First Midterm (A)

Please write down your student ID, your name and the letter in the above parentheses on your answer sheets.

1. (30%) Solving recurrences:
   (a) (5%) $T(n) = 7T(\frac{n}{2}) + \Theta(n^2) = \Theta(g(n))$. Solve $g(n)$.
   (b) (5%) $T(n) = 5T(\frac{n}{5}) + \Theta(n) = \Theta(g(n))$. Solve $g(n)$.
   (c) (5%) $T(n) = 11T(\frac{n}{4}) + \Theta(n^2) = \Theta(g(n))$. Solve $g(n)$.
   (d) (5%) $T(n) = \sqrt{n}T(\sqrt{n}) + \Theta(n) = \Theta(g(n))$. Solve $g(n)$.
   (e) (5%) $T(n) = T(\frac{2n}{3}) + T(\frac{n}{3}) + \Theta(n \log^2 n) = \Theta(g(n))$. Solve $g(n)$.
   (f) (5%) $T(n) = 1 + \frac{1}{n}\sum_{k=1}^{n-1} T(k) = \Theta(g(n))$. Solve $g(n)$.

2. (10%) $T(1) = 1$ and $T(n) = T(\frac{n}{2}) + T(\frac{n}{4}) + 1$ for $n > 1$. Prove or disprove that $T(n) = o(n)$.

3. (4%) Given $12345 \times 32090 = 396151050$ and $87655 \times 35800 = 3138049000$. Compute $1234587655 \times 3209035800$.

4. (10%) Given the preorder traversal $\langle x_1, \ldots, x_n \rangle$ of an n-node binary search tree T.
   (a) (6%) Describe a $O(\log n)$-time algorithm to compute the number of nodes of the left subtree of T.
   (b) (4%) Use your algorithm to compute the size of left subtree of the binary search tree whose preorder traversal is $\langle 4, 1, 3, 2, 9, 7, 5, 6, 8 \rangle$.

5. (10%) Given an integral sequence $\langle a_1, \ldots, a_n \rangle$. Describe a $O(n)$-time algorithm to compute the minimum value of $f(x) = \sum_{i=1}^{n} |x - a_i|$, and prove it is correct.

6. (10%) Given an unsorted array `A[1..n]`. Describe a $O(n)$-time algorithm to construct a maximum binary heap, and prove it is correct.

7. (10%) Rod cutting problem: Suppose cutting a rod of length ℓ costs exactly ℓ dollars and the price of a rod of length ℓ is p_ℓ.

ℓ	1	2	3	4	5	6	7	8	9	10
p_ℓ	2	25	39	42	59	69	70	89	99	100

   (a) (5%) What is the maximum net profit to buy a rod of length 10 for 100 dollars?
   (b) (5%) How to achieve the maximum net profit? Describe an optimal cutting.

8. (10%) Matrix-chain multiplication: Solve the instance $(5, \langle 5, 1, 9, 8, 5, 4 \rangle)$, and give the optimal way to multiply $A_1A_2A_3A_4A_5$.

9. (4%) Find all longest common subsequences of $\langle \texttt{O,K,I,N,A,W,A} \rangle$ and $\langle \texttt{O,K,A,H,A,N,A} \rangle$.

10. (10%) Given an integral sequence $S = \langle s_1, \ldots, s_n \rangle$. Let $L[i]$ denote the length of the longest non-decreasing subsequence of S ending at s_i. Define $\ell = \max_{1 \le i \le n} L[i]$ and $b_k = \min_{i:1 \le i \le n, L[i]=k} s_i$ for $1 \le k \le \ell$. Show that the sequence $\langle b_1, \ldots, b_\ell \rangle$ is non-decreasing. Use this fact to describe a $O(n \log n)$-time algorithm to compute the longest non-decreasing subsequence.

11. (10%) Suppose you have to process n tasks $T_1, \ldots, T_n$. It takes t_i minutes to complete processing task T_i, and no two tasks can be processed simultaneously. The tasks can be processed in any order, so you may process T_p either before or after processing T_q where $1 \le p < q \le n$. Let c_i be the completion time of T_i, i.e., you complete processing task T_i at time c_i. For example, there are two tasks T_1 and T_2. Let $t_1 = 2$ and $t_2 = 4$. If you process T_1 first, then $c_1 = 2$ and $c_2 = 2 + 4 = 6$. If you process T_2 first, then $c_2 = 4$ and $c_1 = 4 + 2 = 6$. Describe an algorithm to schedule the tasks so that the average completion time $\frac{1}{n}\sum_{i=1}^{n} c_i$ is minimized. Prove your algorithm is correct, and analyze its time complexity.

12. (10%) Suppose there are two power plants p_1, p_2 and n houses $h_1, \ldots, h_n$ on an island. In order to supply electricity to every house, we need to connect the power cables properly. A house is properly connected if it is directly connected to a power plant or another house which is properly connected. For $u, v \in \{p_1, p_2\} \cup \{h_1, \ldots, h_n\}$, let $w(u, v)$ be the cost of directly connecting u and v. Give a $O(n^2)$-time algorithm to compute the minimum total cost to connect all houses properly, and show your algorithm is correct.