1 Binary Relation

Definition: Equivalence relation $R \subset X \times X$ on a set X

㊟: 等号関係の一般化

$$\begin{cases} Reflexivity: & xRx \quad (\forall x \in X) \\ Symmetry: & xRy \Rightarrow yRx \quad (\forall x, y \in X) \\ Transitivity: & xRy, yRz \Rightarrow xRz \quad (\forall x, y, z \in X) \end{cases}$$

Example: An example of relation R

1. $=$
2. congruence, similarity (geometry)
3. $x \equiv y \quad (mod\ p)$

Definition: Equivalence class

$$[a] = \{x \in X \mid xRa\} \quad (a \in X, R: \text{relation on a set } X)$$

Definition: Quotient space

㊟ : 同値類集合の集合

$$X/R = \{[a] \mid a \in X\}$$

Definition: Natural projection (Quotient mapping)

$$\gamma : X \longmapsto X/R,\ \gamma(x) = [x] \quad (x \in X)$$

Definition: order relation $R \subset X \times X$ on a set X $(xRy \Leftrightarrow x \leq y)$

㊟: 大小関係の一般化

$$\begin{cases} Reflexivity: & x \leq x \quad (\forall x \in X) \\ Antisymmetry: & x \leq y, y \leq x \Rightarrow x = y \quad (\forall x, y \in X) \\ Transitivity: & x \leq y, y \leq z \Rightarrow x \leq z \quad (\forall x, y, z \in X) \end{cases}$$

Definition: Ordered set $(X, \leq)$

$\leq$ is order relation on a set X

Example: An example of ordered set

1. $(\mathbb{R}, \leq)$, $(\mathbb{R}, <)$ is not ordered set
2. $(2^X, \subset)$

Definition: Ordered subset $(M, \leq_M) \subset (X, \leq)$

$$M \subset X, a \leq_M b \Leftrightarrow a \leq b$$

Definition: Partial order and total order

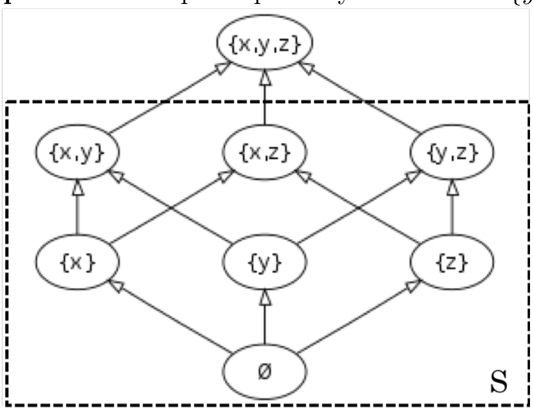
$$\exists (x, y) \in R \quad (x, y \in X) \Rightarrow R \text{ is partial order, } (X, R) \text{ is partially ordered set}$$
$$\forall (x, y) \in R \quad (x, y \in X) \Rightarrow R \text{ is total order, } (X, R) \text{ is totally ordered set}$$

2 Partially Order Set

Definition: Maximum (minimum) , supremum (infimum) and upper bound (lower bound)

$$max\ A = M \Leftrightarrow a \leq M \quad (M \in A,\ \forall a \in A)$$
$$s \text{ is one of upper bounds of } A \Leftrightarrow a \leq s \quad (\forall a \in A)$$
$$sup\ A = M' \Leftrightarrow min\ S = M' \quad (S \text{ is a set of upper bounds of } A)$$
$$\Leftrightarrow \begin{cases} \forall a \in A,\ a \leq M' \\ \forall \epsilon > 0,\ \exists a \in A \text{ s.t. } M' - \epsilon < a \end{cases}$$

Example: An example of partially ordered set. $\{y\} \leq \{x, z\}$ is not defined.



$max\ S : None$, $min\ S : \phi$
$sup\ S : \{x,\ y,\ z\}$, $inf\ S : \phi$

Theorem: Maximum, minimum, supremum and infimum exists uniquely.

Ⓟ: 最大値, 最小値の一意性は順序関係の反対称律を使う.

Ⓟ: 極大値, 極小値の一意性は最大値, 最小値の一意性を使う.

Theorem: If maximum (minimum) of A exists, $max\ A = sup\ A$ ($min\ A = inf\ A$).

Ⓟ: $sup(inf)$ の 2 つめの定義を使う.

Axiom: Existance of supremum and infimum

$$A \subset \mathbb{R},\ A \neq \phi$$
$$A \text{ is bounded above } \Rightarrow sup\ A \text{ exists.}$$
$$A \text{ is bounded below } \Rightarrow inf\ A \text{ exists.}$$

3 Sequence

Definition: Sequence is a mapping $x : \mathbb{N} \longmapsto X$
Definition: Subsequence is a composite mapping $x \circ \iota : \mathbb{N} \longmapsto X$
let $\iota : \mathbb{N} \longmapsto \mathbb{N}$ be the mapping where $i \leq j \Rightarrow \iota(i) \leq \iota(j)$
Example: Overview of Sequence and Subsequence

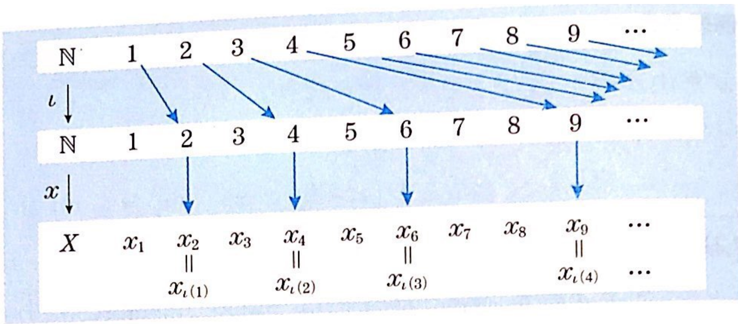


Fig. 1 Sequence

Definition: Limit of sequence
We call α the limit of the sequence $\{x_n\}$ if the following condition holds:

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } n \geq N \Rightarrow |x_n - \alpha| < \epsilon$$

Definition: 'bounded above', 'bounded below'
The sequence $\{x_n\}$ is 'bounded above' if the following condition holds:

$$\exists M \in \mathbb{R},\ \forall i \in \mathbb{N},\ x_i \leq M$$

Definition: Cauchy Sequence
キ: 十分大きな $N \in \mathbb{N}$ を選ぶと $n,\ m \geq N$ において x_n と x_m の差をいくらでも小さくできる列.
We call $\{x_n\}$ a Cauchy Sequence if the following condition holds:

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } n,\ m \geq N \Rightarrow |x_n - x_m| < \epsilon$$

Example: An exapmle of Cauchy Sequence

Theorem: If $\{x_n\},\ \{y_n\}$ are Cauchy Sequences,

$$\{x_n + y_n\},\ \{x_n \cdot y_n\} \text{ are Cauchy Sequences.}$$

Theorem: If a sequence x_n converges to some limit, x_n is a Cauchy Sequence.
Theorem: If a sequence x_n is a Cauchy Sequence, x_n is bounded.

4 Real Number

4.1 Definition of $\mathbb{N}$ and Construction of $\mathbb{Z}$, $\mathbb{Q}$

Axiom: Peano axioms

㊉: 自然数の定義

Define S as a single-vlued successor function. (ex. $S(1) = 2$)

1. 0 is a natural number.
2. For every natural number n, $S(n)$ is a natural number.
3. For every natural number n, $S(n) = 0$ is false.
4. $a,\ b \in \mathbb{N},\ a \neq b \Rightarrow S(a) \neq S(b)$
5. if Φ is a unary preficate such that:
   - $\Phi(0)$ is true.
   - for every natural number n, $\Phi(n)$ being true implies that $\Phi(S(n))$ is true.

   (**Mathematical induction**)

Definition: Integers and rational number

$$\mathbb{Z} = \mathbb{N} \cup -\mathbb{N},\ \mathbb{Q} = \{\frac{b}{a} \mid a,\ b \in \mathbb{Z},\ a \neq 0\}$$

4.2 Construction of $\mathbb{R}$

4.2.1 Dedekind cut

Definition: Dedekind cut of $\mathbb{Q}$

$$A \cup B = \mathbb{Q},\ A \cap B = \phi,\ A \neq \phi,\ B \neq \phi,\ a \in A,\ b \in B \Rightarrow a < b$$

Definition: Real number α is defined as a boundary value $\alpha = \langle A \mid B \rangle$ of Dedekind cut.

㊉: 有理数全体集合のデデキント切断の境界値を実数と定義.

4.2.2 Completion of rational numbers via Cauchy sequences

Definition: Equivalence relation of Cauchy sequences

$$\begin{aligned}\{a_n\} \sim \{b_n\} &\Leftrightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \alpha \\ &\Leftrightarrow \forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } n \geq N \Rightarrow |a_n - b_n| < \epsilon\end{aligned}$$

Definition: $\mathbb{R}$

We can define the bijection $\Phi : (S\ /\ \sim) \longmapsto \mathbb{R}$ (S is the set of all Cauchy Sequences on $\mathbb{Q}$)

$$\Phi([\{a_n\}]) = \lim_{n\to\infty} a_n \in \mathbb{R}$$

㊉: $\mathbb{Q}$ 上のコーシー列の同値類と実数の間に 1 対 1 写像を定義する.

Example: Napier's constant $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n,\ a_n = (1 + \frac{1}{n})^n$ is a Cauchy Sequence.

4.3 Continuity of real numbers

4.3.1 The Density of the Rational Numbers / The Density of the Irrational Numbers

Theorem: The Density of the Rational Numbers

$$\forall x,\ y \in \mathbb{R},\ x < y \Rightarrow \exists r \in \mathbb{Q} \text{ s.t. } x < r < y$$

※ Another expression : $\forall \epsilon > 0,\ a \in \mathbb{R},\ \exists r \in \mathbb{Q},\ |a - r| < \epsilon$

Theorem: The Density of the Irrational Numbers

$$\forall x,\ y \in \mathbb{R},\ x < y \Rightarrow \exists q \in \mathbb{R} \backslash \mathbb{Q} \text{ s.t. } x < q < y$$

Theorem: Archimedean property
㊍: 有理数の稠密性と同値

$$\forall a,\ b \in \mathbb{R},\ 0 < a < b \Rightarrow \exists n \in \mathbb{N} \text{ s.t. } b < na$$

Theorem: $\mathbb{N} \subset \mathbb{R}$ is not bounded above.
㊍: 有理数の稠密性と同値
Theorem: $lim_{n\to\infty}\frac{1}{n} = 0$
㊍: 有理数の稠密性と同値

4.3.2 Completeness of the real numbers

Axiom: Every Cauchy Sequence on $\mathbb{R}$ is convergent.
㊍: 一般：収束列 $\Rightarrow$ Cauchy 列, $\mathbb{R}$：収束列 $\Leftrightarrow$ Cauchy 列.
Axiom: カントールの区間縮小定理

4.3.3 Dedekind Theorem

㊍: 4.3.1 & 4.3.2 と同値.

Definition: Dedekind cut of $\mathbb{R}$

$$A \cup B = \mathbb{R},\ A \cap B = \phi,\ A \neq \phi,\ B \neq \phi,\ a \in A,\ b \in B \Rightarrow a < b$$

Axiom: Dedekind theorem ($\Leftrightarrow$ continuity of real numbers)
For any cut $\langle A \mid B \rangle$ of the set of real numbers there exists only one real number γ s.t:

$$\alpha \in A,\ \beta \in B \Rightarrow \alpha \leq \gamma \leq \beta,\ \gamma \text{ is } max\ A \text{ or } min\ B$$

4.3.4 単調有界数列の収束

4.3.5 ボルツァーノ-ワイアシュトラウスの定理

5 Cardinality

Definition: The cardinality of a set A is denoted by $|A|$, $card\ A$, $\#A$ etc...

$$|A| = \begin{cases} n \in \{0\} \cup \mathbb{N} & : \text{cardinality of finite set} \\ (others) & : \text{cardinality of infinite set} \end{cases} \quad ex. \begin{cases} \aleph_0 & : \text{cardinality of countably finite set} \\ \aleph & : \text{cardinality of the continuum} \end{cases}$$

Definition: $|A| = |B| \Leftrightarrow A \sim B \Leftrightarrow \exists f$ (a bijection function) $: A \longmapsto B$

$R = \{(A, B) \mid |A| = |B|\ \} \subset S \times S$ on a set S is an equivalence relation.

Definition: $|A| \leq |B| \Leftrightarrow \exists f$ (a injective function) $: A \longmapsto B$

$R = \{(A, B) \mid |A| \leq |B|\} \subset S \times S$ on a set S is an order relation.

Ⓟ: 反対称律は Bernstein の定理を用いる.

Theorem: $\aleph_0 < \aleph$

Ⓟ: $\mathbb{N} \not\sim [0, 1)$ をカントールの対角線論法で示す.

Theorem: $|X| < |\mathfrak{P}(X)|$

Ⓟ: $X \subset \mathfrak{P}(X)$ より, $|X| \leq |\mathfrak{P}(X)|$ は明らか.$|X| \neq |\mathfrak{P}(X)|$ を対角線論法で示す.

Theorem: $A \cap B = \phi$ とする.

1. $\cup$

$$|A \cup B| = |A| + |B|$$

$\lvert A \cup B \rvert$	finite	countable	uncountable
finite	finite	countable	uncountable
countable		countable	uncountable
uncountable			uncountable

2. $\times$

$$|A \times B| = |A| \cdot |B|$$

$\lvert A \times B \rvert$	finite	countable	uncountable
finite	finite	countable	uncountable
countable		countable	uncountable
uncountable			uncountable

3. pow

$$|A^B| = |A|^{|B|}$$

$\lvert A^B \rvert$	finite	countable	uncountable
finite	finite	uncountable	uncountable
countable	countable	uncountable	uncountable
uncountable	uncountable	uncountable	uncountable

Example:

$$A = \{1,\ 2,\ 3\},\ |A| = 3$$

$$\aleph_0 = |\mathbb{N}| = |\mathbb{N}^2| = |\mathbb{Z}| = |\mathbb{Q}|$$

$$\aleph = |\mathbb{R}| = |[a, b]| = |(a, b)| = |[a, b)|$$
$$= \aleph_1 = |\mathfrak{P}(\mathbb{N})| \quad \text{(ZFC Axioms)}$$