Proof: Relationship between chi-squared distribution and beta distribution

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Theorem: Let X and Y be independent random variables following chi-squared distributions:

$$X \sim \chi^2(m) \quad \text{and} \quad Y \sim \chi^2(n) \; . \tag{1}$$

Then, the quantity $X/(X+Y)$ follows a beta distribution:

$$\frac{X}{X+Y} \sim \mathrm{Bet}\left(\frac{m}{2}, \frac{n}{2}\right) \; . \tag{2}$$

Proof: The probability density function of the chi-squared distribution is

$$X \sim \chi^2(u) \quad \Rightarrow \quad f_X(x) = \frac{1}{\Gamma\left(\frac{u}{2}\right) \cdot 2^{u/2}} \cdot x^{\frac{u}{2}-1} \cdot e^{-\frac{x}{2}} \; . \tag{3}$$

Define the random variables Z and W as functions of X and Y

$$\begin{aligned} Z &= \frac{X}{X+Y} \\ W &= Y \; , \end{aligned} \tag{4}$$

such that the inverse functions X and Y in terms of Z and W are

$$\begin{aligned} X &= \frac{ZW}{1-Z} \\ Y &= W \; . \end{aligned} \tag{5}$$

This implies the following Jacobian matrix and determinant:

$$\begin{aligned} J &= \begin{bmatrix} \frac{\mathrm{d}X}{\mathrm{d}Z} & \frac{\mathrm{d}X}{\mathrm{d}W} \\ \frac{\mathrm{d}Y}{\mathrm{d}Z} & \frac{\mathrm{d}Y}{\mathrm{d}W} \end{bmatrix} = \begin{bmatrix} \frac{W}{(1-Z)^2} & \frac{Z}{1-Z} \\ 0 & 1 \end{bmatrix} \\ |J| &= \frac{W}{(1-Z)^2} \; . \end{aligned} \tag{6}$$

Because X and Y are independent, the joint density of X and Y is equal to the product of the marginal densities:

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) \; . \tag{7}$$

With the probability density function of an invertible function, the joint density of Z and W can be derived as:

$$f_{Z,W}(z,w) = f_{X,Y}(x,y) \cdot |J| \; . \tag{8}$$

Substituting (5) into (3), and then with (6) into (8), we get:

$$
\begin{aligned}
f_{Z,W}(z,w) &= f_X\left(\frac{zw}{1-z}\right)\cdot f_Y(w)\cdot |J| \\
&= \frac{1}{\Gamma\left(\frac{m}{2}\right)\cdot 2^{m/2}}\cdot\left(\frac{zw}{1-z}\right)^{\frac{m}{2}-1}\cdot e^{-\frac{1}{2}\left(\frac{zw}{1-z}\right)}\cdot\frac{1}{\Gamma\left(\frac{n}{2}\right)\cdot 2^{n/2}}\cdot w^{\frac{n}{2}-1}\cdot e^{-\frac{w}{2}}\cdot\frac{w}{(1-z)^2} \\
&= \frac{1}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)\cdot 2^{m/2}2^{n/2}}\cdot\left(\frac{z}{1-z}\right)^{\frac{m}{2}-1}\left(\frac{1}{(1-z)}\right)^{2}\cdot w^{\frac{m}{2}+\frac{n}{2}-1}e^{-\frac{1}{2}\left(\frac{zw}{1-z}+\frac{w(1-z)}{1-z}\right)} \\
&= \frac{1}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)\cdot 2^{(m+n)/2}}\cdot z^{\frac{m}{2}-1}\cdot(1-z)^{-\frac{m}{2}-1}\cdot w^{\frac{m+n}{2}-1}\cdot e^{-\frac{1}{2}\left(\frac{w}{1-z}\right)}\;.
\end{aligned}
\tag{9}
$$

The marginal density of Z can now be obtained by integrating out W:

$$
\begin{aligned}
f_Z(z) &= \int_0^\infty f_{Z,W}(z,w)\,\mathrm{d}w \\
&= \frac{1}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)\cdot 2^{(m+n)/2}}\cdot z^{\frac{m}{2}-1}\cdot(1-z)^{-\frac{m}{2}-1}\cdot\int_0^\infty w^{\frac{m+n}{2}-1}\cdot e^{-\frac{1}{2}\left(\frac{w}{1-z}\right)}\,\mathrm{d}w \\
&= \frac{1}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)\cdot 2^{(m+n)/2}}\cdot z^{\frac{m}{2}-1}\cdot(1-z)^{-\frac{m}{2}-1}\cdot\frac{\Gamma\left(\frac{m+n}{2}\right)}{\left(\frac{1}{2(1-z)}\right)^{\frac{m+n}{2}}}\cdot \\
&\quad \int_0^\infty \frac{\left(\frac{1}{2(1-z)}\right)^{\frac{m+n}{2}}}{\Gamma\left(\frac{m+n}{2}\right)}\cdot w^{\frac{m+n}{2}-1}\cdot e^{-\frac{1}{2(1-z)}w}\,\mathrm{d}w\;.
\end{aligned}
\tag{10}
$$

At this point, we can recognize that the integrand is equal to the probability density function of a gamma distribution with

$$
a = \frac{m+n}{2} \quad \text{and} \quad b = \frac{1}{2(1-z)}\;,
\tag{11}
$$

and because a probability density function integrates to one, we have:

$$
\begin{aligned}
f_Z(z) &= \frac{1}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)\cdot 2^{(m+n)/2}}\cdot z^{\frac{m}{2}-1}\cdot(1-z)^{-\frac{m}{2}-1}\cdot\frac{\Gamma\left(\frac{m+n}{2}\right)}{\left(\frac{1}{2(1-z)}\right)^{\frac{m+n}{2}}} \\
&= \frac{\Gamma\left(\frac{m+n}{2}\right)\cdot 2^{(m+n)/2}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)\cdot 2^{(m+n)/2}}\cdot z^{\frac{m}{2}-1}\cdot(1-z)^{-\frac{m}{2}+\frac{m+n}{2}-1} \\
&= \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}\cdot z^{\frac{m}{2}-1}\cdot(1-z)^{\frac{n}{2}-1}\;.
\end{aligned}
\tag{12}
$$

With the definition of the beta function, this becomes

$$
f_Z(z) = \frac{1}{\mathrm{B}\left(\frac{m}{2},\frac{n}{2}\right)}\cdot z^{\frac{m}{2}-1}\cdot(1-z)^{\frac{n}{2}-1}
\tag{13}
$$

which is the probability density function of the beta distribution with parameters

$$\alpha = \frac{m}{2} \quad \text{and} \quad \beta = \frac{n}{2} \; , \tag{14}$$

such that

$$Z \sim \mathrm{Bet}\left(\frac{m}{2}, \frac{n}{2}\right) \; . \tag{15}$$

■

Sources:

- Probability Fact (2021): "If X ~ chisq(m) and Y ~ chisq(n) are independent" ; in: *Twitter* , retrieved on 2022-10-17 ; URL: https://twitter.com/ProbFact/status/1450492787854647300 .

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