Sept 27, 2018 — fixed ambient set = universal set

MATH240

◇ Binomial Coefficient

last time **Thm** The number of k-sets of an n-set is $\frac{n!}{k!(n-k)!}$ ($k!$: multiplicity; $n!$: ordered; $k!(n-k)!$: unordered.)

Notation $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ · integer (b/c it counts s/th)

Fact: ① $\binom{n}{k} = \binom{n}{n-k}$ "complementation of k sets to n-k sets."

② $\binom{n}{0} + \binom{n}{1} + \cdots \binom{n}{n} = 2^n$ "all possible subsets of n-set" — $|P(n)|$

proof: partition of subsets of a n-set according to cardinality.

③ $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$

proof: ① algebra

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}$$

$$= \frac{n!}{(k-1)!(n-k)!(n-k+1)} + \frac{n!}{(k-1)!(n-k)!\,k!}$$

$$= \frac{k \cdot n! + (n-k+1)n!}{k!(n-k+1)!} = \frac{n!(n+1)}{k!(n+1-k)!} = \frac{(n+1)!}{k!(n+1-k)!}$$

$$= \binom{n+1}{k} \quad \blacksquare$$

② Counting

$\binom{n+1}{k}$: number of k-subsets of an (n+1) set. ↳ eg. $\{1, \ldots, n+1\}$

left hand side suggests partition.

$\mathcal{C}_1$ = collection of k subsets containing n+1

$\mathcal{C}_2$ = collection of k subsets not containing n+1
= k subsets of $\{1, \ldots, n\}$

so $|\mathcal{C}_2| = \binom{n}{k}$

fix n+1, choose (k-1) elements. → $|\mathcal{C}_1|$ = # of (k-1) subsets of $\{1, \ldots, n\}$ — bijection
$= \binom{n}{k-1}$

So $\binom{n+1}{k} = |\mathcal{C}_1| + |\mathcal{C}_2| = \binom{n}{k} + \binom{n}{k-1}$

◇ Pascal's Triangle

$$\begin{array}{ccccccc|l}
 & & & \binom{0}{0} & & & & 0 \text{ row} \\
 & & \binom{1}{0} & & \binom{1}{1} & & & 1 \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & 2 \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & 3 \\
 & & & & & & & \vdots
\end{array}$$

$\binom{n}{0}$ $\binom{n}{1}$ $\binom{n}{n}$

1 1

1 $\boxed{2}$ 1

1 3 3 1

1 4 $\boxed{6}$ 4 1

1 5 10 10 5 1

1 6 15 $\boxed{20}$ 15 6 1 $\leftarrow$ Sum $= 2^n = 2^6$

- - -

Symmetry: $\binom{n}{k} = \binom{n}{n-k}$

spine ÷ divisibility $C_n = \frac{1}{n}\binom{2n}{n}$

Thm: Binomial Formula

for each positive integer n, and variables x and y, we have:

$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + y^n$$

$$= \boxed{\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k}$$

binomial coefficient

- Proof by induction.
- Proof (by Counting): $(x+y)^n = (x+y)(x+y)\cdots(x+y)$

xxx … xx

xxx … xy

each parenthesis has dof=2 (2 choices)

$\therefore 2^n$ terms of $x^k y^{n-k}$

each term $x^k y^{n-k}$ appears by picking x out of k parentheses and y out the remaining $(n-k)$ parentheses

There are $\binom{n}{k}$ ways of picking k parentheses

$\rightarrow x^k y^{n-k}$ appears $\binom{n}{k}$ times n

Applications: $x=1, y=1$: $2^n = \sum_{k=0}^{n}\binom{n}{k}$

$x=1, y=-1$: $0 = \sum_{k=0}^{n}\binom{n}{k}(-1)^k$ oscillation cancellation.

$x=1$; $(y+1)^n = \sum_{k=0}^{n}\binom{n}{k}y^k$

differentiate w.r.t y: $n(y+1)^{n-1} = \sum_{k=0}^{n}k\binom{n}{k}y^{k-1}$

$\rightarrow$ evaluate at $y=1$: $\sum_{k=0}^{n}k\binom{n}{k} = n2^{n-1}$

◇ Inclusion - Exclusion (Counting Principle)

Two sets: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

Three sets: $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - [|A_1 \cap A_2| + |A_2 \cap A_3| + |A_1 \cap A_3|) + |A_1 \cap A_2 \cap A_3|$

For n sets: $A_1, A_2, \ldots, A_n$ $\left|\bigcup_{k=1}^{n} A_k\right| = |A_1| + |A_2| + \cdots + |A_n| - (|A_1 \cap A_2| + \cdots) + (|A_1 \cap A_2 \cap A_3| + \cdots)$

☆☆ $\boxed{\left|\bigcup_{k=1}^{n} A_k\right| = \sum_{k=1}^{n} \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ (|I|=k)}} (-1)^{k-1} \left|\bigcap_{i \in k} A_i\right|}$