University of Iowa | Homework 1 | CS 3330
Department of CS | Fall 2024

Assigned: Friday 11:59 PM, August 30, 2024
Due: Friday 11:59 PM, September 6, 2024
Reading: Kleinberg and Tardos, Chapter 1.1, Slides of Week 1

1. **[10 points] [A Human Compiler]** Show the execution of the Gale-Shapley algorithm presented in 01StableMatching (including **proposal, acceptance, rejections, and switches**) on input with 3 hospitals and 3 students having the following preferences:

$$h_1 : s_1 > s_2 > s_3 \qquad h_2 : s_2 > s_1 > s_3 \qquad h_3 : s_2 > s_1 > s_3$$
$$s_1 : h_2 > h_3 > h_1 \qquad s_2 : h_1 > h_3 > h_2 \qquad s_3 : h_1 > h_2 > h_3$$

   Let us suppose that whenever your algorithm has a choice of "free" hospital to pick for making a proposal, it picks the one with the **highest** index. Hence, h_3 will make the first proposal.

2. **[Gale-Shapley Execution Times]**: Consider the Gale-Shapley algorithm discussed in class.

   (a) [15 points] Show that there is an input with n hospitals and n med-school students to the Gale-Shapley algorithm that forces it to execute $\Omega(n^2)$ iterations of the **while**-loop. Describe both the input and the algorithm execution.

   (b) [15 points] Prove that there is an input with n hospitals and n med-school students to the Gale-Shapley algorithm where the algorithm terminates after only $O(n)$ iterations of the **while**-loop. Describe both the input and the algorithm execution.

3. **[Matched/Unmatched]**: Consider a stable matching problem instance where a medical school m_1 prefers a student s_1 the most among all the students. Similarly, the student s_1 prefers the medical school m_1 the most among the medical schools.

   (a) [10 points] Prove that the matching produced by the Gale-Shapley Algorithm always includes (s_1, m_1) pair.

   (b) [10 points] Argue that any matching S which does not include the (s_1, m_1) pair is unstable.

   (c) [20 points] Now consider the opposite situation where m_1 prefers the student s_1 the least among all students and s_1 also prefers m_1 the least among all medical schools. Prove that the following statement is **false**:
   Any matching S which pairs m_1 and s_1 is unstable.

4. **[Stable Teammates Problem]**: Let us consider the problem of creating teams for a 3v3 pickup basketball (where each team consists of 3 players) event. Let us assume a total of $3 \times n$ players, where n is an integer greater than 2. We aim to test if we can use the ideas from the Gale-Shapley algorithm to create a 'stable team matching'. First, we ask all the players to rank the remaining players according to their preferences.

   (a) [10 points] Design a variant of the Gale-Shapley algorithm, which leverages the preference list to produce a set of teams, each consisting of 3 players. Provide concise and clear pseudocode.

(b) [10 points] Now, given a matching $\mathcal{M}$ of a set of teams (each consisting of 3 players), let us define an unstable team as a tuple $\mathcal{T} = (a, b, c)$ of players, where all a, b, and c, prefer both teammates in T than both the teammates they are assigned in $\mathcal{M}$. Now, construct an input of 9 players (including their preferences) to the Stable Teammates problem for which there is a matching with an unstable team. Show the unstable team.