CS 3330: Algorithms

Midterm 1, Fall 2024

Instructions

- You must neither give nor receive aid on this exam. To do otherwise is a **violation of the Honor Code.**
- There are five questions. The exam is out of 120 points. Each question specifies the points it is worth.
- The exam will be graded from 0 to 100; i.e., we will cut off the scores at 100.
- You will have 120 minutes to complete the exam.
- Ask the instructor for extra blank papers if you need any.
- Write your answers carefully. Use complete sentences. Illegible texts may not get points.

Good Luck!

First Name	
Last Name	
HawdkID	

The work in this paper is my own. Signature: ______________________

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1. (25 points) Multiple Choice Questions. In each of the following, please select **ALL** options that are true.

   (a) (5 points) Let $f(n) = 10n^7 + 32n^3 + 18n^{-1} + 44$. Select all that applies:

   - ☐ $f(n)$ is $O(n^{19})$
   - ☐ $f(n)$ is $O(n^7 \log(n))$
   - ☐ $f(n)$ is $O(n^{-1})$
   - ☐ $f(n)$ is $O(n^6)$
   - ☐ $f(n)$ is $O(n^8 + \sin(n))$

   (b) (5 points) Select all true statements:

   - ☐ 3^n is $\Theta(2^n)$
   - ☐ $n^{1+0.5\sin(n)}$ is $\Omega(\sqrt{n})$
   - ☐ $\log\log(n^{12})$ is $O(n\log(n))$
   - ☐ $\log_3(n^4)$ is $\Theta\big(\log_2(n^2)\big)$
   - ☐ $1/n$ is $O(\log(n))$

   (c) (5 points) Consider the following statements, where a, b, k, and c are some constants greater than 1. Please select all the true statements.

   - ☐ $f(n)$ is $\Theta(g(n))$ implies $g(n)$ is $\Theta(f(n))$
   - ☐ If $f(n) = \log_c(n)$ and $g(n) = c^n$, then $g(n)$ is $\Omega(f(n))$
   - ☐ $(n-c)^k$ is $\Omega(n^k)$
   - ☐ $[f(n) \times g(n)] + [f(n) - g(n)]$ is $\Theta(\max[f(n), g(n)])$
   - ☐ If $f(n)$ is $O(g(n))$ then there exist $c > 0$ such that $0 \leq f(n) \leq cg(n)$ for infinitely many values of n.

(d) (5 points) Consider a set of preferences between hospitals and medical students. Assume there are n students and n hospitals. We will use the Gale-Shapley algorithm to find a stable matching between them. Select all true statements:

- ☐ Assuming hospitals make proposals, they can benefit by lying about their preferences.
- ☐ The running time of the gale-shapely algorithm for certain instances could potentially be $O(n^3)$.
- ☐ The running time of the gale-shapely algorithm for certain instances could potentially be $O(n \log n)$.
- ☐ A hospital may be paired with a student who is not its best valid partner.
- ☐ A student may be paired with their best valid partner.

(e) (5 points) Consider an undirected (unless mentioned otherwise) graph $G = (V, E)$. Suppose that $|V| = n \geq 3$, and that $|E| \geq 1$. Select all true statements:

- ☐ If G is a tree, it is also a bipartite graph.
- ☐ If G has no cycle and is connected, then it has at least one node with degree two.
- ☐ If G is a tree, then it has n edges.
- ☐ If D is a cutset of G, then it contains even number of edges.
- ☐ Let G be a directed graph. If G is a DAG, then G might be strongly connected.

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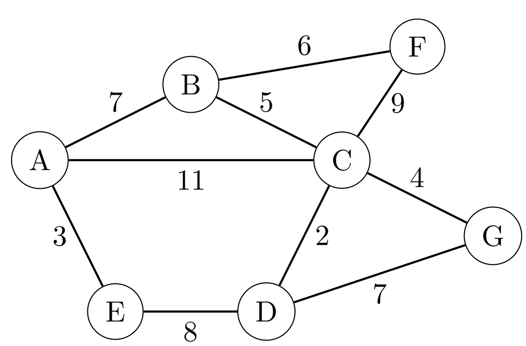
2. (20 points) Execute the following pseudocode of the Prim's algorithm assuming we initiate the function with $s = A$. Fill in the table on the next page at the end of each iteration of the while loop (line 20).

Algorithm 1 Prim's Algorithm

1: $S \leftarrow \emptyset, T \leftarrow \emptyset$
2: $s \leftarrow$ node A
3: $\text{pred}[s] \leftarrow \text{null}; \pi[s] \leftarrow 0$
4: **for all** $v \neq s$ **do**
5:     $\pi[v] \leftarrow \infty, \text{pred}[v] \leftarrow \text{null}$
6: **end for**
7: Create an empty priority queue pq
8: **for all** $v \in V$ **do**
9:     INSERT$(pq, v, \pi[v])$
10: **end for**
11: **while** IS-NOT-EMPTY(pq) **do**
12:     $u \leftarrow$ DEL-MIN(pq)
13:     $S \leftarrow S \cup \{u\}, T \leftarrow T \cup \{\text{pred}[u]\}$
14:     **for all** edge $e = (u, v) \in E$ with $v \notin S$ **do**
15:         **if** $c_e < \pi[v]$ **then**
16:             DECREASE-KEY(pq, v, c_e)
17:             $\pi[v] \leftarrow c_e, \text{pred}[v] \leftarrow e$
18:         **end if**
19:     **end for**
20: **end while**



Iteration	$\pi[A]$	$\pi[B]$	$\pi[C]$	$\pi[D]$	$\pi[E]$	$\pi[F]$	$\pi[G]$
0	0	∞	∞	∞	∞	∞	∞
1							
2							
3							
4							
5							
6							
7							
8							

Iteration	T
0	$\emptyset$
1	
2	
3	
4	
5	
6	
7	
8	

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3. (15 points) Jeff is traveling downstream the Mississippi river from Davenport to New Orleans in his new boat. He can travel m miles before he runs out of fuel once he fills his gas tank to its total capacity. He starts with a full tank. He has a map of all the gas stations along the Mississippi river and wants to minimize the number of gas stops he has to make. Give an **efficient** greedy algorithm to decide which gas stations Jeff should stop at to minimize the number of stops.

   (a) (5 points) Please write down the pseudocode.

   (b) (10 points) Use the **greedy-stays-ahead** argument to prove that your algorithm is optimal.

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4. (30 points) The shortsighted government of the Kingdom of Elandor has built a connected road network that contains no cycles. After receiving numerous complaints from citizens about long travel times between cities, the king has become interested in shortening the travel distance between the two cities that are the furthest apart by constructing a new road. However, the government is unable to determine which two cities are the furthest from each other.

   After hearing about your heroics in conquering the formidable Homework 5 of CS 3330, the king has personally requested your help to solve this problem. You may assume you have access to **BFS**$(\cdot)$ subroutine.

   (a) (5 points) Formulate the problem using graph theory. Describe how you will construct the graph (i.e., define the nodes, edges, and weights).

   (b) (5 points) Design an $O(n^2)$ algorithm to find the two cities that are the furthest apart.

(c) (10 points) Design an $O(m + n)$ algorithm to find the two cities that are the furthest apart.

(d) (10 points) Prove the correctness of your algorithm from part (c).

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5. (30 points) In the newly settled land of **Aquoria**, the government faces a critical challenge: providing a reliable water supply to its n cities. Each city is currently without water, and the government is exploring two options to meet the cities' water needs:

- **Drill a new water well:** constructing a new water wall in city $i \in \{1, 2, \cdots, n\}$, will cost $W(i)$.
- **Lay down pipes between two cites:** adding a pipe between cities i and j to transfer water, if water is available for one of them, will cost $c(i, j)$.

Your objective is to design the water supply infrastructure for Aquoria by either drilling new wells or connecting cities with pipelines, ensuring that every city has access to a water supply at the minimum possible cost.

(a) (5 points) Let's start with modeling the problem. Consider a complete graph with n vertices as the model for this problem. What is the cost associated with the edges and the vertices of this graph?

(b) (5 points) Suppose that the government has decided to construct only one water well in the entire nation. Which city should they choose to minimize their total cost? Once this city has been selected, what class of algorithms should they use to solve the problem? It suffices to name only one algorithm from that class.

(c) (10 points) Suppose that the government decided to drill two water wells in city A and city B. Modify your graph in a way so that the problem falls into the same category as in part (b).

(d) (10 points) Given the cost of edges and the cost of vertices, describe an algorithm in plain English that minimizes the total cost of the underlying structure. (Hint: add a fictitious vertex to the graph and connect it to all the vertices. What should be the cost of edges connected to this vertex?)

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