Perfect Partitions of Some $(\mathbf{0}, \mathbf{1})$-Matrices

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Abstract

For a given regular bipartite graph G, can we partition the set of all perfect matchings of G into subsets such that each subset gives a 1-factorization of G? Or equivalently, given a $(0,1)$-matrix A and the set $\mathcal{P}_A$ of permutation matrices componentwise less than A, can we partition $\mathcal{P}_A$ into subsets so that the matrix sum of elements in each subset is A? If so, we say the graph G or the matrix A has a perfect partition. We focus our attention on a class of regular bipartite graphs, and show the existence of perfect partitions for two particular regular bipartite graphs of the class.

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Chapter 1

Introduction

Given a regular bipartite graph G, we know from Hall's matching theorem that G contains a perfect matching [1]. Further, G remains regular bipartite if any perfect matching is removed; thus G can be decomposed into a set of perfect matchings. This is called a 1-factorization of G. Given the set of all perfect matchings in G, is it possible to partition them into edge-disjoint subsets such that the union of matchings in each set gives G? In other words, any subset covers each edge of G exactly once. If so, we say that G admits a *perfect partition*. This problem was first studied by Brualdi, Chiang, and Li [2] in 2005.

By the *biadjacency matrix* of a bipartite graph G with disjoint vertex sets U and V, we mean the submatrix of the adjacency matrix of G with rows representing vertices in U and columns representing vertices in V. Then another way to view the problem is by examining the biadjacency matrix A of G defined above. We let $\mathcal{P}_A$ denote the set of all permutation matrices which are componentwise less than A. Then A admits a perfect partition, if there exists a partition of $\mathcal{P}_A$ such that the matrix sum of each subset is A. As we can view an $n \times n$ permutation matrix in $\mathcal{P}_A$ as a permutation in the symmetric group S_n, we may also examine the perfect partition problem for sets of permutations in S_n.

Brualdi et al. [2] provided examples of matrices which do not admit a perfect partition: we reproduce one in Figure 1.1.

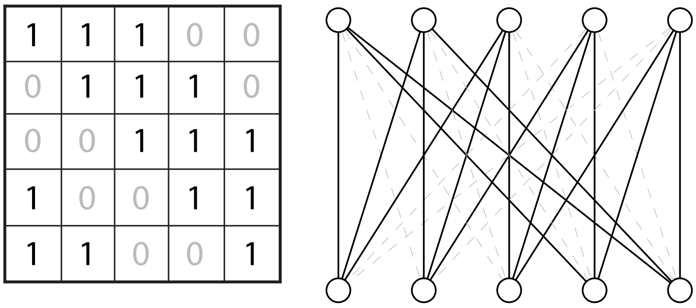


Figure 1.1: Matrix and corresponding bipartite graph with no perfect partition.

In this paper, we focus on matrices given in Defintion 1.0.1.

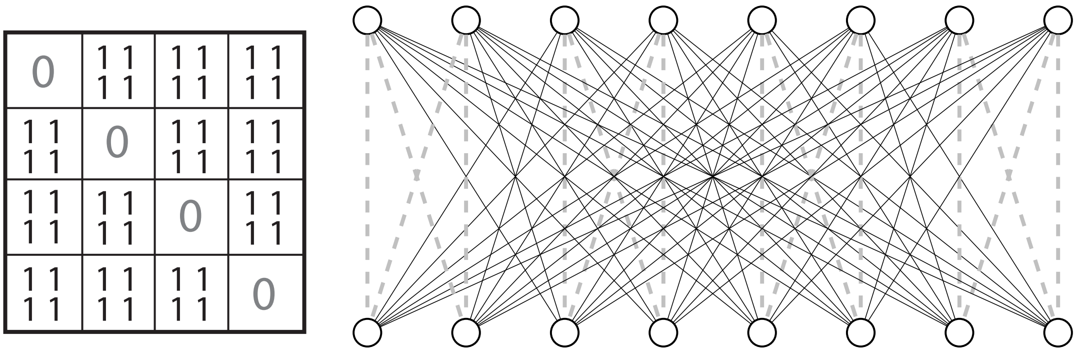


Figure 1.2: Left: $L_{8,2}$. Right: $G(L_{8,2})$.

Definition 1.0.1. *Let $G(L_{n,r}) = K_{n,n} - mK_{r,r}$, and $L_{n,r}$ its biadjacency matrix, where $n = rm$.*

So $L_{n,r}$ is the $n \times n$ matrix with 0s on the $r \times r$ block diagonal, and 1s everywhere else, and $G_{(}L_{n,r})$ is the complete bipartite graph with n vertices in each part, minus n/r separate instances of $K_{r,r}$. Also note that $G_{(}L_{n,r})$ is $(n-r)$-regular. As examples, $L_{8,2}$ and $G(L_{8,2})$ are shown in Figure 1.2.

Brualdi et al. [2] showed that perfect partitions do exist for $L_{n,0}$, $L_{2n,n}$, $L_{4,1}$, $L_{5,1}$, and $L_{6,2}$. The proof for $L_{n,0}$ follows.

Proposition 1.0.2. *There exists a perfect partition for $L_{n,0}$.*

Proof. $L_{n,0}$ is the $n \times n$ matrix consisting of all 1s. Thus $P_{L_{n,0}}$ contains all elements of the symmetric group S_n. Let H be the cyclic subgroup generated by the element $(1\,2\,\cdots\,n)$. Then the $(n-1)!$ cosets of H in S_n form a perfect partition for $L_{n,0}$. $\square$

The problem of determining whether $L_{n,r}$ admits a perfect partition for $r > 0$ is more difficult, as the group structure of S_n seems to be unhelpful. For example, the perfect partitions for $L_{4,1}$ and $L_{5,1}$ given in [2] do not appear directly related to S_n, and there is no clear pattern between them to suggest a solution for $L_{n,1}$. Brualdi et al. [2] left the question of whether $L_{6,1}$ admits a perfect partition an open problem, after several unsuccessful attempts.

In this paper, we show the existence of a perfect partition for $L_{6,1}$ and $L_{8,2}$. It is interesting to note that, again, the stategies used for both are not obvious extensions of $L_{5,1}$ and $L_{6,2}$. Thus our results for $L_{6,1}$ and $L_{8,2}$ are unlikely to help solve $L_{n,1}$, $n \geq 7$ or $L_{2n,2}$, $n \geq 5$. Furthermore, our strategies do not use properties of the group S_n.

Chapter 2

Perfect Partition of $L_{6,1}$

Recall that $L_{6,1}$ refers to the following matrix:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Thus $\mathcal{P}_{L_{6,1}}$ contains all permutation matrices which exclude 1s on the main diagonal. Viewing these as permutations in S_6, they are exactly the permutations containing no fixed points, or the derangements on 6 elements. There are 265 such elements, so we must partition 265 elements into 53 subsets of size 5 with the desired property.

Lemma 2.0.3. *$\mathcal{P}_{L_{6,1}}$ can be partitioned based on cycle decomposition as follows. Let C_6 denote the set of 6-cycles, $C_{2,4}$ the set of elements which are the product of a 2- and 4-cycle, $C_{3,3}$ the set of elements which are the product of three 3-cycles,* *and $C_{2,2,2}$ the set of elements which are the product of three transpositions. Then $P_{L_{6,1}} = C_6 \cup C_{2,4} \cup C_{3,3} \cup C_{2,2,2}$, and furthermore:*

$$|C_6| = 120, \quad |C_{2,4}| = 90, \quad |C'_{2,4}| = 30, \quad |C''_{2,4}| = 60, \quad |C_{3,3}| = 40, \quad |C_{2,2,2}| = 15.$$

Proof. We will count the number of elements in each subset by enumerating the elements in cycle notation. To ensure that each element is counted exactly once, we require that all the first element to appear in any cycle is the minimum element in that cycle.

To count $|C_6|$, we place 1 in the first position, and then we can rearrange the other five elements freely to get $|C_6| = 5! = 120$.

For $|C'_{2,4}|$, place 1 in the 2-cycle. This gives five choices for the other element in the 2-cycle. Place the minimum remaining element in the 4-cycle, and then arrange the other three to get $|C'_{2,4}| = 5 \cdot 3! = 30$. For $|C''_{2,4}|$, place 1 in the 4-cycle. Then select the other three elements to appear in the 4-cycle and arrange them. Then the 2-cycle is determined, so that $|C''_{2,4}| = \binom{5}{3} \cdot 3! = 60$. Furthermore, $|C_{2,4}| = |C'_{2,4}| + |C''_{2,4}| = 90$.

To count $|C_{3,3}|$, place 1 in the first cycle. Then select two elements and arrange them. Place the minimum remaining element in the second cycle and then arrange the remaining two elements. Thus $|C_{3,3}| = \binom{5}{2} \cdot 2! \cdot 2! = 40$.

For $|C_{2,2,2}|$, place 1 in the first 2-cycle. Select one of five choices for the other element in the first 2-cycle. Place the minimum element in the second 2-cycle, and then select the other element in the second 2-cycle from three choices. Thus $|C_{2,2,2}| = 5 \cdot 3 = 15$. $\square$

To generate perfect partitions for $L_{6,1}$, we will use the following partitioning scheme:

Type I. 30 subsets, where each contains four elements from C_6 and one element from $C'_{2,4}$, using all 30 $C'_{2,4}$ elements and all 120 C_6 elements.

Type II. 16 subsets, where each contains three elements from $C''_{2,4}$ and two elements from $C_{3,3}$, using 48 $C''_{2,4}$ elements and 32 $C_{3,3}$ elements.

Type III. 3 subsets, where each contains four elements from $C''_{2,4}$ and one element from $C_{2,2,2}$, using 12 $C''_{2,4}$ elements and 3 $C_{2,2,2}$ elements.

Type IV. 4 subsets, where each contains two elements from $C_{3,3}$ and three elements from $C_{2,2,2}$, using 8 $C_{3,3}$ elements and 12 $C_{2,2,2}$ elements.

To generate subsets of Type II, given, $\sigma = (1\,x\,y)(a\,b\,c) \in C_{3,3}$, we will include σ^{-1} in the same subset. Then, to form a valid 5-element subset of Type II, we must have either 3 $C''_{2,4}$ elements of the form $(1\,\cdot\,y\,\cdot)(x\,\cdot)$, or three elements of the form $(1\,\cdot\,x\,\cdot)(y\,\cdot)$. (Here, $\cdot$ refers to an undetermined element). That is, the elements a, b, and c cannot appear in any 2-cycle in this subset. For example, given element $\sigma = (1\,2\,3)(4\,5\,6)$, the possible subsets will be as follows:

$$\{(1\,2\,3)(4\,5\,6), (1\,3\,2)(4\,6\,5), (1\,\cdot\,3\,\cdot)(2\,\cdot), (1\,\cdot\,3\,\cdot)(2\,\cdot), (1\,\cdot\,3\,\cdot)(2\,\cdot)\}$$
$$\{(1\,2\,3)(4\,5\,6), (1\,3\,2)(4\,6\,5), (1\,\cdot\,2\,\cdot)(3\,\cdot), (1\,\cdot\,2\,\cdot)(3\,\cdot), (1\,\cdot\,2\,\cdot)(3\,\cdot)\}$$

To determine which subset type to choose, we introduce the notion of a *class.*

Definition 2.0.4. *Let $\sigma = (1\,x\,y)(a\,b\,c) \in C_{3,3}$ with $a < b$ and $a < c$. Then the class of σ is y if $b < c$ and x otherwise.*

Thus element $(1\,2\,3)(4\,5\,6)$ has class 3 because $5 < 6$. Observe that the class of σ coincides with the class of σ^{-1}, so that $(1\,3\,2)(4\,6\,5)$ has class 3 as well.

Now suppose $\sigma = (1\,x\,y)(a\,b\,c)$ has class y. Then we choose the subset for σ where each $C''_{2,4}$ element contains y in the 2-cycle:

$$\{(1\,x\,y)(a\,b\,c), (1\,y\,x)(a\,c\,b), (1\,\cdot\,x\,\cdot)(y\,\cdot), (1\,\cdot\,x\,\cdot)(y\,\cdot), (1\,\cdot\,x\,\cdot)(y\,\cdot)\}$$

For $\sigma = (1\,2\,3)(4\,5\,6)$, which has class 3, we choose the following subset:

$$\{(1\,2\,3)(4\,5\,6), (1\,3\,2)(4\,6\,5), (1\,\cdot\,2\,\cdot)(3\,\cdot), (1\,\cdot\,2\,\cdot)(3\,\cdot), (1\,\cdot\,2\,\cdot)(3\,\cdot)\}$$

There are two possible ways to finish determining the $C''_{2,4}$ elements:

$$\{(1\,x\,y)(a\,b\,c), (1\,y\,x)(a\,c\,b), (1\,\underline{a}\,x\,\underline{b})(y\,\underline{c}), (1\,\underline{c}\,x\,\underline{a})(y\,\underline{b}), (1\,\underline{b}\,x\,\underline{c})(y\,\underline{a})\} \tag{2.1}$$

$$\{(1\,x\,y)(a\,b\,c), (1\,y\,x)(a\,c\,b), (1\,\underline{a}\,x\,\underline{c})(y\,\underline{b}), (1\,\underline{b}\,x\,\underline{a})(y\,\underline{c}), (1\,\underline{c}\,x\,\underline{b})(y\,\underline{a})\} \tag{2.2}$$

We introduce the concept of a *pattern* here to choose a subset.

Definition 2.0.5. *Given $\sigma = (1\,x\,y)(a\,b\,c)$, the pattern associated with σ and σ^{-1} is a 3-cycle with value chosen as either $(a\,b\,c)$ or $(a\,c\,b)$.*

If we associate pattern $(a\,b\,c)$ with σ, σ^{-1}, then we select subset (2.1); otherwise we select subset subset (2.2). To illustrate this rule, observe that a pattern has three equivalent expressions in cycle notation. Then each expression determines one $C''_{2,4}$ element in the subset. For example, the expressions of the cycle $(a\,b\,c)$ are $(a\,b\,c)$, $(b\,c\,a)$, and $(c\,a\,b)$. The underlined elements in (2.1) are determined according to these three expressions.

Thus, given a pair of elements $\sigma, \sigma^{-1} \in C_{3,3}$, the combination of class and pattern uniquely determines three $C''_{2,4}$ elements. Although the class of σ, σ^{-1} is determined by definition, the associated pattern is not – we must select a pattern (from two possibilities) in order to determine the $C''_{2,4}$ elements for the subset.

If we select patterns for all pairs of $\sigma, \sigma^{-1} \in C_{3,3}$, we can generate subsets containing all elements of $C_{3,3}$ with elements of $C''_{2,4}$. However, care must be taken in selecting the patterns to ensure that no $C''_{2,4}$ element appears in multiple subsets. For example, suppose that we have $\sigma_1 = (1\,2\,3)(4\,5\,6)$ (which is class 3) and the selected pattern $(4\,5\,6)$. Then one $C''_{2,4}$ element in the subset containing σ_1 will be $(1\,4\,2\,5)(3\,6)$. However, if $\sigma_2 = (1\,2\,6)(3\,4\,5)$, a class 6 element, has pattern $(3\,4\,5)$, then $(1\,4\,2\,5)(3\,6)$ will appear in the subset containing σ_2 as well. Thus σ_2 must have pattern $(3\,5\,4)$ to ensure no repeating $C''_{2,4}$ element. Therefore, selecting pattern $(4\,5\,6)$ for $(1\,2\,3)(4\,5\,6)$ determines that the pattern for $(1\,2\,6)(3\,4\,5)$ is $(3\,5\,4)$. Note that this pattern is only determined by element $(1\,4\,2\,5)(3\,6)$ – the other two $C''_{2,4}$ elements in the subset determine other patterns in a similar fashion.

Since selecting one pattern determined three others in the previous example, it is natural to ask if those three will further determine other patterns, and ultimately how many patterns will be determined by choosing a single pattern. We examine this in the general case. Let $1 < a < b < c < d$, and x different from all of these. Suppose we choose pattern $(b\,c\,d)$ for element $(1\,x\,a)(b\,c\,d)$. The chain of determined patterns is illustrated below, where the underline denotes the selected pattern.

$$\{(1\,x\,a)\underline{(b\,c\,d)}, (1\,a\,x)(b\,d\,c), (1\,b\,x\,c)(a\,d), (1\,d\,x\,b)(a\,c), (1\,c\,x\,d)(a\,b)\} \tag{2.3}$$

$$\{(1\,x\,b)(a\,c\,d), (1\,b\,x)\underline{(a\,d\,c)}, (1\,a\,x\,d)(b\,c), (1\,c\,x\,a)(b\,d), (1\,d\,x\,c)(b\,a)\} \tag{2.4}$$

$$\{(1\,x\,c)\underline{(a\,b\,d)}, (1\,c\,x)(a\,d\,b), (1\,a\,x\,b)(c\,d), (1\,d\,x\,a)(c\,b), (1\,b\,x\,d)(c\,a)\} \tag{2.5}$$

$$\{(1\,x\,d)(a\,b\,c), (1\,d\,x)\underline{(a\,c\,b)}, (1\,a\,x\,c)(d\,b), (1\,b\,x\,a)(d\,c), (1\,c\,x\,b)(d\,a)\} \tag{2.6}$$

Thus the selection of any one pattern will determine exactly three others, and which three are determined is known. We can describe the elements whose patterns are determined using the following definition.

Definition 2.0.6. *Let* $\sigma = (1\,x\,y)(a\,b\,c) \in C_{3,3}$ *with* $a < b$ *and* $a < c$. *Then the section of* σ *is* x *if* $b < c$ *and* y *otherwise.*

Compare with Definition 2.0.4, *class*. Thus selecting 1 pattern for a section y element will determine the patterns for all section y elements, and no other elements. Note that the section and class of an element are never equal, and furthermore, the set of all elements in section y will contain four σ, σ^{-1} pairs with each pair in a different class.

We are now ready to generate a perfect partition for $L_{6,1}$.

Theorem 2.0.7. *There exists a perfect partition of* $L_{6,1}$.

Proof. To generate Type I subsets, define a map $f : C'_{2,4} \to (C_6)^4$ as follows:

$$f\,(\sigma) = ((1\,x_3\,x_2\,x_5\,x_4\,x_6), (1\,x_4\,x_2\,x_6\,x_5\,x_3), (1\,x_5\,x_2\,x_3\,x_6\,x_4), (1\,x_6\,x_2\,x_4\,x_3\,x_5))$$

where $\sigma = (1\,x_2)(x_3\,x_4\,x_5\,x_6)$. We show that $f(C'_{2,4})$ covers all elements of C_6 exactly once. Given $\tau = (1\,y_2\,y_3\,y_4\,y_5\,y_6) \in C_6$, we find that the only $C'_{2,4}$ elements whose images under f contain τ are $(1\,y_3)(y_2\,y_5\,y_4\,y_6)$, $(1\,y_3)(y_6\,y_2\,y_5\,y_4)$, $(1\,y_3)(y_4\,y_6\,y_2\,y_5)$, $(1\,y_3)(y_5\,y_4\,y_6\,y_2)$, but these are all the same element. Thus each C_6 element is covered by exactly one $C'_{2,4}$ element, so that the 30 elements of $C'_{2,4}$ cover all 120 C_6 elements.

For the remaining elements, we will begin by generating subsets of Type IV. Select a class $y \in [6]\backslash\{1\}$. We will use the class y elements of $C_{3,3}$ for Type IV subsets. Let $a < b < c < d$. Then the following gives the four subsets of Type IV.

$$\{(1\,y\,a)(b\,c\,d), (1\,a\,y)(b\,d\,c), (1\,b)(a\,d)(c\,y), (1\,c)(a\,b)(d\,y), (1\,d)(a\,c)(b\,y)\},$$

$$\{(1\,y\,b)(a\,c\,d), (1\,b\,y)(a\,d\,c), (1\,a)(b\,c)(d\,y), (1\,c)(a\,y)(b\,d), (1\,d)(a\,b)(c\,y)\},$$

$$\{(1\,y\,c)(a\,b\,d), (1\,c\,y)(a\,d\,b), (1\,a)(b\,y)(c\,d), (1\,b)(a\,c)(y\,d), (1\,d)(a\,y)(b\,c)\},$$

$$\{(1\,y\,d)(a\,b\,c), (1\,d\,y)(a\,c\,b), (1\,a)(b\,d)(c\,y), (1\,b)(a\,y)(c\,d), (1\,c)(a\,d)(y\,b)\}.$$

Now select the pattern for one class y $C_{3,3}$ element. With this pattern, we can generate three $C''_{2,4}$ elements. Given one $C_{2,2,2}$ element of the form $(1\,y)(\cdot\,\cdot)(\cdot\,\cdot)$ and one $C''_{2,4}$ element, we can determine the three other $C''_{2,4}$ elements in a subset to make a Type III subset. Thus, we let each of these three $C''_{2,4}$ elements pair with one $C_{2,2,2}$ element to determine the three sets for Type III.

Each of these determined $C''_{2,4}$ elements are associated with the $C_{3,3}$ elements used for the Type IV subsets. We can find the patterns used for the $C_{3,3}$ elements based on the given $C''_{2,4}$ elements. We have shown that the selection of a pattern determines all other patterns in a section. Here, we have the selected the patterns for four $C_{3,3}$ element-inverse pairs, with each in a different section: thus the selected patterns generate Type II subsets without repeating $C''_{2,4}$ elements. $\square$

Chapter 3

Perfect Partition of $L_{8,2}$

We begin by exstablishing some notation:

M_n: the set of $n \times n$ real matrices,

$\{E_{11}, E_{12}, \ldots, E_{nn}\}$: standard basis for M_n,

$J_n \in M_n$: the matrix with all entries equal to one,

$O_n \in M_n$: the matrix with all entries equal to zero,

$C(i, j)$: Swap columns i and j in matrix,

$R(i, j)$: Swap rows i and j in matrix,

$\mathcal{S}(A)$: the set of permutation matrices P in M_n such that $A - P$ is nonnegative for a given zero one matrix $A \in M_n$.

Lemma 3.0.8. *Suppose a matrix $P \in \mathcal{S}(L_{8,2})$ is witten in block form $P = (P_{ij})_{1 \le i,j \le 4}$ so that $P_{ij} \in M_2$ for every pair (i, j). Then either none, one, two, or four of the P_{ij} blocks are invertible, i.e., two of the four entries equal to 1. Thus, $\mathcal{S}(L_{8,2})$ can be partitioned into $\mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_4$, where $\mathcal{S}_k$ consists of matrices $P = (P_{ij})_{1 \le i,j \le 4}$ in $\mathcal{S}(L_{8,2})$ such that exactly k of the submatrices P_{ij} are invertible. Moreover, we have:*

$$|\mathcal{S}_0| = 2^8 9, \quad |\mathcal{S}_1| = 2^9 3, \quad |\mathcal{S}_2| = 2^8 3, \quad |\mathcal{S}_4| = 2^4 9.$$

Proof. $\mathcal{S}_0$ contains the matrices for which no submatrices are invertible. Then all submatrices contain no more than one 1, and each row and column of blocks contain exactly two submatrices containing exactly one 1. Denote such a submatrix by E. For the first block column, there are $\binom{3}{2} = 3$ possible choices for which submatrices are E. This selection determines that the block not chosen must be O_2, so the other two non-diagonal entries in that row must be E. The first block row also allows $\binom{3}{2} = 3$ possible choices for which submatrices are X, then the other selections of E are determined. Thus, there are $3 \cdot 3 = 9$ combinations of E. Each E may one of $E_{11}, E_{12}, E_{21}, E_{22}$. There are 4 ways to select two E blocks, 2 ways to select the next four E blocks, and 1 way to select the last two. Therefore, $|\mathcal{S}_0| = 2^8 9$.

$\mathcal{S}_1$ contains the matrices for which exactly one submatrix is invertible. In the 2×2 case, this is only when a block is I_2 or R_2. Denote such a matrix by X. Then there are 12 non-diagonal positions for which the first X may be placed. All other submatrices E and O_2 are determined. The single X may be I_2 or R_2, so it can be chosen in 2 ways. One E block can be chosen in 4 ways, the next four E blocks can be each chosen in 2 ways, and the last is determined. Thus, $|\mathcal{S}_1| = 2^8 12 = 2^9 3$.

$\mathcal{S}_2$ contains the matrices for which exactly two submatrices are invertible. Denote such submatrices by X. Then the first X can be placed in one of 12 non-diagonal positions. This placement allows only 2 ways to choose the other X. Since order does not matter, we have $\frac{12 \cdot 2}{2} = 12$ ways to choose the placement of two X blocks. There are two choices for each X block, 4 choices for the first E block, 2 choices for the next two E blocks, and 1 choice for the last E block. Thus, $|\mathcal{S}_2| = 2^6 12 = 2^8 3$.

$\mathcal{S}_4$ contains the matrices for which exactly four submatrices are invertible. Denote those four by X; then all other submatrices must be O_2. There are 3 ways to place one X in the first block column. Then find the column whose diagonal position is in the same row as the X in the first column. There are 3 ways to place one X in this column. All other X blocks are then determined, so there are $3 \cdot 3 = 9$ ways to place the X blocks. There are 2 ways to choose each X, so $|\mathcal{S}_4| = 2^4 9$. $\square$

Theorem 3.0.9. *There exists a perfect partition for $L_{8,2}$.*

Proof. We will use the following partitioning scheme for $L_{8,2}$:

Type I. Pick two matrices from $\mathcal{S}_0$ and four matrices from $\mathcal{S}_1$ to form subsets.

Type II. Pick four matrices from $\mathcal{S}_0$ and two matrices from $\mathcal{S}_2$ to form subsets.

Type III. Pick six matrices from $\mathcal{S}_4$ to form subsets.

Type I: two matrices from $\mathcal{S}_0$ and four matrices from $\mathcal{S}_1$

In block form, P has 9 perfect matchings which form a perfect partition of 3 subsets:

$$\{(1,2)(3,4),(1,3,2,4),(1,4,2,3)\},$$
$$\{(1,3)(2,4),(1,2,3,4),(1,4,3,2)\},$$
$$\{(1,4)(2,3),(1,2,4,3),(1,3,4,2)\}.$$

$$\begin{array}{|c|c|c|c|}\hline O & O & E & E \\ \hline O & O & E & E \\ \hline E & E & O & O \\ \hline E & E & O & O \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|}\hline O & E & O & E \\ \hline E & O & E & O \\ \hline O & E & O & E \\ \hline E & O & E & O \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|}\hline O & E & E & O \\ \hline E & O & O & E \\ \hline E & O & O & E \\ \hline O & E & E & O \\ \hline \end{array}$$

Figure 3.1: Forms of the elements of $\mathcal{S}_0^1$: $(1,2)(3,4),(1,3)(2,4)$, or $(1,4)(2,3)$.

Let $\mathcal{S}_0^1 \subset \mathcal{S}_0$ be the subset containing all matrices whose non-diagonal zero blocks form a perfect matching of the form $(1,2)(3,4),(1,3)(2,4)$, or $(1,4)(2,3)$.

We will use 2 matrices in $A_1, A_2 \in \mathcal{S}_0^1$ and 4 matrices in $B_i \in \mathcal{S}_1$ with $i \in [4]$ to form partitions.

For a block form $(1,i)(j,k)$, and a given matrix $P \in \mathcal{S}_0^1$, take $Q \in \mathcal{S}_0^1$ so that in the sum $A = P + Q$, the blocks $P_{1i}, P_{i1}, P_{jk}, P_{kj}$ are all O_2, and all other non-diagonal blocks are invertible (described another way, choose $E_1 = P_{1j}, E_2 = P_{ik}, E_3 = P_{j1}, E_4 = P_{ki}$ freely, then P and Q are determined to have the desired A).

$$\begin{array}{|c|c|c|c|}\hline O & \begin{smallmatrix}0&0\\0&0\end{smallmatrix} & \begin{smallmatrix}1&0\\0&0\end{smallmatrix} & \begin{smallmatrix}0&0\\1&0\end{smallmatrix} \\ \hline \begin{smallmatrix}0&0\\0&0\end{smallmatrix} & O & \begin{smallmatrix}0&1\\0&0\end{smallmatrix} & \begin{smallmatrix}0&0\\0&1\end{smallmatrix} \\ \hline \begin{smallmatrix}1&0\\0&0\end{smallmatrix} & \begin{smallmatrix}0&0\\1&0\end{smallmatrix} & O & \begin{smallmatrix}0&0\\0&0\end{smallmatrix} \\ \hline \begin{smallmatrix}0&0\\1&0\end{smallmatrix} & \begin{smallmatrix}0&1\\0&0\end{smallmatrix} & \begin{smallmatrix}0&0\\0&0\end{smallmatrix} & O \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|}\hline O & \begin{smallmatrix}0&0\\0&0\end{smallmatrix} & \begin{smallmatrix}0&0\\0&1\end{smallmatrix} & \begin{smallmatrix}0&1\\0&0\end{smallmatrix} \\ \hline \begin{smallmatrix}0&0\\0&0\end{smallmatrix} & O & \begin{smallmatrix}0&0\\1&0\end{smallmatrix} & \begin{smallmatrix}1&0\\0&0\end{smallmatrix} \\ \hline \begin{smallmatrix}0&0\\0&1\end{smallmatrix} & \begin{smallmatrix}0&1\\0&0\end{smallmatrix} & O & \begin{smallmatrix}0&0\\0&0\end{smallmatrix} \\ \hline \begin{smallmatrix}0&1\\0&0\end{smallmatrix} & \begin{smallmatrix}0&0\\1&0\end{smallmatrix} & \begin{smallmatrix}0&0\\0&0\end{smallmatrix} & O \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|}\hline O & \begin{smallmatrix}0&0\\0&0\end{smallmatrix} & \begin{smallmatrix}1&0\\0&1\end{smallmatrix} & \begin{smallmatrix}0&1\\1&0\end{smallmatrix} \\ \hline \begin{smallmatrix}0&0\\0&0\end{smallmatrix} & O & \begin{smallmatrix}0&1\\1&0\end{smallmatrix} & \begin{smallmatrix}1&0\\0&1\end{smallmatrix} \\ \hline \begin{smallmatrix}1&0\\0&1\end{smallmatrix} & \begin{smallmatrix}0&1\\1&0\end{smallmatrix} & O & \begin{smallmatrix}0&0\\0&0\end{smallmatrix} \\ \hline \begin{smallmatrix}0&1\\1&0\end{smallmatrix} & \begin{smallmatrix}0&1\\1&0\end{smallmatrix} & \begin{smallmatrix}0&0\\0&0\end{smallmatrix} & O \\ \hline \end{array}$$

Figure 3.2: Left: arbitrary choice of P in block form $(1\,2)(3\,4)$. Center: Q as determined by P and A. Right: $A = P + Q$.

We then choose $S, T, U, V \in \mathcal{S}_1$ so that the only invertible block in B_i is

$S_{1i}, T_{i1}, U_{jk}, V_{kj}$, respectively. Furthermore, let blocks T_{1j} and V_{j1} be uniquely determined by P_{1k} and P_{ji} by the following rule:

$$\begin{cases} \text{If } P_{1k} = E_{1,n}, \text{ then } T_{1j} &= E_{1,n} \\ \text{If } P_{1k} = E_{2,n}, \text{ then } T_{1j} &= E_{2,n} \\ \text{If } P_{ji} = E_{1,n}, \text{ then } V_{j1} &= E_{1,n} \\ \text{If } P_{ji} = E_{2,n}, \text{ then } V_{j1} &= E_{2,n} \end{cases}$$

We show that S, T, U, V are uniquely determined.

First, note that the following blocks are determined as follows:

$$T_{1j} \leftrightarrow T_{1k} \leftrightarrow V_{1k} \leftrightarrow V_{ik} \leftrightarrow S_{ik} \leftrightarrow S_{ij} \leftrightarrow U_{ij} \leftrightarrow U_{1j} \leftrightarrow T_{1j} \tag{3.1}$$

$$V_{j1} \leftrightarrow V_{ji} \leftrightarrow T_{ji} \leftrightarrow T_{ki} \leftrightarrow U_{ki} \leftrightarrow U_{k1} \leftrightarrow S_{k1} \leftrightarrow S_{j1} \leftrightarrow V_{j1} \tag{3.2}$$

Now consider the block $H_1 = S_{jk}$, which is determined by the cycles given by (3.1) and (3.2) (specifically, by S_{ik} and S_{j1}). Then S_{jk}, together with the invertible block U_{jk}, determines T_{jk}: if $S_{jk} = E_{a,b}$, then $T_{jk} = E_{3-a,3-b}$.

We show that S_{jk} with (3.1) and (3.2) determine T_{jk} in the same way. Given $S_{jk} = E_{a,b}$:

1. By comparing rows, we have $S_{j1} = E_{3-a,n}; V_{j1} = E_{a,n}; V_{ji} = E_{3-a,n}; T_{ji} = E_{a,n}; T_{jk} = E_{3-a,n}$.

2. By comparing columns, we have $S_{ik} = E_{n,3-b}; V_{ik} = E_{n,b}; V_{1k} = E_{n,3-b}; T_{1k} = E_{n,b}; T_{jk} = E_{n,3-b}$.

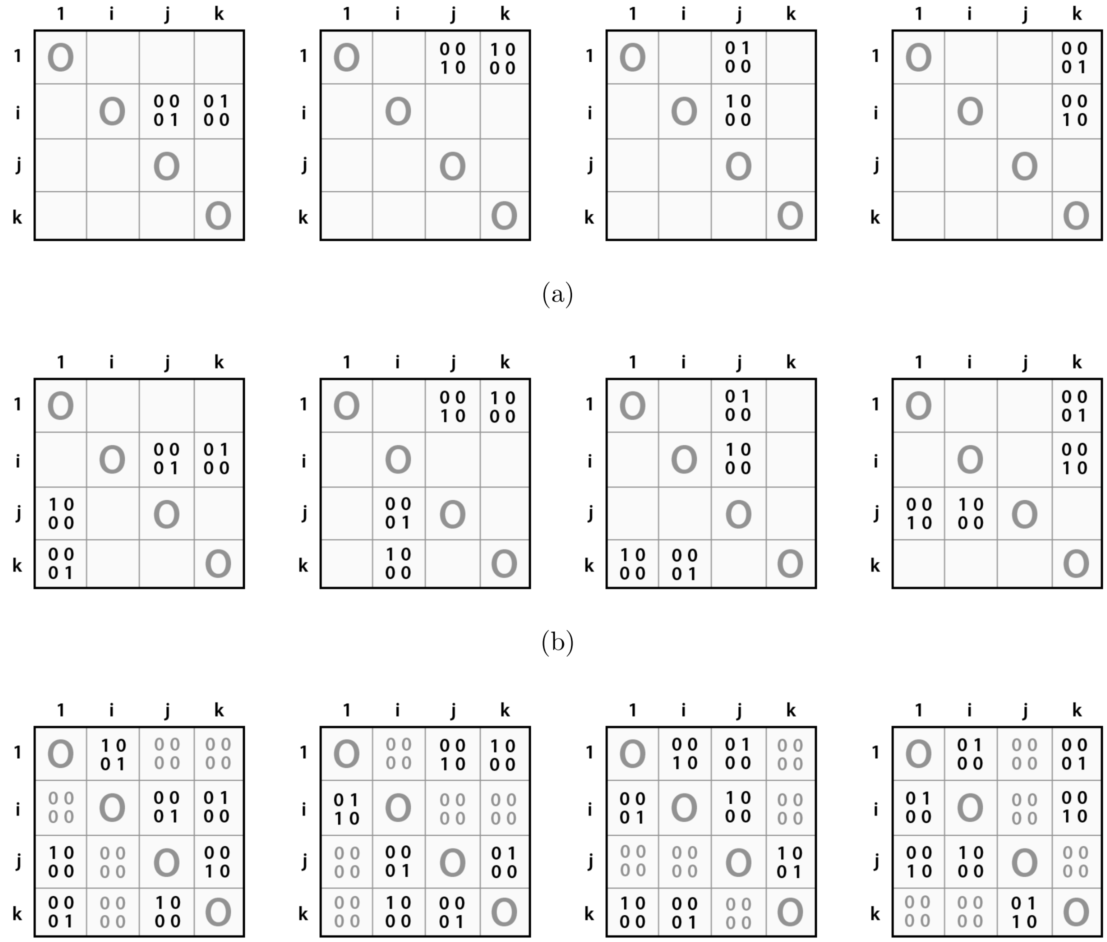
Thus $T_{jk} = E_{3-a,3-b}$ as determined by S_{jk} and by cycles (3.1), (3.2). Symmetric results follow for all remaining singular blocks of S, T, U, V.

Note that for each choice of E_1, E_2, E_3, E_4, all six matrices are uniquely determined. By symmetry of P and Q, there are $3 \cdot (4 \cdot 4 \cdot 4 \cdot 4 \cdot \frac{1}{2}) = 3 \cdot 2^7$ choices of different pairs $\{P, Q\}$.

For each pair $\{P, Q\}$ in $\mathcal{S}_0^1$, four different elements of $\mathcal{S}_1$ are used, so we actually use up all elements in $\mathcal{S}_1$.

Type II: four matrices from $\mathcal{S}_0 - \mathcal{S}_0^1$ and two matrices from $\mathcal{S}_2$

In block form, P has nine perfect matchings which forms perfect partition of three subsets. In partitions for $\mathcal{S}_0^1$ and $\mathcal{S}_1$, three perfect matchings were used. This partition will use the remaining six. Therefore, no element of $\mathcal{S}_0^1$ (that was used in the above partition) will be used in this partition.



(a)

(b)

(c)

Figure 3.3: Left to right: S, T, U, V as determined by the example P, Q in Figure 3.2.

For a perfect matching $(1, i, j, k)$, (whose inverse is $(1, k, j, i)$)
Choose a pair $A_1, A_1' \in \mathcal{S}_0 - \mathcal{S}_0^1$ so that

1. A_1 and A_1' have the same blocks at $(1, j), (j, 1), (k, i), (i, k)$;

2. the blocks of A_1 at $(1, i), (i, j), (j, k), (k, 1)$ are 0_2;

3. the blocks of A_1' at $(1, k), (k, j), (j, i), (i, 1)$ are 0_2.

We get A_2 so that $A_1 + A_2$ only has zero blocks or invertible blocks, and get A_3, A_4 from A_2 by applying operations $\{R(1, 2), R(3, 4), R(5, 6), R(7, 8)\}$ and $\{C(1, 2), C(3, 4), C(5, 6), C(7, 8)\}$, respectively.

Let $B_1, B_2 \in \mathcal{S}_2$ so that $B_1 + B_2 = L(8, 2) - \sum_{i=1}^{4} A_i$. Because of the structure of matrices in $\mathcal{S}_2$, B_1 and B_2 are determined if the sum is known.

Similarly we get A_2', A_3', A_4' and B_1', B_2'.

Now we show that every matrix in $\mathcal{S}_2 \cup (\mathcal{S}_0 - \mathcal{S}_0^1)$ appears exactly once in the above construction. Note that for each choices of blocks at positions $(1,j),(j,1),(k,i),(i,k)$, we get two different partitions of $L(8,2)$, with eight matrices in $\mathcal{S}_0$ and four in $\mathcal{S}_2$. We can partition matrices in $\mathcal{S}_0 - \mathcal{S}_0^1$ into sets of eight matrices, and each set uses four matrices in $\mathcal{S}_2$. So in total $\frac{1}{2} \cdot |\mathcal{S}_0 - \mathcal{S}_0^1| = 3 \cdot 2^8$ matrices in $\mathcal{S}_2$ are used, that is, we use up all matrices in $\mathcal{S}_2$.

Type III: six matrices from $\mathcal{S}_4$

In the block form, P has 9 perfect matchings which form a perfect partition into three subsets. Each block J_2 in the perfect matching can be decomposed into two invertible submatrices I_2 and R_2, so we will have three subsets summing to $L_{8,2}$.

For each of the subsets, we can apply one of the 7 operations $\{R(1,2)\}$, $\{R(3,4)\}$, $\{R(5,6)\}$, $\{R(7,8)\}$ $\{R(1,2), R(3,4)\}$, $\{R(1,2), R(5,6)\}$, $\{R(1,2), R(7,8)\}$ to get a different partition. In such a way, we use up all $18 \cdot 8 = 9 \cdot 2^4$ matrics in $\mathcal{S}_4$ to form perfect partitions. $\square$

Chapter 4

Conclusions and Future Work

We conclude with some open questions, which appear in [2].

Problem 4.0.10. *Does $L_{n,1}$ have a perfect partition for $n \geq 7$?*

Problem 4.0.11. *Does $L_{2n,2}$ have a perfect partition for $n \geq 5$?*

We used the perfect partition of $L_{4,1}$ for part of the perfect partition for $L_{8,2}$, which suggests a possible strategy for $L_{4n,n}$, or more generally, $L_{2^n,2^k}$.

Problem 4.0.12. *Does $L_{n,r}$ have a perfect partition?*

Currently, we have not seen values for n, r such that $L_{n,r}$ does not admit a perfect partition. However, this problem is largely unexplored for $r \geq 3$.

Problem 4.0.13. *Which (0-1)-matrices admit a perfect partition?*

This is the perfect partition problem for $(0, 1)$-matrices not necessarily included $L_{n,r}$. As the question is only interesting when the number of permutation matrices is divisible by the number of required elements in a subset, we find Problem 4.0.12 more interesting in general, as this divisibility is always satisfied for $L_{n,r}$.

Bibliography

[1] J.A. Bondy and U.S.R. Murty. *Graph Theory*. Springer, 2008.

[2] Richard A. Brualdi, Hanley Chiang, and Chi-Kwong Li. A partition problem for sets of permutation matrices. *Bulletin of ICA*, 43:1–11, 2005.

Appendix A

Sample Perfect Partition of $L_{6,1}$

We provide a sample perfect partion of $L_{6,1}$ in the following four tables.

Table A.1: Subsets of Type I. Has 30 subsets, where each uses four from C_6 and one from $C'_{2,4}$. Total used: 120 from T_1 and 30 from T_2.

1	(1 2)(3 4 5 6)	(1 3 2 5 4 6)	(1 4 2 6 5 3)	(1 5 2 3 6 4)	(1 6 2 4 3 5)
2	(1 2)(3 4 6 5)	(1 3 2 6 4 5)	(1 4 2 5 6 3)	(1 6 2 3 5 4)	(1 5 2 4 3 6)
3	(1 2)(3 5 6 4)	(1 3 2 6 5 4)	(1 5 2 4 6 3)	(1 6 2 3 4 5)	(1 4 2 5 3 6)
4	(1 2)(3 5 4 6)	(1 3 2 4 5 6)	(1 5 2 6 4 3)	(1 4 2 3 6 5)	(1 6 2 5 3 4)
5	(1 2)(3 6 5 4)	(1 3 2 5 6 4)	(1 6 2 4 5 3)	(1 5 2 3 4 6)	(1 4 2 6 3 5)
6	(1 2)(3 6 4 5)	(1 3 2 4 6 5)	(1 6 2 5 4 3)	(1 4 2 3 5 6)	(1 5 2 6 3 4)
7	(1 3)(2 4 5 6)	(1 2 3 5 4 6)	(1 4 3 6 5 2)	(1 5 3 2 6 4)	(1 6 3 4 2 5)
8	(1 3)(2 4 6 5)	(1 2 3 6 4 5)	(1 4 3 5 6 2)	(1 6 3 2 5 4)	(1 5 3 4 2 6)
9	(1 3)(2 5 6 4)	(1 2 3 6 5 4)	(1 5 3 4 6 2)	(1 6 3 2 4 5)	(1 4 3 5 2 6)
10	(1 3)(2 5 4 6)	(1 2 3 4 5 6)	(1 5 3 6 4 2)	(1 4 3 2 6 5)	(1 6 3 5 2 4)
11	(1 3)(2 6 5 4)	(1 2 3 5 6 4)	(1 6 3 4 5 2)	(1 5 3 2 4 6)	(1 4 3 6 2 5)
12	(1 3)(2 6 4 5)	(1 2 3 4 6 5)	(1 6 3 5 4 2)	(1 4 3 2 5 6)	(1 5 3 6 2 4)
13	(1 4)(2 3 5 6)	(1 2 4 5 3 6)	(1 3 4 6 5 2)	(1 5 4 2 6 3)	(1 6 4 3 2 5)
14	(1 4)(2 3 6 5)	(1 2 4 6 3 5)	(1 3 4 5 6 2)	(1 6 4 2 5 3)	(1 5 4 3 2 6)
15	(1 4)(2 5 6 3)	(1 2 4 6 5 3)	(1 5 4 3 6 2)	(1 6 4 2 3 5)	(1 3 4 5 2 6)
16	(1 4)(2 5 3 6)	(1 2 4 3 5 6)	(1 5 4 6 3 2)	(1 3 4 2 6 5)	(1 6 4 5 2 3)
17	(1 4)(2 6 5 3)	(1 2 4 5 6 3)	(1 6 4 3 5 2)	(1 5 4 2 3 6)	(1 3 4 6 2 5)
18	(1 4)(2 6 3 5)	(1 2 4 3 6 5)	(1 6 4 5 3 2)	(1 3 4 2 5 6)	(1 5 4 6 2 3)
19	(1 5)(2 3 4 6)	(1 2 5 4 3 6)	(1 3 5 6 4 2)	(1 4 5 2 6 3)	(1 6 5 3 2 4)
20	(1 5)(2 3 6 4)	(1 2 5 6 3 4)	(1 3 5 4 6 2)	(1 6 5 2 4 3)	(1 4 5 3 2 6)
21	(1 5)(2 4 6 3)	(1 2 5 6 4 3)	(1 4 5 3 6 2)	(1 6 5 2 3 4)	(1 3 5 4 2 6)
22	(1 5)(2 4 3 6)	(1 2 5 3 4 6)	(1 4 5 6 3 2)	(1 3 5 2 6 4)	(1 6 5 4 2 3)
23	(1 5)(2 6 4 3)	(1 2 5 4 6 3)	(1 6 5 3 4 2)	(1 4 5 2 3 6)	(1 3 5 6 2 4)
24	(1 5)(2 6 3 4)	(1 2 5 3 6 4)	(1 6 5 4 3 2)	(1 3 5 2 4 6)	(1 4 5 6 2 3)
25	(1 6)(2 3 4 5)	(1 2 6 4 3 5)	(1 3 6 5 4 2)	(1 4 6 2 5 3)	(1 5 6 3 2 4)
26	(1 6)(2 3 5 4)	(1 2 6 5 3 4)	(1 3 6 4 5 2)	(1 5 6 2 4 3)	(1 4 6 3 2 5)
27	(1 6)(2 4 5 3)	(1 2 6 5 4 3)	(1 4 6 3 5 2)	(1 5 6 2 3 4)	(1 3 6 4 2 5)
28	(1 6)(2 4 3 5)	(1 2 6 3 4 5)	(1 4 6 5 3 2)	(1 3 6 2 5 4)	(1 5 6 4 2 3)
29	(1 6)(2 5 4 3)	(1 2 6 4 5 3)	(1 5 6 3 4 2)	(1 4 6 2 3 5)	(1 3 6 5 2 4)
30	(1 6)(2 5 3 4)	(1 2 6 3 5 4)	(1 5 6 4 3 2)	(1 3 6 2 4 5)	(1 4 6 5 2 3)

Table A.2: Subsets of Type II. Has 16 subsets, where each uses three from $C''_{2,4}$ and two from $C_{3,3}$. Total used: 48 from $C_{2,4}$ and 32 from $C_{3,3}$.

31	(1 2 3)(4 5 6)	(1 3 2)(4 6 5)	(1 4 2 5)(3 6)	(1 5 2 6)(3 4)	(1 6 2 4)(3 5)
32	(1 2 3)(4 6 5)	(1 3 2)(4 5 6)	(1 4 3 6)(2 5)	(1 5 3 4)(2 6)	(1 6 3 5)(2 4)
33	(1 2 4)(3 5 6)	(1 4 2)(3 6 5)	(1 3 2 6)(4 5)	(1 5 2 3)(4 6)	(1 6 2 5)(3 4)
34	(1 2 4)(3 6 5)	(1 4 2)(3 5 6)	(1 3 4 5)(2 6)	(1 5 4 6)(2 3)	(1 6 4 3)(2 5)
35	(1 2 5)(3 6 4)	(1 5 2)(3 4 6)	(1 3 5 6)(2 4)	(1 4 5 3)(2 6)	(1 6 5 4)(2 3)
36	(1 2 6)(3 4 5)	(1 6 2)(3 5 4)	(1 3 2 5)(4 6)	(1 4 2 3)(5 6)	(1 5 2 4)(3 6)
37	(1 2 6)(3 5 4)	(1 6 2)(3 4 5)	(1 3 6 4)(2 5)	(1 4 6 5)(2 3)	(1 5 6 3)(2 4)
38	(1 3 6)(2 4 5)	(1 6 3)(2 5 4)	(1 2 3 4)(5 6)	(1 4 3 5)(2 6)	(1 5 3 2)(4 6)
39	(1 3 4)(2 5 6)	(1 4 3)(2 6 5)	(1 2 3 5)(4 6)	(1 5 3 6)(2 4)	(1 6 3 2)(4 5)
40	(1 3 6)(2 5 4)	(1 6 3)(2 4 5)	(1 2 6 5)(3 4)	(1 4 6 2)(3 5)	(1 5 6 4)(2 3)
41	(1 3 4)(2 6 5)	(1 4 3)(2 5 6)	(1 2 4 6)(3 5)	(1 5 4 2)(3 6)	(1 6 4 5)(2 3)
42	(1 3 5)(2 6 4)	(1 5 3)(2 4 6)	(1 2 5 4)(3 6)	(1 4 5 6)(2 3)	(1 6 5 2)(3 4)
43	(1 4 6)(2 3 5)	(1 6 4)(2 5 3)	(1 2 4 5)(3 6)	(1 3 4 2)(5 6)	(1 5 4 3)(2 6)
44	(1 4 6)(2 5 3)	(1 6 4)(2 3 5)	(1 2 6 3)(4 5)	(1 3 6 5)(2 4)	(1 5 6 2)(3 4)
45	(1 4 5)(2 6 3)	(1 5 4)(2 3 6)	(1 2 5 6)(3 4)	(1 3 5 2)(4 6)	(1 6 5 3)(2 4)
46	(1 5 6)(2 3 4)	(1 6 5)(2 4 3)	(1 2 5 3)(4 6)	(1 3 5 4)(2 6)	(1 4 5 2)(3 6)

Table A.3: Subsets of Type III. Has 3 subsets, where each uses four from $C''_{2,4}$ and one from $C_{2,2,2}$. Total used: 12 from $C''_{2,4}$ and 3 from $C_{2,2,2}$.

47	(1 5)(2 3)(4 6)	(1 2 4 3)(5 6)	(1 3 6 2)(4 5)	(1 4 2 6)(3 5)	(1 6 3 4)(2 5)
48	(1 5)(2 4)(3 6)	(1 2 6 4)(3 5)	(1 3 4 6)(2 5)	(1 4 3 2)(5 6)	(1 6 2 3)(4 5)
49	(1 5)(2 6)(3 4)	(1 2 3 6)(4 5)	(1 3 2 4)(5 6)	(1 4 6 3)(2 5)	(1 6 4 2)(3 5)

Table A.4: Subsets of Type IV. Has 4 subsets, where each uses two from $C_{3,3}$ and three from $C_{2,2,2}$, using 8 from $C_{3,3}$ and 12 from $C_{2,2,2}$.

50	(1 2)(3 4)(5 6)	(1 3)(2 5)(4 6)	(1 6)(2 4)(3 5)	(1 4 5)(2 3 6)	(1 5 4)(2 6 3)
51	(1 2)(3 5)(4 6)	(1 3)(2 6)(4 5)	(1 4)(2 5)(3 6)	(1 5 6)(2 4 3)	(1 6 5)(2 3 4)
52	(1 2)(3 6)(4 5)	(1 4)(2 3)(5 6)	(1 6)(2 5)(3 4)	(1 3 5)(2 4 6)	(1 5 3)(2 6 4)
53	(1 3)(2 4)(5 6)	(1 4)(2 6)(3 5)	(1 6)(2 3)(4 5)	(1 2 5)(3 4 6)	(1 5 2)(3 6 4)