25th September

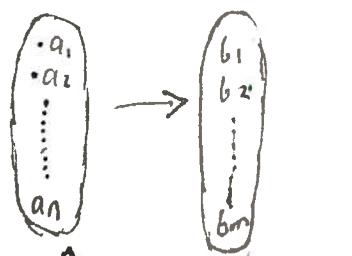
Lets count!

n-set : a set with n elements ($n = 0, 1, 2, \ldots$)

for example $\{1, 2, 3, \ldots, n\}$

Recall the powerset of an n-set is a 2^n-set

Problem 1: Count the number of functions from an n-set to an m-set



A   B

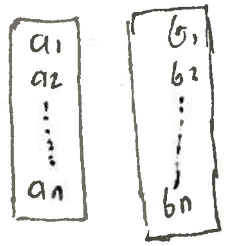
$|A| = n$ $|B| = m$

- $f(a_1)$ has m possible values
- $f(a_2)$ " " " "
- $f(a_3)$ " " " "

$\boxed{\text{So there are } \underbrace{m \cdot m \cdot m \cdots}_{n \text{ time}} = m^n \text{ functions}}$

Note: In particular there are 2^n functions from an n-set to $\{0, 1\}$

Problem 2 Count the number of bijections from an n-set to an m-set



$|A| = n$ $|B| = n$

- $f(a_1)$ has n possible values then,

  $f(a_2)$ has $n-1$ possible values

  [Factorial n]

  ⋮

  $f(a_n)$ has 1 possible values

The injectivity are the limiting factor, showing the available values in B, hence its trivially injective by construction

- But it is also surjective, since f takes n distinct values amoung $[b_1, b_2 \ldots b_n]$ So all $b_1, b_2 \ldots b_n$ are taken as values

$\Rightarrow$ there are $n(n-1) \cdots 1 = \boxed{n!}$ bijections

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Remark: A bijection from $\{1 \ldots n\}$ to itself is called permutation

eg $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$

The set of permutations of $\{1 \ldots n\}$ is denoted S_n,

it appears in def of a determinant

Remark: the statement

There are $n!$ bijection from an n-set to another n-set

$\boxed{\text{INDUCTION}}$

Base Case $n = 1$, there are $1!$

Bijection to 1-set from 1-set

$\Rightarrow$

Induction step

$A = \{a_1, \ldots, a_n, \boxed{a_{n+1}}\}$ ← assume known for n, prove $n+1$

$B = \{b_1, \ldots, b_n, b_{n+1}\}$

Partition the set of Bijections according to the values of a_{n+1} (among b's) for each value b_k, there are $n!$. Bijections from $A \setminus \{a_{n+1}\}$ to $B \setminus \{b_k\}$ then there are

$$\underbrace{\{n! + n! + n! \cdots n!\}}_{n+1 \text{ times}} = (n+1)n!$$

$$= \boxed{(n+1)!} \quad \square$$

Correction here,

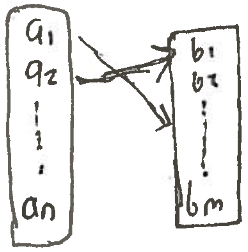
Injections can only happen for n<=m

so 0 if N>m
and mPn if n<m

and n if n = m

Problem 3

Count the number of injections from a n-set to an m-set if $n > m$



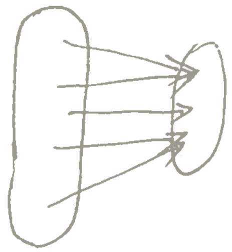
$|A| = n$ $\quad$ $|b| = m$

$$\begin{cases} 0 \\ m(m-1) \cdots\cdots [m-(n-1)] \\ \qquad\qquad\qquad (m-n+1) \end{cases}$$

$\hookrightarrow \dfrac{m!}{(m-n)!}$ $\square$ Combination

Problem 4

Count the number of surjections from a n-set to a m-set



$n \quad m$

$$\begin{cases} 0 & \text{if } n < m \\ ??? & \text{later} \end{cases}$$

inclusion / exclusion

Problem 5:

Count the number of k-subsets from a n-set $\{k = 0, 1 \ldots n\}$ ☆

Example 2-Subsets of $\{1,2,3,4\}$ ordered pairs of distinct elements

	(1,2)	(1,3)	(1,4)	(2,3)	(2,4)	(3,4)
shadows >	(2,1)	(3,1)	(4,1)	(3,2)	(4,2)	(4,3)
	(1&2)	(1&3)	(1&4)	(2&3)	(2&4)	(3&4)

$4 \cdot 3 = 12$

Note different ordered pair (inverse and org order) give the same subset.

☆ $\left[(a_{i_1}, \ldots\ldots, a_{i_k}) \right]$ k tuple — ordered list of k distinct elements from $\{1, \ldots, n\}$

$\left[\{a_{i_1}, \ldots\ldots a_{i_k}\} \right]$ K-set — unordered ~~li~~ of k distinct elements from $\{1 \ldots n\}$

There are $\frac{n!}{(n-k)!}$ ordered list of distinct k-elements from $\{1 \ldots n\}$

Each k-set arises from $k!$ - tuples

$\Rightarrow$ there are $\boxed{\frac{n!}{(n-k)!\,k!}}$ k-subsets $\square$

TRY proving by INDUCTION

Notation $\quad \frac{n!}{(k!)(n-k)!} = \binom{n}{k} \Rightarrow$ n chose k

${}_nC_k$

$\Rightarrow \binom{n}{0} \binom{n}{1} \cdots \binom{n}{n-1} \binom{n}{n}$

$\binom{n}{0} \to 1$ #0 sets

$\binom{n}{1} \to$ 1 element sub, n

$\binom{n}{n-1} \to$ n-1 sets, $n-1$

$\binom{n}{n} \to 1$ # n-sets

We notice of symetry

$\binom{n}{k} = \binom{n}{n-k}$

Proof (1) algebra

(2) Bijection (equi counting without counting)

$\{$k-set$\}$ & $\{$n-k set$\}$ $\Rightarrow$ [Compliment!!]