1. 求实值函数在任一点的梯度及 Hesse 矩阵 $f(x) = f(x_1, x_2) = (x_1 - x_2)^2 + 4x_1x_2 + e^{x1+x2}$。

解.

$$
\begin{aligned}
\partial_1 f(\boldsymbol{x}) &= 2(x_1 - x_2) + 4x_2 + e^{x_1+x_2} \\
\partial_2 f(\boldsymbol{x}) &= -2(x_1 - x_2) + 4x_1 + e^{x_1+x_2} \\
\nabla f(\boldsymbol{x}) &= (2x_1 + 2x_2 + e^{x_1+x_2}, 2x_1 + 2x_2 + e^{x_1+x_2})^T \\
\partial_{1,1} f(\boldsymbol{x}) &= 2 + e^{x_1+x_2} \\
\partial_{1,2} f(\boldsymbol{x}) &= 2 + e^{x_1+x_2} \\
\partial_{2,1} f(\boldsymbol{x}) &= 2 + e^{x_1+x_2} \\
\partial_{2,2} f(\boldsymbol{x}) &= 2 + e^{x_1+x_2} \\
\nabla^2 f(\boldsymbol{x}) &= \begin{pmatrix} 2 + e^{x_1+x_2} & 2 + e^{x_1+x_2} \\ 2 + e^{x_1+x_2} & 2 + e^{x_1+x_2} \end{pmatrix}
\end{aligned}
$$

2. 求二次函数在任一点的梯度以及 Hesse 矩阵 $f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}^T\boldsymbol{x} + c$, 其中 $\boldsymbol{A} \in \mathbb{R}^{n\times n}, \boldsymbol{b} \in \mathbb{R}^n, c \in \mathbb{R}$。

解.

$$
\begin{aligned}
f(\boldsymbol{x}+\boldsymbol{\epsilon}) - f(\boldsymbol{x}) =& \frac{1}{2}(\boldsymbol{x}+\boldsymbol{\epsilon})^T\boldsymbol{A}(\boldsymbol{x}+\boldsymbol{\epsilon}) + \boldsymbol{b}^T(\boldsymbol{x}+\boldsymbol{\epsilon}) + c \\
& - \frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}^T\boldsymbol{x} - c \\
=& \frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{\epsilon} + \frac{1}{2}\boldsymbol{\epsilon}^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}^T\boldsymbol{\epsilon} + \frac{1}{2}\boldsymbol{\epsilon}^T\boldsymbol{A}\boldsymbol{\epsilon} \\
=& (\frac{1}{2}\boldsymbol{x}^T\boldsymbol{A} + \frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}^T + \boldsymbol{b}^T)\boldsymbol{\epsilon} + \frac{1}{2}\boldsymbol{\epsilon}^T\boldsymbol{A}\boldsymbol{\epsilon}
\end{aligned}
$$

由 $f(\boldsymbol{x}+\boldsymbol{\epsilon}) = f(\boldsymbol{x}) + \nabla^T f(\boldsymbol{x})\boldsymbol{\epsilon} + \boldsymbol{\epsilon}^T\nabla^2 f(\boldsymbol{x})\boldsymbol{\epsilon} + o(\|\boldsymbol{\epsilon}\|^2)$，可得

$$
\begin{aligned}
\nabla f(\boldsymbol{x}) &= \frac{1}{2}(\boldsymbol{A}^T\boldsymbol{x} + \boldsymbol{A}\boldsymbol{x}) + \boldsymbol{b} \\
\nabla^2 f(\boldsymbol{x}) &= \boldsymbol{A}
\end{aligned}
$$

3. 求向量值函数 $\boldsymbol{f}(\boldsymbol{x})$ 在任一点的导数（即 Jacobi 矩阵）。

$$
\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}(x_1, x_2) = \begin{bmatrix} \sin x_1 + \cos x_2 \\ e^{2x_1+x_2} \\ 2x_1^2 + x_1x_2 \end{bmatrix}
$$

解.

$$
\begin{aligned}
\boldsymbol{J} = \nabla_{\boldsymbol{x}}\boldsymbol{f} = \frac{\mathrm{d}\boldsymbol{f}(\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}} &= \begin{bmatrix} \frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial x_1} & \frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial x_2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial f_1(\boldsymbol{x})}{\partial x_1} & \frac{\partial f_1(\boldsymbol{x})}{\partial x_2} \\ \frac{\partial f_2(\boldsymbol{x})}{\partial x_1} & \frac{\partial f_2(\boldsymbol{x})}{\partial x_2} \\ \frac{\partial f_3(\boldsymbol{x})}{\partial x_1} & \frac{\partial f_3(\boldsymbol{x})}{\partial x_2} \end{bmatrix} \\
&= \begin{bmatrix} \cos x_1 & -\sin x_2 \\ 2e^{2x_1+x_2} & e^{2x_1+x_2} \\ 4x_1 + x_2 & x_1 \end{bmatrix}
\end{aligned}
$$

4. 用定义验证下列集合为凸集。

$$S = \{(x_1, x_2) | x_1 + 2x_2 \geq 1, x_1 - x_2 \geq 1\}$$

解. 任取集合中的两个点 $\boldsymbol{x} = (x_1, x_2)$ 和 $\boldsymbol{y} = (y_1, y_2)$，满足

$$\begin{cases} x_1 + 2x_2 & \geq 1 \\ x_1 - x_2 & \geq 1 \end{cases}$$
$$\begin{cases} y_1 + 2y_2 & \geq 1 \\ y_1 - y_2 & \geq 1 \end{cases}$$

对于 $\forall \theta \in [0, 1]$，点 $\boldsymbol{z} = (z_1, z_2) = \theta \boldsymbol{x} + (1-\theta)\boldsymbol{y} = (\theta x_1 + (1-\theta)y_1, \theta x_2 + (1-\theta)y_2)$，都满足

$$\begin{aligned} z_1 + 2z_2 &= \theta x_1 + (1-\theta)y_1 + 2\theta x_2 + 2(1-\theta)y_2 \\ &= \theta(x_1 + 2x_2) + (1-\theta)(y_1 + 2y_2) \\ &\geq \theta + (1-\theta) \\ &\geq 1 \\ z_1 - z_2 &= \theta x_1 + (1-\theta)y_1 - \theta x_2 - (1-\theta)y_2 \\ &= \theta(x_1 - x_2) + (1-\theta)(y_1 - y_2) \\ &\geq \theta + (1-\theta) \\ &\geq 1 \end{aligned}$$

所以 $\boldsymbol{z}$ 也在集合 S 中，所以集合 S 是一个凸集。