1. 用外点罚函数方法求解如下问题:

$$\begin{cases} \min & x_1^2 + x_2^2 \\ \text{s.t.} & 2x_1 + x_2 \le 2 \\ & x_2 \ge 1 \end{cases}$$

Answer. $F(x,\sigma) = x_1^2 + x_2^2 + \sigma\left(\max(0, 2x_1 + x_2 - 2)^2 + \max(0, 1 - x_2)^2\right)$

$$\begin{aligned} \partial_{x_1} F &= 2x_1 + 4\sigma \max(0, 2x_1 + x_2 - 2) \\ \partial_{x_2} F &= 2x_2 + 2\sigma \max(0, 2x_1 + x_2 - 2) - 2\sigma \max(0, 1 - x_2) \end{aligned}$$

令 $\partial_{x_1} F = \partial_{x_2} F = 0$, 当 $\sigma \to +\infty$ 时,$\bar{x} = (0,1)^t$,$f^* = 1$.

2. 考虑如下问题, 定义障碍函数为 $P(x,\pi) = x_1 x_2 - \frac{\ln g(x)}{\pi}$

$$\begin{cases} \min & x_1 x_2 \\ \text{s.t.} & g(x) = -2x_1 + x_2 \ge 3 \end{cases}$$

用内点罚函数方法求解此问题.

Answer. 令 $\partial_{x_1} P = \partial_{x_2} P = 0$, 发现无解，故不存在最优点.

3. 用乘子罚函数方法求解如下问题:

$$\begin{cases} \min & x_1 + \frac{1}{3}(x_2 + 1)^2 \\ \text{s.t.} & x_1 \ge 0 \\ & x_2 \ge 1 \end{cases}$$

Answer. 定义增广 Lagrange 函数

$$\Phi(x, w, \sigma) = f(x) + \frac{1}{2\sigma}\left[(\max\{0, w_1 - \sigma g_1(x)\})^2 - w_1^2 + (\max\{0, w_2 - \sigma g_2(x)\})^2 - w_2^2\right]$$

$$\frac{\partial \Phi}{\partial x_1} = \begin{cases} 1 - (w_1 - \sigma x_1), & x_1 \le \frac{w_1}{\sigma} \\ 1, & x_1 > \frac{w_1}{\sigma} \end{cases}$$

$$\frac{\partial \Phi}{\partial x_2} = \begin{cases} \frac{2}{3}(x_2 + 1) - \left[w_2 - \sigma(x_2 - 1)\right], & x_2 - 1 \le \frac{w_2}{\sigma} \\ \frac{2}{3}(x_2 + 1), & x_2 - 1 > \frac{w_2}{\sigma} \end{cases}$$

第 k 次迭代中, 令 $\frac{\partial \Phi}{\partial x_1} = \frac{\partial \Phi}{\partial x_2} = 0$, 解得 $x^{(k)} = (\frac{w_1^{(k)} - 1}{\sigma}, \frac{3w_2^{(k)} + 3\sigma - 2}{2 + 3\sigma})^t$, 修正乘子 $w^{(k)}$, 得乘子 $\bar{w} = (1, \frac{4}{3})^t$, 最优解 $\bar{x} = (0, 1)$, $f_{\min} = \frac{4}{3}$.