Delauney Triangulation

Part I Theory

Banyas Miron

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Kyiv Algorithms Club

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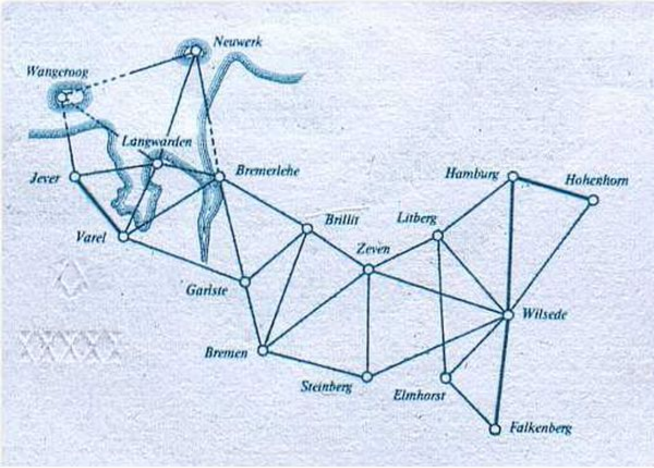
History

Carl Friedrich Gauss

Boris Delaunay

History

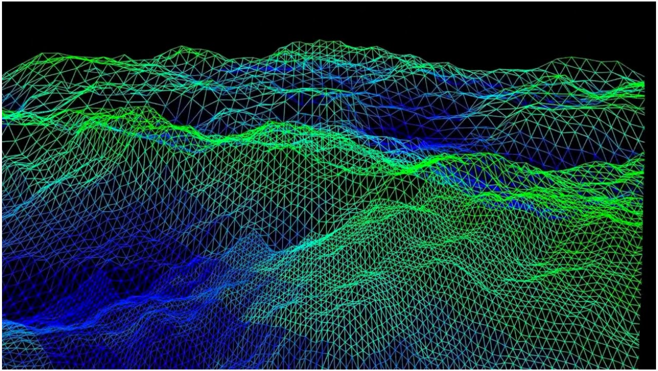


Triangulation of Kingdom of Hannover performed by Gauss (1817-1821)

Application

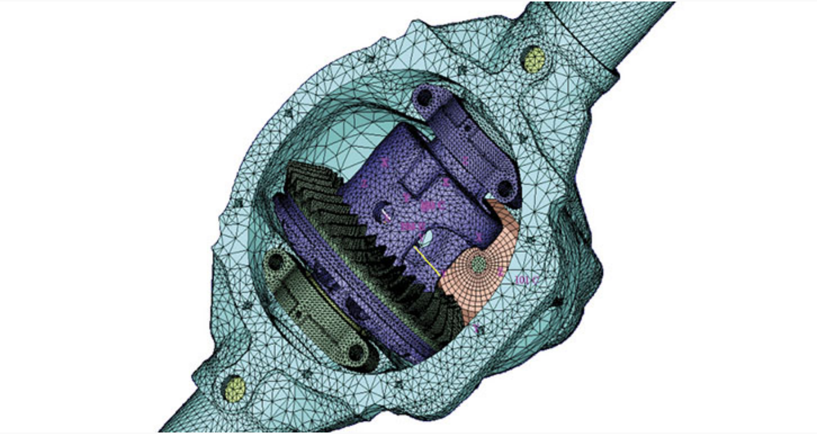
- Geodesy
- Engineering
- Image processing
- 3D scanning

Application



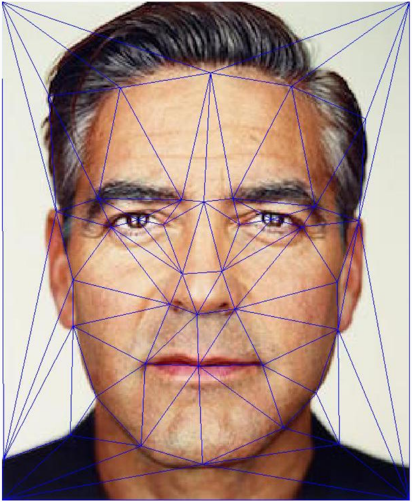
3D terrain triangulation

Application



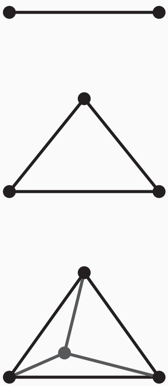
3D mesh of a vehicle differential

Application



Face recognition

Basic Defintion



A simplex is a generalization of the notion of a triangle to arbitrary dimensions.

Let $P = \{p_1 ..., p_n\}$ be a set of points in $\mathbb{R}^d$.

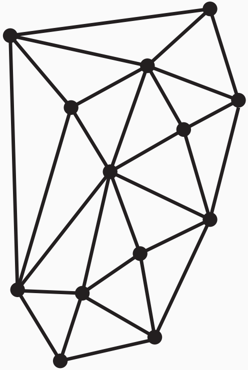
$\sum_{i=1}^{n} \lambda_i p_i$ represents a linear combination of points in P.

If, for $\sum_{i=1}^{n} \lambda_i = 1$ and $\forall i, \lambda_i \geq 0$, such combination is called a convex hull.

The convex hull of a finite number of points in $\mathbb{R}^d$ is called a polytope.

The convex hull of $k+1$ points not in an affine space of dimension $k-1$ is called a simplex or more generally a k-simplex.

Triangulation



Let the convex hull of P, denoted as $Conv(P)$, defines a domain Ω in $\mathbb{R}^d$

The set of simplexes $\mathcal{T}_r$ is a triangulation of Ω if

- The vertices of the elements in $\mathcal{T}_r$ is exactly P.
- $\Omega = \bigcup_{T \in \mathcal{T}_r} T$.
- The intersection of the interior of any two elements is an empty set.
- The intersection of two elements in $\mathcal{T}_r$ is either reduced to the empty set or a vertex, an edge, or a face (for $d = 3$).

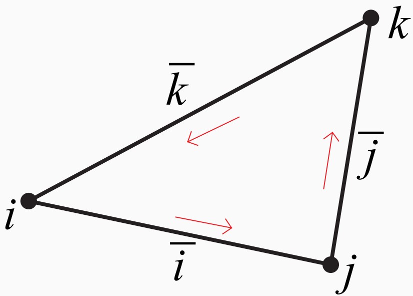
Data Structures

Triangulation Representation

A triangulation can be represented by using

- Nodes with neighbors
- Double edges
- Points and triangles
- Points, edges and triangles

Triangle



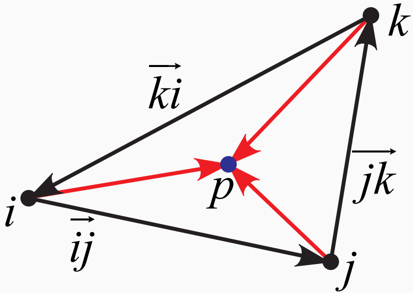
For storing a triangulation we will use a triangle data structure T_{ijk} with points i,j,k that have a counterclockwise orientation.

The edges $\bar{i}$, $\bar{j}$, $\bar{k}$ have the same orientation.

For all edges we will store adjacent triangles $T^{(i)}$, $T^{(j)}$, $T^{(k)}$.

For convenience, the triangle has a local cyclic indexing.

Triangle



A point p is inside of the triangle T_{ijk} iff the triangle edges and p makes counter-clockwise turn, i.e.

$$\vec{ij} \times \vec{ip} > 0,$$

$$\vec{jk} \times \vec{jp} > 0,$$

$$\vec{ki} \times \vec{kp} > 0.$$

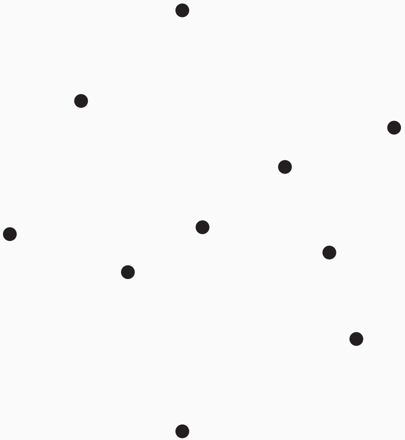
Simple Algorithm

Graham's scan

Graham's algorithm provides an efficient way for finding the convex hull of a set of points.

- *Step 1* Choose a point p_0 with the lowest x coordinate. Add p_0 to convex hull H.
- *Step 2* Sort the remaining points $\{p_1, ...p_n\}$ in angular order about p_0.
- *Step 3* Iterate points. For each point p_i:
    - If adding p_i to H results in making a "left turn", add p_i to the convex hull.
    - If adding p_i to H results in making a "right turn", remove elements from H until adding p_i makes a left turn, then add p_i to the convex hull.

Graham's Scan Visualization

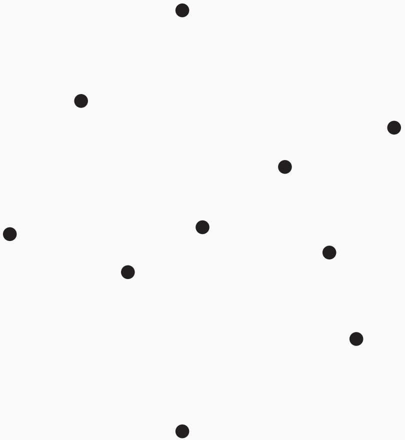


Simple Triangulation Algorithm

A simple algorithm of a triangulation can be performed with using Graham's scan.

- *Step 1* Sort P in ascending y coordinate order.
- *Step 2* Add triangle T_{012} to triangulation $\mathcal{T}_r$.
- *Step 3* Iterate points starting from p_3. For each point p_i:
  - Build convex hull H_i for points $p_0, ..., p_i$.
  - Compare H_i with H_{i-1} and find a part of H_{i-1} not in H_i with points $p_k, ..., p_k + m$.
  - Add triangles $T_{ijj+1}, j = \overline{k, m-1}$ to $\mathcal{T}_r$.

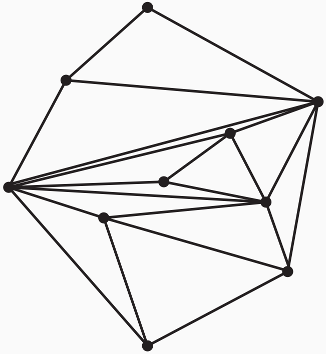
Visualization of a Simple Triangulation

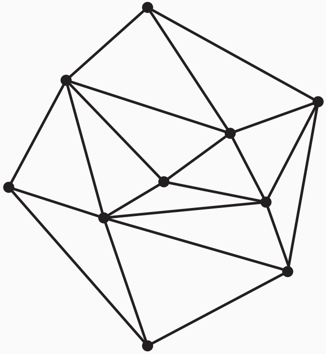


Optimal Triangulation

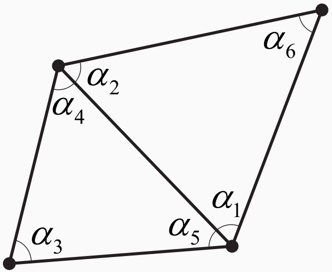
Non-unique triangulation

But which triangulation?





Angle Vector of Triangulation



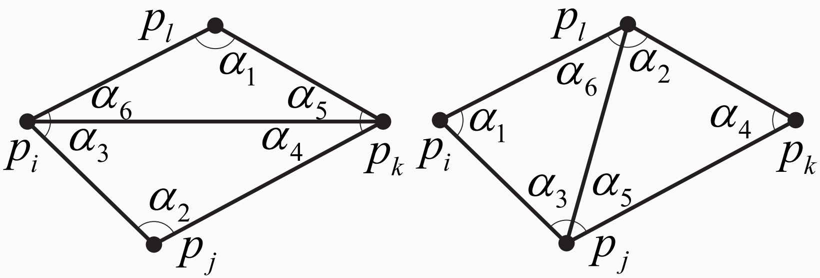
Let $\mathcal{T}_r$ be a triangulation of P with m triangles. Its angle vector is $\mathcal{A}(\mathcal{T}_r) = (\alpha_1, ..., \alpha_{3m})$ where $\alpha_1, ..., \alpha_{3m}$ are the angles of $\mathcal{T}_r$ sorted by increasing value.

Let $\mathcal{T}_r'$ another triangulation of P. We define $\mathcal{A}(\mathcal{T}_r) > \mathcal{A}(\mathcal{T}_r')$ if $\mathcal{A}(\mathcal{T}_r)$ is lexicographically larger than $\mathcal{A}(\mathcal{T}_r')$.

$\mathcal{T}_r$ is optimal if $\mathcal{A}(\mathcal{T}_r) \geq \mathcal{A}(\mathcal{T}_r')$ for all triangulations $\mathcal{A}(\mathcal{T}_r')$ of P.

Delauney Triangulation

Basic Idea. Edge Flipping

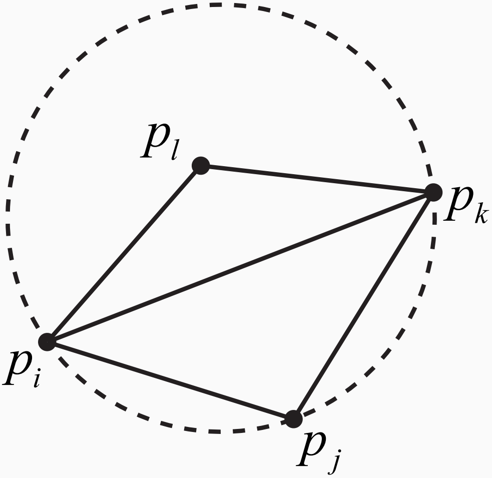


Flipping of an edge leads to changing in angle vector:

$\alpha_1, ..., \alpha_6$ are replaced by $\alpha'_1, ..., \alpha'_6$.

The edge $\overline{p_i p_j}$ is illegal if $\min_{1 \le i \le 6} \alpha_i < \min_{1 \le i \le 6} \alpha'_i$

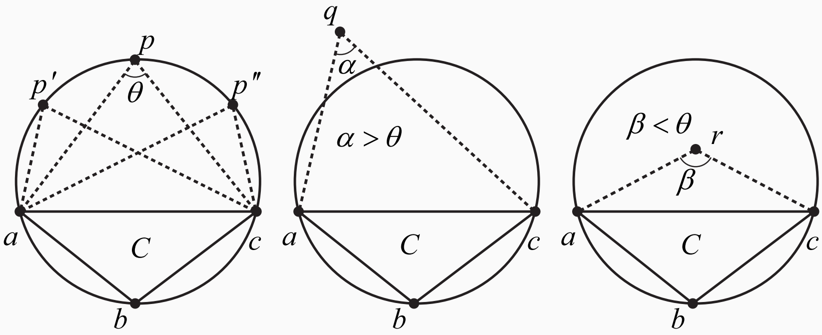
Characteristic of Illegal Edges



How do we determine if an edge is illegal?

The edge $\overline{p_i p_j}$ is illegal iff $\overline{p_l}$ lies in the interior of the circle C.

Thales' Theorem



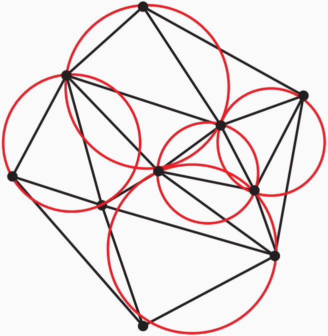
Let C be a circle, $\overline{ab}$ is a chord of C, and p, p', p'', q, r points lying on the same side of $\overline{ab}$. Suppose that p, p', p'', q lie on C. Then

$$\angle apc = \angle ap'c = \angle ap''c = \theta,$$

$$\angle aqc = \alpha, \angle arc' = \beta,$$

$$\alpha < \theta < \beta$$

Delauney triangulation

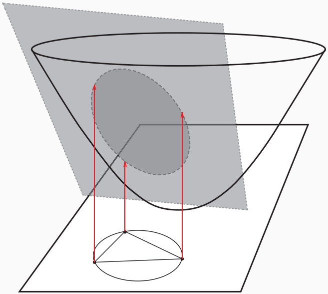


A legal triangulation is a triangulation that does not contain any illegal edge.

$\mathcal{T}_r$ is a Delaunay triangulation of P if the open (balls) discs circumscribed associated with its elements are empty.

Any angle-optimal triangulation of P is a Delaunay triangulation of P. Furthermore, any Delaunay triangulation of P maximizes the minimum angle over all triangulations of P.

Parabolic lifting

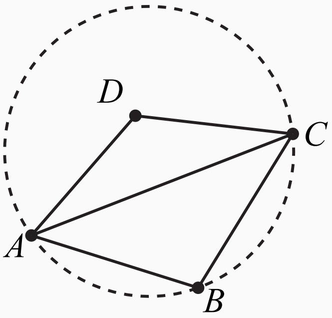


Consider the (parabolic) lifting map. For a point $p = (x, y) \in \mathbb{R}^2$ lifting $\ell(p)$ is the point

$$\ell(p) = (x, y, x^2 + y^2) \in \mathbb{R}^3$$

Let $C \subseteq \mathbb{R}^2$ be a circle of positive radius. The "lifted circle" $\ell(C)$ is contained in a unique plane $h_C in \mathbb{R}^3$. Moreover, a point $p \in \mathbb{R}^2$ is strictly inside (outside) of C iff $\ell(p)$ below(above) h_C

Parabolic lifting



Let T_{ABC} be a triangle with vertices A, B, C and D is a point outside T_{ABC}. Then the condition that D lies in circumscribed circle can be written as

$$\begin{vmatrix} x_A & y_A & x_A^2 + y_A^2 & 1 \\ x_B & y_B & x_B^2 + y_B^2 & 1 \\ x_C & y_C & x_C^2 + y_C^2 & 1 \\ x_D & y_D & x_D^2 + y_D^2 & 1 \end{vmatrix} > 0$$

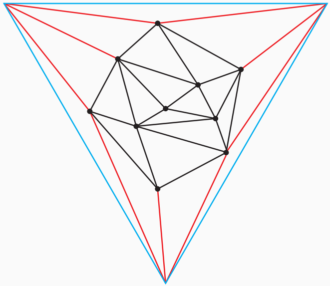
.

Robust algorithm

The robust algorithm of Delaunay triangulation based on two-pass approach.

- *Step 1* Build any a non-optimal triangulation.
- *Step 2* Iterate the triangulation while it contains a pair of triangles that have an illegal edge.
- *Step 2* If an illegal is founded, flip it.

Incremental algorithm



Incremental triangulation algorithms are based on sequential addition of points to a triangulation.

- *Step 1* Build a super triangle that contains P.
- *Step 2* Add a point to the triangulation:
  - Find triangle that contains the point.
  - If the point lies on edge, divide two adjacent triangles into four parts.
  - If the point lies in triangle interior, divide triangle into three parts.
  - Improve triangulation.
- *Step 3* Remove triangles that contains the vertices of the super triangle.