总成绩 (100%) = 考勤 (10%) + 习题 (30%) + 测验 (60%)，习题每个点至少完成一半

课程概况：

- 绪论 2h
- 凸集和凸函数 3h
- 凸优化问题 6h
- 拉格朗日乘子 5h
- 凸优化应用 6h
- 无约束凸优化问题求解 5h
- 有约束凸优化问题求解 4h
- 课程测试 2h

1 绪论

Remark 1.1. f 是凸函数，$\nabla f_x = 0 \Leftrightarrow$ 点 x 是最小值点

2 凸集和凸函数

Remark 2.1. 凸集：$\forall x, y \in C, \theta \in [0,1] \Longrightarrow \theta x + (1-\theta)y \in C$。空集、点、线段都是凸集。

Remark 2.2. 凸集的例子：

- 超平面：$\{x : a^t x = b\}$，$a \in \mathbb{R}^n - \{0\}$ 是法向量
- 半平面：$\{x : a^t x \leq b\}$，$a \in \mathbb{R}^n - \{0\}$
- 欧几里得球：$B(x_c, r) = \{x : \|x - x_c\| < r\} = \{x_c + ru : \|u\| < 1\}$
- 椭球：$\{x : (x - x_c)^t P^{-1}(x - x_c) < 1\}, P \in S^n_{++}$ 或 $\{x_c + Au : \|u\| < 1\}, A \in S^n_{++}$
- 多面体：$\{x : Ax \preceq b, Cx = d\}, A \in \mathbb{R}^{m\times n}, C \in \mathbb{R}^{p\times n}, b \in \mathbb{R}^m, d \in \mathbb{R}^p$，多面体是半空间和超平面的交集，任意个数的凸集的交集是凸集。

Remark 2.3. 凸集 S：是 S 中所有点的凸组合的最小集合

- 如果 C 是一个凸集，则 $aC + b = \{ax + b : x \in C\}, a \in \mathbb{R}, b \in \mathbb{R}^n$ 也是一个凸集
- 对于仿射函数 $f(x) = Ax + b, A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$

- 凸集在 f 下的像是凸集

$$S \subset \mathbb{R}^n \text{ convex } \Longrightarrow f(S) = \{f(x) : x \in S\} \subset \mathbb{R}^m \text{ convex}$$

- 凸集在 f 下的逆像是凸集

$$C \subset \mathbb{R}^m \text{ convex } \Longrightarrow f^{-1}(C) = \{x : f(x) \in C\} \subset \mathbb{R}^n \text{ convex}$$

- 两个凸集可以用一个超平面分离（证明困难）
- 支撑超平面：$\{x : a^t x = a^t x_0\}, a \in \mathbb{R}^n - \{0\}, a^t x \leq a^t x_0, \forall x \in C$，其中 C 是一个凸集，x_0 是凸集上一边界点
- 支撑超平面定理：如果 C 是凸的，则在 C 的每个边界点上都存在一个支撑超平面

Remark 2.4. 凸函数：$f : \mathbb{R}^n \to \mathbb{R}$ 的定义域 $\mathrm{dom}(f)$ 是一个凸集，满足 $\forall x, y \in \mathrm{dom}(f), \theta \in [0, 1]$

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

f 是一个凸函数，则 f 在定义域的任何内点都是连续的，并且 f 是局部有界的：$\exists B(x, r) \subset \mathrm{dom}(f)$

$$z = \theta x + (1-\theta)y \Longrightarrow \frac{f(z) - f(x)}{1-\theta} \leq \frac{f(y) - f(z)}{\theta} \Longrightarrow \frac{f(z) - f(x)}{\|z - x\|} \leq \frac{f(y) - f(z)}{\|y - z\|}$$

Remark 2.5. 一些凸函数的例子：

- 仿射函数：$a^t x + b, \forall a \in \mathbb{R}^n, b \in \mathbb{R}$
- 幂函数 x^α on $\mathbb{R}_{++} = (0, \infty), \alpha \geq 1$ or $\alpha \leq 0$
- l^p 范数：$\|x\|_p = \sum_{i=1}^n (|x_i|^p)^{\frac{1}{p}}$ on $\mathbb{R}^n$ for $p \geq 1$

Remark 2.6. 凸函数的性质

- f is convex $\Longrightarrow \alpha f$ for $\alpha > 0$ is convex.
- $f_1, \ldots, f_m$ are convex $\Longrightarrow f_1 + \cdots + f_m$ is convex.
- Composition with affine function: f is convex $\Longrightarrow f(Ax + b)$ is convex.
- $f_1, \ldots, f_m$ are convex $\Longrightarrow f(x) = \max\{f_1(x), \ldots, f_m(x)\}$ is convex.
  - 分段线性函数 (piecewise-linear function): $f(x) = \max_{1 \leq i \leq m} \{a_i^t x + b_i\}$

Remark 2.7. 严格凸函数 strictly convex function，$f : \mathbb{R}^n \to \mathbb{R}$

- dom(f) is convex set
- $\forall x \neq y \in \mathrm{dom}(f), \theta \in (0,1)$

$$f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta)f(y)$$

l^p-norm and l^∞-norm are not strictly convex.

Remark 2.8. 强凸函数 strongly convex function，$f : \mathbb{R}^n \to \mathbb{R}$

- dom(f) is convex set
- $\exists m > 0$, satisfy $f(x) - \frac{m}{2}\|x\|^2$ is convex.
- 判定方法：$\nabla^2 f \succ 0$
- 谱分解：$A = P^t Q P$，其中 Q 的对角线是 A 的特征值
- $\nabla^2 f - mI \succeq 0 \Longrightarrow u^t(\nabla^2 f - mI)u \geq 0 \Longrightarrow u^t\nabla^2 f u \geq m\|u\|^2 \Longrightarrow u^t P^t Q P u \geq m\|u\|^2 \Longrightarrow \forall \lambda \geq m, \lambda$ 是 $\nabla^2 f(x)$ 的特征值
- $f(x)$ is convex $\Longrightarrow f(x) + \frac{m}{2}\|x\|^2$ is strictly convex.

strongly convex $\Longrightarrow$ strictly convex $\Longrightarrow$ convex

Remark 2.9. 凸函数判定：

- First-order condition: **differentiable** f with **convex domain** is convex iff $\forall x, y \in \mathrm{dom}(f)$

$$f(y) \geq f(x) + \nabla f(x)^t(y-x)$$

  - *Proof.*
    - f is convex $\Longleftrightarrow \frac{f(z)-f(x)}{z-x} \leq \frac{f(y)-f(z)}{y-z} \Longrightarrow$ with $z \to x, \frac{f(y)-f(x)}{y-x} \geq f'(x) \Longrightarrow f(y) \geq f(x) + f'(x)(y-x)$
    - $f(y) \geq f(x) + f'(x)(y-x) \Longrightarrow \frac{f(y)-f(z)}{y-z} \geq f'(z) \geq \frac{f(x)-f(z)}{x-z} \Longrightarrow$ by $z = \theta x + (1-\theta)y, f$ is convex

  $f(x^*) = 0 \Longleftrightarrow x^*$ is global minimum of f.

  f is strictly convex $\Longleftrightarrow \forall x \neq y \in \mathrm{dom}(f), f(y) > f(x) + \nabla f(x)^t(y-x)$

- Second-order conditions: for **twice differentiable** f with **convex domain** is convex iff $x \in \mathrm{dom}(f)$

$$\nabla^2 f(x) \succeq 0$$

  - if $\nabla^2 f(x) \succ 0$ for $\forall x \in \mathrm{dom}(f) \Longrightarrow f$ is strictly convex.

  - $\exists m > 0$, satisfy $\nabla^2 f(x) \succeq mI$ for $\forall x \in \operatorname{dom}(f) \iff f$ is strongly convex.

- Restriction of a convex function to a line:
  - $f : \mathbb{R}^n \to \mathbb{R}$ is convex iff the function $g : \mathbb{R} \to \mathbb{R}$

$$g(t) = f(x + tv), \quad \operatorname{dom}(g) = \{t : x + tv \in \operatorname{dom}(f)\}$$

    is convex for any $x \in \operatorname{dom}(f)$ and $v \in \mathbb{R}^n$
  - *Proof.* $g(t) = f(x + t(y - x))$ is convex $\iff g(\theta) \le \theta g(0) + (1 - \theta)g(1) \iff f(\theta x + (1 - \theta y)) \le \theta f(x) + (1 - \theta)f(y) \iff f$ is convex. □

Remark 2.10. $X \in \mathbb{S}^n_{++} \implies X = P^t Q P = P^t \operatorname{diag}(q_1, \ldots, q_n)P \implies X^\alpha \triangleq P^t \operatorname{diag}(q_1^\alpha, \ldots, q_n^\alpha)P$, satisfy $X^\alpha X^\beta = X^{\alpha+\beta}$, $X^0 = I$.

Remark 2.11.

- sublevel set(下水平集) of $f : \mathbb{R}^n \to \mathbb{R}$

$$C_\alpha = \{x\ in \operatorname{dom}(f) : f(x) \le \alpha\}$$

  sublevel set of convex functions are convex.

- Epigraph(上境图) set of $f : \mathbb{R}^n \to \mathbb{R}$

$$\operatorname{epi}(f) = \left\{(x, t) \in \mathbb{R}^{n+1} : x \in \operatorname{dom}(f), t \ge f(x)\right\}$$

  f is convex iff epi is convex set.

3 凸优化问题

Remark 3.1. Optimization problem:

$$\begin{cases} \text{minimize} & f(x) \\ \text{subject to} & f_i(x) \le 0, \quad 1 \le i \le m \\ & h_i(x) = 0, \quad 1 \le i \le p \end{cases}$$

- feasible set $X \subset \mathcal{D}$

$$\mathcal{D} = \operatorname{dom}(f) \cap \left(\cap_{i=1}^m \operatorname{dom}(f_i)\right) \cap \left(\cap_{i=1}^p \operatorname{dom}(h_i)\right)$$

- optimal value: $p^* = \inf\{f(x) : x \text{ is feasible}\}$

- a feasible x is an optimal solution(minimizer) if $f(x) = p^*$

Remark 3.2. Convex optimization problem(COP):

$$\begin{cases} \text{minimize} & f(x) \\ \text{subject to} & f_i(x) \leq 0, \quad 1 \leq i \leq m \\ & Ax = b \end{cases}$$

- objective function f is convex
- inequality constraints $f_1., \ldots, f_m$ are convex
- equality constraints are affine: $A \in \mathbb{R}^{p\times n}, b \in \mathbb{R}^p$
- the feasible set X of COP is convex
  - $X = \text{dom}(f) \cap (\cap_{i=1}^m X_i) \cap \text{hyperplanes}$
    * $X_i = \{x \in \text{dom}(f_i) : f_i(x) \leq 0\}$
  - so a COP is actually an unconstrained COP defined on a convex set.
- any local minimum of a COP is globally optimal.
  - x^* is a local minimum: a solution of the COP in $B(x^*, r) \cap X$
  - $\forall y \in X$, take $\theta \to 0$, satisfy $z = \theta x^* + (1-\theta)y \in B(x^*, r)$, by convexity, $\theta f(x^*) + (1-\theta)f(y) \geq f(z) \geq f(x^*)$, thus $f(y) \geq f(x^*)$.
- the set of optimal solutions is convex.

Remark 3.3. Important examples:

- Linear program(LP):

$$\begin{cases} \text{minimize} & c^t x \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{cases}$$

  - convex problem with affine object over a polyhedron
  - standard from

$$\begin{cases} \text{minimize} & c^t x \\ \text{subject to} & x \succeq 0 \\ & Ax = b \end{cases}$$

- Quadratic program(QP):

$$\begin{cases} \text{minimize} & \frac{1}{2}x^t P x + q^t x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{cases}$$

  $P \in \mathcal{S}_+^n$, convex problem with quadratic object over a polyhedron.

- Quadratically constrained quadratic program(QCQP)

$$\begin{cases} \text{minimize} & \frac{1}{2}x^t P x + q^t x + r \\ \text{subject to} & \frac{1}{2}x^t P_i x + q_i^t x + r \preceq 0, \quad 1 \le i \le m \\ & Ax = b \end{cases}$$

  - $P, P_i \in \mathbb{S}_+^n$, objective and constraints are convex quadratic.
  - if $P_1, \ldots, P_m \in \mathbb{S}_{++}^n$, feasible region is intersection of m ellipsoids andan affine set.

Remark 3.4. For **differentiable** f, $x \in X$ is optimal iff

$$\nabla f(x)^t (y - x) \ge 0, \quad \forall y \in X$$

- if x is optimal, let $g(\theta) = f(x + \theta(y - x))$

$$0 \le \lim_{\theta \downarrow 0} \frac{f(x + \theta(y - x)) - f(x)}{\theta} = g'(0) = \nabla f(x)^t (y - x)$$

- conversely, by convexity(First order condition)

$$f(y) \ge f(x) + \nabla f(x)^t (y - x) \Longrightarrow \nabla f(x)^t (y - x) \ge 0$$

Important, for COP:

$$x \text{ optimal} \Longleftrightarrow \nabla f(x)^t (y - x) \ge 0, \forall y \in X$$

Remark 3.5. Unconstrained COP, with differentiable f

$$\text{minimize}\, f(x)$$

- $x \in \operatorname{dom}(f)$ (open set!) is optimal iff $\nabla f(x) = 0$
  - if x is optimal, $\nabla f(x)^t (y - x) \ge 0$ for any feasible y, take $y = x - \lambda \nabla f(x)$ for sufficient small $\lambda > 0$, thus $\nabla f(x) = 0$
  - conversely, $\nabla f(x)^t (y - x) = 0$

– Intuitive interpretation: x is optimal, then $\langle \nabla f(x), y-x\rangle \geq 0$. if $\nabla f(x) \neq 0$, $\exists y$ satisfy $\langle \nabla f(x), y-x\rangle < 0$, so $\nabla f(x) = 0$.

Remark 3.6. Equality constrained COP, with differentiable f, $A \in \mathbb{R}^{p\times n}, b \in \mathbb{R}^p$

$$\begin{cases} \text{minimize} & f(x) \\ \text{subject to} & Ax = b \end{cases}$$

- $x \in \operatorname{dom}(f)$ is optimal iff:

$$Ax = b, \text{ and there exists } v \in \mathbb{R}^p, \text{ s.t. } \nabla f(x) = A^t v, \nabla f(x) \in \mathcal{R}(A^t)$$

Proof. x optimal $\iff Ax = b, \nabla f(x) = A^t v$.

– "$\Longleftarrow$": $\forall y \in X$, $Ay = b \Longrightarrow (\nabla f(x))^t (y-x) = v^t A(y-x) = v^t(b-b) = 0 \Longrightarrow$, x is optimal.

– $\mathcal{N}(A) = \mathcal{R}(A^t)^\perp, \mathcal{N}(A)^\perp = \mathcal{R}(A^\perp)$

– "$\Longrightarrow$": $\forall u \in \mathcal{N}(A)$, $x + \theta u \in \operatorname{dom}(f)$ for $\theta \to 0$. make $y = x + \theta u$, $\nabla f(x)(y-x) \geq 0 \Longrightarrow \theta\langle \nabla f(x), u\rangle \geq 0$ satisfy for all $\theta \to 0$, so $\langle \nabla f(x), u\rangle = 0$. As a result, $\nabla f(x)^t \in \mathcal{N}(A)^\perp$, then $\exists v \in \mathbb{R}^p$ s.t. $\nabla f(x) = A^t v$.

□

Remark 3.7. Equality constrained QP: $P \in \mathbb{S}^n_+, q \in \mathbb{R}^n, r \in \mathbb{R}, A \in \mathbb{R}^{p\times n}$ with $\operatorname{rank}(A) = p$, $b \in \mathbb{R}^p$

$$\begin{cases} \text{minimize} & \frac{1}{2}x^t P x + q^t x + r \\ \text{subject to} & Ax = b \end{cases}$$

- x^* is optimal $\iff \exists v^* \in \mathbb{R}^p$ s.t.

$$\begin{bmatrix} P & A^t \\ A & 0 \end{bmatrix} \begin{bmatrix} x^* \\ v^* \end{bmatrix} = \begin{bmatrix} -q \\ b \end{bmatrix}$$

- coefficient matrix is called KKT matrix.

- KKT matrix is nonsingular $\iff$ "$Ax = 0, x \neq 0 \Longrightarrow x^t P x > 0$"

Remark 3.8. Inequality constrained COP, with differentiable $f, f_1, \ldots, f_m$

$$\begin{cases} \text{minimize} & f(x) \\ \text{subject to} & f_i(x) \leq 0, \quad ,1 \leq i \leq m \end{cases}$$

- sufficient condition: for a feasible x, if $\lambda_i \geq 0$ for $i \in [1, m]$ and $\nabla f(x) + \sum_{i=1}^m \lambda_i \nabla f_i(x) = 0$, $\lambda_1 f_1(x) = \ldots = \lambda_m f_m(x) = 0$, x is optimal if the following is established
  - $f_i(x) \neq 0 \Longrightarrow \lambda_i = 0$
  - $f_i(x) = 0 \Longrightarrow \nabla f_i(x)^t(y - x) \leq 0$ for any feasible y
  - for any feasible y, $\nabla f(x)^t(y - x) = -\sum_{i=1}^m \lambda_i \nabla f_i(x)^t(y - x) \geq 0$
- the converse is false.

Remark 3.9. COP over nonnegative orthant, with differentiable f

$$\begin{cases} \text{minimize} & f(x) \\ \text{subject to} & x \succeq 0 \end{cases}$$

- $x \in \mathrm{dom}(f)$ is optimal iff

$$x \succeq 0, \qquad \begin{cases} \nabla f(x)_i \geq 0 & \text{if } x_i = 0 \\ \nabla f(x)_i = 0 & \text{if } x_i > 0 \end{cases}$$

- *Proof.* by observe
  - x is optimal $\Longrightarrow \nabla f(x)^t(y - x) \geq 0$ holds **for all feasible** y
  - for $x_i > 0 \Longrightarrow y_i - x_i$ can be positive or negative, so $\nabla f(x)_i = 0$
  - for $x_i = 0 \Longrightarrow y_i - x_i \geq 0, \nabla f(x)_i \geq 0$

□

Remark 3.10.

$$\lim_{\varepsilon \to 0} \frac{f(x + a\varepsilon) - f(x)}{\varepsilon} = \langle \nabla f(x), a \rangle, \qquad \varepsilon \to x + a\varepsilon \to f(x + a\varepsilon)$$

Remark 3.11. COP over a simplex, with differentiable f

$$\begin{cases} \text{minimize} & f(x) \\ \text{subject to} & x \succeq 0, \sum_{i=1}^n x_i = 1 \end{cases}$$

- $x \in \mathrm{dom}(f)$ is optimal iff

$$\partial_j f(x) \geq \partial_i f(x) \text{ for all } 1 \leq j \leq n \text{ when } x_i > 0$$

- *Proof.* by observe

  - if x is optimal, $\nabla f(x)^t(y-x) \geq 0$ holds for all feasible y
  - for $x_i > 0 \Longrightarrow y_i - x_i$ can be positive or negative, so $\partial_j f(x) \geq \partial_i f(x) = 0$ for any j
  - the converse is obvious

$$\nabla f(x)^t(y-x) = \sum_{x_i>0} \partial_i f(x)(y_i - x_i) + \sum_{x_i=0} \partial_i f(x)(y_i - x_i) \geq C \sum_{i=1}^{n} (y_i - x_i) = 0$$

□

4 拉格朗日乘子

Remark 4.1. Standard form optimization problem (not necessarily convex)

$$\begin{cases} \text{minimize} & f(x) \\ \text{subject to} & f_i(x) \leq 0, \quad 1 \leq i \leq m \\ & h_i(x) = 0, \quad 1 \leq i \leq p \end{cases}$$

$x \in \mathcal{D} = \operatorname{dom}(f) \cap (\cap_{i=1}^{m} \operatorname{dom}(f_i)) \cap (\cap_{i=1}^{p} \operatorname{dom}(h_i))$, optimal value denoted p^*. $x \in \mathcal{D}$ **do not need to satisfy constraints.**

- Lagrange function, $L : \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^p \mapsto \mathbb{R}$

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \mu_i h_i(x)$$

- Lagrange dual function, $g : \mathbb{R}^m \times \mathbb{R}^p \mapsto \mathbb{R}$

$$g(\lambda, \mu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \mu)$$

  - g is concave, can be $-\infty$ for some (λ, μ)
  - lower bound property: if $\lambda \succeq 0$, then $g(\lambda, \mu) \leq p^*$

Remark 4.2. Standard form LP:

$$\begin{cases} \text{minimize} & c^t x \\ \text{subject to} & x \succeq 0 \\ & Ax = b \end{cases}$$

- Lagrangian:

$$L(x, \lambda, \mu) = c^t x - \lambda^t x + \mu^t (Ax - b) = -\mu^t b + (c + A^t \mu - \lambda)^t x$$

- dual function:

$$
\begin{aligned}
g(\lambda, \mu) &= \inf_x \left(-\mu^t b + (c + A^t\mu - \lambda)^t x\right) \\
&= \begin{cases} -\mu^t b, & \text{if } c + A^t\mu - \lambda = 0 \\ -\infty, & \text{otherwise} \end{cases}
\end{aligned}
$$

- lower bound property: $p^* \geq -\mu^t b$ if $c + A^t\mu \succeq 0$.

Remark 4.3. Tow-way partitioning, for $W \in \mathbb{S}^n$

$$
\begin{cases} \text{minimize} & x^t W x \\ \text{subject to} & x_i^2 = 1, \quad i = 1, \ldots, n \end{cases}
$$

- a nonconvex problem, feasible set contains 2^n discrete points
- Lagrangian: $L(x, \mu) = f(x) + \sum_{i=1}^n \mu_i(x_i^2 - 1)$
- dual function:

$$
\begin{aligned}
g(\mu) &= \inf_x \ (x^t W x + \sum_{i=1}^n \mu_i(x_i^2 - 1)) \\
&= \inf_x \ x^t(W + \operatorname{diag}(\mu))x - \sum_{i=1}^n \mu_i \\
&= \begin{cases} -\sum_{i=1}^n \mu_i, & \text{if } W + \operatorname{diag}(\mu) \succeq 0 \\ -\infty, & \text{otherwise} \end{cases}
\end{aligned}
$$

- lower bound property: $p^* \geq -\sum_{i=1}^n \mu_i$ if $W + \operatorname{diag}(\mu) \succeq 0$.
  - $p^* \geq n\lambda_{\min}(W)$, where $\lambda_{min}(W)$ is the smallest eigenvalue of W.
  - *Proof.* $W + \operatorname{diag}(\mu) = P^t Q P + \theta I = P^t(Q + \theta I)P \succeq 0$, let $\operatorname{diag}(\mu) = \theta I$, so $\lambda_i + \theta \geq 0, \theta = -\lambda_{\min}(W)$, $p^* \geq n\lambda_{\min}(W)$. □

Remark 4.4. Lagrange dual problem

$$
\begin{cases} \text{maximize} & g(\lambda, \mu) \\ \text{subject to} & \lambda \succeq 0 \end{cases}
$$

- COP, optimal value denoted d^*
- finds best lower bound on $p*$, obtained from Lagrange dual function.
- week duality: $d^* \leq p^*$

- always holds (for both convex and nonconvex problems)
- can be used to find nontrivial lower bounds for difficult problems

- strong duality: $d^* = p^*$
  - does not hold in general, but usually holds for COP
  - an example that the strong duality does not hold

$$\begin{cases} \text{minimize} & e^{-x} \\ \text{subject to} & x^2/y \le 0 \end{cases}$$

* $\mathcal{D} = \{(x, y) : x \in \mathbb{R}, y > 0\}, g(\lambda) = 0 \Longrightarrow d^* = 0 < p^* = 1$

5 凸优化应用

6 无约束凸优化问题求解

7 有约束凸优化问题求解