University of Iowa | Homework 0 (Ungraded) | CS 3330
Department of CS | Fall 2024

Reading: Review of Discrete Math and Data Structures

1. **[Induction]**:

   (a) [10 points] Using induction, prove that for any $r \neq 1$ and any integer $k \geq 1$, we have

   $$1 + r + r^2 + \cdots + r^{k-1} = \frac{1 - r^k}{1 - r}.$$

   Using the above equality, show that if $|r| < 1$, then

   $$1 + r + r^2 + \cdots = \frac{1}{1 - r}.$$

   (b) [10 points] Using induction, prove that for any integer $n \geq 1$, we have:

   $$1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n! = (n + 1)! - 1$$

2. **[Proof by Contradiction]**:

   (a) [10 points] Prove or disprove the following statements: if $a \times b \geq 30$ then either $a \geq 5$ or $b \geq 6$.

   (b) [15 points] Prove that there is no way to arrange 5 people in a circle such that no two people who were initially adjacent are still adjacent.

3. **[Set Theory]**: [20 points] Let $A_1, A_2, \ldots, A_n$ be a collection of sets. Prove the generalized DeMorgan's laws:

   $$\left(\bigcup_{i=1}^{n} A_i \right)^c = \bigcap_{i=1}^{n} A_i^c$$

   and

   $$\left(\bigcap_{i=1}^{n} A_i \right)^c = \bigcup_{i=1}^{n} A_i^c.$$

4. **[Graph Theory]**:

   (a) [15 points] Prove that a connected graph G with n vertices and $n - 1$ edges is a tree. Conclude that for any connected undirected graph, the number of edges is at least $n - 1$.

   (b) [10 points] Show that the total number of edges in a graph G is equal to half the summation of the degrees of all vertices. Specifically, prove that

   $$|E| = \frac{1}{2} \sum_{v \in V} d_v,$$

   where the degree d_v of a vertex v is the number of neighbors of v in the graph G.

   (c) [10 points] Use induction to prove that a non-empty binary tree with n nodes has height at least $\log_2 n$