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Eigenvectors and Eigenvalues

Explained Visually

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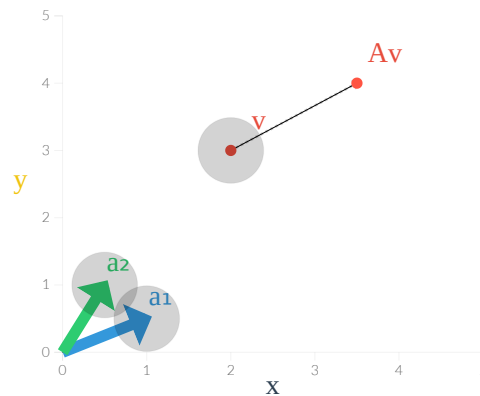
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By [Victor Powell](#) and [Lewis Lehe](#)

Eigenvalues/vectors are instrumental to understanding electrical circuits, mechanical systems, ecology and even Google's [PageRank](#) algorithm. Let's see if visualization can make these ideas more intuitive.

To begin, let \mathbf{v} be a vector (shown as a point) and \mathbf{A} be a matrix with columns \mathbf{a}_1 and \mathbf{a}_2 (shown as arrows). If we multiply \mathbf{v} by \mathbf{A} , then \mathbf{A} sends \mathbf{v} to a new vector \mathbf{Av} .



$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_{1,x} & \mathbf{a}_{2,x} \\ \mathbf{a}_{1,y} & \mathbf{a}_{2,y} \end{bmatrix} = \begin{bmatrix} 1.00 & 0.50 \\ 0.50 & 1.00 \end{bmatrix}$$

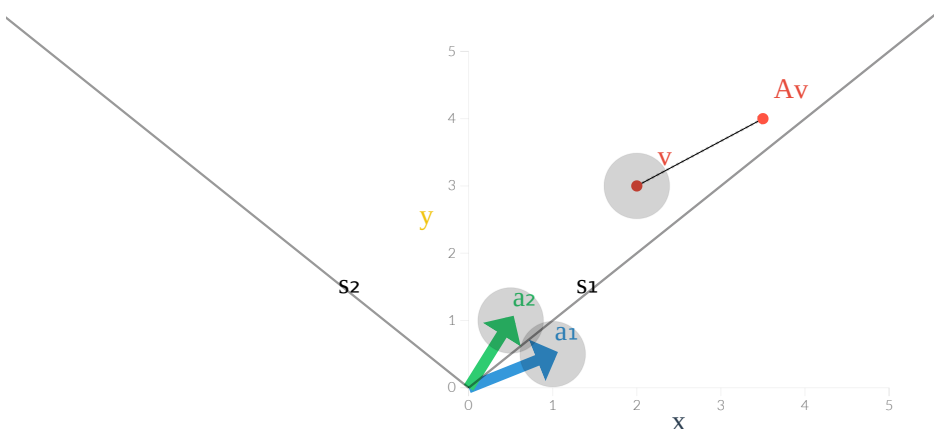
$$\mathbf{v} = \begin{bmatrix} 2.00 \\ 3.00 \end{bmatrix}$$

$$\mathbf{Av} = \begin{bmatrix} 3.50 \\ 4.00 \end{bmatrix}$$

If you can draw a line through the three points $(0,0)$, \mathbf{v} and \mathbf{Av} , then \mathbf{Av} is just \mathbf{v} multiplied by a number λ ; that is, $\mathbf{Av} = \lambda \mathbf{v}$. In this case, we call λ an **eigenvalue** and \mathbf{v} an **eigenvector**. For example, here $(1, 2)$ is an eigenvector and 5 an eigenvalue.

$$\mathbf{Av} = \begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \lambda \mathbf{v}.$$

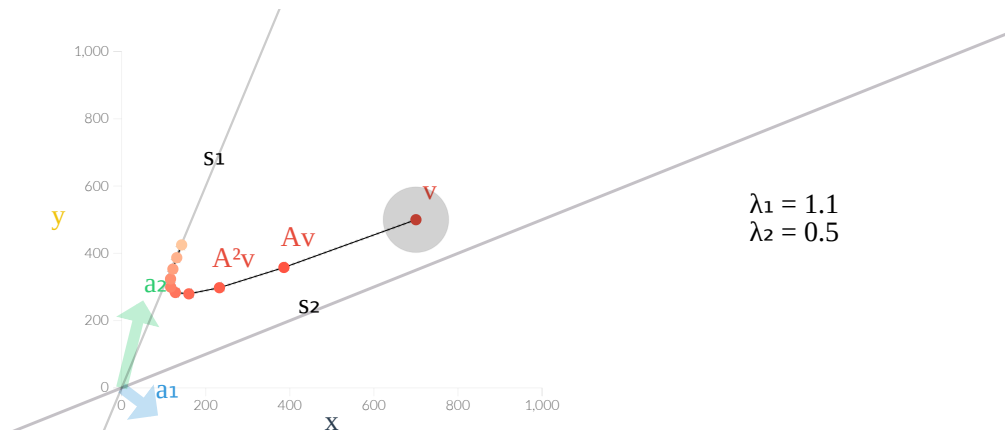
Below, change the columns of \mathbf{A} and drag \mathbf{v} to be an eigenvector. Note three facts: First, every point on the same line as an eigenvector is an eigenvector. Those lines are **eigenspaces**, and each has an associated eigenvalue. Second, if you place \mathbf{v} on an eigenspace (either \mathbf{s}_1 or \mathbf{s}_2) with associated eigenvalue $\lambda < 1$, then \mathbf{Av} is closer to $(0,0)$ than \mathbf{v} ; but when $\lambda > 1$, it's farther. Third, both eigenspaces depend on both columns of \mathbf{A} : it is not as though \mathbf{a}_1 only affects \mathbf{s}_1 .



$$\begin{aligned} \lambda_1 &= 1.5 \\ \lambda_2 &= 0.5 \end{aligned}$$

What are eigenvalues/vectors good for?

If you keep multiplying v by A , you get a sequence v, Av, A^2v , etc. Eigenspaces attract that sequence and eigenvalues tell you whether it ends up at $(0, 0)$ or far away. Therefore, eigenvectors/values tell us about systems that evolve step-by-step.



Let's explore some applications and properties of these sequences.

Fibonacci Sequence

Suppose you have some amoebas in a petri dish. Every minute, all adult amoebas produce one child amoeba, and all child amoebas grow into adults (Note: this is not really how amoebas reproduce.). So if t is a minute, the equation of this system is

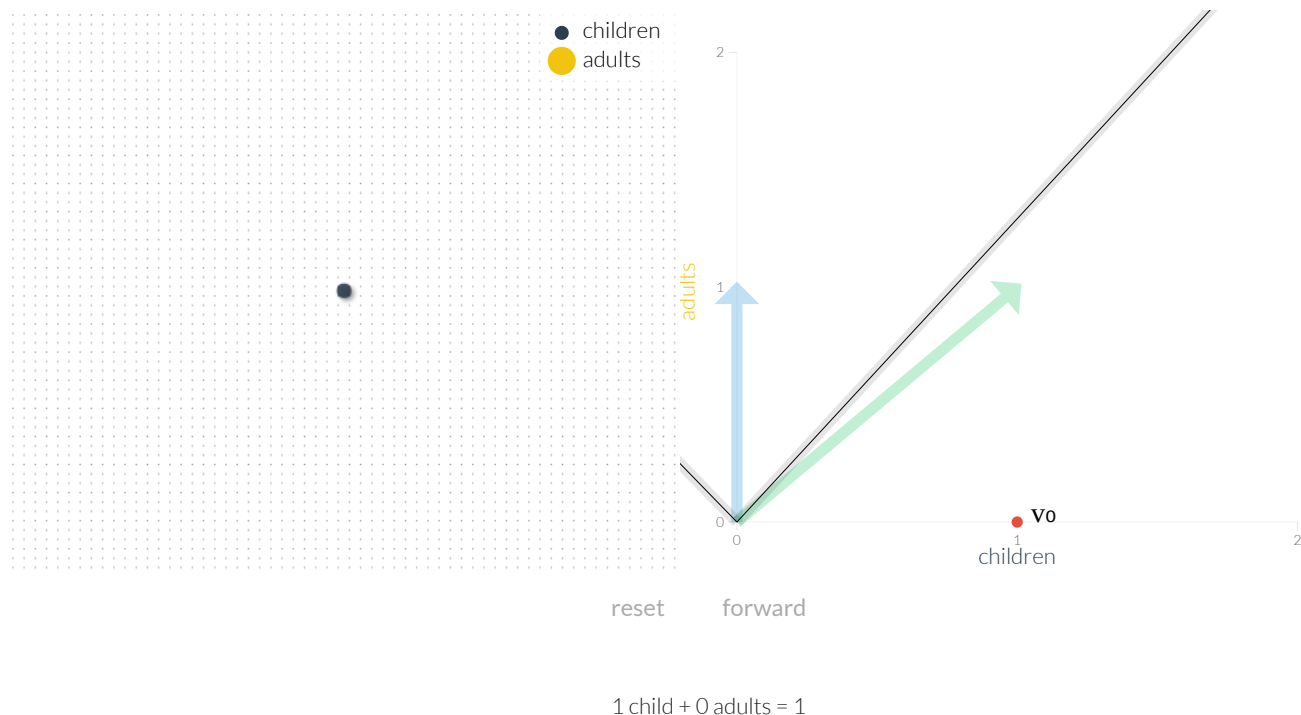
$$\begin{aligned} \text{adults}_{t+1} &= \text{adults}_t + \text{children}_t \\ \text{children}_{t+1} &= \text{adults}_t \end{aligned}$$

which we can rewrite in matrix form like

$$v_{t+1} = A \cdot v_t$$

$$\begin{pmatrix} \text{adults}_{t+1} \\ \text{children}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \text{adults}_t \\ \text{children}_t \end{pmatrix}$$

Below, press "Forward" to step ahead a minute. The total population is the [Fibonacci Sequence](#).

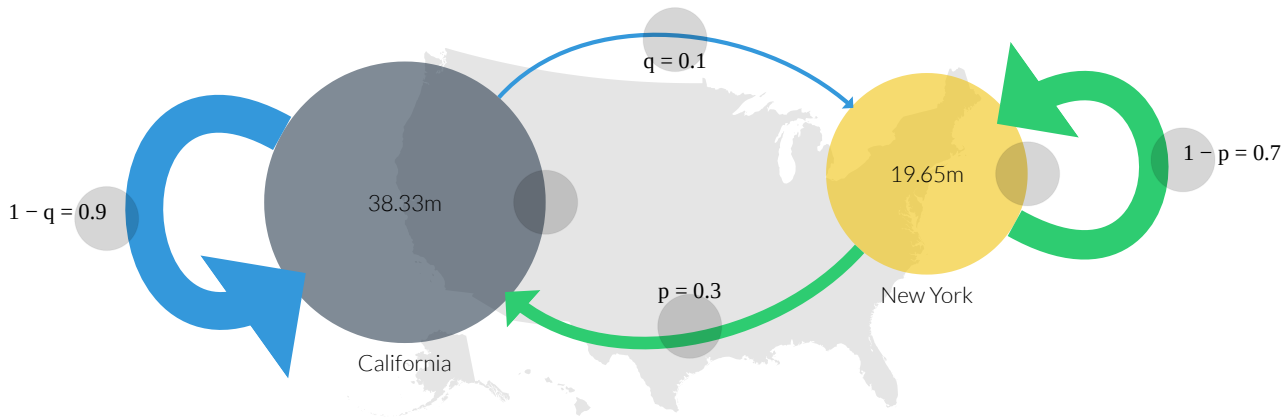


1 1 2 3 5 8 13 21 34 55 89 144 233

As you can see, the system goes toward the grey line, which is an eigenspace with $\lambda = (1 + \sqrt{5})/2 > 1$.

Steady States

Suppose that, every year, a fraction p of New Yorkers move to California and a fraction q of Californians move to New York. Drag the circles to decide these fractions and the number starting in each state.

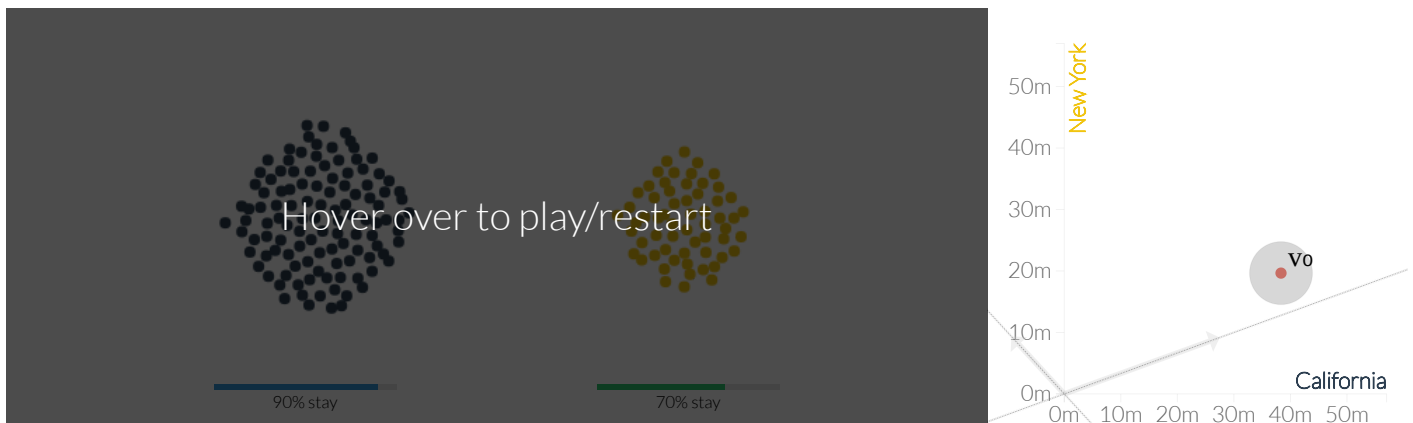


To understand the system better, we can start by writing it in matrix terms like:

$$v_{t+1} = Av_t$$

$$\begin{pmatrix} \text{New York}_{t+1} \\ \text{California}_{t+1} \end{pmatrix} = \begin{pmatrix} 1-p & q \\ p & 1-q \end{pmatrix} \cdot \begin{pmatrix} \text{New York}_t \\ \text{California}_t \end{pmatrix}$$

It turns out that a matrix like A , whose entries are positive and whose columns add up to one (try it!), is called a [Markov matrix](#), and it always has $\lambda = 1$ as its largest eigenvalue. That means there's a value of v_t for which $Av_t = \lambda v_t = 1v_t = v_t$. At this "steady state," the same number of people move in each direction, and the populations stay the same forever. Hover over the animation to see the system go to the steady state.



For more on Markov matrices, check out our explanation of [Markov Chains](#).

Complex eigenvalues

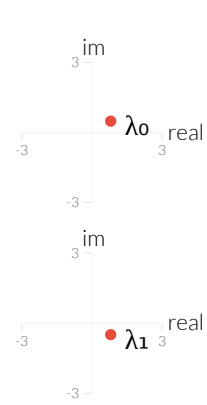
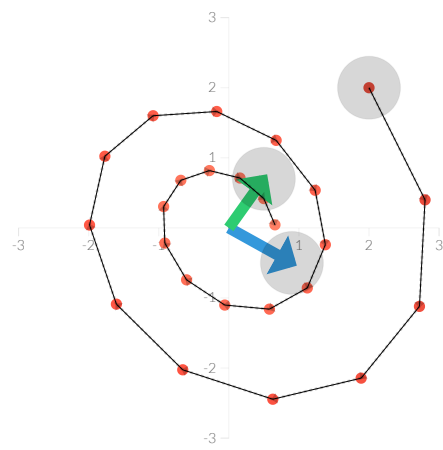
So far we've only looked at systems with real eigenvalues. But looking at the equation $Av = \lambda v$, who's to say λ and v can't have some imaginary part? That it can't be a [complex](#) number? For example,

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} = (1+i) \cdot \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Here, $1+i$ is an eigenvalue and $(1, i)$ is an eigenvector.

If a matrix has complex eigenvalues, its sequence spirals around $(0, 0)$. To see this, drag A 's columns (the arrows) around until you get a spiral. The eigenvalues are plotted in the real/imaginary plane to the right. You'll see that whenever the eigenvalues have an imaginary part, the system spirals, no matter where you start things off.

steps: 



Learning more

We've really only scratched the surface of what linear algebra is all about. To learn more, check out the legendary Gilbert Strang's [Linear Algebra](#) course at MIT's Open Courseware site. To get more practice with applications of eigenvalues/vectors, also ceck out the excellent [Differential Equations](#) course.

For more explanations, visit the Explained Visually [project homepage](#).

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