MATH 323: DAY 16
OCTOBER 31 TUE

COVERING:

- Introducing Continous Random Variables.
- Cumulative Density Function vs Probabilty Density Function
- Relationship between PDF AND CDF
    - Idea of Area under the graph
- Expected Value and Variance of PDF using integrals

31st (11) October Math 323.

Continous Distributions

Formal Def:- A R.V x is said to be continous or have continous distribution if its C.D.F (cumulative Distribution function), $F_x X$ is a continous function of x for all $-\infty < x < \infty$

o i.e There are no Jumps.

Important consequence of this

$P(X=x) = 0$ for all $-\infty < x \infty$

Reason?: In general, for any R.V x, the proba
$P(X=x) = P(X \leq x) - P(X < x)$

We know all C.D.F (cont or discrete) are cont from the Right.

o A continous C.D.F is continous from the Right & Left.

Now $P(X<x) = \lim_{y \uparrow x} P(X \leq y) = \lim_{y \uparrow x} F_x(y)$

$\therefore P(X=x) = F_x(x) - F_x(x) = 0$ for all x

Note : That $P(X=x)=0$ does not mean that all events must have Prob of 0 if X is continous.

i.e $P(X \in A) = 0$ is not true

Point: There are uncountably many points in an interval.
eg interval $[1,3)$ = $\{x \in R : x \in [1,3)\}$

$P[X \in [1,3)] \neq$ "sum" of prob of all x in $[1,3)$.

Axiom3 can't apply to uncountable union's as there are infinite points.

In analogy, even though every point on the real line has length 0, the length of an interval $\neq 0$. We dont calculate total length of the interval by adding up the lengths of its uncountably many points.

↔

Question is how do we calculate the probabilities that X falls in an interval? To do this, we need an analogue of the probability function $(P_X(x) = P(X=x))$ defined for discrete random variables.

Definition Let X be a continous RV. Then we define the probability density function (P.D.F) of X denoted by f_X, to be any function with properties

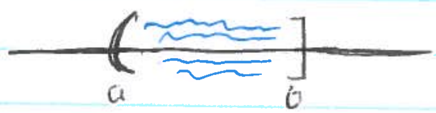
$$P(X \leq x) = F_X(x) = \int_{-\infty}^{x} f_X(y)\, dy.$$

i.e a PDF is a function that has the property that when integrated from $-\infty$ to x, it produces the property that $P[X \leq x]$

Notes on PDF

i) $\int_{-\infty}^{\infty} f_x(y)\, dy = 1$

ii) For all C.D.F's (cont or discrete), the prob $P[a < x \leq b]$
$= F_x(b) - F_x(a)$



hence $P[a < X \leq b] = \int_a^b f_x(y)\, dy$

$$= F_x(b) - F_x(a) \quad (*)$$

Note that x is continous, $P[a < x \leq b] = P[a \leq x < b]$
$= P[a < x < b] = P[a \leq x \leq b]$

Since $P[x = a] = P[x = b]$

iii) Relationship between C.D.F & P.D.F.

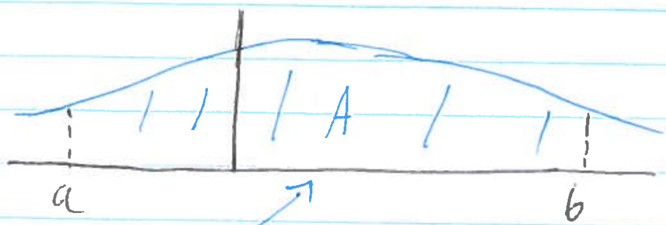
By definition, if given a P.D.F, you have a C.D.F as seen above in (*). Conversely if you have C.D.F, you can recover the P.D.F. In fact

$$f_x(x) = F'_x(x) = \frac{d}{dx} F_x(x)$$

Fundemental theorem of Calculus.

Consequence is that if x is a continous R.V. Then its probability distribution is uniquely determined by P.D.F or C.D.F

iv) Of course if $f_x(x)$ is a P.d.F then the area under the curve between a & b is $P[a < x \leq b]$

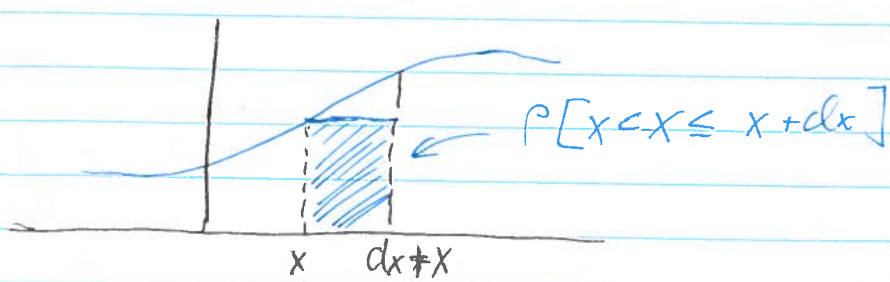


$A = \int_a^b f_x(x)\,dx$ = Area under the curve

v) Where as the $P_x(x) = P[X = n]$ is a probability if X is discrete, $f_x(x)$, the P.D.F does not give $P[X = x]$ if X is continous

infact although $f_x(x) \geq 0$, it need not satisfy $f_x(x) \leq 1$. Infact often $f_x(x) \geq 1$, as long as it is ≥ 0 and integrates to 1. (over $-\infty$ to ∞) it is P.d.F of some R.V.

However we have $f_x(x)\,dx \approx P[x < X \leq x + dx]$ for a small dx.



Area of Rectangle = $f(x) \cdot$ width = $f(x)\,dx$

$\approx A$ when dx is small.

vi) if you're given any function $g(x)$ such that $g(x) \geq 0$ and $\int_{-\infty}^{\infty} g(x)\,dx = c < \infty$ {finite}

then you can convert $g(x)$ into a P.D.F by standardizing it as follows:-

o Let $f_X(x) = g(x) \Big/ \int_{-\infty}^{\infty} g(x)\,dx$

Then $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$.

vii) if X is continous, then $P[X \in R]$, where R is set of all rational numbers, is equal to $\longrightarrow 0$

Rational no's are countable (infinite but countable).

Since they are countable and each rational has Prob $= 0$ if X is continous, then with axiom 3, sum of all these $= 0$ as well.

$\longleftrightarrow$

Expected value and variance of a continous R.V.

Definition: if X is a continous R.V then we define its expected value (mean) to be

$$E(X) = \mu = \int_{-\infty}^{\infty} x\, f_X(x)\,dx$$

In particular $\mathcal{E}(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$

Variance $= \mathcal{E}[X-\mu]^2 = \mathcal{E}[X^2] - \cancel{\mathcal{E}[\mu]}\ \mu^2$

$$= \int_{-\infty}^{\infty} x^2 f_X(x)\, dx - \left[\int_{-\infty}^{\infty} x\, f_X(x)\, dx\right]^2$$

OR $\left(\int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\, dx\right)$

(+) We say the Expected value of RV exist if $\mathcal{E}(|X|)$ is finite OR $\mathcal{E}[|X|] < \infty$

i.e $\int_{-\infty}^{\infty} |x|\, f_X(x)\, dx < \infty$

- We shall see examples of Cont. RV that do not have finite Expectation.