25th September

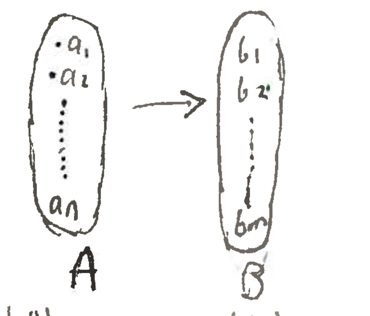
Lets count!

n-set : a set with n elements $(n=0,1,2\ldots)$

for example $\{1,2,3 \ldots n\}$

Recall the powerset of an n-set is a 2^n-set

Problem 1: Count the number of functions from an n-set to an m-set



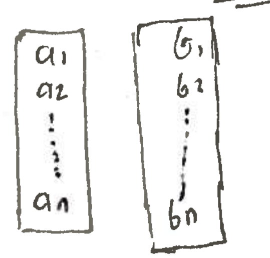
$|A|=n$ $|B|=m$

- $f(a_1)$ has m possible values
- $f(a_2)$ " " " "
- $f(a_3)$ " " " "

$\boxed{\text{So there are } \underbrace{m\cdot m\cdot m \ldots}_{n \text{ time}} = m^n \text{ functions}}$

Note: In particular there are 2^n functions from an n-set to $\{0,1\}$

Problem 2 Count the number of bijections from an n-set to an m-set



$|A|=n$ $|B|=n$

- $f(a_1)$ has n possible values then,

  $f(a_2)$ has $n-1$ possible values

  [Factorial n]

  $\vdots$

  $f(a_n)$ has 1 possible values

The injectivity are the limiting factor, shorting the available values in B, hence its trivially injective by construction

- But it is also surjective, since f takes n distinct values amoung $[b_1, b_2 \ldots b_n]$ So all $b_1, b_2 \ldots b_n$ are taken as values

$\Rightarrow$ there are $n(n-1)\ldots 1 = \boxed{n!}$ bijections

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Remark: A bijection from $\{1 \ldots n\}$ to itself is called permutation

eg $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$

The set of permutations of $\{1 \ldots n\}$ is denoted S_n, it appears in def of a determinant

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Remark: the statement

There are $n!$ bijection from an n-set to another n-set

$\boxed{\text{INDUCTION}}$

Base Case $n=1$, there are $1!$

Bijection to 1-set from 1-set

$\Rrightarrow$

Induction step $A=\{a_1,\dots\dots a_n,\boxed{a_{n+1}}\}$ ← assume known for n prove $n+1$

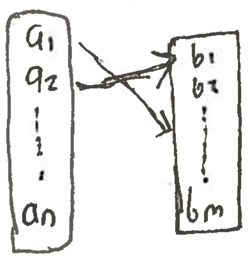
$B=\{b_1,\dots b_n,b_{n+1}\}$

Partition the set of Bijections according to the values of a_{n+1} (among b's) for each value b_k, there are $n!$. Bijections from $A\setminus\{a_{n+1}\}$ to $B\setminus\{b_k\}$ then there are

$$\underbrace{\{n!+n!+n!\cdots n!\}}_{n+1\ \text{times}} = (n+1)n! = \boxed{(n+1)!} \quad \square$$

Problem 3

Count the number of injections from a n-set to an m-set if $n>m$



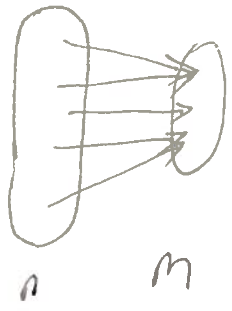
$|A|=n$ $|B|=m$

$$\begin{cases} 0 \\ m(m-1)\cdots\cdots[m-(n-1)] \\ \qquad\qquad (m-n+1) \end{cases}$$

$\to \dfrac{m!}{(m-n)!}$ $\square$ Combination

Problem 4

Count the number of surjections from a n-set to a m-set



$$\begin{cases} 0 & \text{if } n<m \\ ??? & \text{later} \end{cases}$$

inclusion / exclusion

Problem 5

: Count the number of k-subsets from a n-set $\{k=0,1\dots n\}$ ☆

Example 2-Subsets of $\{1,2,3,4\}$ ordered pairs of distinct elements

	$(1,2)$	$(1,3)$	$(1,4)$	$(2,3)$	$(2,4)$	$(3,4)$
shadows >	$(2,1)$	$(3,1)$	$(4,1)$	$(3,2)$	$(4,2)$	$(4,3)$
	$(1\&2)$	$(1\&3)$	$(1\&4)$	$(2\&3)$	$(2\&4)$	$(3\&4)$

$4\cdot 3=12$

Note different ordered pair (inverse and orig order) give the same subset.

☆ $[(a_{i_1},\dots\dots,a_{i_k})]$ k tuple — ordered list of k distinct elements from $\{1,\dots\dots,n\}$

$[\{a_{i_1},\dots\dots a_{i_k}\}]$ K-set — unordered list of k distinct elements from $\{1\dots n\}$

There are $\frac{n!}{(n-k)!}$ ordered list of distinct k-elements from $\{1 \cdots n\}$

Each ~~k~~ k-set arises from $k!$ - tuples.

$\Rightarrow$ there are $\boxed{\frac{n!}{(n-k)!\,k!}}$ k-subsets $\square$

TRY proving by INDUCTION

Notation $\frac{n!}{(k!)(n-k)!} = \binom{n}{k} \Rightarrow$ n chose k

${}_nC_k$

$\Rightarrow \binom{n}{0} \binom{n}{1} \cdots\cdots \binom{n}{n-1} \binom{n}{n}$

$\binom{n}{0} \downarrow 1$ #0 sets

$\binom{n}{1} \downarrow$ 1 element sub n

$\binom{n}{n-1} \downarrow$ n-1 sets $n-1$

$\binom{n}{n} \downarrow 1$ # n-sets

We notice of symetry

$\binom{n}{k} = \binom{n}{n-k}$

Proof (1) algebra

(2) Bijection (equi counting without counting)

$\{k\text{-set}\}$ & $\{n-k \text{ set}\} \Rightarrow$ [Compliment!!]