1 Topological space

Definition: Characterizations of the category of topological space

(I). Definition via open sets (開集合系)
all pairs $(X,\ \mathfrak{O})$ of set X together with a collection $\mathfrak{O}$ of subsets of X satisfying:
1. $\phi,\ X \in \mathfrak{O}$
2. $(O_\lambda)_{\lambda\in\Lambda},\ |\Lambda| < \aleph_0,\ \forall\lambda \in \Lambda(O_\lambda \in \mathfrak{O}) \Rightarrow \cup_{\lambda\in\Lambda} O_\lambda \in \mathfrak{O}$
3. $(O_\lambda)_{\lambda\in\Lambda},\ \forall\lambda \in \Lambda(O_\lambda \in \mathfrak{O}) \Rightarrow \cap_{\lambda\in\Lambda} O_\lambda \in \mathfrak{O}$

(II). Definition via closed sets (閉集合系)
all pairs $(X,\ \mathfrak{C})$ of set X together with a collection $\mathfrak{C}$ of subsets of X satisfying:
1. $\phi,\ X \in \mathfrak{C}$
2. $(C_\lambda)_{\lambda\in\Lambda},\ \forall\lambda \in \Lambda(C_\lambda \in \mathfrak{C}) \Rightarrow \cup_{\lambda\in\Lambda} C_\lambda \in \mathfrak{C}$
3. $(C_\lambda)_{\lambda\in\Lambda},\ |\Lambda| < \aleph_0,\ \forall\lambda \in \Lambda(C_\lambda \in \mathfrak{C}) \Rightarrow \cap_{\lambda\in\Lambda} C_\lambda \in \mathfrak{C}$

(III). Definition via interior operators (開核作用子)
all pairs $(X,\ int)$ of set X together with ans interior operator $int : \mathfrak{P}(X) \to \mathfrak{P}(X)$ satisfying:
1. $int(\phi) = \phi \quad (\Leftrightarrow int(X) = X)$
2. $\forall M \in \mathfrak{P}(X),\ int(M) \subset M$
3. $\forall M, N \in \mathfrak{P}(X),\ int(M \cap N) = int(M) \cap int(N)$
4. $\forall M \in \mathfrak{P}(X),\ int(int(M)) = int(M)$

(IV). Definition via closure operators (閉包作用子)
all pairs $(X,\ cl)$ of set X together with ans closure operator $cl : \mathfrak{P}(X) \to \mathfrak{P}(X)$ satisfying:
1. $cl(\phi) = \phi \quad (\Leftrightarrow cl(X) = X)$
2. $\forall M \in \mathfrak{P}(X),\ M \subset cl(M)$
3. $\forall M, N \in \mathfrak{P}(X),\ cl(M \cup N) = cl(M) \cup cl(N)$
4. $\forall M \in \mathfrak{P}(X),\ cl(cl(M)) = cl(M)$

(V). Definition via neighbourhoods (近傍系) all pairs $(X,\ V)$ of set X together with a neighbourhood function $V : X \to \mathfrak{P}(X)$
1. $\forall V \in V(x),\ x \in V$
2. $\forall U \subseteq X,\ V \in V(x),\ V \subset U \Rightarrow U \in V(x)$
3. $(V_\lambda)_{\lambda\in\Lambda}, |\Lambda| < \aleph_0,\ \forall\lambda \in \Lambda(V_\lambda \in V(x)) \Rightarrow \cap_{\lambda\in\Lambda} V_\lambda \in V(x)$
4. $\forall V \in V(x),\ \exists W \in V(x) : \forall y \in W,\ V \in V(y)$, ($W = V^i$ meet the requirement.)

Definition: Comparison of topologies (位相の比較)

$$\mathfrak{O}_1 \subset \mathfrak{O}_2 \equiv \text{“ } \mathfrak{O}_1 \text{ is coarser than } \mathfrak{O}_2 \text{ ”} \equiv \text{“ } \mathfrak{O}_2 \text{ is finer than } \mathfrak{O}_1 \text{ ”},\ (\mathfrak{O},\ \subset) \text{ is an order set.}$$

Definition: Topology generation (位相の生成)
Let X be a set, $\mathfrak{X}$ is a subset of $\mathfrak{P}(X)$ i.e. $\mathfrak{X} \subset \mathfrak{P}(X)$

The corsest topology including $\mathfrak{X}$ is denoted by $\mathfrak{O}(\mathfrak{X})$, it is called the topology generated by $\mathfrak{X}$.

The following is how to generate any topology from $\mathfrak{X}$:
1. $\mathfrak{X}_\circ = \cap U \quad (U \in \mathfrak{X})$
2. $\mathfrak{O}(\mathfrak{X}) = \cup V \quad (V \in \mathfrak{X}_\circ)$

Theorem: the supremum and infimum of topology set

$$\inf\{\mathfrak{O}_\lambda \mid \lambda \in \Lambda\} = \cap_{\lambda\in\Lambda}\mathfrak{O}_\lambda,\ \sup\{\mathfrak{O}_\lambda \mid \lambda \in \Lambda\} = \mathfrak{O}(\cup_{\lambda\in\Lambda}\mathfrak{O}_\lambda)$$

Definition: Induced topology (誘導位相)
1. induced from a set Y.
Let X be a set, $(Y,\ \mathfrak{O}_Y)$ be a topological space, f is continuous $X \mapsto Y$.
The induced topology on X is defined by:

$$\mathfrak{O}_X = \{f^{-1}(U) \mid U \in \mathfrak{O}_Y\}$$

キ f が連続関数となるように誘導位相 $\mathfrak{O}_X$ 定める.f が連続となるような X の位相の中では最弱.

2. induced from a collection of sets $(Y_\lambda)_{\lambda\in\Lambda}$
Let X be a set, $(Y_\lambda,\ \mathfrak{O}_\lambda)_{\lambda\in\Lambda}$ be a collection of a topology space, f_λ is continuous $X \mapsto Y_\lambda$.
The induced topology on X is defined by:

$$\sup\{\mathfrak{O}_\lambda \mid \lambda \in \Lambda\},\ \mathfrak{O}_\lambda = \{f_\lambda^{-1}(U) \mid U \in \mathfrak{O}_\lambda\}.$$

㊟ f_λ が全て連続となるような X の位相の中では最弱の位相を求めている.

Definition: Subspace topology (部分位相)
Let $(Y,\ \mathfrak{O}_Y)$ be a topological space, X be a subset of Y i.e. $X \subset Y$
The subspace topology on X is defined by:

$$\mathfrak{O}_X = \{U \cap Y \mid U \in \mathfrak{O}_Y\}$$

㊟ 部分集合に与える自然な位相. 包含写像による誘導位相.

Definition: Cartesian topology (直積位相)
Let $(X_\lambda,\ \mathfrak{O}_\lambda)_{\lambda\in\Lambda}$ be a collection of topological space.
The cartesian topology $\mathfrak{O}$ is induced on $\prod_{\lambda\in\Lambda} X_\lambda$ by projection $pr_\lambda : \prod_{\lambda\in\Lambda} X_\lambda \to X_\lambda$. The Cartesian topology on $\prod_{\lambda\in\Lambda} X_\lambda$ is defined by:

$$\sup\{\mathfrak{O}_\lambda^* \mid \lambda \in \Lambda\} = \mathfrak{O}(\cup_{\lambda\in\Lambda}\mathfrak{O}_\lambda^*)$$
$$\mathfrak{O}_\lambda^* = \{pr_\lambda^{-1}(O_\lambda) \mid O_\lambda \in \mathfrak{O}_\lambda\} = \prod_{\iota\in(\Lambda-\lambda)} X_\iota \times O_\lambda \quad (O_\lambda \in \mathfrak{O}_\lambda)$$

㊟ 直積集合に与える自然な位相. 射影による誘導位相.

Definition: Quotient space (商位相)
Let X/R be a quotient set, $(X,\ \mathfrak{O}_X)$ be a topological space, π is natural projection $X \mapsto X/R$
The Quotient space $\mathfrak{O}$ is defined by:

$$\mathfrak{O}_{X/R} = \{\pi(U) \mid \pi^{-1}(U) \in \mathfrak{O}_X\}.$$

㊟ 商集合に与える自然な位相. 自然な射影による誘導位相.

Definition: Points in topological space.

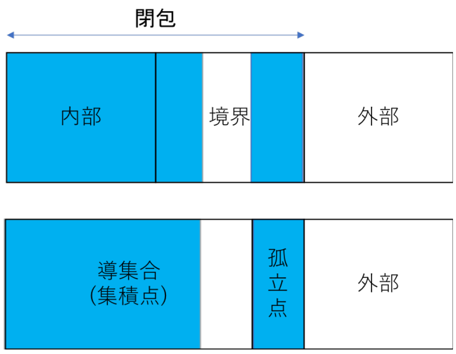

Fig. 1 Points in topological space

Definition: Base and Subbase
Let $(X,\ \mathfrak{O})$ be a topological space, $\mathfrak{B}$ be a collection of open set.
The Base is defined by:

$$\forall O \in \mathfrak{O},\ O = \cup_{\lambda\in\Lambda} B_\lambda \quad (B_\lambda \in \mathfrak{B})$$

Let X be a set, the base of the corsest topology $\mathfrak{O}$ on X including $\mathfrak{B}$ satisfying:

$$\cup\mathfrak{B} = X,\ \forall B_1,\ B_2 \in \mathfrak{B},\ \exists\mathfrak{B}' \subset \mathfrak{B}\ s.t.\ B_1 \cap B_2 = \cup\mathfrak{B}'$$

The subbase of the $\mathfrak{O}$ is satisfying:

$$\cup \mathfrak{B} = X$$

Definition: Separation axioms

キ 位相が弱すぎないための条件

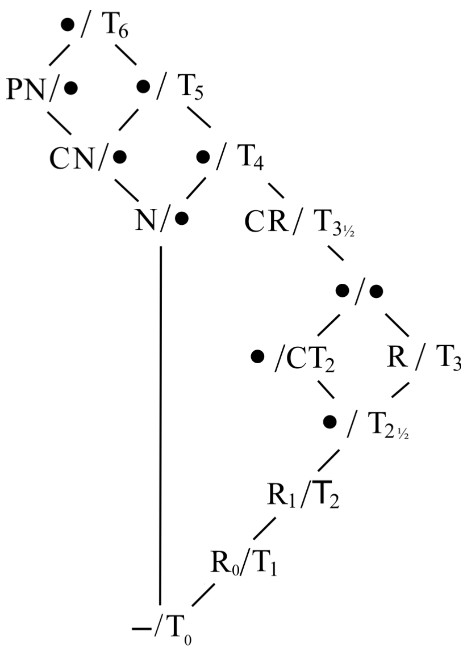

Fig. 2 Separation Axioms

1. T_0 : Kolmogorov space $(X,\ \mathfrak{O})$

$$\forall x, y \in X,\ (\exists V \in V(x)\ s.t.\ x \in V,\ \ y \notin V) \vee (\exists V \in V(y)\ s.t.\ x \notin V,\ \ y \in V)$$

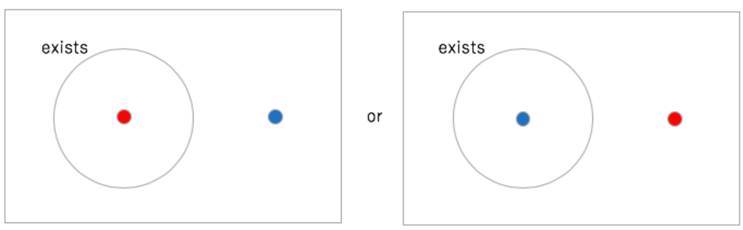

Fig. 3 T_0 space

2. T_1 : T_1 space $(X,\ \mathfrak{O})$

$$\forall x,\ y \in X,\ (\exists V \in V(x)\ s.t.\ x \in V,\ \ y \notin V) \wedge (\exists V \in V(y)\ s.t.\ x \notin V,\ \ y \in V)$$

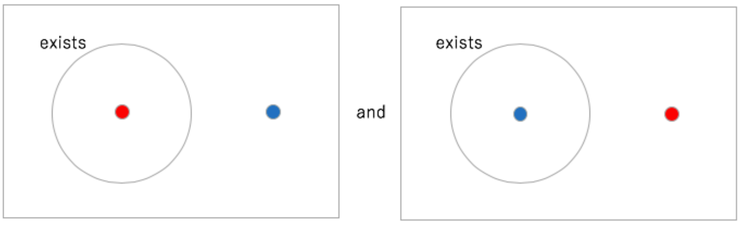

Fig. 4 T_1 space

3. T_2 : Hausdorff space $(X,\ \mathfrak{O})$

$$\forall x,\ y \in X,\ \exists V_x \in V(x),\ \exists V_y \in V(y)\ s.t.\ V_x \cap V_y \neq \phi$$

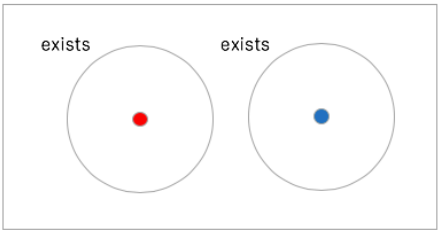

Fig. 5 T_2 space

4. $T_{2\frac{1}{2}}$: Urysohn space $(X,\ \mathfrak{O})$

$$\forall x,\ y \in X,\ \exists V_x \in V^*(x),\ \exists V_y \in V^*(y)\ s.t.\ V_x \cap V_y \neq \phi$$

$$V^*(x) : \text{closed neighborhood of } x,\ V^*(y) : \text{closed neighborhood of } y$$

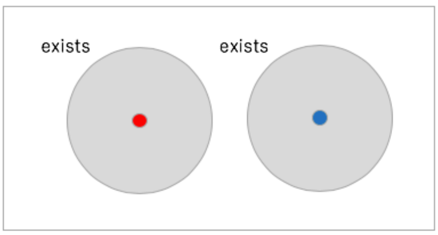

Fig. 6 $T_{2\frac{1}{2}}$ space

5. completely T_2 : Completely Hausdorff space $(X,\ \mathfrak{O})$

$$\forall x,\ y \in X,\ \exists f : X \mapsto [0,\ 1],\ s.t.\ f(x) = 0,\ f(y) = 1,\ f \text{ is continuous function.}$$

6. T_3 : Regular Hausdorff space $(X,\ \mathfrak{O})$
Regular Hausdorff space is a topological space that is both regular and a Hausdorff space.
Regular space :

$$\forall F \text{ (closed set) },\ \forall x \notin F,\ \exists O_1,\ O_2\ s.t.\ x \in O_1,\ F \subset O_2,\ O_1 \cap O_2 = \phi$$

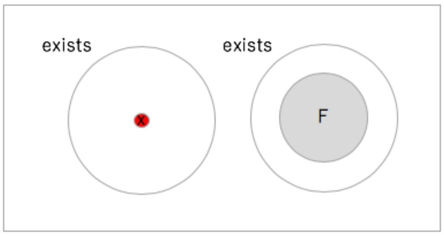

Fig. 7　Regular space

7. $T_{3\frac{1}{2}}$: Tychonoff space (completely regular Hausdorff space) $(X,\ \mathfrak{O})$
   Regular Hausdorff space is a topological space that is both conmpletely regular and a Hausdorff space.
   Completely regular space : $\forall A \subseteq X,\ A$ is closed set, $\forall x \in X \backslash A,$
   $$\exists f : X \mapsto \mathbb{R}\ s.t.\ f(x) = 1,\ f(a) = 0 (\forall a \in A),\ f \text{ is continuous function.}$$
8. T_4 : Normal Hausdorff space $(X,\ \mathfrak{O})$
   Normal Hausdorff space is a topological space that is both normal and a Hausdorff space.
   Normal space :
   $$\forall E,\ F \text{ :closed set, } \exists O_1,\ O_2\ s.t.\ E \subset O_1,\ F \subset O_2,\ O_1 \cap O_2 = \phi$$

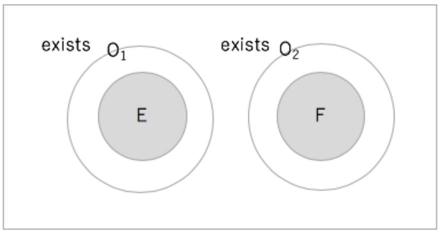

Fig. 8　Normal space

9. T_5 : Completely Normal Hausdorff space $(X,\ \mathfrak{O})$
   Completely Normal Hausdorff space is a topological space that is both completely normal and Hausdorff space.
   Completely Normal space :
   $$\forall A \subset X,\ (A,\ \mathfrak{O}_A) \text{ is normal space.}$$

Definition:　Metrizable space (距離化可能空間)
A topological space $(X,\ \mathfrak{O})$ is said to be metrizable if there is a metric:
$$d : X \times X \mapsto [0,\ \infty)$$
such that the topology induced by d is $\mathfrak{O}$.

Theorem:　Urysohn's metrization theorem (距離化可能定理)

Hausdorff second-countable regular space is metrizable.