Indirect Model Reference Adaptive Control

Abstract: *This report analyses Indirect Adaptive Control. A basic theory of adaptive control is provided and then we move on to Indirect Adaptive control. The Indirect Adaptive Control is divided into the adaptive observer and the tuner and the controller. The design as well as stability proofs for adaptive observer are provided. The tuner and the controller are briefly explained as they are partially covered in the adaptive observer section. Additionally this type of controller is simulated on a second order plant which has to follow a first order reference model. The results are presented for various inputs for varying parameters of plant.*

Introduction: Adaptive control generally refers to the control of partially known systems. In real life we rarely have fixed parameters for a given system. These parameters are likely to change in face of disturbances or change in environment. The main focus of adaptive control is to design a feedback system such that the plant output tracks the reference output of the model. Interest in adaptive systems generated during the 1950s for autopilot systems in Aircrafts. The first major breakthrough in this field was by Whitaker whose 1961 paper became the famous MIT rule. However few of the major drawbacks of this approach was its difficult analysis and as well as cases of instability. A modification of this rule using Lyapunovs method was also presented by Parks.

Currently there are 3 major methods for adaptive control, which are Gain Scheduling, Self-Tuning Regulators and Model Reference Adaptive Systems. Self-Tuning Regulators and MRAS are pretty similar. One of the few major differences between them is the fact that STRs are used for stochastic process while MRAS are used for deterministic processes. In MRAS there are 2 categories of adaptive control: Direct Adatptive Control and Indirect Adaptive Control. In Indirect Adaptive control we estimate the plant parameters using an adaptive observer. The estimated measurements are then used by the tuning and the control circuit to generate the desired input to track the output reference. In Direct Control we directly adjust the control parameters.

Indirect Adaptive Control:

In Direct Control we control adjust the control parameters in a feedback loop depending upon the error between plant and the model outputs. In Indirect control we estimate the plant parameters and then adjust the control on the basis of these estimates. Here we give an initial basic overview of indirect adaptive control using a scalar system.

1) **Identification Problem:**

Consider a scalar single input-output case. The plant is represented as

$$\dot{x_p}(t) = a_p(t)x_p(t) + k_p(t)u(t)$$

And the reference model is represented as

$$\dot{x_m} = a_m x_m(t) + k_m r(t)$$

Here when the plant parameter k_p is known while the other parameter i.e. a_p is unknown. We can define new parameters

$$\theta(t) \triangleq \frac{a_m - \widehat{a_p}(t)}{k_p} \quad \text{and} \quad k^* \triangleq \frac{k_m}{k_p}$$

So that if u(t) is chosen as

$$u(t) = \theta(t)x_p(t) + k^* r(t)$$

Along with an adaptive law to update $\theta(t)$ will give:

$$\lim_{t\to\infty} \left| x_p(t) - x_m(t) \right| = 0$$

To analyze this consider an estimator of the form:

$$\dot{\widehat{x_p}} = a_m \widehat{x_p} + \left(\widehat{a_p}(t) - a_m\right)x_p + k_p u$$

The output identification error and the parameter estimation error is given as:

$$e_i(t) \triangleq x_p(t) - \widehat{x_p}(t) \quad \text{and} \quad \widetilde{a_p}(t) \triangleq \widehat{a_p}(t) - a_p$$

2) Stability Analysis:

Stability analysis is done using Lyapunov's theorem. As this system is linear, a quadratic function is chosen for stability analysis.

$$V(e_i, \widetilde{a_p}) \triangleq \frac{1}{2}[e_i^2 + \widetilde{a_p}^2]$$

The time derivative of this equation when evaluated along the trajectories of the two errors gives:

$$\dot{V} = a_m e_i^2 - e_i \widetilde{a_p} x_p + \widetilde{a_p}\dot{\widetilde{a_p}}$$

For stability we requires $\dot{V} \le 0$. This is only possible when

$$\dot{\widetilde{a_p}} = e_i x_p \quad \text{which gives us} \quad \dot{V} = a_m e_i^2 \le 0$$

Hence $e_i(t)$ and $a_p(t)$is uniformly bounded. Additionally this also gives us the adaptive law $\dot{\widetilde{a_p}} = e_i x_p$. Additionally with the given control input we can express the estimator as

$$\dot{\widehat{x_p}} = a_m \widehat{x_p}(t) + k_m r(t)$$

Additionally the reference model is expressed as

$$\dot{x_m} = a_m x_m(t) + k_m r(t)$$

Hence if we choose $\widehat{x_p}(t_0) = x_m(t_0)$, then using the above two equations we get

$$\widehat{x_p}(t) = x_m(t) \qquad \forall\, t \ge t_0$$

As x_m is uniformly bounded we get $\widehat{x_p}$ is uniformly bounded. And as $x_p = \widehat{x_p} + e_i,\ x_p$ is also bounded. Hence $e_i(t)$ is also uniformly bounded. Now $e_i \in \mathcal{L}^2$ therefore

$$\lim_{t\to\infty} \left| x_p(t) - \widehat{x_p}(t) \right| = \lim_{t\to\infty} \left| x_p(t) - x_m(t) \right| = 0$$

If the adaptive law is given by

$$\dot{\widetilde{a_p}} = \dot{\widehat{a_p}} = e_i x_p$$

The proofs described above was for scalar equations. However they can readily be extended to vector equations. Assuming that all the states in the system can be accessed we can write the plant as:

$$\dot{x_p}(t) = A_p(t)x_p(t) + B_p(t)u(t)$$

Where A_p, B_p are unknown and $A_p \in \boldsymbol{R}^{n\times n}$ and $B_p \in \boldsymbol{R}^{n\times p}$. Here we assume that A_p is asymptotically stable and u is bounded. This assumption is valid as we are trying to solve an identification problem. When we have an unstable plant in a control problem, for a properly designed system, the controller and plant together will form an asymptotically stable system. The estimator used in this case is, analogous to the scalar case, of the form

$$\dot{\widehat{x_p}} = A_m\widehat{x_p} + \left(\widehat{A_p}(t) - A_m\right)x_p + \widehat{B_p}u$$

Where, A_m is a stable (nxn) matrix. We adjust $\widehat{A_p}$ and $\widehat{B_p}$ to reduce the identifier error to zero. The state errors and the parameter errors are defined as

$$e_i(t) \triangleq x_p(t) - \widehat{x_p}(t)\,,\ \Phi(t) \triangleq \widehat{A_p}(t) - A_p,\ \ \Psi(t) \triangleq \widehat{B_p}(t) - B_p$$

The error equation is given by

$$\dot{e_i}(t) = A_m e_i(t) + \Phi(t)x_p(t) + \Psi(t)u(t)$$

Now the objective is to adjust $\widehat{A_p}(t)$ and $\widehat{B_p}(t)$ so that $e_i(t)$, $\Phi(t)$ and $\Psi(t)$ tend to zero as $t \to \infty$. We choose the adaptive laws by solving Lyapunov's equation.

$$V(e_i, \Phi, \Psi) \triangleq {e_i}^T P e_i + Tr(\Phi^T\Phi + \Psi^T\Psi)$$

Solving the equations like in the scalar case using the adaptive laws

$$\dot{\widehat{A_p}} = \dot{\Phi}(t) = -Pe_i x_p^T(t)$$

$$\dot{\widehat{B_p}} = \dot{\Psi}(t) = -Pe_i u^T(t)$$

We get

$$\dot{V} = -e_i^T Q_0 e_i \leq 0$$

3) Adaptive Observers:

The case we considered earlier we assumed that we have all the states available for measurement. However this is not necessarily always true. It might be expensive to measure states or sometimes the states may be simply unmeasurable. Hence we would like to estimate the states using only input and output and then design a controller to track the output of the reference model.

Representation:

Consider a n[th] order Single Input Single Output LTI system. It can be represented as:

$$\dot{x_p} = A_p x_p + bu$$

$$y_p = h^T x_p$$

We can describe an input-output equivalent to the LTI system as

$$\dot{x_1} = -\lambda x_1 + \theta^T \omega$$

$$\dot{\omega_1} = \Lambda \omega_1 + lu$$

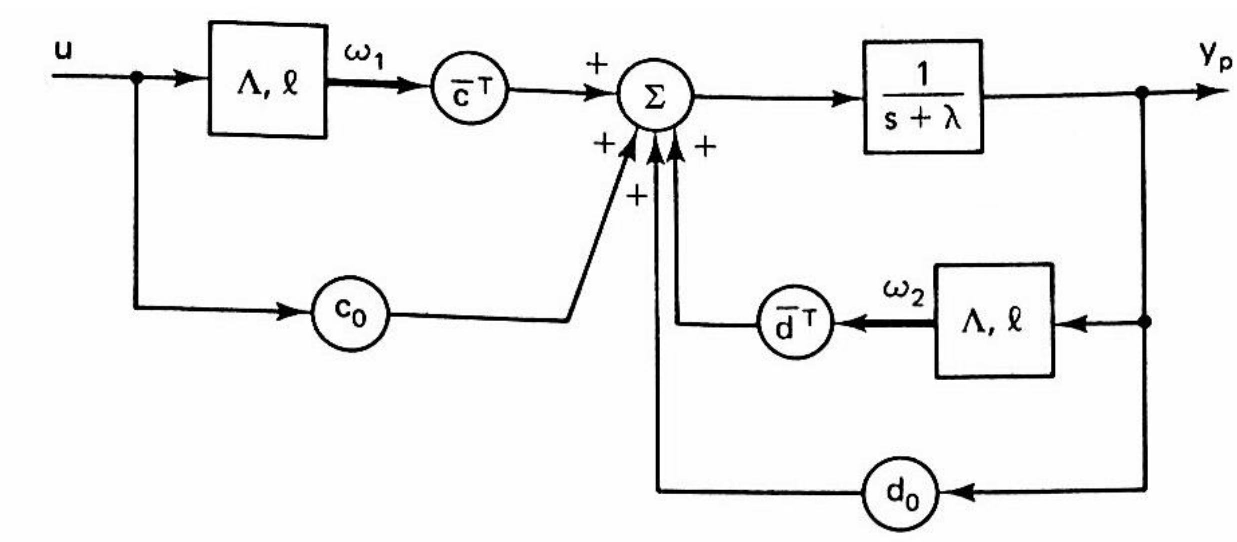
$$\dot{\omega_2} = \Lambda \omega_2 + lu$$

$$y_p = x_1$$

We define θ as the control vector

$$\theta \triangleq \begin{bmatrix} c_0 \\ \bar{c} \\ d \\ \bar{d} \end{bmatrix} \text{ and } \omega \triangleq \begin{bmatrix} u \\ \omega_1 \\ y_p \\ \omega_2 \end{bmatrix}$$

Here $\lambda > 0$ is a scalar, (Λ, l) is controllable and Λ is a (n-1)x(n-1) asymptotically stable matrix.



Defining the vectors as

$$\theta_1 \triangleq \begin{bmatrix} c_0 \\ \bar{c} \end{bmatrix}, \quad \theta_2 \triangleq \begin{bmatrix} d_0 \\ \bar{d} \end{bmatrix}, \quad \overline{\omega}_1 \triangleq \begin{bmatrix} u \\ \omega_1 \end{bmatrix}, \quad \text{and} \quad \overline{\omega}_2 \triangleq \begin{bmatrix} y_p \\ \omega_2 \end{bmatrix}$$

The transfer function from u to just before the summation block is given by

$$\bar{c}^T [sI - \Lambda]^{-1} \ell + c_0 \quad \triangleq \quad \frac{P(s)}{R(s)}$$

Here P(s) is a (n-1)th degree polynomial and R(s) is the characteristic polynomial of the asymptotically stable matrix Λ. Similarly the transfer function from y_p to $\theta_2^T \overline{\omega_2}$ is given by

$$\bar{d}^T [sI - \Lambda]^{-1} \ell + d_0 \quad \triangleq \quad \frac{Q(s)}{R(s)}$$

Here Q(s) is an (n-1)th degree polynomial.
It is seen that P(s) and Q(s) are dependent upon the elements of θ_1 and θ_2 respectively. The transfer function from u to the output y_p can be expressed as

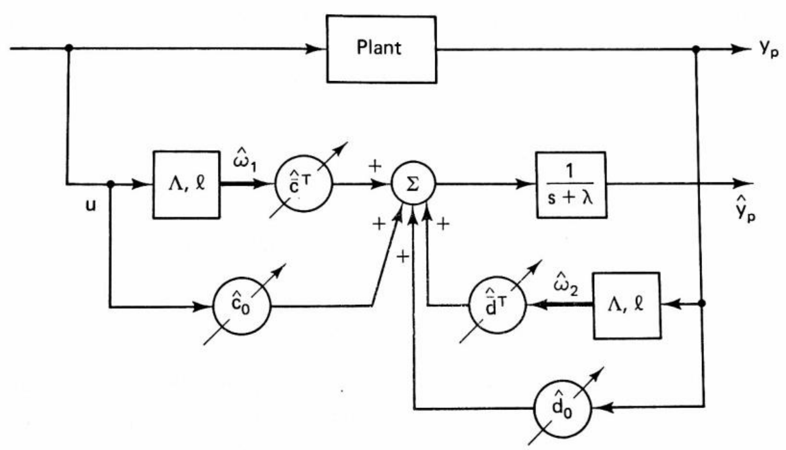
$$\begin{aligned} W_p(s) &= \frac{P(s)}{R(s)} \cdot \frac{\frac{1}{s+\lambda}}{1 - \frac{Q(s)}{R(s)(s+\lambda)}} \\ &= \frac{P(s)}{(s+\lambda)R(s) - Q(s)} \triangleq \frac{Z_p(s)}{R_p(s)}. \end{aligned}$$

Thus any LTI system can be represented as shown above. The parameter vector $\theta_1$is determined by the zeroes of the transfer function while θ_2 is determined by the poles. The adaptive observer in this form is called the parallel observer and is represented by the equations:

$$\begin{aligned} \dot{\hat{x}}_1 &= -\lambda \hat{x}_1 + \hat{\theta}^T \hat{\omega} \\ \dot{\hat{\omega}}_1 &= \Lambda \hat{\omega}_1 + \ell u \\ \dot{\hat{\omega}}_2 &= \Lambda \hat{\omega}_2 + \ell \hat{y}_p \\ \hat{y}_p &= \hat{x}_1 \end{aligned}$$

The adaptive observer shown above is the parallel observer. However the observer that we will be using for our simulation is slightly different and is called a Series Parallel Observer. The difference between this and the parallel observer is that it makes use of the plant output y_p instead of the estimated output $\hat{y}_p$

$$
\begin{aligned}
\dot{\widehat{\omega}}_1 &= \Lambda\widehat{\omega}_1 + \ell u \\
\dot{\widehat{\omega}}_2 &= \Lambda\widehat{\omega}_2 + \ell y_p \\
\dot{\widehat{x}}_1 &= -\lambda\widehat{x}_1 + \widehat{\theta}^T\widehat{\omega} \\
\widehat{y}_p &= \widehat{x}_1
\end{aligned}
$$



Series Parallel adaptive observer

Here the parameters are given as:

$$
\widehat{\theta}(t) = \begin{bmatrix} \widehat{c}_0(t) \\ \widehat{\bar{c}}(t) \\ \widehat{d}_0(t) \\ \widehat{\bar{d}}(t) \end{bmatrix}, \qquad \widehat{\omega}(t) = \begin{bmatrix} u(t) \\ \widehat{\omega}_1(t) \\ y_p(t) \\ \widehat{\omega}_2(t) \end{bmatrix}
$$

We adjust $\hat{\theta}(t)$ such that all the signals remain bounded while the error between the plant and the observer output tends to zero. Doing stability analysis using Lyapunov's theorm it can be found that the required adaptive law is

$$\dot{\hat{\theta}} = -e_1\widehat{\omega}$$

4) Tuner and Controller:

The function of the tuner and the controller is to make the system perfectly track the output reference asymptotically. The tuner tunes the parameters θ of the controller while the controller adjusts the input to ensure output tracking.

It can be shown that if the system is represented as:

$$\begin{aligned}
\dot{\omega}_1(t) &= \Lambda\omega_1(t) + \ell u(t) \\
\dot{\omega}_2(t) &= \Lambda\omega_2(t) + \ell y_p(t) \\
\omega(t) &\triangleq \left[r(t),\ \omega_1^T(t),\ y_p(t),\ \omega_2^T(t)\right]^T \\
\theta(t) &\triangleq \left[k(t), \theta_1^T(t), \theta_0(t), \theta_2^T(t)\right]^T \\
u(t) &= \theta^T(t)\omega(t)
\end{aligned}$$

Where Λ is an asymptotically stable matrix and $\lambda(s) = \det[sI - \Lambda]$

For constant values of θ the overall transfer function can be written as:

$$W_o(s) = \frac{k_c k_p Z_p(s)\lambda(s)}{(\lambda(s) - C(s))\,R_p(s) - k_p Z_p(s) D(s)}.$$

Where λ(s) is a monic polynomial of degree (n-1) and C(s) and D(s) are polynomials of degree (n-2) and (n-1) respectively. θ_1 determines the coefficients of C(s) while θ_0 and θ_2 determine the coefficients of D(s)

If we define $C^*(s)$ and $D^*(s)$ as

$$\lambda(s) - C^*(s) = Z_p(s), \qquad R_p(s) - k_p D^*(s) = R_m(s).$$

And let $\lambda(s) = Z_m(s)$ (this can be done if the relative degree of the reference model is 1)

Then θ^* exists such that

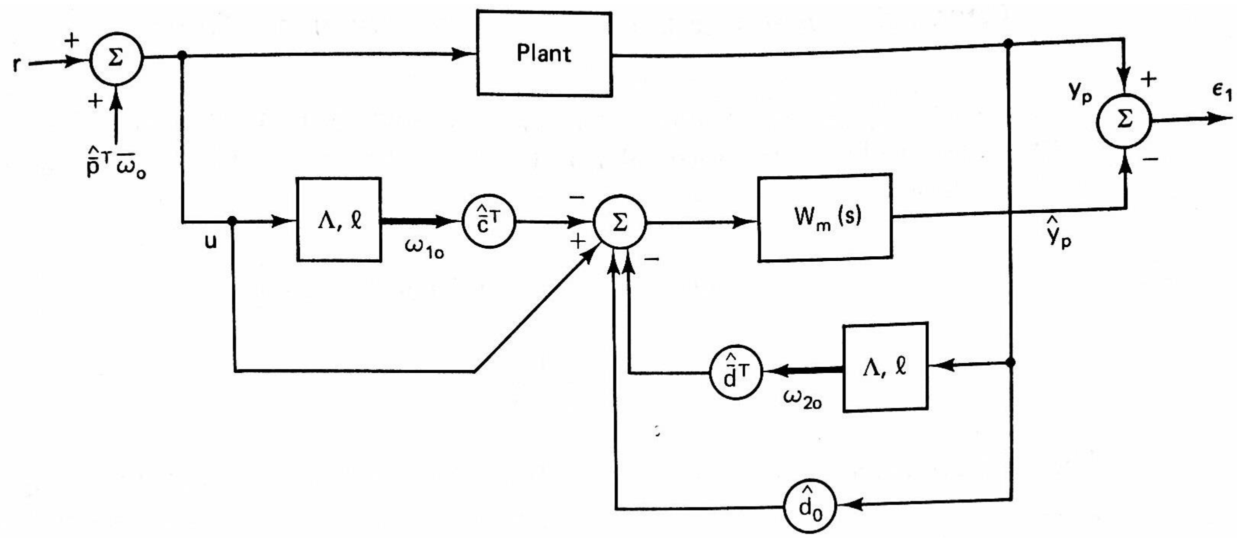
$$k^* = \frac{k_m}{k_p}, \quad \theta_1^{*T}(sI - \Lambda)^{-1}\ell = \frac{C^*(s)}{\lambda(s)}, \text{ and } \theta_0^* + \theta_2^{*T}(sI - \Lambda)^{-1}\ell = \frac{D^*(s)}{\lambda(s)}.$$

The transfer function in this case becomes:

$$W_o(s) = k_m \frac{Z_p(s) Z_m(s)}{Z_p(s)\left[R_p(s) - k_p D^*(s)\right]} = W_m(s).$$

Thus if the system is represented in the given form we can create controllers such that they transfer function of the controller and plant will be the same as the transfer function of the reference model.

5) Simulation:



The final control algorithm combines the observer, tuner and controller described previously. The structure is shown above

Assumptions:

1) The relative degree of W_p and W_m is one.
2) k_p=k_m=1 i.e the numerator and denominator of both polynomials W_p and W_m is monic.
3) The numerator of both W_p and W_m is also Hurwitz.

Adaptive Observer:

$$\begin{aligned} \dot{\omega}_{1o} &= \Lambda\omega_{1o} + \ell u \qquad \overline{\omega}_o = [\omega_{1o}^T, y_p, \omega_{2o}^T]^T \\ \dot{\omega}_{2o} &= \Lambda\omega_{2o} + \ell y_p \\ \widehat{y}_p &= W_m(s)\left[u - \widehat{\overline{p}}^T \overline{\omega}_o\right] \qquad \widehat{\overline{p}} = [\widehat{\overline{c}}^T, \widehat{d}_0, \widehat{\overline{d}}^T]^T \end{aligned}$$

Here the adaptive law to minimize the Identification error is

$$\dot{\hat{p}} = -e_1\overline{\omega_0}$$

The adaptive controller is given by:

$$u = r + \hat{p}^T(t)\,\overline{\omega_0}$$

Parameters:

We choose:

Λ = -1 l = 1 $W_m(s) = \frac{1}{(s+3)}$

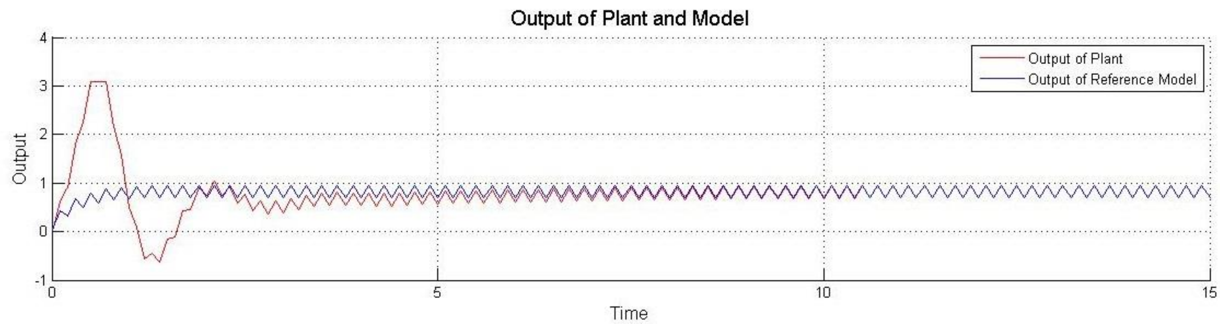
The plant is $W_p(s) = \frac{(s+5)}{(s^2+a_1 s+a_0)}$ where the unknown elements lie in a compact set

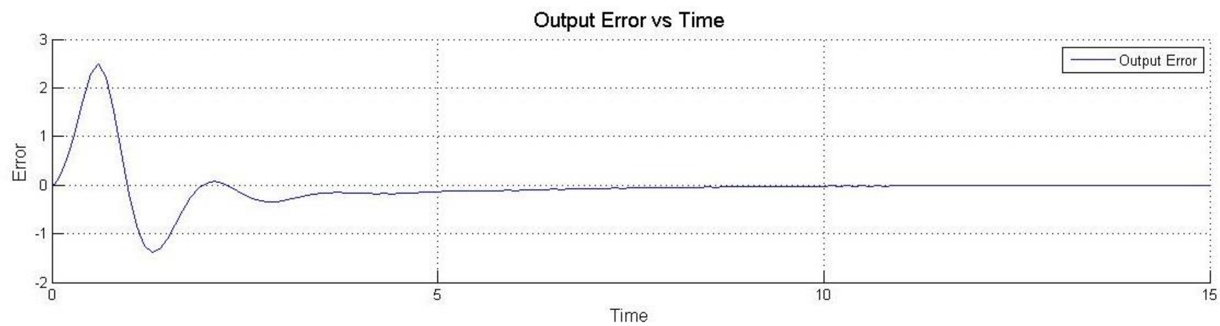
$$S = \{-0.6 \le a_1 \le 3.4, -2 \le a_0 \le 2\}$$

Simulation Results:

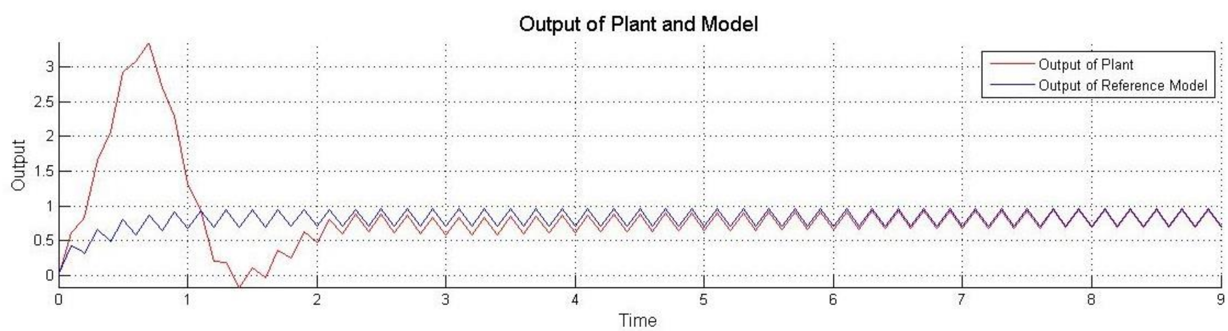
Square Wave Input:

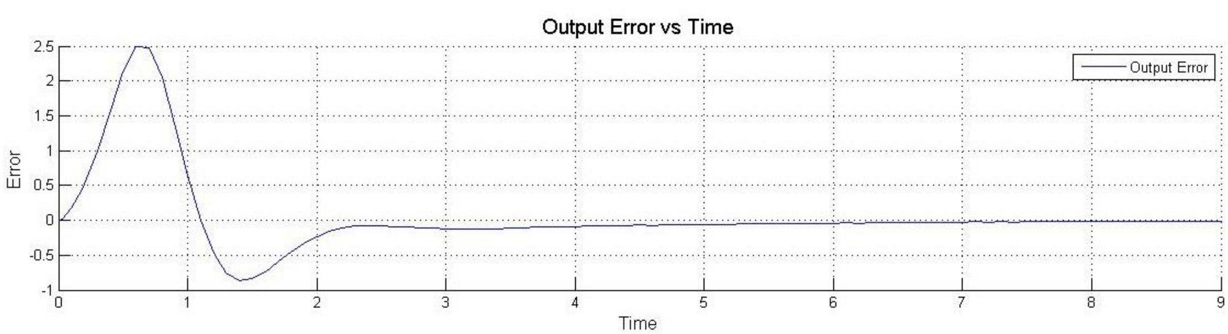
1) Unstable and oscillatory Plant: $a_1 = -0.5$ and $a_0 = 2$



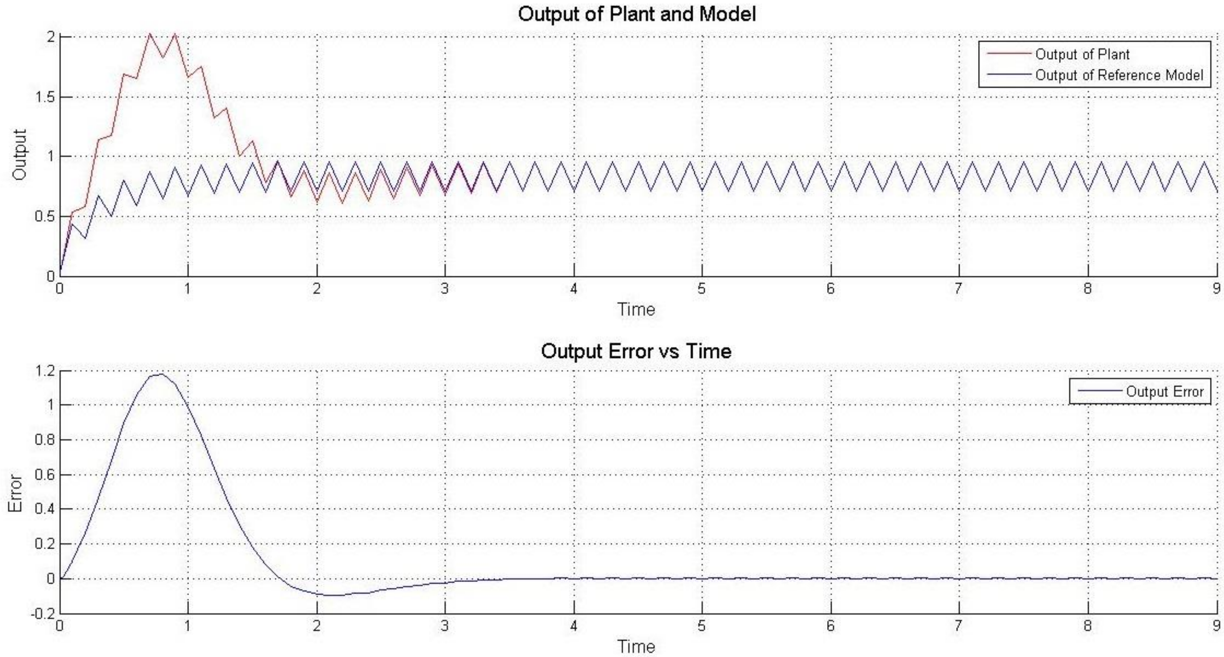


2) Unstable and nonoscillatory plant: $a_1 = 0.5$ and $a_0 = -2$



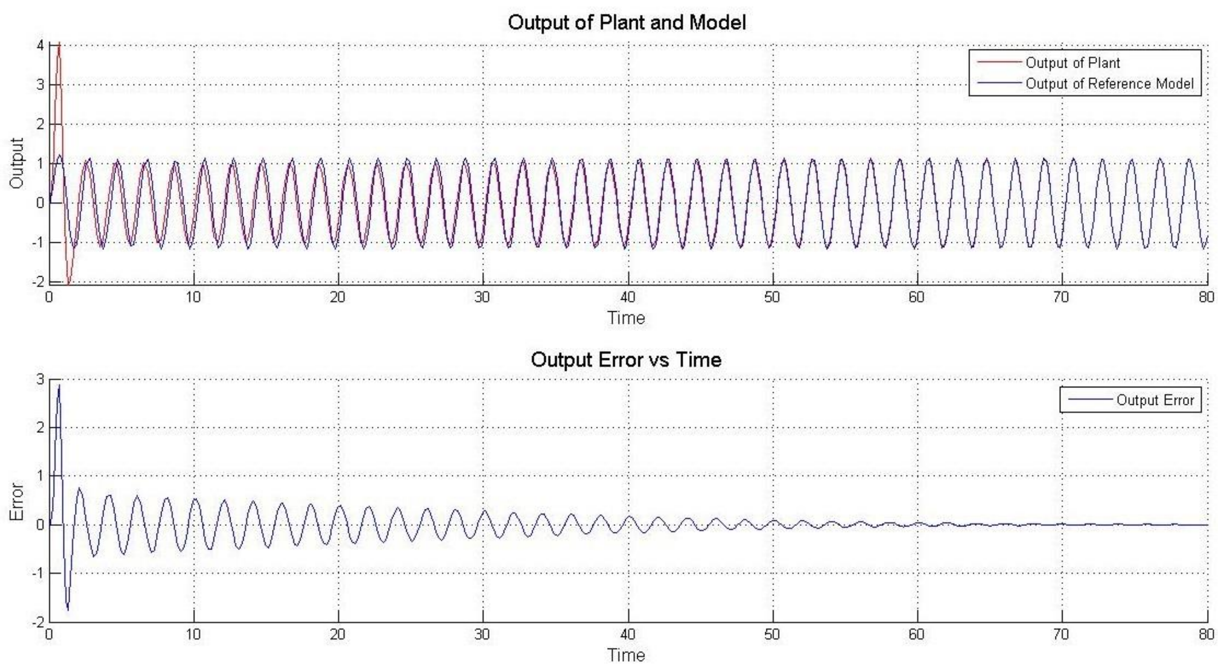


3) Stable and nonoscillatory plant: $a_1 = 3.4$ and $a_0 = 2$

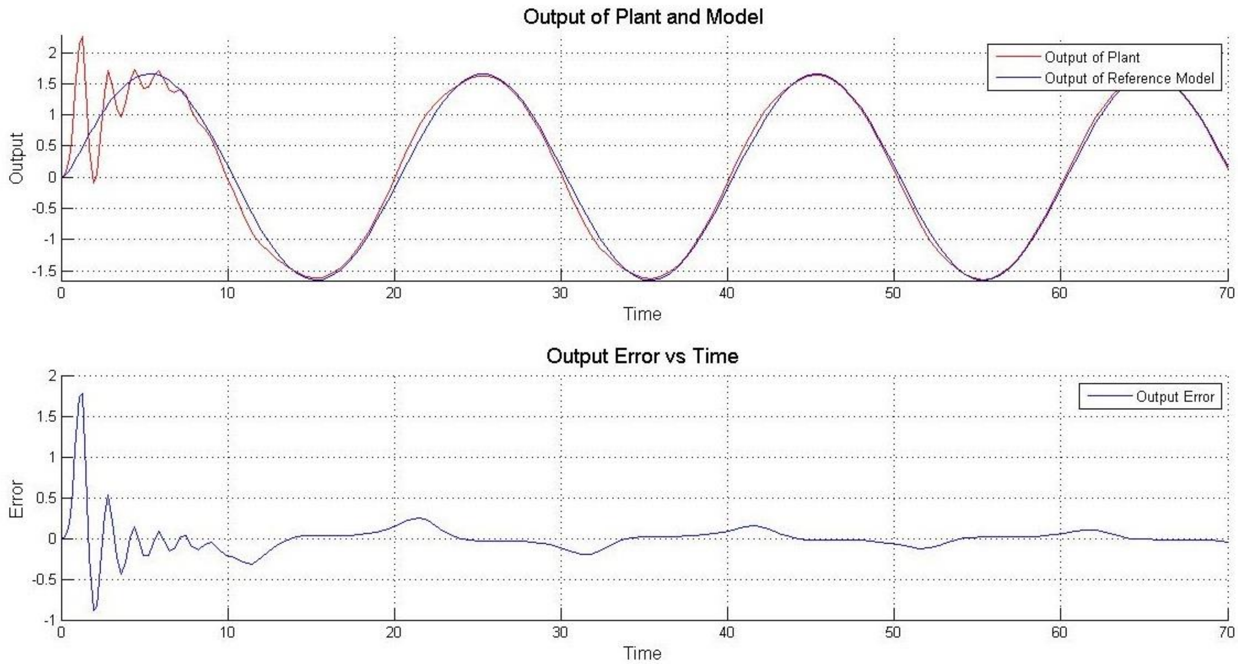


Sinusoidal Input:

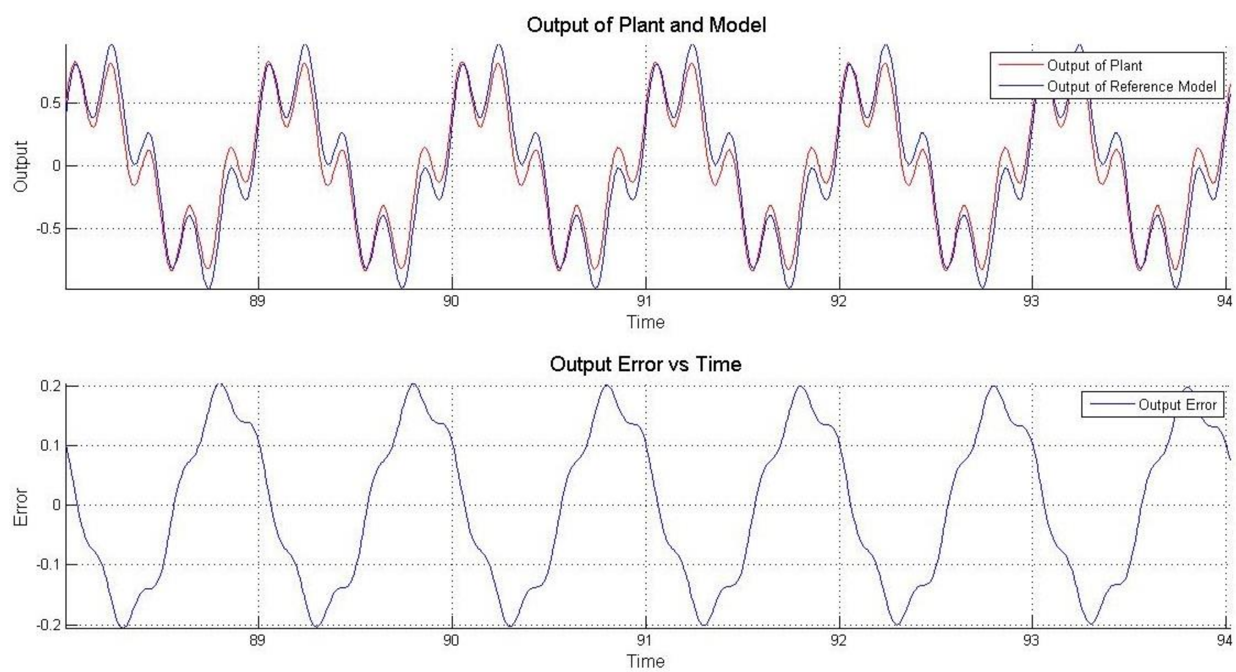
1) Sinusoidal input to an unstable and oscillatory plant: $a_1 = -0.5$ and $a_0 = 2$



2) Low frequency sinusoidal input to an unstable and oscillatory plant:

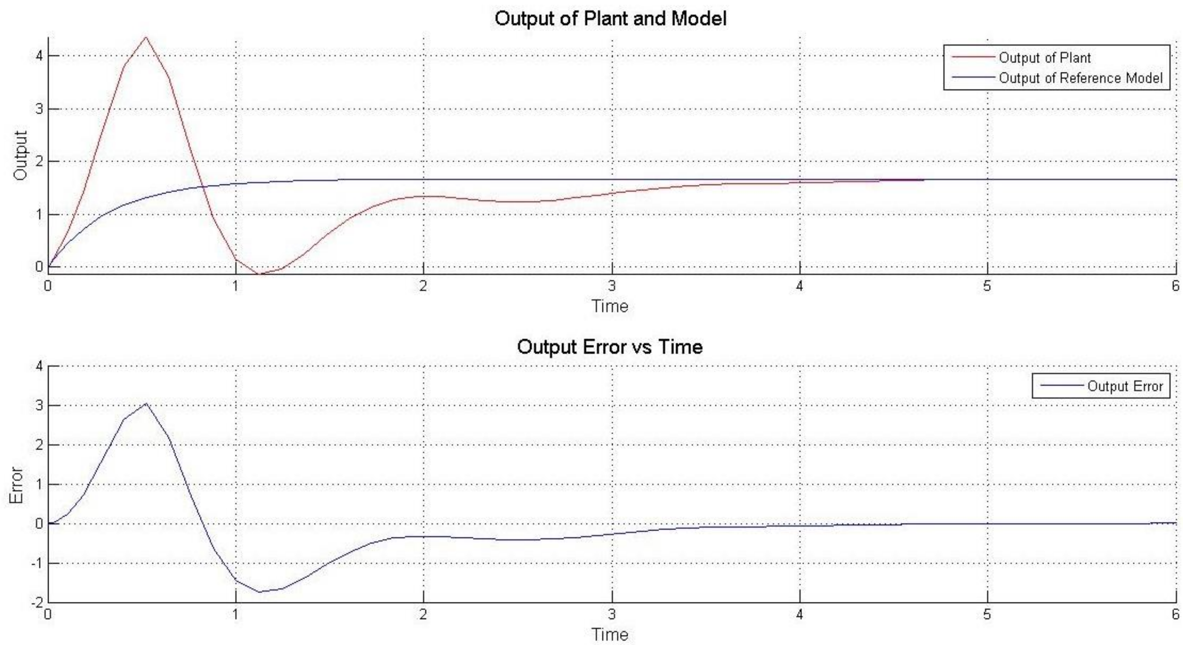


3) Combination of Low and High frequency inputs to an unstable and oscillatory plant:

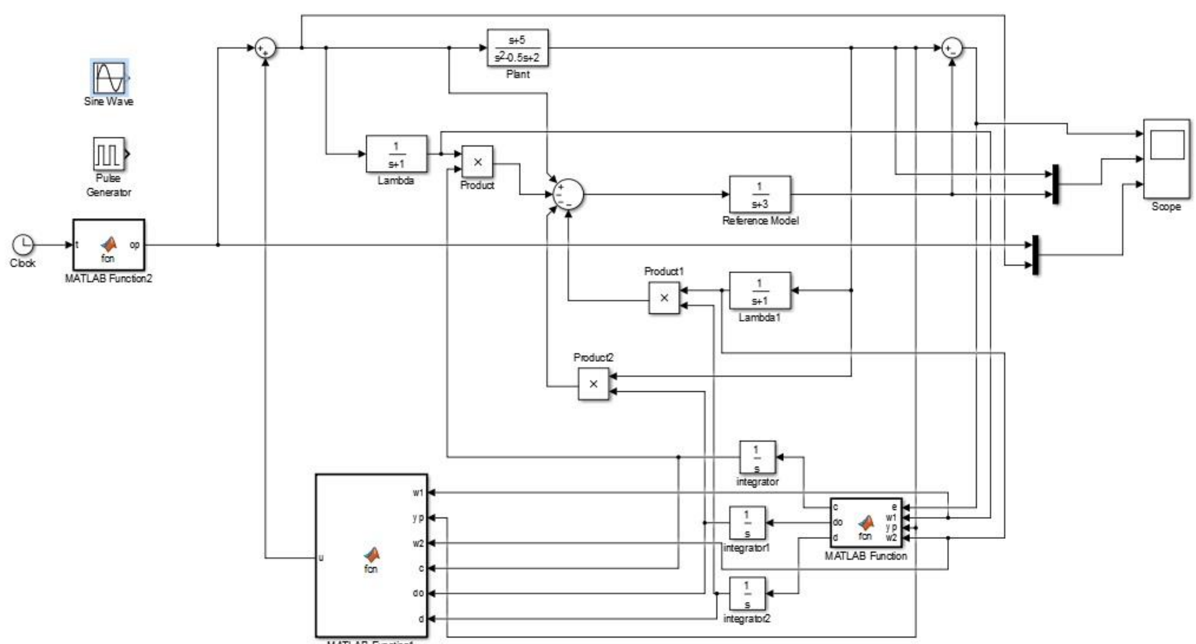


Cosntant Input:

Constant Input to an unstable and oscillatory plant:



Simulation Model:



References:

[1] K. S. Narendra and J. Balakrishnan, "Improving transient response of adaptive control systems using multiple models and switching," *IEEE Transactions on Automatic Control*, vol. 39, no. 9, pp. 1861–1866, Sep. 1994.

[2] R. H. Middleton, G. C. Goodwin, D. J. Hill, and D. Q. Mayne, "Design issues in adaptive control," *IEEE Transactions on Automatic Control*, vol. 33, no. 1, pp. 50–58, Jan. 1988.

[3] K. S. Narendra and J. Balakrishnan, "Adaptive control using multiple models," *IEEE Transactions on Automatic Control*, vol. 42, no. 2, pp. 171–187, Feb. 1997.

[4] P. Parks, "Liapunov redesign of model reference adaptive control systems," *IEEE Transactions on Automatic Control*, vol. 11, no. 3, pp. 362–367, Jul. 1966.

[5] C. Hang and P. Parks, "Comparative studies of model reference adaptive control systems," *IEEE Transactions on Automatic Control*, vol. 18, no. 5, pp. 419–428, Oct. 1973.

[6] K. S. Narendra and A. M. Annaswamy, Stable Adaptive Systems. Englewood Cliffs, NJ: Prentice-Hall, 1989.