20th September

CONT. Pigeonhole

Example Consider a list of n integers, show that there is a sublist whose sum is divisible by n.

Let $a_1, \dots, a_n$ be the given n integers

In list of all sums, one of them is can be divided by the assigned number N

consider:-

$S_1 = a_1$
$S_2 = a_1 + a_2$
$S_n = a_1 + \dots a_n$

If one of S_i's is divisible by n, done, otherwise the n partial sums $S_1 \dots S_n$ have remainder 1 or 2 .... or n-1 upon division by n

Pigeonhole implies that two distinct $i < j$; Select S_i and S_j with same remainder..

Then $S_J - S_i = a_{i+1} + \dots + a_J$ divisible by n

done!!

Induction

To prove a statement S_n depends on positive number $n = 1, 2, 3 \dots$

The proof by induction then has two steps

1) Base Case: verify S_1
2) Inductive Step $\underbrace{\text{Assume } S_n}_{\text{inductive hypothesis}}$; prove S_{n+1}

Example: $1 + 2 + \dots + n = \frac{n(n+1)}{2}$ $\quad n = 1, 2, 3 \dots$

Example 2: $1^3 + 2^3 + \dots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$ $\quad n = 1, 2, 3 \dots$

Proving 2

Base Case: For $n=1$, $1^3 = \left(\frac{1 \cdot 2}{2}\right)^2$ holds

Inductive Step: Assume $1^3 + 2^3 \dots n^3 = \left(n\left[\frac{n+1}{2}\right]\right)^2 = (S_n)$

then $1^3 + 2^3 + \dots n^3 + (n+1)^3$

$$= \left[\frac{n(n+1)}{2}\right]^3 + (n+1)^3$$

$x^2 + 4n + 4$

$$= \left(\frac{n+1}{2}\right)^2 \left(\frac{n^2 + 4(n+1)}{(n+2)^2}\right) = \left[\frac{(n+1)(n+2)}{2}\right]^2$$

So S_{n+1} proven

Example: If $|A| = n$, then $|P(A)| = 2^n$ for $n = 1, 2, 3 \ldots$

Base Case: if $n = 0$ then $A = \emptyset$, $P(A) = \{\emptyset\}$ so

$$|P(A)| = 1 = 2^0$$

inductive Step: Assume $|P(A)| = 2^n$ when $|A| = n$

Show: $|P(B)| = 2^{n+1}$ whenever $|B| = n+1$

Fix an element in B, say $*$ Set: $A = B \setminus \{*\}$

then $P(B)$ can be partitioned into two parts

$$P_1(B) = \{ S \subseteq B : * \in S \}$$
$$P_2(B) = \{ S \subseteq B : * \notin S \}$$

So obviously, since $P_2(B)$ is without $*$, as mentioned before that means $P_2(B) = P(A)$

So $|P_2(B)| = |P(A)| = 2^n$ by inductive hypothesis

Note that $P_2(B) = P(A)$

and $P_1(B) \Longrightarrow P(A)$

$$S \Longrightarrow S \setminus \{*\}$$

is Bijective (can just add a star to inverse)

So $|P_1(B)| = |P(A)| = 2^n$

So $2^n + 2^n = \underline{2^{n+1}}$ proven

Example Prove $2^n \geq n^2$ for $n = 4, 5 \ldots\ldots$

Base Case for $n = 4$, $2^4 = 16 \geq 16 = 4^2$

inductive Step assume $2^n \geq n^2$ inductive step $2 \cdot 2^n \geq 2n^2$

to show $2^{n+1} \geq (n+1)^2$

to check $2n^2 \geq (n+1)^2$

$$n^2 \geq 2n+1$$

which holds since $n - 1 \geq 3$

Show that $n(n+1)$ is odd for all $n = 1, 2, 3$

↳ if you dont check Base case, answer will be wrong

↳ In this Base case will show wrong answer

but you can prove the Induction, which is wrong

↳ $(n+1)(n+2)$

$$= \underbrace{n(n+1)}_{\text{odd}} + \underbrace{2(n+1)}_{\text{even}} \quad \text{so odd}$$

Induction = true

~~tr~~ Base case = false

hence Statement = false

Strong / complete Induction :-

To prove: a statement S_n depending on $n = 1, 2, \cdots 3$

Proof by Strong Induction

① Base Case : Verify S_1

② Strong-Inductive Step : Assume $\overbrace{S_1, S_2 \cdots\cdots S_n}^{\text{multiple}}$

Prove S_{n+1}

Example : Every positive integer n can be written as a Sum of distinct powers of 2

Proof by Strong Induction

Base Case $n = 1$ can be written as $1 = 2^0$

Strong Induction Step Assume known for $1, 2, 3 \cdots\cdots n$

Prove for $n+1$

Case 1 $n+1$ is even

$n+1 = 2k$ for some $k \leq n$ $(i_1 > i_2 \cdots\cdots >$

by inductive hyp $k = 2^{i_1} + 2^{i_2}, 2^{i_3} \cdots 2^{i_s}$

=

$n+1 = 2(2^{i1} + 2^{i2} + \dots 2^{is})$

$= 2^{i1+1} + 2^{i2+1} \dots 2^{is+1}$

Case 2 $n+1$ is odd

So $n+1 = 2k+1$ for $k \le n$

$k = 2^{i1} + 2^{i2} + \dots 2^{is}$

$n+1 = 2^{i1+1} + 2^{i2+1} + \dots 2^{is+1} + 2^{0}$ 1

[Just replaced K in 2K+1 and expanded]

Hidden Induction

Not real name .... more of subtlety

Example Find $1 + 2 + 2^2 + \dots 2^n$

- Don't know the formula but have to come up with it yourself
- Try = $n = 1, 2, 3$ = $3, 7, 15$

Propose $1 + 2 + 2^2 \dots + 2^n$ ↳ $2^{n+1} - 1$

$2^{n+1} - 1$ can be inductively proved.