puzzle: prove by induction.

Notation : $\frac{n!}{k!(n-k)!} = \binom{n}{k} \rightarrow$ n choose k, $(k = 0, 1, 2, \ldots, n)$.

$\binom{n}{0}\ \binom{n}{1}\ \binom{n}{2} \ldots \binom{n}{n-1}\ \binom{n}{n}$

$\binom{n}{1}, \binom{n}{n-1} = n$

The number of zero sets = 1

number of n-sets = 1

Proposition : $\binom{n}{k} = \binom{n}{n-k}$.

Proof : ① Algebra.

② Bijection between $\{k\text{-sets}\} \rightarrow \{n-k \text{ sets}\}$.

↳ complement. → equi-counting. Count 2 things that are in bijection.

$\boxed{\text{Discrete : Lecture \#8}}$

↳ Binomial coefficients

dividing the multiplicity

ordered (perm)

Th: The number of k-subsets of an ambient n-set is $\frac{n!}{k!(n-k)!}$

Notation: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, ${}_nC_k$

n and k are integers, because they are counting something.

Discrete: Lec#8 (cont.)

$\binom{n}{k} = \binom{n}{n-k}$ → Think about what they are counting. **FACT 1**

↳ equi-counting, one complement of another.

$\int \frac{1}{1+\cos x} dx$

FACT 2 $\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$

↳ Counting all subsets.

$\sqrt{\cos x} = u$

↳ Partition the subsets of an n-set according to the cardinalities.

$du = \frac{1}{2}$

Fact 3 $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k} = \frac{(n+1)\,n!}{k!(n-k+1)!}$

↳ $\frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} = \frac{k\,n!}{k!(n-k+1)!} + \frac{n!\,(n-k+1)}{k!(n-k+1)!}$

(2) Counting. $\binom{n+1}{k}$ = number of k-subsets of an n+1 set.

LHS: (expect a partition). e.g.: $\{1, \dots, n+1\}$.

Separate into $\mathcal{C}_1$ = collection of k-subsets containing n+1

$\mathcal{C}_2$ = collection of k-subsets NOT containing n+1.

$\mathcal{C}_2$ = k-subsets of 1 up to n.

$\therefore |\mathcal{C}_2| = \binom{n}{k}$

→ To pick $\mathcal{C}_1$ means to pick k-1 elements AND put in (n+1) in the end

So, $|\mathcal{C}_1|$ = # k-1 subsets of 1 to n. $= \binom{n}{k-1}$

So. $\binom{n+1}{k} = |\mathcal{C}_1| + |\mathcal{C}_2| = \binom{n}{k-1} + \binom{n}{k}$

Pascal's Triangle

Row	
0	$\binom{0}{0}$
1	$\binom{1}{0}\ \binom{1}{1}$
2	$\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}$
3	$\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}$
4	$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}$
⋮	

from Fact 3: add two above numbers, get number below!

↳
```
      1
     1 1
    1 2 1
   1 3 3 1
  1 4 6 4 1
 1 5 10 10 5 1
      ⋮
```

Lots of symmetries :)

↳ Read on spire, $\frac{1}{n}\binom{2n}{n}$ = Catalan sequence.

dividing by n

Discrete: Lec#8 (cont.)

Th: (Binomial Formula)

For each positive integer n, and variables x and y, we have:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 \cdots \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

$\hookrightarrow (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$, $k = 0, 1, \ldots, n$.

Proof: (by counting).

$$(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ times}} \quad \left\{ \begin{array}{l} xxxx\ldots xx \\ xxxx\ldots xy \end{array} \right.$$

$\hookrightarrow$ 2^n terms of the form $x^k y^{n-k}$ $\rightarrow$ Uncollected.

Each term $x^k y^{n-k}$ appears by picking x out of k parentheses and y out of the remaining $n-k$ parentheses.

There are $\binom{n}{k}$ ways of picking k parentheses. $\binom{n}{k}$ = binomial coefficient.

$\hookrightarrow$ $x^k y^{n-k}$ appears $\binom{n}{k}$ times. ∎

Applications:

$x=1, y=1$, $2^n = \sum_{k=0}^{n} \binom{n}{k}$.

$x=1, y=-1$, $0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$.

$x=1$, $(y+1)^n = \sum_{k=0}^{n} \binom{n}{k} y^k$

differentiate wrt y:

$n(y+1)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k} y^{k-1}$

Inclusion - Exclusion

Accounts for overlap.

Two sets: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

Three sets: $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| \overbrace{- |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3|}^{\text{over-counted}}$

$+ |A_1 \cap A_2 \cap A_3|$. ← add middle part.

for n sets: $A_1, A_2, \ldots, A_n$

$\bigcup_{k=1}^{n} A_k = |A_1| + |A_2| + \ldots + |A_n| - (|A_1 \cap A_2| + \ldots +)$.

$+ (|A_1 \cap A_2 \cap A_3| + \ldots) \ldots$ And so on!

$$\boxed{= \sum_{k=1}^{n} \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I| = k}} (-1)^{k-1} \left| \bigcap_{i \in I} A_i \right|}$$