Delauney Triangulation

Part I Application

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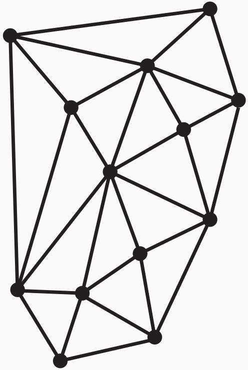
Table of Contents

Basic Ideas

Definition

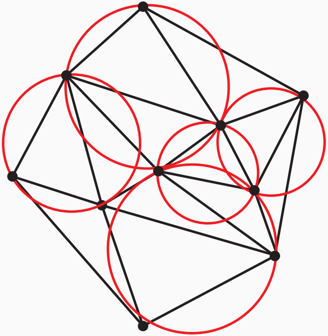


Let the convex hull of a set P of points defines a domain Ω in $\mathbb{R}^d$

The set of simplexes $\mathcal{T}_r$ is a triangulation of Ω if

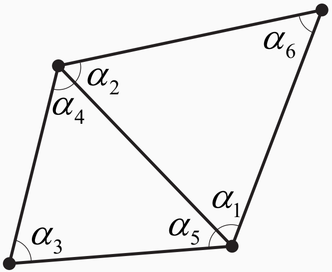
- The vertices of the elements in $\mathcal{T}_r$ is exactly P.
- $\Omega = \bigcup_{T \in \mathcal{T}_r} T$.
- The intersection of the interior of any two elements is an empty set.
- The intersection of two elements in $\mathcal{T}_r$ is either reduced to the empty set or a vertex, an edge, or a face (for $d = 3$).

Delaunay Triangulation



A Delaunay triangulation $\mathcal{DT}_r$ of a set P of points in a plane is a triangulation such that no point in P is inside the circumcircle of any triangle in $\mathcal{T}_r$

Delauney Triangulation Properties

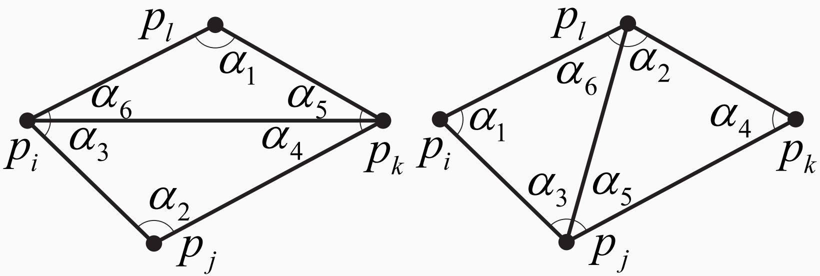


An angle vector of triangulation $\mathcal{T}_r$ is $\mathcal{A}(\mathcal{T}_r) = (\alpha_1, ..., \alpha_{3m})$ where $\alpha_1, ..., \alpha_{3m}$ are the angles of $\mathcal{T}_r$ sorted by increasing value.

Any angle-optimal in a lexicographically sense triangulation of P is a Delaunay triangulation of P.

Furthermore, any Delaunay triangulation of P maximizes the minimum angle over all triangulations of P.

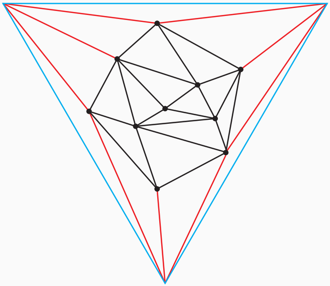
Edge Flipping



Flipping of an edge leads to changing in angle vector:

$\alpha_1, ..., \alpha_6$ are replaced by $\alpha'_1, ..., \alpha'_6$.

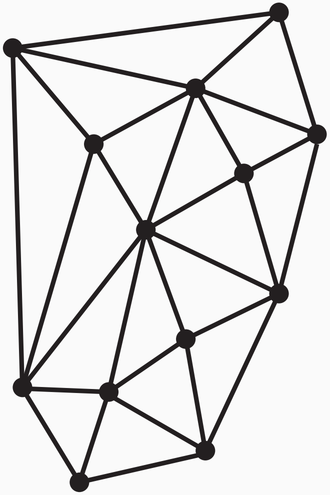
Incremental Algorithm



Incremental triangulation algorithms are based on sequential addition of points to a triangulation.

- *Step 1* Build a super triangle that contains P.
- *Step 2* Add a point to the triangulation:
    - Find triangle that contains the point.
    - If the point lies on edge, divide two adjacent triangles into four parts.
    - If the point lies in triangle interior, divide triangle into three parts.
    - Improve triangulation.
- *Step 3* Remove triangles that contains the vertices of the super triangle.

Euclidean Graph



Planar graph is a graph that can be embedded in the plain, i.e., it can be drawn on the plane in such a way that its edges intersect at their endpoints.

A Euclidean graph is a planar graph in which the vertices are embedded as a points in the Euclidean plane, and the edges are embedded as non-crossing line segments.

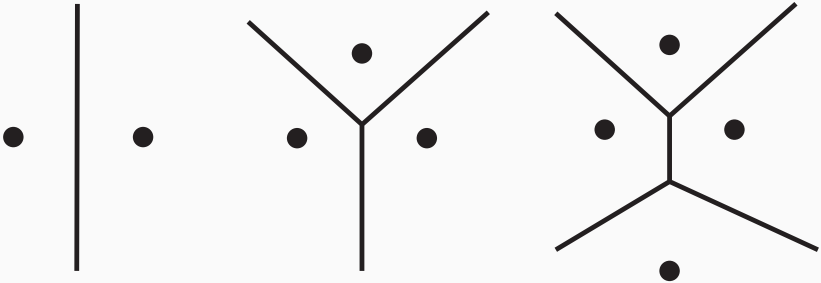
Voronoi Diagram

Problem

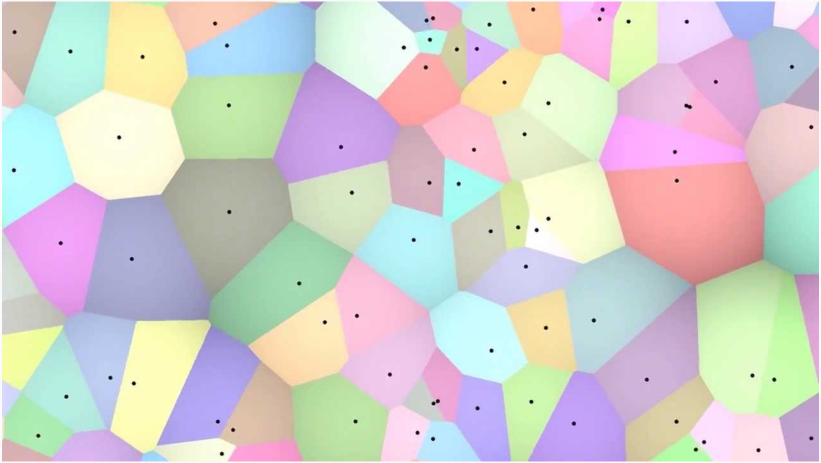
Given a set of point $P = p_1, .., i_n$ and a point p in a plane.

Goal. Find the closest point p_i to the given point p.

Simple cases



General case



Voronoi Diargram

Voronoi diagram of a set of point P is a partitioning of a plane into regions (tiles) R_k that

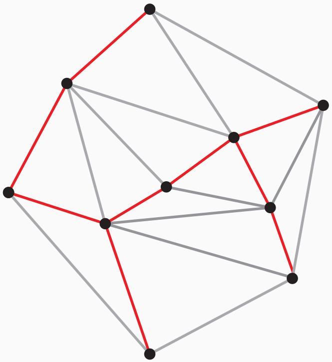
$$R_k = \{x \in \mathbb{R}^2 | d(x, p_k) \leq d(x, p_j) \forall j \neq k\}$$

Dual graph of a plane graph G is a graph that has a vertex for each face of G and an edge whenever two faces of G are separated from each other by an edge, and self-loop when the same face appears on both sides of an edge.

The Voronoi diagram is a dual graph to a Delaunay triangulation with vertices in a centers of circumcircles.

Minimal Spanning Tree

Minimum Spanning Tree



A Minimum spanning tree is a subset of the edges of a connected, edge-weighted undirected graph that connects all the vertices together, without any cycles and with the minimum possible total edge weight.

Algorithm

Prim's Algorithm

- Init a tree with a single vertex, chosen arbitrarily from the graph.
- Grow the tree by one edge: of the edges that connect the tree to vertices not yet in the tree, find the minimum weight edge, and transfer it to the tree.

Kruskal's algorithm

- Create a forest F, where each vertex in the graph is a separate tree.
- Create a set S containing all the edges in the graph.
- While S is nonempty and F is not yet spanning
  - remove an edge with minimum weight from S,
  - if the removed edge connects two different trees then add it to the forest F, combining two trees into a single tree.

Pub Crawl Problem

Given a set P of points in the plane, the Euclidean Traveling Salesperson Problem is to compute a tour (cycle) that visits all points of P and has minimum length.

A brute-force algorithm has a complexity $O(n!)$.

Approximation algorithms

If an algorithm A solves an optimization problem always within a factor k of the optimum, then A is called an k-approximation algorithm.

If an instance I of ETSP has an optimal solution of length L, then a k-approximation algorithm will find a tour of length $\leq k \cdot L$.

Visualization