Page X

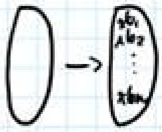
October 2, 2018 8:14 AM

Inclusion - Exclusion

$A_1, \dots, A_n$ finite sets

$$\left| \bigcup_{k=1}^{n} A_k \right| = \sum_{k=1}^{n} \sum_{\substack{I \subseteq \{1,\dots,n\} \\ |I|=k}} (-1)^{k-1} \left| \bigcap_{i \in I} A_i \right|$$

Problem 4 Count the number of surjections from an m-set to an n-set



X_k: set of maps from A to B that do not take on the value b_k,

$|A| = m$ & $|B| = n$ for $k = 1, \dots, n$

$\bigcup_{k=1}^{n} X_k$ = set of maps $A \to B$ that miss some element in B.

= set of non surjections.

$$\Rightarrow \left| \bigcup_{k=1}^{n} X_k \right| = n^m - \# \text{ of surjections}$$

LHS of Inc-Excl.

for $I \subseteq \{1, \dots, n\}$ with $|I| = k$

$\bigcap_{i \in I} X_i$: set of maps $A \to B$ miss all values in $\{b_i : i \in I\}$

$= \text{set of maps } A \to B \setminus \underbrace{\{b_i : i \in I\}}_{\text{size } k}$

$$\Rightarrow \left| \bigcap_{i \in I} X_i \right| = (n-k)^m$$

$$\sum_{\substack{I \subseteq \{1,\dots,n\} \\ |I|=k}} (-1)^{k-1} \left| \bigcap_{i \in I} X_i \right| = \sum_{\substack{I \subseteq \{1,\dots,n\} \\ |I|=k}} (-1)^{k-1} (n-k)^m$$

$$= (-1)^{k-1} (n-k)^m \binom{n}{k}$$

RHS of Inc-Exc. $\sum_{k=1}^{n} \binom{n}{k} (-1)^{k} (n-k)^m$

$$n^m - \# \text{Surjections} = \sum_{k=1}^{n} \binom{n}{k} (-1)^k (n-k)^m$$

$$\# \text{ Surjections} : n^m - \sum_{k=1}^{n} (-1)^k (n-k)^m = \boxed{\sum_{k=0}^{n} (-1)^k (n-k)^m}$$

$(k = n-k)$

Recurrence Relations

Sequences $a_1, a_2, a_3, \dots$ or $(a_n)_{n \geq 1}$

- arithmetic sequence

$3, 8, 13, 18, \dots : (a_n)_{n \geq 1}$, $a_{n+1} = a_n + 5$ with $a_1 = 3$

$\boxed{a_n = 3 + 5(n-1) = 5n - 2}$ → Prove by induction.

- geometric sequence

$3, 6, 12, 24, \dots (a_n)_{n \geq 1}$, $a_{n+1} = 2a_n$, $a_1 = 3$

$\boxed{a_n = 3 \cdot 2^{n-1}}$

Fibonacci Sequence

$1, 1, 2, 3, 5, 8, 13, 21, 34, \dots$

$\boxed{(F_n)_{n \geq 1}}$, $\boxed{F_{n+1} = F_n + F_{n-1}}$ $F_1 = 1, F_2 = 1$

Example Count # of binary n-strings without consecutive 1's.

$\xrightarrow{\text{name}}$ 'special n-strings'.

Let S_n be the # of special n-strings.

S_{n+1}? Partition special $(n+1)$-strings according to the last digit.

Case 0: last digit is 0; these have the form.

$\underbrace{XXX\dots X}_{\text{special } n\text{-string.}}0$ (Bijection between n and $n+1$ special n-string)

now get S_n special $(n+1)$-strings in this case

Case 1: last digit is 1; these have form

$\underbrace{XXXX\dots X}_{\text{special } (n-1) \text{ string}}01$

here get S_{n-1} special $(n+1)$-strings.

$\Rightarrow S_{n+1} = S_n + S_{n-1}$ with $S_1 = 2, S_2 = 3$

$$\boxed{S_n = F_{n+2}}$$

Q: What's formula for F_n

$$\boxed{F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}}$$

* has a geometric behaviour.

Theorem Let $(X_n)_{n \geq 2}$, be a sequence given by

$$X_{n+1} = aX_n + bX_{n-1}$$

with initial values X_1 and X_2.

Consider the characteristic equation $x^2 = ax + b$

(i) Of these there are two distinct roots, r_1 and r_2, then

$$X_n = c r_1^n + d r_2^n$$

where c and d are determined by X_1 and X_2.

(ii) If there is one repeated root, r, then

$$X_n = c r^n + d n r^n$$

where c and d are determined by X_1 and X_2