: Overall performance across five competition-level benchmarks (MATH-500, AIME 2024, AMC, OlmpiadBench, and Minerva Math). TAPO significantly outperforms existing RL methods, especially on the challenging AIME and AMC benchmark (†99% and †41% over GRPO).

While a very recent and concurrent work LUFFY [16] introduces off-policy guidance to enhance on-policy learning, it necessitates supervision signals from a computationally expensive strong policy model (DeepSeek-R1 [2] in the paper). Moreover, the substantial capability gap between the external strong policy and the policy model being trained may lead to training instability issues.

To address these limitations, we propose **TAPO**, a Thought-Augmented **Policy Optimization** framework for LLM reasoning. Building upon conventional RL methods like GRPO [6], TAPO introduces high-level thought patterns that effectively bridge external guidance and model internal reasoning capabilities during training. Specifically, we design "thought library", a general repository storing high-level thought templates abstracted from 500 prior samples. Each template represents an abstract problem-solving strategy for a category of problems and serves as reasoning guidance. For each incoming question during GRPO sampling, we adaptively identify and apply relevant thought templates from this library to enhance the reasoning process. This dynamic integration of external guidance with internal model abilities enables the system to internalize more generalizable and explainable reasoning behaviors, stabilize model learning, and produce more powerful reasoning models.

Extensive experiments demonstrate that TAPO significantly outperforms GRPO across diverse datasets, achieving an average improvement of +12.0 points, including gains of 99% on AIME, 41% on AMC, and 17% on Minerva Math. As shown in Figure 1, our method also surpasses other powerful RL approaches. Moreover, TAPO proves effective across various model scales and architectures while exhibiting strong generalization to out-of-distribution reasoning tasks. Notably, our method can achieve stable learning on Llama3.2-3B-Base, which has been previously documented to struggle with standard GRPO training [8, 11]. Further analysis confirms that introducing external guidance enhances both model output explainability and readability. Our core contributions are:

- Novel RL framework: We propose Thought-Augmented Policy Optimization (TAPO), which
 enhances model reasoning by integrating external high-level thought guidance.
- Remarkable Performance: TAPO significantly outperforms GRPO by 99% on AIME, 41% on AMC, and 17% on Minerva Math, as well as previous powerful RL methods.
- Superior Generalization and Enhanced Output Quality: TAPO extends effectively to out-ofdistribution tasks, various model types while improving output explainability and readability.

2 Thought-Augmented Policy Optimization

In this paper, we aim to investigate RL for LLMs with external guidance, as shown in Figure 2. We first introduce the popular RL method GRPO [6] (Section 2.1), then present our extended GRPO framework incorporating high-level thought guidance (Section 2.2), and finally describe how to construct a thought library that provides external guidance for RL training (Section 2.3).

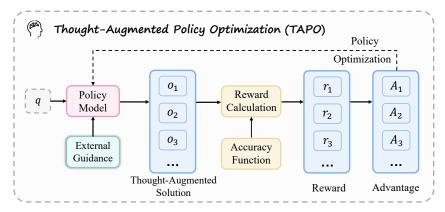


Figure 2: Flowchart of TAPO: Enhancing policy model capabilities through integration of external guidance (high-level thought patterns). This thought augmentation establishes an optimal balance between model internal exploration and external strategy exploitation.

2.1 Group Relative Policy Optimization (GRPO)

LLMs as Markov Decision Processes The generation process of LLMs can be formulated as a token-level Markov Decision Process (MDP) $\mathcal{M}(\mathcal{S}, \mathcal{A}, r, p_{\mathcal{Q}})$ [11, 17], where \mathcal{S} represents states (observation sequences) and \mathcal{A} represents the action space (vocabulary). At each step t, the state $s_t \in \mathcal{S}$ consists of the concatenation of the input question \mathbf{q} and all tokens generated so far $\mathbf{o}_{< t}$. This state serves as input to the policy model $\pi_{\theta}(\cdot|s_t)$. Specifically, the policy processes $s_t = (\mathbf{q}, \mathbf{o}_{< t}) = (q_1, q_2, \ldots, q_l, o_1, o_2, \ldots, o_{t-1})$, where q_i denotes the i-th token of question \mathbf{q} and $o_{j,< t}$ represents the token generated by π_{θ} at step j. The policy then samples the next token from the vocabulary \mathcal{A} . In the RL framework, the entropy-regularized objective [18] to be optimized is:

$$\mathcal{J}(\pi_{\theta}) = \mathbb{E}_{\mathbf{q} \sim p_{\mathcal{Q}}} \left[\mathbb{E}_{\mathbf{o} \sim \pi_{\theta}(\cdot|\mathbf{q})} [R(\mathbf{q}, \mathbf{o})] - \beta \cdot \mathbb{D}_{KL} [\pi_{\theta}(\cdot|\mathbf{q})) || \pi_{\text{ref}}(\cdot|\mathbf{q})] \right], \tag{1}$$

where $R(\mathbf{q}, \mathbf{o}) = \sum_{t=1}^{|\mathbf{o}|} r(s_t, o_t)$ denotes the return [17] of the trajectory $(\mathbf{q}; \mathbf{o}), r(\cdot)$ represents the reward model, and π_{ref} is used to denote a reference policy. The KL regularization term is usually adopted to prevent the policy model π_{θ} from deviating too far from the reference model π_{ref} .

GRPO Traditional RL approaches such as Proximal Policy Optimization (PPO) [19] employ policy gradient methods to optimize the objective in Equation 1. GRPO [2] offers an elegant simplification of PPO by eliminating the need for an additional reward model. It assigns a scalar reward to each trajectory and normalizes these rewards across the group. Specifically, let $\pi_{\theta_{\text{old}}}$ denote the policy model before updating. Given an input question \mathbf{q} and G outputs $\{\mathbf{o}_1,\ldots,\mathbf{o}_G\}$ generated by $\pi_{\theta_{\text{old}}}$, the normalized reward $A_{i,t}$ is shared across all tokens in \mathbf{o}_i :

$$A_{i,t} = \frac{r(\mathbf{o}_i) - \operatorname{mean}(\{r(\mathbf{o}_i) \mid \mathbf{o}_i \sim \pi_{\theta_{\text{old}}}(\cdot \mid s_t), i = 1, 2, \dots, G\})}{\operatorname{std}(\{r(\mathbf{o}_i) \mid \mathbf{o}_i \sim \pi_{\theta_{\text{old}}}(\cdot \mid s_t), i = 1, 2, \dots, G\})}.$$
 (2)

Then, the GRPO objective function is shown below:

$$\mathcal{J}_{\text{GRPO}}(\pi_{\theta}) = \frac{1}{G} \sum_{i=1}^{G} \frac{1}{|\mathbf{o}_i|} \sum_{t=1}^{|\mathbf{o}_i|} \left\{ \min \left[\rho_{i,t} A_{i,t}, \hat{\rho}_{i,t} A_{i,t} \right] - \beta \cdot \mathbb{D}_{KL} \left[\pi_{\theta}(\cdot|\mathbf{q}) \right) || \pi_{\text{ref}}(\cdot|\mathbf{q}) \right] \right\}$$

with probability ratio
$$\rho_{i,t} = \frac{\pi_{\theta}(o_{i,t}|\mathbf{q}, \mathbf{o}_{i,< t})}{\pi_{\theta_{\text{old}}}(o_{i,t}|\mathbf{q}, \mathbf{o}_{i,< t})}$$
 and clipped ratio $\hat{\rho}_{i,t} = \text{clip}(\rho_{i,t}; 1 - \epsilon, 1 + \epsilon)$.

In practical implementations of GRPO, $r(\cdot)$ typically represents a rule-based verifier.

2.2 Extending GRPO with High-Level External Guidance

In this subsection, we formally extend GRPO by incorporating high-level external guidance to enhance the model's internal reasoning capabilities. For multi-step reasoning tasks, it is typically

easier for weak models to generate one correct step than to complete the entire reasoning steps in a single inference. We leverage this characteristic through our guidance mechanism.

We define guidance to be a function g that transforms an input question \mathbf{q} into a thought-augmented form $\mathbf{q}^{\mathrm{aug}}$. When prompted with guidance, the model generates a partial solution which is then combined with the original question \mathbf{q} to create a question with hints $\mathbf{q}^{\mathrm{aug}}$. Specifically, for a specific guidance g_j , we have $\mathbf{q}_j^{\mathrm{aug}} = g_j(\mathbf{q})$. Then, for each augmented question $\mathbf{q}_j^{\mathrm{aug}}$, we sample a micro group of G_j outputs $\mathbf{o}_{j,1}, \mathbf{o}_{j,2}, \ldots, \mathbf{o}_{j,G_j}$ from the old policy model, representing diverse reasoning paths facilitated by the guidance. Section 2.3 provides details of this guidance mechanism.

Given multiple diverse guidances $g_1, ..., g_j$, the GRPO objective can be reformulated as:

$$\tilde{\mathcal{J}}_{\text{GRPO}}(\pi_{\theta}) = \frac{1}{\sum_{i=1}^{|g|} G_{i}} \sum_{i=1}^{|g|} \sum_{j=1}^{G_{i}} \frac{1}{|\mathbf{o}_{i}|} \sum_{t=1}^{|\mathbf{o}_{i}|} \left\{ \min \left[\rho_{i,j,t} A_{i,t}, \hat{\rho}_{i,j,t} A_{i,t} \right] - \beta \cdot \mathbb{D}_{KL} \left[\pi_{\theta} || \pi_{\text{ref}} \right] \right\} ,$$
probability ratio
$$\rho_{i,j,t} = \frac{\pi_{\theta}(o_{i,j,t} | \mathbf{q}_{j}^{\text{aug}}, \mathbf{o}_{i,j,< t})}{\pi_{\theta_{\text{old}}}(o_{i,j,t} | \mathbf{q}_{j}^{\text{aug}}, \mathbf{o}_{i,j,< t})} , \text{clipped ratio } \hat{\rho}_{i,j,t} = \text{clip}(\rho_{i,j,t}; 1 - \epsilon, 1 + \epsilon) .$$

$$(4)$$

For simplicity, we denote the objective for each thought guidance as:

$$\mathcal{J}_i(\pi_{\theta}) = \frac{1}{G_i} \sum_{j=1}^{G_i} \frac{1}{|\mathbf{o}_i|} \sum_{t=1}^{|\mathbf{o}_i|} \left\{ \min \left[\rho_{i,j,t} A_{i,t}, \hat{\rho}_{i,j,t} A_{i,t} \right] - \beta \cdot \mathbb{D}_{KL} \left[\pi_{\theta}(\cdot|\mathbf{q}) \right) || \pi_{\text{ref}}(\cdot|\mathbf{q}) \right] \right\}, \quad (5)$$

then, the extended GRPO objective (Equation 4) can be written as:

$$\tilde{\mathcal{J}}_{GRPO}(\pi_{\theta}) = \frac{1}{\sum_{i=1}^{|g|} G_i} \sum_{i=1}^{|g|} G_i \mathcal{J}_i(\pi_{\theta}). \tag{6}$$

This formulation can be viewed as a weighted sum of each guidance-specific objective, with G_i serving as the weight for each term. Consequently, we are optimizing the model under multiple diverse high-level thought guidances simultaneously. As demonstrated in DAPO [10], prompts yielding binary accuracy values (0 or 1) produce no gradients, impeding learning and reducing sample efficiency. Let $p_j \ (\le 1)$ denote the probability of obtaining zero accuracy when sampling from $\pi_{\theta_{\text{old}}}(\mathbf{q}_j^{\text{aug}})$. The probability of obtaining at least one positive sample in the training group becomes $1 - \Pi_i^{|g|} p_i \ (\ge 1 - p_j, \forall j)$. Thus, we can infer that training with such grouped samples, guided by more diverse instructions, will lead to more stable model learning.

In our implementation, we generate equal outputs for each guidance $(G_1 = G_2 = \cdots = G_{|g|})$, assigning uniform learning weights. Future work could explore different or dynamic weight allocation strategies based on the learning process. Notably, when |g| = 1 and questions remain unaugmented $(\mathbf{q} = \mathbf{q}^{\mathrm{aug}})$, Equation 4 degenerates to the vanilla GRPO objective (Equation 3). Therefore, GRPO can be viewed as a special case of our more general framework.

2.3 Thought Library and Augmented Reasoning

In this subsection, we describe how to construct our thought library, which provides external guidance for RL training in Section 2.2. As illustrated in previous work [20, 21], humans typically solve complex reasoning tasks by applying universal guidelines ("thought patterns") induced from similar problems rather than starting from scratch. These high-level thought patterns help address unfamiliar tasks by leveraging previously successful reasoning strategies. Inspired by prior work [22, 23], we introduce "thought library", a lightweight hub of high-level thought templates abstracted from just 500 seed data—that adaptively provides relevant thought patterns during GRPO sampling.

Thought Library Starting with a small set of seed samples $S = \{\mathbf{s}_1, \dots, \mathbf{s}_s\}$, we employ Monte Carlo Tree Search (MCTS) [24–26] to generate solution trees. For each question $\mathbf{s}_i \in S$, a predefined action set $A = \{a_1, \dots, a_A\}$ and model π , MCTS build a search tree \mathcal{T}_i where: the root node represents question \mathbf{s}_i , each edge denotes an action $a \in A$, and each child node n contains partial solutions generated by π under the corresponding action. A path from the root to a leaf node $\mathbf{n}_{j,d}$ forms a solution trajectory $\mathbf{t}_j = (\mathbf{s}_i, a_{j,1}, \mathbf{n}_{j,1}, \dots, a_{j,d}, \mathbf{n}_{j,d})$. Each intermediate node $\mathbf{n}_{j,l}$ is generated based on the cumulative context of its parent nodes and the current action, i.e. $\mathbf{n}_{j,l} = \mathbf{n}_{j,l} = \mathbf{n}_{j,l}$

 $\pi([\mathbf{s}_i, a_{j,1}, \mathbf{n}_{j,1}, \dots, a_{j,l}])$. Through this process, we obtain a diverse set of solution traces $\mathbb{T} = \{\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_t\}$. The MCTS algorithm will assign a final reward $R(\mathbf{t}_j|\mathbf{s}_i)$ to each trace $\mathbf{t}_j \in \mathbb{T}$. Further MCTS details are provided in Appendix A.1.

Through the above process, for each seed question s_i , we obtain multiple solution traces. To identify the optimal trajectory for s_i , we utilize a simple path selection metric proposed in HiAR-ICL [23]:

$$Score(\mathbf{s}_i, \mathbf{t}_j) = b \cdot R(\mathbf{t}_j | \mathbf{s}_i) - (1 - b) \cdot C(\mathbf{t}_j), \tag{7}$$

where $C(\mathbf{t}_i)$ represents trajectory complexity (action count), and b (set to 0.95) balances solution quality against complexity. This scoring function selects trajectories that maximize accuracy while maintaining procedural conciseness among multiple potential solutions. For each question $\mathbf{s}_i \in \mathcal{S}$, we select the optimal solution trace $\mathbf{t}_{i, \mathsf{best}}$ that maximizes this score. Since each node in $\mathbf{t}_{i,\mathrm{best}}$ corresponds to an instantiated action $a_{i,l} \in \mathcal{A}$, we retain the more general action-trace as a high-level thought pattern $T_j = (a_1, \ldots, a_d)$, and aggregate these patterns to construct our thought library $\mathcal{L} = {\hat{T}_1, \dots, \hat{T}_s}$. This aggregation is guided by the Problem Condition Complexity (PCC) [27, 28], which represents the number of known prior conditions in q and can be calculated by the model π . Ultimately, each thought template in our library contains both a high-level thought pattern (e.g., $a_1 \rightarrow a_2 \rightarrow a_4$) and the average PCC of questions sharing this pattern:

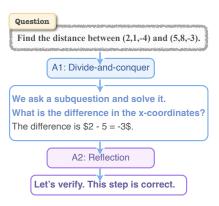


Figure 3: Schematic diagram of action-chain-structured solution trajectory.

 $\hat{T}_j = (PCC_{T_j}, T_j)$. These templates represent generalized problem-solving strategies for similar problems and serve as the external guidance described in Section 2.2. Detailed implementations are provided in Appendix A.2.

Reasoning with Guidance Drawing on meta-reasoning principles [29, 30], we adaptively identify the most relevant high-level thought patterns for each new problem. During GRPO sampling for a new incoming question \mathbf{q}_t , we compute its PCC metric and calculate the absolute distance $d_j = |\text{PCC}_{q_t} - \text{PCC}_{T_j}|$ for each $\hat{T}_j \in \mathcal{L}$. We then sort them to obtain the k most similar templates $\{\hat{T}_{i_1}, \dots, \hat{T}_{i_k}\}$ that best align with the question's complexity. The thought patterns on these templates, which are sequences of actions, guide the step-by-step reasoning process for question \mathbf{q}_t .

Notably, we use MCTS to build the reasoning tree, abstract it to a high-level thought library, and then match new questions with it. In fact, such a thought library could also be obtained in other ways. For example, human experts could write a general solution approach for each category of problems and then match new questions with existing problem types. We leave this for future work.

3 Experimental Setup

Training Datasets To keep the training recipe simple, we select training data exclusively from the training set of MATH [31] datasets. Following prior studies [32, 11], we only use MATH level 3-5 problems for training, yielding 5.5K examples. We randomly sample 500 instances to generate high-level thought patterns, with the remaining 5K examples for training in this work.

Evaluation We mainly focus on several widely used math reasoning benchmarks, including MATH500 [31], AIME 2024 [33], AMC [33], Minerva [34], OlympiadBench [35], GSM8K [36], College Math [37], and Gaokao23 [38]. Since our RL training focus on math reasoning, we further assess the generalization capability on three out-of-distribution benchmarks: GPQA-Diamond [39] (science sraduate knowledge), ARC-C [40] (open-domain reasoning), and MMLU-Pro [41] (questions from academic exams and textbooks). Following common practice and previous work [11, 13], we use greedy decoding during evaluation. We also limit the sampling budget to 3000 tokens.

Baseline Methods We benchmark TAPO with the following baselines on Qwen2.5-Math-7B: (1) *GRPO* [6], a simplified PPO variant using identical 5k training samples as TAPO; (2) *SimpleRL-Zero* [32], which applies GRPO to approximately 24k math samples from GSM8K [36] and MATH [31]; (3) *OpenReasoner-Zero* [42], employing PPO with 129k samples from diverse sources

Table 1: Main results (%) on five competition-level reasoning benchmarks based on Qwen2.5-Math-7B-Base. The best results on each benchmark are highlighted in **bold**. TAPO outperforms RL baselines and shows significant gains over GRPO, with relative improvements provided.

Model	MATH500 ↑	AIME24↑	AMC ↑	Minerva ↑	Olympiad ↑	Avg. ↑
Qwen2.5-Math [45]	50.8	13.3	42.5	12.1	17.2	27.2
Qwen2.5-Math-Instruct [45]	81.0	13.3	55.0	32.7	38.8	44.1
SimpleRL-Zero [32] OpenReasoner-Zero [47] PRIME-Zero [43] Oat-Zero [11]	74.6	26.7	60.0	27.6	35.8	44.9
	81.0	16.7	57.5	32.7	43.2	46.2
	79.0	20.0	60.0	36.4	40.6	47.2
	79.6	30.0	60.0	34.2	39.9	48.7
LUFFY [16] GRPO [6] TAPO (Ours) △ (↑)	83.0 76.2 83.4 +9.4%	26.7 16.7 33.3 +99.4%	70.0 55.0 77.5 +40.9%	32.7 32.7 38.2 +16.8%	38.1 46.2 +21.2%	51.7 43.8 55.8 +27.4%

including AIME; (4) *PRIME-Zero* [43], utilizing implicit process rewards with policy rollouts and outcome labels on 150k NuminaMath [33] queries; (5) *Oat-Zero* [11], which introduces Dr.GRPO to mitigate length bias, trained on 8k MATH questions; and (6) *LUFFY* [16], featuring mixed-policy GRPO that incorporates DeepSeek-R1's outputs, trained on 45k samples from OpenR1-Math-220k [44].

Implementation Details Follow previous work [43, 32, 11], we primarily use Qwen2.5-Math-7B [45] as the default model. Additionally, we apply TAPO to Qwen2.5-Math-1.5B [45], Qwen2.5-Math-1.5B-Instruct [45], Qwen2.5-7b-instruct [46], Llama-3.2-3B-Instruct [5], and Llama-3.1-8B-Instruct [5] to showcase its adaptability across different model scales and types.

For RL training, we follow the Open-R1 [44] pipeline. Following previous work [11, 16], we remove the KL loss term by setting $\beta=0$ and employ the Dr.GRPO loss. Our training configuration includes a batch size of 128, generating 16 samples per prompt. In our implementation, we set the number of guidance to 2 (i.e., |g|=2) by default. We generate an equal number of rollouts for each guidance, which means $G_1=G_2=16/2=8$. The reward function is a binary accuracy metric verified by Math-Verify. We train for 500 steps for all experiments. All training experiments are conducted using 8 A100 GPUs. More implementation details are provided in Appendix B.

4 Results and Discussion

The section presents the results of TAPO from four aspects: §4.1 Main Results, §4.2 Training Dynamics, §4.3 Ablation Study and Discussion, and §4.4 Case Study.

4.1 Main Results

Reasoning Benchmark Performance Following previous works [11, 16], Table 1 presents the main results across five competition-level reasoning benchmarks. We compare TAPO with multiple representative RL methods, as described in Baseline Methods. To ensure a fair comparison, all baselines are based on Qwen2.5-Math-7B. Our evaluation results reveals three key insights:

- TAPO achieves an average score of 55.8, significantly outperforming existing powerful RL methods by a margin of +4.1 points over the best baseline, clearly demonstrating the benefit of integrating high-level external guidance with model internal reasoning capabilities.
- On challenging datasets like AMC (+7.5 points over best baseline), TAPO significantly outperforms other methods. By adaptively integrating action-chain structured thought patterns during GRPO training, our method enables more nuanced problem decomposition, generates higher-quality training samples, and thus facilitates more effective model learning.
- Compared to its GRPO counterpart, TAPO consistently outperforms across all benchmarks, achieving a notable improvement of +12.0 points on average. This consistent performance gain provides a more robust and effective alternative for RL training.

Table 2: Accuracy (%) results of different LLMs across eight benchmarks. The best result	s in each
box are highlighted in bold . We provide the relative improvement of our method compared t	o GRPO.

Method	AIME24↑	AMC↑	MATH500↑	GSM8K↑	Minerva↑	Olympiad↑	CollegeMath ↑	Gaokao23↑	Avg.↑
Qwen2.5-1.5B-Math [45]									
CoT GRPO Ours △ (↑)	10.0 13.3 16.7 +25.6%	42.5 40.0 55.0 +37.5%	59.0 66.4 69.0 +3.9%	74.6 74.7 84.2 +12.7%	24.3 25.0 31.6 +26.4%	27.6 30.1 33.6 +11.7%	39.5 40.5 47.3 +16.8%	49.6 52.7 54.8 +4.0%	40.9 42.8 49.0 +14.5%
				Qwen2.5-1.5	B-Math-Instr	uct [45]			
CoT GRPO Ours △ (↑)	6.7 13.3 16.7 +25.6%	47.5 52.5 55.0 +4.8%	68.2 76.8 76.0 -1.0%	76.8 85.9 86.5 +1.0%	28.3 28.3 29.4 +3.9%	36.9 36.7 39.7 +8.2%	47.1 45.9 48.3 +5.3%	63.1 65.2 65.7 +1.0%	46.8 50.6 52.2 +3.2%
				Qwen2.5	-7B-Instruct	[46]			
CoT GRPO Ours △ (↑)	13.3 13.3 16.7 +25.6%	47.5 57.5 67.5 +17.4%	73.2 76.6 78.0 +1.9%	90.0 90.1 91.5 +1.6%	30.5 32.4 36.8 +13.6%	38.8 36.1 40.6 +12.5%	46.9 44.5 50.6 +13.7%	64.2 62.9 65.2 +3.7%	50.5 51.6 55.8 +8.1%
Llama-3.2-3B-Instruct [5]									
CoT GRPO Ours △ (↑)	6.7 3.3 6.7 +103.0%	20.0 25.0 27.5 +10.0%	38.3 47.8 48.8 +2.1%	69.3 75.2 78.8 +4.8%	11.8 17.6 18.4 +4.6%	12.6 14.5 16.0 +10.4%	23.8 34.1 32.5 -4.7%	33.5 40.8 43.1 +5.7%	27.6 32.2 34.0 +5.6%
Llama-3.1-8B-Instruct [5]									
CoT GRPO Ours △ (↑)	3.3 3.3 6.7 +103.0%	20.0 22.5 30.0 +33.3%	36.6 45.0 52.2 +16.0%	77.2 82.9 85.2 +2.8%	16.2 21.0 26.8 +27.7%	15.9 16.1 17.3 +7.5%	13.3 31.7 34.1 +7.6%	29.9 40.8 42.6 +4.5%	26.5 32.9 36.7 +11.6%

Out-of-Distribution Generalization Recent studies have highlighted the critical impact of distributional bias on LLMs' reliability [48, 49]. Despite impressive in-distribution (ID) performance, these models substantially underperform when confronted with out-of-distribution (OOD) data [50, 51]. To assess TAPO's OOD generalization capabilities, we evaluate on three challenging benchmarks: ARC-C, GPQA-Diamond, and MMLU-Pro. Given that all compared methods were trained on mathematical data, this setup provides a robust OOD evaluation. As shown in Figure 4, TAPO outperforms GRPO by 13.7% on average across OOD tasks. These results highlight TAPO's effectiveness in using high-level external guidance to enhance OOD generalization.

Extension to More Models To demonstrate TAPO's effectiveness across different scales and model types, we extend TAPO to several weaker models: Qwen2.5-1.5B-Math, Qwen2.5-1.5B-Math-Instruct, Qwen2.5-7B-Instruct, Llama-3.2-3B-Instruct and Llama-3.1-8B-Instruct. As shown in Table 2, TAPO achieves significant improvements across all models. Taking Qwen2.5-1.5B-Math as an example, TAPO achieves an average improvement of 14.5%. A similar trend is also observed on Qwen2.5-1.5B-Math-Instruct, Qwen2.5-7B-Instruct, Llama-3.2-3B-Instruct, and Llama-3.1-8B-Instruct, where TAPO achieves improvements of 3.2%, 8.1%, 5.6%, and 11.6%, respectively.

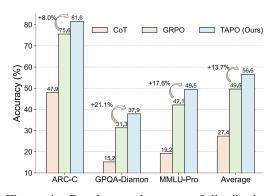


Figure 4: Results on three out-of-distribution benchmark datasets (Qwen2.5-Math-7B-Base).

4.2 Training Dynamics

In this section, we aim to explore the behavioral differences between TAPO and GRPO through the training reward curves. We conduct experiments on Qwen-2.5-7B-Math-Base and Llama3.2-3B-Base.

More Stable Model Learning As illustrated in Figure 5, TAPO consistently achieves higher overall training rewards than GRPO for both models. While this advantage appears modest for Qwen2.5-

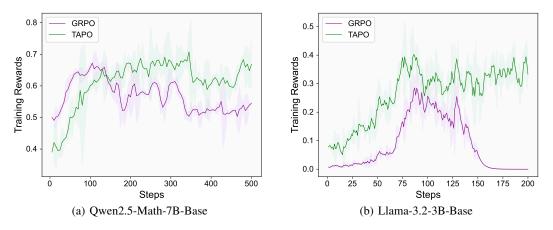


Figure 5: Training Reward Curve on Qwen2.5-Math-7B-Base and Llama-3.2-3B-Base.

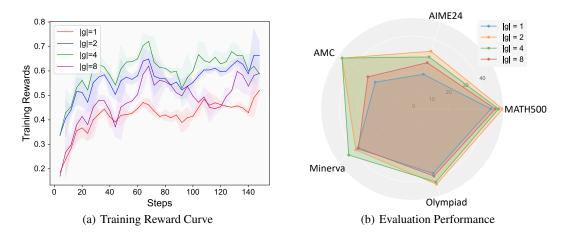


Figure 6: Ablation results with different numbers of thought patterns (external guidance).

Math-7B-Base, it becomes substantially more pronounced when using the weaker Llama3.2-3B-Base. Notably, GRPO tends to collapse during later training stages. This observation aligns with previous findings [8, 11], which demonstrate that directly applying GRPO to weak models like Llama3.2-3B-Base frequently leads to training collapse across both mathematical tasks and reasoning games. In contrast, TAPO maintains stable training dynamics even with such weak models.

The enhanced stability of TAPO stems from its strategic integration of thought patterns into the sampling process. By breaking down complex problems into manageable subtasks, TAPO provides explicit external guidance. This equips weaker models with advanced task decomposition capabilities usually exclusive to stronger models. As a result, weaker models can learn from difficult examples that typically fail to produce positive outcomes under standard GRPO training. As shown in Section 2, this mechanism directly contributes to enhanced training stability and robust model performance.

4.3 Ablation Study and Discussion

We investigate the effect of external guidance quantity (|g| = 1, 2, 4, 8) on training dynamics with fixed total rollouts. Figure 6 shows training rewards and evaluation results across varying thought pattern quantities. For clearer visualization, we uniformly adjust performance on AMC and MATH by subtracting a fixed value without affecting conclusions. We observe that more diverse guidance (|g| = 1, 2, 4) typically yields higher training rewards. However, when guidance becomes excessive (|g| = 8), performance declines slightly while still outperforming minimal guidance (|g| = 1). This may be because: (1) Smaller |g| makes it harder to sample positive examples; (2) Larger |g| increases the likelihood of sampling positive examples, but excessive quantities can overwhelm the

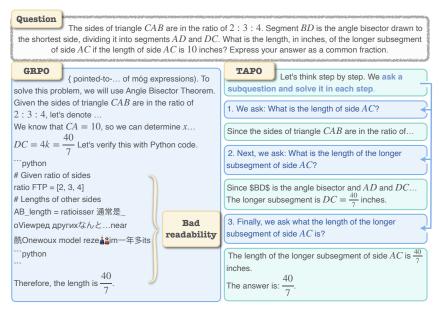


Figure 7: Comparison of GRPO and TAPO for a simple geometry problem from the MATH dataset.

model's limited learning capacity with too many patterns. In other words, when pattern quantity becomes excessive, the model faces too many learning tasks simultaneously, exceeding its processing capabilities. Therefore, designing an effective thought-augmented framework requires balancing pattern quantity and model capacity to ensure stable learning while maintaining generalization ability. Evaluation performance in Figure 6(b) follows a similar trend, highlighting the importance of balancing guidance quantity with model capacity.

4.4 Case Study

We compare the reasoning processes of TAPO and GRPO on a geometry problem in Figure 7. GRPO produces less readable outputs with reasoning interspersed with code and inconsistent language. In contrast, TAPO first identifies its solution strategy (e.g., divide-and-conquer), then systematically addresses each subproblem with clear solutions. This demonstrates how TAPO training enhances both the readability and interpretability of the model's reasoning process.

5 Related Work

RL for LLMs Recent advances in LLM reasoning, such as OpenAI-o1 [1], DeepSeek-R1 [2], and Kimi-k1.5 [3], have shifted focus from Chain-of-Thought (CoT) [7] and supervised fine-tuning (SFT) [52, 53] to reinforcement learning (RL). Contemporary research have primarily focused on: (1) addressing inherent limitations of GRPO [10–12], such as length bias and KL divergence constraints; and (2) improving data organization and reducing data dependency [13–15]. However, these methods typically bias the model's output distribution toward reward-maximizing paths without introducing external knowledge, narrowing reasoning capabilities compared to base models [9]. While the recent concurrent work LUFFY [16] introduces off-policy guidance to enhance on-policy learning, it still relies on supervision signals from a strong policy (DeepSeek-R1). Moreover, the substantial capability gap between the external strong policy and the policy model being trained potentially increases training instability. In contrast, TAPO incorporates external high-level thought guidance to augment the model's intrinsic capabilities without strong policy. By integrating diverse thought patterns, TAPO enables more stable learning and enhanced reasoning performance.

Reasoning with Guidance A common approach to enhancing model response quality involves augmenting input questions with external prompts [54–58]. This methodology has been widely applied to reasoning tasks with varying implementation strategies. Some research adaptively searches for suitable exemplars to perform few-shot COT prompting [59], while others focus on decomposing complex reasoning tasks into simpler, sequential subtasks [60, 61]. The partial solutions derived

through few-shot prompting or task decomposition are subsequently concatenated with the original problem as guiding hints, thereby reducing problem complexity. However, these approaches typically necessitate meticulous prompt design and exhibit strong dependencies on example quality. Although recent works have advanced from specific examples toward more abstract high-level thought patterns [22, 23, 62], they primarily enhance the reasoning capabilities of fixed models through guidance. Moreover, few studies have investigated how to effectively integrate high-level guidance with RL training paradigms. Our work bridges this gap by introducing external abstract problem-solving guidance into RL training, achieving superior performance while maintaining flexibility.

6 Conclusion

In this paper, we introduce TAPO (Thought-Augmented Policy Optimization), a novel RL framework that addresses fundamental limitations in current approaches for training reasoning models. By incorporating external high-level thought patterns into policy optimization, TAPO effectively bridges model-internal exploration with structured external guidance. Unlike conventional methods that merely bias toward reward-maximizing trajectories, our approach adaptively integrates abstract reasoning strategies during training, enhancing model capabilities across diverse problems. Extensive experiments demonstrate TAPO's significant improvements over GRPO, with gains of 99% on AIME, 40% on AMC, and 17% on Minerva Math. Our method maintains effectiveness across various model scales and architectures, including weak models that typically struggle with standard GRPO training. Moreover, TAPO produces models with enhanced output explainability and readability. These results establish TAPO as a promising direction for developing more powerful, generalizable, and interpretable reasoning systems, opening avenues for future research on integrating high-level thought patterns into model training across broader reasoning domains.

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Technical Appendix of TAPO

The supplementary material provides in-depth insights into our TAPO method, covering additional algorithm details ($\S A$), experimental details ($\S B$), and case study ($\S C$). The appendix is organized as follows:

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A More Details about TAPO

In this section, we provide a comprehensive elaboration of the TAPO algorithm's technical details. We further describe the specific implementation of the Monte Carlo Tree Search algorithm, the construction process of our thought library, and the adaptive retrieval and instantiation mechanisms for thought patterns during reasoning.

A.1 Monte Carlo Tree Search (MCTS)

As a heuristic search algorithm, MCTS has demonstrated remarkable success in complex reasoning and decision-making environments [63–65]. The algorithm conceptualizes search spaces as tree structures and has achieved significant breakthroughs across various domains, most notably in game-playing AI such as AlphaGo and AlphaZero [66]. As described in Section 2.3 in the main text, we employ MCTS to generate solution trees based on a small set of 500 seed samples.

To implement MCTS effectively, we first define a predefined action set. Understanding human complex reasoning is crucial for modeling cognitive processes [67]. Existing studies distinguish between two cognitive systems: System 1 and System 2 [20, 68]. While "System 1" represents fast, intuitive, yet error-prone thinking, "System 2" involves slow, deliberative thinking with superior performance. With the emergence of advanced models like OpenAI's o1, developing efficient "System 2" approaches to emulate human cognitive processes has gained significant research attention [69, 2].

Inspired by this and following previous work [23, 25], we introduce five human-like reasoning actions to bridge the gap between model reasoning and human cognition:

- Divide and Conquer (DC, a₁): Approaching complex problems by breaking them into manageable sub-problems for easier resolution.
- Self-Reflection (SR, a₂): Assessing and refining prior solutions during the reasoning process to
 ensure correctness.
- System Analysis (SA, a₃): Analyzing the overall structure of the problem and identifying the constraints and conditions before addressing it, thereby clarifying task requirements effectively.
- One-Step Thought (OST, a_4): Aiming to address a single aspect of the problem through a focused and concise reasoning step.
- Chain-of-Thought (CoT, a₅): Adopting a sequential reasoning process that builds a series of connected logical steps.

Based on the above predefined action set $\mathcal{A}=\{a_1,...,a_A\}$ and model π , for each question $\mathbf{s}_i\in\mathcal{S}$, MCTS builds a search tree \mathcal{T}_i where: the root node represents question \mathbf{s}_i , each edge denotes an action $a\in\mathcal{A}$, and each child node n contains partial solutions generated by π under the corresponding action. A path from the root to a leaf node $\mathbf{n}_{j,d}$ forms a solution trajectory $\mathbf{t}_j=(\mathbf{s}_i,a_{j,1},\mathbf{n}_{j,1},\ldots,a_{j,d},\mathbf{n}_{j,d})$. Each intermediate node $\mathbf{n}_{j,l}$ is generated based on the cumulative context of its parent nodes and the current action, i.e. $\mathbf{n}_{j,l}=\pi([\mathbf{s}_i,a_{j,1},\mathbf{n}_{j,1},\ldots,a_{j,l}])$. Specifically, the MCTS algorithm involves an iterative search process with four key steps: selection, expansion, simulation, and backpropagation:

(1) Selection. This operation identifies optimal nodes for expansion. Starting from the root node, a child node is chosen at each tree level until reaching a leaf node, defined as achieving maximum tree depth or arriving at an answer here. To balance the exploration and exploitation, we employ the well-known Upper Confidence Bounds applied to Trees (UCT) [70] for node selection:

$$UCT(s) = Q(s) + w\sqrt{\frac{\ln N(p)}{N(s)}}$$
(8)

where Q(s) is the reward value for node s, N(s) is the visit count, p is the parent node, and w is the exploration weight. The node with the highest UCT value is selected for subsequent phases, balancing exploration and exploitation.

- (2) Expansion. The selected node s is expanded by sampling n actions from π and generating corresponding reasoning outcomes. These n child nodes are then added to the tree.
- (3) Simulation. Starting from the selected node, we iteratively sample and expand nodes until reaching a terminal state (maximum depth or final answer node). To enhance efficiency, we implement an

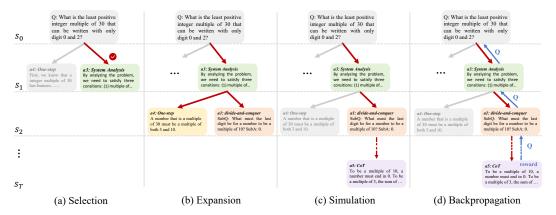


Figure 8: An illustration of four phases in an iteration of MCTS for complex reasoning tasks.

early termination strategy based on self-consistency [71]. This strategy exploits the observation that repeatedly sampled actions at the same state likely indicate successful task completion. If the model's consistency score exceeds a threshold c, i.e., SC(s) > c, the simulation terminates early

(4) Backpropagation. Upon simulation completion, node information is updated along the simulation path $s_0, ... s_d$. Visit counts are incremented $(N(s) \leftarrow N(s) + 1)$, and node value Q(s) is propagated backward to its parent node p using the following equation:

$$Q(p) \leftarrow (1 - \alpha)Q(p) + \alpha Q(s) \tag{9}$$

where α is a discount factor for future rewards. For terminal nodes, following prior work [25], we adopt the likelihood (confidence) of self-consistency majority voting as the reward value, enabling supervision-free generalization.

Through the above four process, we obtain a diverse set of solution traces $\mathbb{T} = \{\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_t\}$ for question $s_i \in \mathcal{S}$. The MCTS algorithm will assign a final reward $R(\mathbf{t}_i|\mathbf{s}_i)$ to each trace $\mathbf{t}_i \in \mathbb{T}$.

Figure 8 illustrates the four phases in an iteration, expanding the tree and then updating reward values.

A.2 Thought Library Construction

For each question $s_i \in \mathcal{S}$, we obtain its solution tree through MCTS, which provides multiple concrete solution paths for s_i . As described in Section 2.3 in the main text, we then need to identify the best reasoning path for s_i and abstract it into generalizable thought patterns. To identify the optimal trajectory for each question s_i , we employ a balanced scoring metric proposed in HiAR-ICL [23]:

$$Score(\mathbf{s}_i, \mathbf{t}_j) = b \cdot R(\mathbf{t}_j | \mathbf{s}_i) - (1 - b) \cdot C(\mathbf{t}_j), \qquad (10)$$

where $C(\mathbf{t}_j)$ represents trajectory complexity (action count), and b (set to 0.95) balances solution quality against complexity. This scoring function selects trajectories that maximize accuracy while maintaining procedural conciseness among multiple potential solutions.

For each question $\mathbf{s}_i \in \mathcal{S}$, we select the optimal solution trace $\mathbf{t}_{i,\text{best}}$ that maximizes this score. Since each node in $\mathbf{t}_{i,\text{best}}$ corresponds to an instantiated action $a_{i,l} \in \mathcal{A}$, we extract the general action sequence as a high-level thought pattern $T_j = (a_1, \dots, a_d)$. For instance, a successful solution might follow the pattern $a_1 \to a_2 \to a_4$ (System Analysis \to One-Step Thought \to Divide and Conquer).

To organize these extracted patterns effectively, we introduce Problem Condition Complexity (PCC) [27, 28] as a categorization metric. PCC quantifies the number of known prior conditions in a question s_i and can be calculated by the model π . Similar problems tend to share similar PCC values, making this metric effective for pattern aggregation.

Through this process, each question $\mathbf{s}_i \in \mathcal{S}$ is associated with its optimal thought pattern, with some questions naturally sharing identical patterns (e.g. $a_1 \to a_2 \to a_4$). Our final thought library $\mathcal{L} = \{\hat{T}_1, \dots, \hat{T}_s\}$ consists of entries where each thought template \hat{T}_j contains both a high-level

thought pattern and the average PCC of questions sharing this pattern: $\hat{T}_j = (\text{PCC}_{T_j}, T_j)$. These templates represent generalized problem-solving strategies and serve as external guidance for similar problems encountered during GRPO training.

A.3 Adaptive Retrieval and Instantiation of Thought Patterns

When encountering a new problem during GRPO training, we employ an adaptive retrieval mechanism to identify and apply the most relevant reasoning strategies from our thought library. This approach is grounded in meta-reasoning principles [29, 30], which emphasize the importance of selecting appropriate problem-solving strategies based on problem characteristics.

Adaptive Retrieval For each new incoming question \mathbf{q}_t encountered during GRPO sampling, we first compute its PCC metric. This complexity measure serves as a fingerprint that characterizes the question's structure and difficulty. We then compare this value against the PCC values of all templates in our thought library \mathcal{L} by calculating the absolute distance for each template $\hat{T}_i \in \mathcal{L}$:

$$d_j = |PCC_{q_t} - PCC_{T_j}| \tag{11}$$

This distance metric quantifies how similar the current question's complexity is to those problems from which each thought pattern was derived.

Pattern Selection and Application After computing these distances, we rank the templates and select the k most similar ones $\{\hat{T}_{i_1},\ldots,\hat{T}_{i_k}\}$ that minimize this distance measure. These selected templates contain high-level thought patterns that have proven effective for problems with similar complexity profiles.

The retrieved thought patterns, which are sequences of abstract reasoning actions (a_1,\ldots,a_d) , guide the step-by-step reasoning process for question \mathbf{q}_t . During GRPO sampling, these patterns serve as external guidance that effectively balances between exploiting known successful strategies and allowing for model-internal exploration. This adaptive retrieval mechanism ensures that the model leverages appropriate reasoning strategies based on problem characteristics, rather than attempting to apply a one-size-fits-all approach. By dynamically matching problems with relevant thought patterns, our framework enables more targeted and effective sampling across diverse problem types.

B Experimental Details

In addition to the implementation details presented in the main text, we provide supplementary experimental details here. During training, we generate with rollout parameters of temperature=0.8 and top-p=0.95, and a maximum generation of 1500 tokens. The reward function is a binary accuracy metric verified by Math-Verify, defined as $r(\mathbf{o}) = \mathbb{1}\{\mathbf{o} \text{ contains the correct final answer}\}$. Moreover, we employ a cosine learning rate decay with warm-up. The maximum learning rate is set at 3×10^{-6} , and the warm-up ratio is set at 0.1. We use the same system prompt for all experiments, as shown in Figure 9.

System Prompt: A conversation between User and Assistant. The user asks a question, and the Assistant solves it step by step. The assistant first thinks about the reasoning process in the mind and then provides the user with the answer, i.e., reasoning process here ... the answer is: \boxed{answer}.

Figure 9: The system prompt used for all experiments.

C Case Study

To further analyze the improvements of TAPO over conventional GRPO, we compare their reasoning processes on representative mathematical problems from the MATH dataset in Figure 10 and Figure 11.

In Figure 10, we observe how both methods approach an arithmetic mean problem. GRPO produces a solution with scattered notation and repetitive statements, particularly evident in its final steps where it repeatedly states "The problem involves ... final answer is 46." In contrast, TAPO demonstrates a more structured approach by explicitly introducing a "step by step" thinking process. The thought-augmented process methodically builds upon each reasoning step, clearly identifying the relationship between the original sum (984), the highest score (98), and the remaining scores to derive the lowest score (46). This structured approach results in a more readable and logically coherent solution.

Figure 11 presents a more challenging problem involving complex numbers and Vieta's formulas. Here, the limitations of GRPO become more pronounced. While GRPO initially applies the correct formula, its reasoning process deteriorates into incoherent text fragments and coding artifacts (e.g., "Tre localVEC?" and various non-mathematical expressions). This demonstrates how GRPO struggles with maintaining coherent reasoning for complex problems. In contrast, TAPO maintains its structured approach throughout, clearly stating the problem context, applying Vieta's formulas with proper explanation, and presenting a clean, coherent solution without extraneous text or errors.

D Future Directions

While TAPO demonstrates significant improvements in reasoning performance, our current work primarily focuses on mathematical reasoning and knowledge-intensive reasoning tasks. In future research, we plan to explore the application of thought-augmented reinforcement learning to more diverse tasks and domains, particularly retrieval-augmented generation and multimodal reasoning scenarios that integrate visual and textual information. In summary, TAPO establishes a strong foundation for effectively integrating external knowledge with reinforcement learning for enhanced reasoning capabilities across various AI applications.

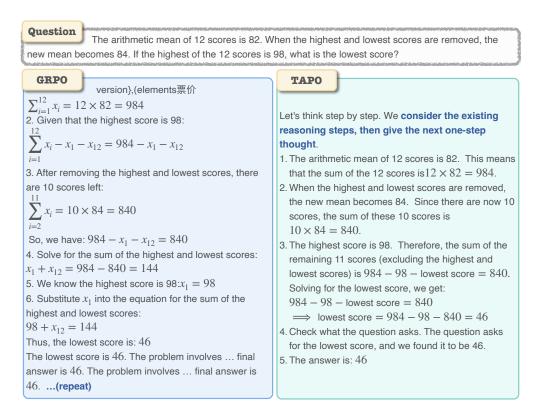


Figure 10: Comparison of GRPO and TAPO for a simple algorithm problem from the MATH dataset.

Let \$a, b, c\$ be distinct complex numbers such that
$$2b^3 + 7a^2 - 8a + 5 = 0$$
, Compute the value of the product $2c^3 + 7c^2 - 8c + 5 = 0$.

GRPO To solve for the product abc , we start by noting that a , b , and c are the roots of the polynomial equation $2x^3 + 7x^2 - 8x + 5 = 0$.

According to Vieta's formulas, for a cubic polynomial of the form $ax^3 + bx^2 + cx + d = 0$, the product of the roots taken one at a time (i.e., the product of the roots) is given by $-\frac{d}{a}$. Here, the polynomial is $2x^3 + 7x^2 - 8x + 5 = 0$, so we have $a = 2$, $b = 7$, $c = -8$, and $d = 5$. Therefore, the product of the roots is $abc = -\frac{5}{2}$.

Thus, the value of the product abc is $abc = -\frac{5}{2}$.

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The local/VEC?

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Figure 11: Comparison of GRPO and TAPO for a difficult algorithm problem from the MATH dataset.