# Computational Exercise #1

Astrophysical Fluids — PHYS 643

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## Description of Files

- **hw1.pdf** : contains the write-up for this assignment
- **hw1.cc** : the C++ file where the brunt of the RK4 simulation is located
- **pc_mass_radius.txt** : a file containing data mass, radius, density, Fermi energy, and polytropic index data from the simulation
- **mass_radius_relation_plotter.py** : a python script used to plot the mass-radius relation
  - *mass_radius_relation.pdf* : a plot of the mass-radius relation
- **density_profile_plotter.py** : a python script used to plot the density profiles
  - *density_profile_0019.pdf, density_profile_0199.pdf, density_profile_0099.pdf, density_profile_0499.pdf* : a few select plots demonstrating the density profile variation for different masses
  - *density_profile.gif* : a gif demonstrating the density profile evolution for a wide range of masses
- **fermi_energy_plotter.py** : a python script used to plot the Fermi energy vs mass and density
  - *fermi_energy_density.pdf, fermi_energy_mass.pdf* : plots of the Fermi energy vs mass and density
- **polytop_index_plotter.py** : a python script used to plot the polytropic index vs mass and density
  - *gamma_density.pdf, gamma_mass.pdf* : plots of the polytropic index vs mass and density

## Problem Description

The goal of this computational exercise was to investigate the properties of $T = 0$ white dwarf stars by integrating the equations of hydrostatic balance. Hydrostatic balance occurs when the pressure gradient from the interior of the star balances the force of gravity pulling inward. In this case, the momentum equation is given by:

$$\frac{D\vec{u}}{Dt} = -\frac{\nabla P}{\rho} + \vec{g} = 0 \implies \nabla P = \vec{g}\rho$$

In the case of a white dwarf, we can assume spherical symmetry. This yields the equation of hydrostatic equilibrium that we know and love:

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2} \tag{1}$$

where

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

and $P(r)$ is the pressure, $\rho(r)$ is the density, and $m(r)$ is the mass at radius $r$ inside the star. In this case we have the boundary conditions: $m = 0$ at $r = 0$ and $P = \rho = 0$ and $r = R$, where $R$ is the radius of the white dwarf.

**Equation of State**

To solve these equations numerically, we need to find a relation between $P$ and $\rho$ so that we change the integration variable in the first equation from $P$ to $\rho$. This relation between $P$ and $\rho$ is known as the equation of state. White dwarfs, neutron stars, incompressible bodies like rocky moons or planets, and isothermal spheres like globular clusters can all be represented by the relatively simple relation: $P \propto \rho^\gamma$. Such systems are known as polytropes.

White dwarfs are composed of mostly degenerate electron matter. In low-mass white dwarfs, the electrons are **non-relativistic**. In this limit, $\gamma = 5/3$ and the Fermi energy is given by:

$$E_F = \frac{p_F^2}{2m_e} \quad \text{where} \quad p_F = \hbar k_F = \hbar \left(3\pi^2 n_e\right)^{1/3} \tag{2}$$

and the pressure is given by:

$$P = \frac{2}{5} n_e E_F \tag{3}$$

The polytropic equation the non-relativistic case is given by: $P = K_{nr}\rho^{5/3}$. Using equations 2 and 3, we can solve for $K_{nr}$:

$$K_{nr} = \frac{P}{\rho^{5/3}} = \frac{2}{5}\frac{n_e E_F}{\rho^{5/3}} = \frac{2}{5}\frac{n_e}{\rho^{5/3}} \cdot \frac{p_F^2}{2m_e} = \frac{2}{5}\frac{n_e}{2m_e\rho^{5/3}} \cdot \hbar^2 \left(3\pi^2 n_e\right)^{2/3} = \frac{\hbar^2 \left(3\pi^2\right)^{2/3}}{5m_e}\left(\frac{n_e}{\rho}\right)^{5/3}$$

Now we take:

$$Y_e = \frac{n_e m_p}{\rho} \implies \rho = \frac{n_e m_p}{Y_e} \tag{4}$$

Plugging this back in we find:

$$\begin{aligned} K_{nr} &= \frac{\hbar^2 \left(3\pi^2\right)^{2/3}}{5m_e}\left(\frac{n_e \cdot Y_e}{n_e \cdot m_p}\right)^{5/3} = \frac{\hbar^2 \left(3\pi^2\right)^{2/3}}{5m_e m_p^{5/3}} Y_e^{5/3} \\ &= \frac{\left(1.0546 \cdot 10^{-27}\,\mathrm{erg\,s}\right)^2 \left(3\pi^2\right)^{2/3}}{5\left(9.109 \cdot 10^{-28}\,\mathrm{g}\right)\left(1.673 \cdot 10^{-24}\,\mathrm{g}\right)^{5/3}} Y_e^{5/3} \\ &\implies K_{nr} = \left(9.9 \cdot 10^{12}\,\mathrm{cgs}\right) Y_e^{5/3} \end{aligned}$$

Therefore, the polytropic relation for low-mass white dwarfs is given by:

$$\boxed{P_{nr} = K_{nr}\rho^{5/3} = \left(9.9 \cdot 10^{12}\,\text{cgs}\right) Y_e^{5/3}\rho^{5/3}} \tag{5}$$

However, as white dwarf mass increases, the electrons eventually become **relativistic**. In this limit, $\gamma = 4/3$ and the Fermi energy is given by:

$$E_F = p_F c \quad \text{where} \quad p_F = \hbar k_F = \hbar\left(3\pi^2 n_e\right)^{1/3} \approx m_e c \tag{6}$$

and the pressure is given by:

$$P = \frac{1}{4} n_e E_F \tag{7}$$

The polytropic equation the relativistic case is given by: $P = K_r\rho^{4/3}$. Using equations 6 and 7, we can solve for $K_r$:

$$K_r = \frac{P}{\rho^{4/3}} = \frac{n_e E_F}{4\rho^{4/3}} = \frac{n_e c p_F}{4\rho^{4/3}} = \frac{n_e c\hbar\left(3\pi^2 n_e\right)^{1/3}}{4\rho^{4/3}} = \frac{\hbar c\left(3\pi^2\right)^{1/3}}{4\rho^{4/3}}\left(\frac{n_e}{\rho}\right)^{4/3}$$

Again we plug in equation 4:

$$\begin{aligned}
K_r &= \frac{\hbar c\left(3\pi^2\right)^{1/3}}{4\rho^{4/3}}\left(\frac{n_e \cdot Y_e}{n_e \cdot m_p}\right)^{4/3} = \frac{\hbar c\left(3\pi^2\right)^{1/3}}{4\rho^{4/3}} Y_e^{4/3} \\
&= \frac{\left(1.0546 \cdot 10^{-27}\,\text{erg s}\right)\left(2.99 \cdot 10^{10}\,\text{cm s}^{-1}\right)\left(3\pi^2\right)^{1/3}}{4\left(1.673 \cdot 10^{-24}\,\text{g}\right)^{4/3}} Y_e^{4/3} \\
&\implies K_r = \left(1.2 \cdot 10^{15}\,\text{cgs}\right) Y_e^{4/3}
\end{aligned}$$

Therefore, the polytropic relation for high-mass white dwarfs is given by:

$$\boxed{P_r = K_r\rho^{4/3} = \left(1.2 \cdot 10^{15}\,\text{cgs}\right) Y_e^{4/3}\rho^{4/3}} \tag{8}$$

For our simulations, we will assume a carbon/oxygen white dwarf for which $Y_e \approx 0.5$.

To take into account the change from non-relativistic to relativistic degenerate electron matter, we can use the analytic fitting formula for the equation of state derived by Paczynksi (1983):

$$P^{-2} \approx P_{nr}^{-2} + P_r^{-2} \implies \ln(P) \approx -\frac{1}{2}\ln\left(P_{nr}^{-2} + P_r^{-2}\right)$$

Using this, we have that:

$$\frac{d\ln(P)}{dr} = \frac{d\ln(\rho)}{dr}\left[\frac{5}{3}\left(\frac{P}{P_{nr}}\right)^2 + \frac{4}{3}\left(\frac{P}{P_r}\right)^2\right] \tag{9}$$

To show that this is true, we first calculate a few derivatives:

$$\frac{d\ln(P)}{dr} = \frac{1}{P}\frac{dP}{dr} \qquad \frac{d\ln(\rho)}{dr} = \frac{1}{\rho}\frac{d\rho}{dr}$$

$$\frac{dP_{nr}}{dr} = \frac{5}{3}\frac{P_{nr}}{\rho}\frac{d\rho}{dr} = \frac{5}{3}P_{nr}\frac{d\ln(\rho)}{dr} \qquad \frac{dP_r}{dr} = \frac{4}{3}\frac{P_r}{\rho}\frac{d\rho}{dr} = \frac{4}{3}P_r\frac{d\ln(\rho)}{dr}$$

Then we have:

$$\frac{d\left(P_{nr}^{-2}\right)}{dr} = -2P_{nr}^{-3} \cdot \frac{dP_{nr}}{dr} = -\frac{2}{P_{nr}^3} \cdot \frac{5}{3} P_{nr} \frac{d\ln(\rho)}{dr} = -2 \cdot \frac{5}{3} \cdot \frac{1}{P_{nr}^2} \frac{d\ln(\rho)}{dr}$$
$$\frac{d\left(P_{r}^{-2}\right)}{dr} = -2P_{r}^{-3} \cdot \frac{dP_{r}}{dr} = -\frac{2}{P_{r}^3} \cdot \frac{4}{3} P_{r} \frac{d\ln(\rho)}{dr} = -2 \cdot \frac{4}{3} \cdot \frac{1}{P_{r}^2} \frac{d\ln(\rho)}{dr}$$

and finally:

$$\begin{aligned}
\frac{d\ln(P)}{dr} &= -\frac{1}{2}\left(\frac{1}{P_{nr}^{-2} + P_r^{-2}}\right)\left(\frac{d\left(P_{nr}^{-2}\right)}{dr} + \frac{d\left(P_r^{-2}\right)}{dr}\right) \\
&= -\frac{1}{2}P^2\left(-2 \cdot \frac{5}{3} \cdot \frac{1}{P_{nr}^2}\frac{d\ln(\rho)}{dr} - 2 \cdot \frac{4}{3} \cdot \frac{1}{P_r^2}\frac{d\ln(\rho)}{dr}\right) \\
\implies \frac{d\ln(P)}{dr} &= \frac{d\ln(\rho)}{dr}\left[\frac{5}{3}\left(\frac{P}{P_{nr}}\right)^2 + \frac{4}{3}\left(\frac{P}{P_r}\right)^2\right]
\end{aligned}$$

Now we can use this relation to change the integration variable in equation 1 from $P$ to $\rho$:

$$\frac{1}{P}\frac{dP}{dr} = \frac{1}{\rho}\frac{d\rho}{dr}\left[\frac{5}{3}\left(\frac{P}{P_{nr}}\right)^2 + \frac{4}{3}\left(\frac{P}{P_r}\right)^2\right] = \frac{1}{\rho}\frac{d\rho}{dr} \cdot A(\rho)$$

Plugging this into equation 1:

$$\frac{dP}{dr} = \frac{P}{\rho}\frac{d\rho}{dr} \cdot A(\rho) = -\frac{Gm\rho}{r^2} \implies \boxed{\frac{d\rho}{dr} = \frac{-Gm\rho^2}{Pr^2 \cdot A(\rho)}} \tag{10}$$

where $P$ is defined by the Paczynksi (1983) relation.

**Approximations**

In this section, we will derive approximations for the $T = 0$ white dwarf mass-radius relation. To do this, we will use the following properties:

$$\alpha_\gamma = \frac{P_c}{GM^2/R^4} \qquad \beta_\gamma = \frac{\rho_c}{\langle\rho\rangle} = \frac{4\pi R^3 \rho_c}{3M}$$

We can look up the values of these quantities for different polytropic indices. For $\gamma = 5/3$, $\alpha_{5/3} = 0.77$ and $\beta_{5/3} = 5.99$. For $\gamma = 4/3$, $\alpha_{4/3} = 11.1$ and $\beta_{4/3} = 54.2$.

Now, in the low-mass, non-relativistic case:

$$\alpha_{5/3} = \frac{P_c}{GM^2/R^4} = \frac{K_{nr}\rho_c^{5/3}}{GM^2/R^4} = \frac{K_{nr}}{GM^2/R^4} \cdot \left(\frac{3M\beta_{5/3}}{4\pi R^3}\right)^{5/3} \tag{11}$$

$$\implies R_{5/3} = M^{-1/3}\left(\frac{K_{nr}}{\alpha_{5/3}G}\right)\left(\frac{3M\beta_{5/3}}{4\pi}\right)^{5/3} = \boxed{9 \cdot 10^8\,\text{cm}\left(\frac{M}{M_\odot}\right)^{-1/3}\left(\frac{Y_e}{0.5}\right)^{5/3}} \tag{12}$$

So we can say that, at low masses, the mass-radius relation is approximately: $R \sim M^{-1/3}$. As mass increases, the radius of the white dwarf decreases.

We can play the same sort of game for the high mass, relativistic case:

$$\alpha_{4/3} = \frac{P_c}{GM^2/R^4} = \frac{K_r \rho_c^{4/3}}{GM^2/R^4} = \frac{K_r}{GM^2/R^4} \cdot \left( \frac{3M\beta_{4/3}}{4\pi R^3} \right)^{4/3}$$

In this case, the radius cancels out and we find a formula for the Chandrasekhar mass:

$$\begin{aligned} M^{2/3} = \frac{M^{6/3}}{M^{4/3}} &= \left( \frac{K_r}{\alpha_{4/3} G} \right) \left( \frac{3\beta_{4/3}}{4\pi} \right)^{4/3} \\ \Longrightarrow & \boxed{M_{Ch} = \left( \frac{K_r}{\alpha_{4/3} G} \right)^{3/2} \left( \frac{3\beta_{4/3}}{4\pi} \right)^{2}} = 1.45 M_\odot \left( \frac{Y_e}{0.5} \right)^2 \end{aligned}$$

Finally, if we complete the same calculations assuming the Paczynksi (1983) equation of state then we find:

$$R \approx R_{5/3} \left[ 1 - \left( \frac{M}{M_{Ch}} \right)^{4/3} \right]^{1/2} = \boxed{8.7 \cdot 10^8 \text{ cm} \left( \frac{M}{M_\odot} \right)^{-1/3} \left[ 1 - \left( \frac{M}{M_{Ch}} \right)^{4/3} \right]^{1/2}} \tag{13}$$

where we have let $Y_e = 0.5$.

## Algorithm

As shown in the previous section, we now need to numerically integrate the two coupled differential equations:

$$\frac{d\rho}{dr} = \frac{-Gm\rho^2}{Pr^2 \cdot A(\rho)} \qquad \text{and} \qquad \frac{dm}{dr} = 4\pi r^2 \rho \tag{14}$$

To accomplish this, we have decided to implement 4th order Runge-Kutta algorithm. According to this algorithm, each integration step is given by:

$$\begin{aligned} \rho_{n+1} &= \rho_n + \frac{1}{6} \left( k_0 + 2k_1 + 2k_2 + k_3 \right) \\ m_{n+1} &= m_n + \frac{1}{6} \left( \ell_0 + 2\ell_1 + 2\ell_2 + \ell_3 \right) \end{aligned}$$

where the RK4 increments $k_i$ and $\ell_i$ are calculated in the order:

$$
\begin{aligned}
k_0 &= dr \cdot \frac{d\rho}{dr}\,(r_n, \rho_n, m_n) \\
\ell_0 &= dr \cdot \frac{dm}{dr}\,(r_n, \rho_n) \\
k_1 &= dr \cdot \frac{d\rho}{dr}\,(r_n + 0.5dr, \rho_n + 0.5k_0, m_n + 0.5\ell_0) \\
\ell_1 &= dr \cdot \frac{dm}{dr}\,(r_n + 0.5dr, \rho_n + 0.5k_0) \\
k_2 &= dr \cdot \frac{d\rho}{dr}\,(r_n + 0.5dr, \rho_n + 0.5k_1, m_n + 0.5\ell_1) \\
\ell_2 &= dr \cdot \frac{dm}{dr}\,(r_n + 0.5dr, \rho_n + 0.5k_1) \\
k_3 &= dr \cdot \frac{d\rho}{dr}\,(r_n + dr, \rho_n + k_2, m_n + \ell_2) \\
\ell_3 &= dr \cdot \frac{dm}{dr}\,(r_n + dr, \rho_n + k_2)
\end{aligned}
$$

In these increments, $dr$ is the radius stepsize, $r_n$ is the radius, $\rho_n$ is the density, and $m_n$ is the mass of the current integration step, and $d\rho/dr(r, \rho, m)$ and $dm/dr(r, \rho)$ are given in equation 14.

**We implemented our code in C++. It is located in the file "hw1.cc"**

# Results

With the framework described above, we simulate 5000 white dwarfs with central densities ranging logarithmically from $\rho_c = 10$ to $10^{20}\,\mathrm{g\,cm^{-3}}$, using a constant integration stepsize of $dr = 10^5\,\mathrm{cm} = 1\,\mathrm{km}$ for each simulation. To avoid dividing by zero, we set our initial simulation radius to $r_0 = 1\,\mathrm{cm}$. Each simulation finishes once $\rho \approx 0$. The final mass and radius are taken to be the mass and radius of the white dwarf.

### Mass-Radius Relation

Mass-radius relation data for all 5000 white dwarfs are saved in a file called "pc_mass_radius.txt". Use the script called "mass_radius_relation_plotter.py" to make a plot of the mass-radius relation according to our simulations. The resulting plot is shown in Figure 1. As can be seen from the figure, the radius rapidly decreases as mass increases, as anticipated. The maximum mass reached is approximately the Chandrasekhar mass $M_{Ch} \approx 1.44\,M_\odot$. Our simulation is largely in agreement with the mass-radius relation approximations described by equations 12 and 13.

### Investigating the Density Profile

Density profile data ($\rho$ vs radius) are saved for 500 of these simulations in files named "rk4_pc_0000.txt". We make mass vs radius plots for each of these datasets. The code for creating these plots is located in "density_profile_plotter.py" Examples of these plots are shown in Figure 2.

To demonstrate how the density profile changes with mass, we combine all of these plots into a single gif called "density_profile.gif" (click to view on GitHub). As expected, we can see that as the mass increases, the density of the star increases and the radius decreases.

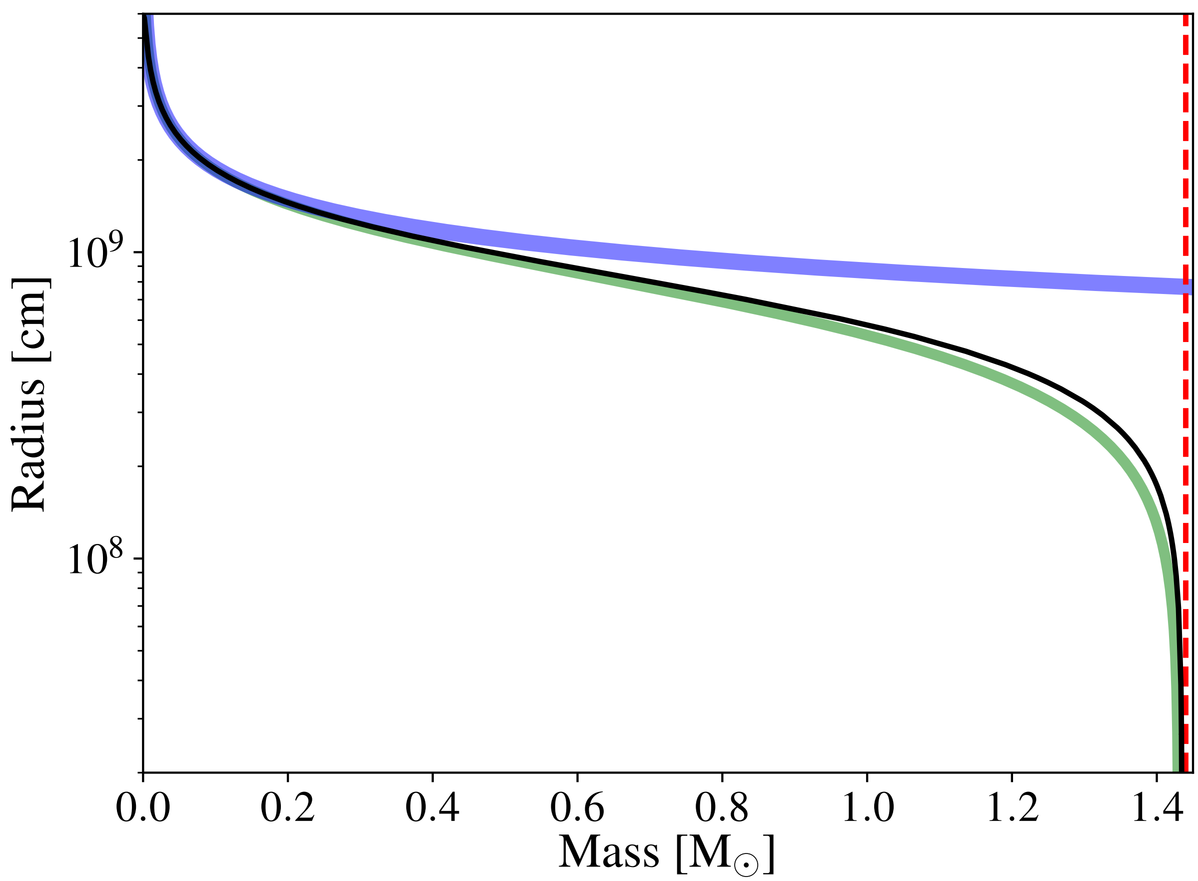


Figure 1: The mass-radius relation for $T = 0$ white dwarfs. The black line represents the data obtained from the numerical integration implemented in this assignment. The wider green line represents the general approximation described by equation 13. The widest blue line represents the low-mass mass-radius approximation described by equation 12.

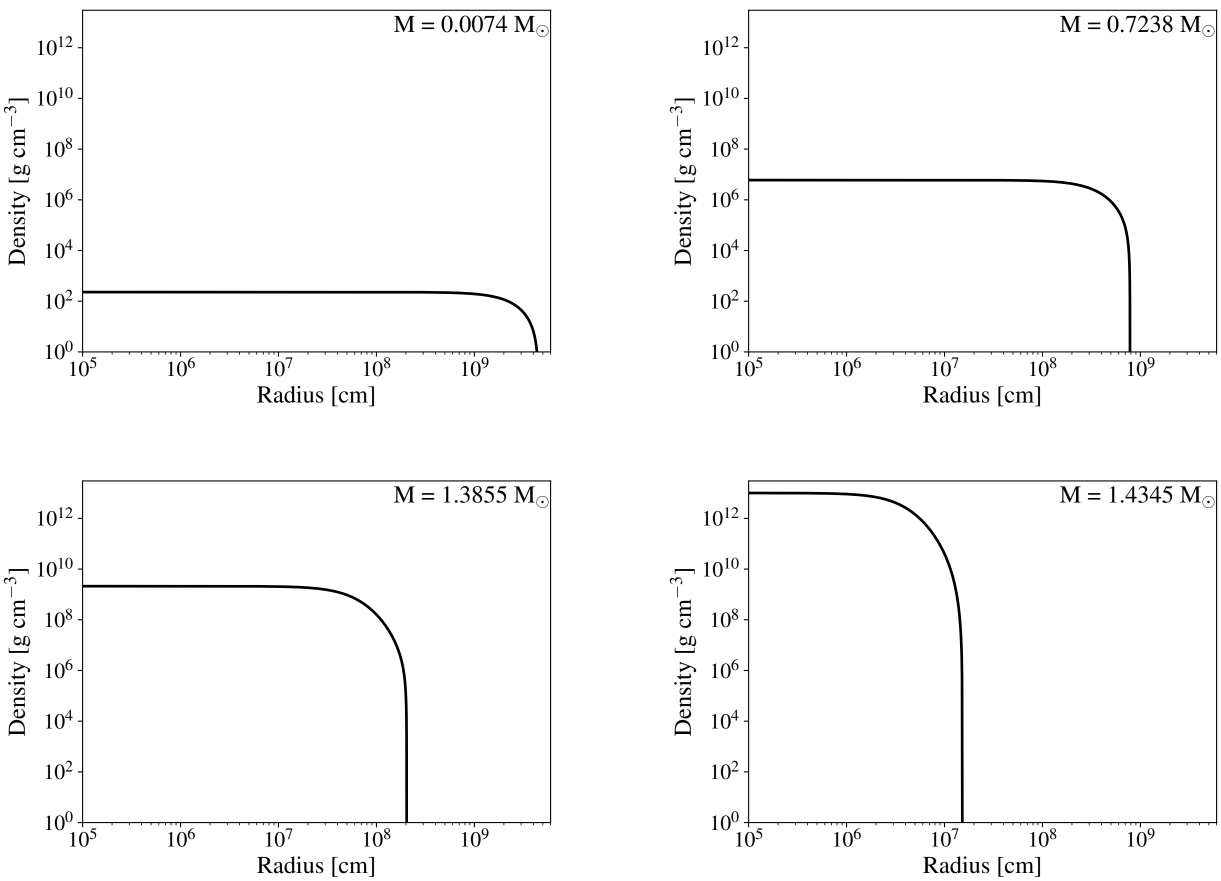


Figure 2: Density profiles for white dwarfs of masses $M = 0.0074, 0.7238, 1.3855, 1.4345\, M_\odot$.

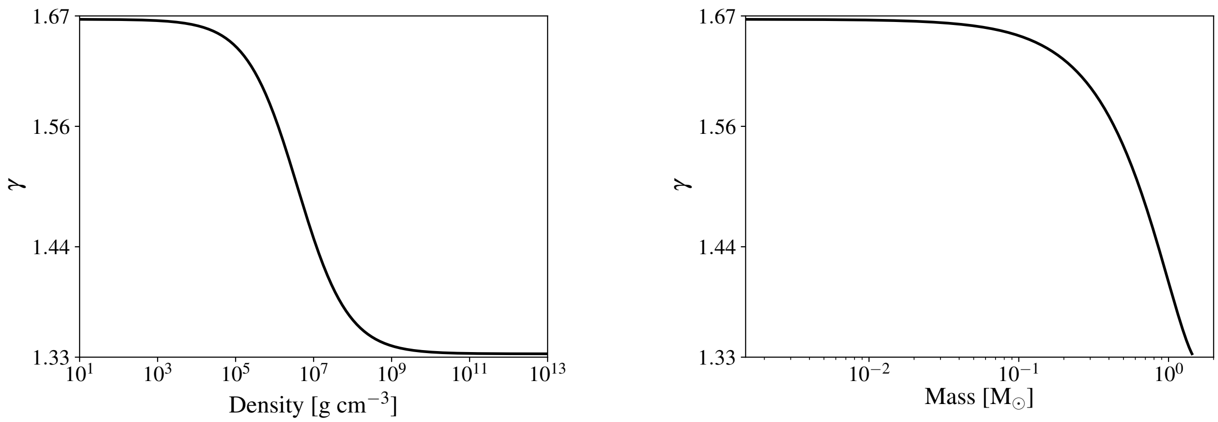


Figure 3: The polytropic index vs mass and radius.

### The Extent to Which Electrons are Relativistic

While completing the simulations, we also calculated the polytropic index $\gamma$ and Fermi energy $E_F$ at the center of the white dwarf as a function of mass and central density. To calculate $\gamma$, we first recognize the following about the polytropic relation:

$$P_c = K\rho_c^\gamma \implies \ln(P_c) = \gamma \ln(\rho_c) + \ln(K)$$

Therefore, $\gamma$ is just the slope of the linear line formed between $\ln(P_c)$ and $\ln(\rho_c)$. We output the values of $\ln(P_c)$ and $\ln(\rho_c)$ from our simulation. Then, in "polytop_index_plotter.py", we calculate:

$$\gamma_i = \frac{\ln(P_{c,i+1}) - \ln(P_{c,i})}{\ln(\rho_{c,i+1}) - \ln(\rho_{c,i})} \tag{15}$$

With these values, we then plot $\gamma$ as a function of mass and central density. The resulting plots are shown in Figure 3. As expected, $\gamma$ ranges from $\sim 5/3$ at low masses (where the electrons are non-relativistic) to $\sim 4/3$ at high masses (where the electrons are relativistic).

To calculate the Fermi energy, we used:

$$E_F = m_e c^2 \left(\sqrt{1+x^2} - 1\right) \qquad \text{where} \qquad x = \frac{p_F}{m_e c} \approx \left(\frac{\rho Y_e}{10^6\,\mathrm{g\,cm^{-3}}}\right)^{1/3}$$

The resulting plots are shown in Figure 4 (see the code for these plots in "fermi_energy_plotter.py"). As labelled in the plot, the slope of the Fermi energy starts to change at around $M \approx 0.5\,M_\odot$, which corresponds to a central density of $\rho_c \sim 10^6\,\mathrm{g\,cm^{-3}}$. This indicates that the electrons are starting to become degenerate at about these values.

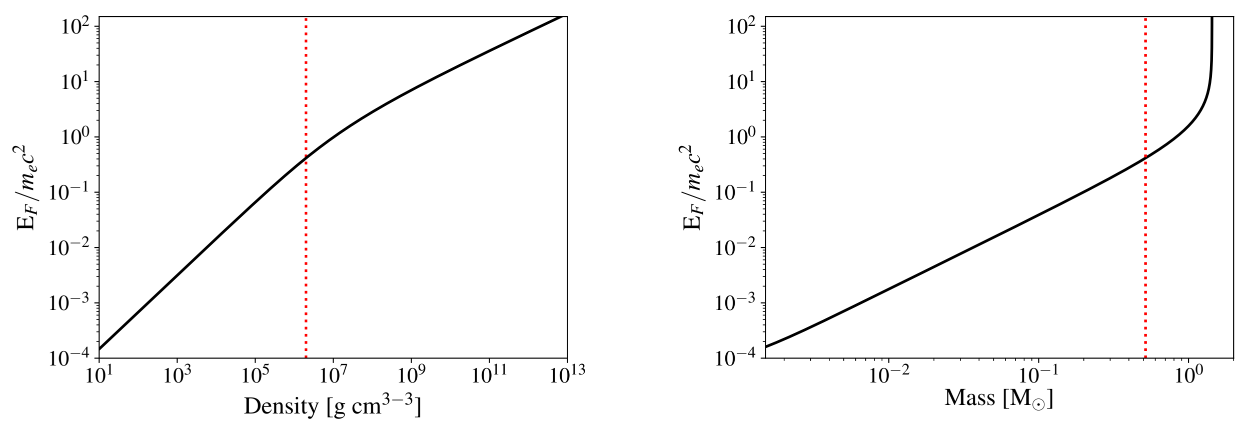


Figure 4: The Fermi energy vs mass and radius. The red line represents $\rho_c \sim 10^6\,\mathrm{g\,cm^{-3}}$ and $M \approx 0.5\,M_\odot$.