**1** (P413 1.(1)). 用外点法求解下列问题:

$$\begin{cases} \min & x_1^2 + x_2^2 \\ \text{s.t.} & x_2 = 1 \end{cases}$$

**Answer.** $F(x, \sigma) = x_1^2 + x_2^2 + \sigma(x_2 - 1)^2$

$$\begin{aligned} \partial_{x_1} F &= 2x_1 \\ \partial_{x_2} F &= 2x_2 + 2\sigma(x_2 - 1) \end{aligned}$$

令 $\partial_{x_1} F = \partial_{x_2} F = 0$，解得 $x_1 = 0, x_2 = \frac{\sigma}{\sigma+1}$，当 $\sigma \to +\infty$ 时，$\bar{x} = (0, 1)^t, f^* = 1$.

**2** (P413 1.(2)). 用外点法求解下列问题:

$$\begin{cases} \min & x_1^2 + x_2^2 \\ \text{s.t.} & x_1 + x_2 - 1 = 0 \end{cases}$$

**Answer.** $F(x, \sigma) = x_1^2 + x_2^2 + \sigma(x_1 + x_2 - 1)^2$

$$\begin{aligned} \partial_{x_1} F &= 2x_1 + 2\sigma(x_1 + x_2 - 1) \\ \partial_{x_2} F &= 2x_2 + 2\sigma(x_1 + x_2 - 1) \end{aligned}$$

令 $\partial_{x_1} F = \partial_{x_2} F = 0$，解得 $x_1 = x_2 = \frac{\sigma}{2\sigma+1}$，当 $\sigma \to +\infty$ 时，$\bar{x} = (\frac{1}{2}, \frac{1}{2})^t$，$f^* = \frac{1}{2}$.

**3** (P413 1.(3)). 用外点法求解下列问题:

$$\begin{cases} \min & -x_1 - x_2 \\ \text{s.t.} & 1 - x_1^2 - x_2^2 = 0 \end{cases}$$

**Answer.** $F(x, \sigma) = -x_1 - x_2 + \sigma(1 - x_1^2 - x_2^2)^2$

$$\begin{aligned} \partial_{x_1} F &= -1 - 2\sigma(1 - x_1^2 - x_2^2)2x_1 \\ \partial_{x_2} F &= -1 - 2\sigma(1 - x_1^2 - x_2^2)2x_2 \end{aligned}$$

令 $\partial_{x_1} F = \partial_{x_2} F = 0$，解得 $x_1 = x_2, x_1(2x_1^2 - 1) = \frac{1}{4\sigma}$，当 $\sigma \to +\infty$ 时，$\bar{x} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^t$，$f^* = -\frac{1}{\sqrt{2}}$.

**4** (P413 1.(4)). 用外点法求解下列问题:

$$\begin{cases} \min & x_1^2 + x_2^2 \\ \text{s.t.} & 2x_1 + x_2 - 2 \le 0 \\ & x_2 \ge 1 \end{cases}$$

**Answer.** $F(x,\sigma)=x_1^2+x_2^2+\sigma\left(\max(0,2x_1+x_2-2)^2+\max(0,1-x_2)^2\right)$

$$\begin{aligned}\partial_{x_1}F&=2x_1+4\sigma\max(0,2x_1+x_2-2)\\ \partial_{x_2}F&=2x_2+2\sigma\max(0,2x_1+x_2-2)-2\sigma\max(0,1-x_2)\end{aligned}$$

令 $\partial_{x_1}F=\partial_{x_2}F=0$，当 $\sigma\to+\infty$ 时，$\bar{x}=(0,1)^t$，$f^*=1$.

**5** (P413 2)**.** 考虑下列非线性规划问题：

$$\begin{cases}\min & x_1^3+x_2^3\\ \text{s.t.} & x_1+x_2=1\end{cases}$$

(1) 求问题的最优解

(2) 定义罚函数

$$F(x,\sigma)=x_1^3+x_2^3+\sigma(x_1+x_2-1)^2$$

讨论能否通过求解无约束问题

$$\min F(x,\sigma)$$

来获得原来约束问题的最优解？为什么？

**Answer.** (1) 将 $x_2=1-x_1$ 代人目标函数，化成无约束问题：

$$\min f(x_1)=3x_1^2-3x_1+1$$

令 $f'(x_1)=6x_1-3=0$，得到 $x_1=x_2=\frac{1}{2}$，即 $\bar{x}=(\frac{1}{2},\frac{1}{2})^t, f^*=\frac{1}{4}$.

(2) 不能通过解 $\min F(x,\sigma)$ 来获得约束问题的最优解，因为不满足所有无约束问题最优解含于紧集的条件.

**6.** 考虑约束优化问题 (P1)

$$\begin{cases}\min & -x_1x_2x_3\\ \text{s.t.} & x_1+2x_2+3x_3=90\end{cases}$$

(1) 定义外罚函数为

$$G(x,c)=-x_1x_2x_3+\frac{c}{2}(x_1+2x_2+3x_3-90)^2$$

试用外罚函数法求解 (P1)

(2) 利用最优性条件判断 (1) 中得到的极限点是否是 (P1) 的局部最优解？

**Answer.** (1) $G(x,c) = -x_1x_2x_3 + \frac{c}{2}(x_1+2x_2+3x_3-90)^2$

$$\partial_{x_1}G = -x_2x_3 + c(x_1+2x_2+3x_3-90)$$
$$\partial_{x_1}G = -x_1x_3 + 2c(x_1+2x_2+3x_3-90)$$
$$\partial_{x_1}G = -x_1x_2 + 3c(x_1+2x_2+3x_3-90)$$

解得 $x_1 = 9c - \sqrt{81c^2-540c}$，$x_2 = \frac{9c-\sqrt{81c^2-540c}}{2}$，$x_3 = \frac{9c-\sqrt{81c^2-540c}}{3}$. 令 $c \to +\infty$，则 $\bar{x} = (30,15,10)^t$，$f^* = -4500$.

(2) 由 KKT 条件

$$x_1+2x_2+3x_3 = 90$$
$$-x_2x_3 + \mu = 0$$
$$-x_1x_3 + \mu = 0$$
$$-x_1x_2 + \mu = 0$$

可得，(1) 中的点 $\bar{x} = (30,15,10)^t$ 是局部最优解.