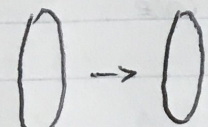
Discrete: Oct 2.

<u>Inclusion - Exclusion</u> $\to$ Sophisticated counting argument. God's Plan

Let $A_1 \dots A_n$ finite sets.

$$\hookrightarrow \left|\bigcup_{k=1}^{n} A_k\right| = \sum_{k=1}^{n} \sum_{\substack{I\subseteq\{1,\dots,n\} \\ |I|=k}} (-1)^{k-1} \left|\bigcap_{i\in I} A_i\right| .$$

<u>Problem</u>: m-set to n-set. $|\text{Surjections}| = ?$



$|A| = m \quad |B| = n$

Let X_k : set of maps $A \to B$ does not take on b_k. for $k \in \{1,\dots,n\}$.

$\hookrightarrow$ map that avoids k.

Then, $\bigcup_{k=1}^{n} X_k$ = set of maps : $A \to B$ that miss <u>some</u> elements in b.

$\hookrightarrow$ set of <u>non</u> surjections!

Hence, $\left|\bigcup_{k=1}^{n} X_k\right| = n^m$ - # surjections.

Fix $I \subseteq \{1,\dots,n\}$ with $|I| = k$.

$\bigcap_{i\in I} X_i$ = set of maps $A\to B$ that miss ALL values in $\{b_i, i\in I\}$.

= set of maps $A \to B \setminus \{b_i, i \in I\}$.

$= (n-k)^m \to (|B| - |\{b_i, i\in I\}|)^m$

Hence, $\sum_{\substack{I\subseteq\{1,\dots,n\} \\ |I|=k}} (-1)^{k-1} \left|\bigcap_{i\in I} X_i\right| = \sum_{\substack{I\subseteq\{1,\dots,n\} \\ |I|=k}} (-1)^{k-1} (n-k)^m$.

RHS of incl/excl $\to$ $= (-1)^{k-1}(n-k)^m \binom{n}{k}$.

n^m - # surjections $= \sum_{k=1}^{n} (-1)^{k-1}(n-k)^m \binom{n}{k}$.

Hence, # surjections $= n^m + \sum_{k=1}^{n} (-1)^k (n-k)^m \binom{n}{k} = \sum_{k=0}^{n} (-1)^k (n-k)^m \binom{n}{k}$.

or, let $k := n-k \to$

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^m .$$

<u>Recurrence Relations</u> Sequences $a_1, a_2 \dots, a_n = \{a_n\}_{n\geq 1}$

- Arithmetic sequence: $3, 8, 13, 18 \dots \to \{a_n\}_{n\geq 1} = a_{n+1} = a_n + 5, \ a_1 = 3$.
  - $\hookrightarrow a_n = a_1 + (n-1)d$
  - $a_n = 5n - 2$. $\leftarrow$ (prove this by induction).
- Geometric Sequence: $3, 6, 12, 24 \dots \ \{a_n\}_{n\geq 1}$ given by $a_n = 2a_{n-1}, \ a_1 = 3$
  - $a_n = 3 \cdot 2^{n-1}$

- Fibonacci Sequence: 1, 1, 2, 3, ~~4~~5, 8 ... $\{F_n\}_{n\geq 1}$ given by $F_{n+1} = F_n + F_{n-1}$; $F_1 = F_2 = 1$

Example: Count number of binary n-strings w/o consecutive 1's.

→ "special" n-strings.

S_n = # of binary n-strings

S_{n+1} = partition special $n+1$ strings according to last digit.

case 0: Last digit is 0; $\underbrace{**\ldots*}_{\text{another "special" } n\text{-string}}0$

Now, get S_{n+1} special $n+1$ strings in this case.

case 1: Last digit is 1; $\underbrace{**\ldots*}_{\text{Special } n-1 \text{ string}}01$

→ get S_{n-1} special $n+1$ strings

↳ $S_{n+1} = S_n + S_{n-1}$ → Fib type relation!

↳ $S_1 = 2$, $S_2 = 3$. → Delayed Fib!

$S_n = F_{n+2}$.

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Thm: $\{a_n\}_{n\geq 1}$ given by $x_{n+1} = ax_n + bx_{n-1}$, init. values x_1 and x_2.

Consider $x^2 = ax + b$. → characteristic equation.

1. Distinct roots, r_1, r_2,

   → $x_n = cr_1^n + dr_2^n$ where c, d found from x_1, x_2.

2. Repeated root, then:

   $x_n = cr^n + dnr^n$ "

   linear growth