Day 14. Math 323 October 19^{th}

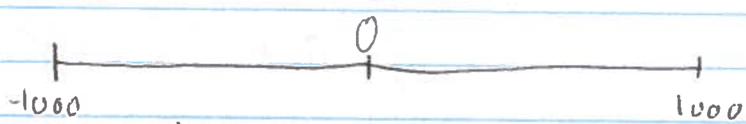
More of $\mathcal{E}(x)$

4. The expected value of a random variable can also be thought of as an avg of x that you would observe in an infinite no of trials, in which a realized value of X is observed at each trial. (law of large no's)

5) $\mathcal{E}(x)$ need not take on any of the values in the range of X. It is 'An Average'

6) A related comment is that despite the name, $E(x)$ is not what you expect to observe when you carry out experiment.

Thus you would certainly not expect to observe 0, (which is the $\mathcal{E}(x)$ of of R.V X with the following Prob function):



$P(X=-1000) = 1/2$ $\qquad$ $P(X=1000) = 1/2$

so $\frac{1}{2}(-1000) + (+1000)(\frac{1}{2})$
$= 0$

You wouldn't expect to observe $\underline{\underline{0}}$.

7) The following simple properties hold for $\mathcal{E}(x)$

a) if c is constant, the $\mathcal{E}(cX) = c\,\mathcal{E}(x)$

Proof $\mathcal{E}(cX) = \sum_x cx\,P(X=x) = c \sum_x x\,P_x(x)$

6) if $x_1, x_2, \ldots, x_n$ are any set of n r.v-s, then

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(x_i)$$ (later Justification)

8) We say a R.V X has an expectation, if $E(|X|) < \infty$ [is finite]. There are R.V's that don't have a finite expectation

Example of a R.V whose expected value is ∞

Let $X = n^2$ with probability $1/n$
$= n^3$ with " $1 - 1/n$

where $n = 2, 3, \ldots$

We have $\mu_n = E(X) = \sum_{n=1}^{\infty} n^2 \cdot 1/n + \sum_{n=1}^{\infty} n^3 \cdot (1 - 1/n)$

$$= \sum_{n=1}^{\infty} n + \sum_{n=1}^{\infty} n^3 \left(1 - \frac{1}{n}\right)$$

$$> \sum_{n=1}^{\infty} n = +\infty \quad \square$$

Example of insurance premiums:-

Suppose that you wish to insure your bike for \$1000. Suppose that insurance company knows 5% of such bikes are stolen in a given year. 95% are not

What premium should insurance company charge such that Expected gain = 0

Solution Let X be the gain of the company. then:

$$P[X = c] = .95 \quad \text{and} \quad P[X = \underbrace{-1000}_{\text{stolen}} + c] = .05$$

We seek c, such that $E(x) = 0$

Now $E(x) = c \cdot 0.95 + (c - 1000) \cdot .05$

We have $E(X) = 0$ iff $c = \$\,\underline{50}$ Ans

Meaning If company were to charge premium of 50\$ for all such bikes, (different customers), avg gain at end of year = 0

The previous example showing the Expected value is not what you expect to see can be demonstrated that although to R.V's can have same means $\longrightarrow$ they can be very different

o

-1000 0 1000

$P(X=-1000) = 1/2$ $P(X=1000) = 1/2$

-1 0 1

$P(X=-1) = 1/2$ $P(X=1) = 1/2$

Both end with 0, but very different R.V's

o We see that X has greater variability than does . We need a second measure of spread to accompany E(x) in order to more accurately summarize a distribution

o To this end, we have

Definition:- Let X be a R.V. We define variance of X as follows denoting it by Var(x) or σ_x^2

$$Var(x) = \sigma_x^2 = E[(X-\mu)^2]$$

$$= \sum_{x} (x-\mu)^2 P[X=x]$$

Notes

1. $Var(x)$ is the average of the squared deviation of X from its mean - the avg squared distance of X from μ.

2) Because $Var(x)$ is called units of X squared we often use $\sqrt{Var(x)} = \sigma$ to describe spread

σ is called standard deviation of a r.v. σ^2 and σ are mathematically equivalent.

3) An alternative expression for the $Var(x)$ is $Var(x) = E(X^2) - \mu^2$

Proof

$$Var(x) = E[(X-\mu)^2]$$
$$= E[x^2 - 2\mu x + \mu^2]$$
$$= E(X^2) - 2\mu E(X) + \mu^2$$
$$= E(X^2) - 2\mu^2 + \mu^2$$
$$= E(X^2) - \mu^2$$

Note $E(X^2) = \sum_{\text{all } x} x^2 \, P[X = x]$

Also
- $E(x^2) - \mu^2 \leq 0 \longrightarrow E(x)^2 \leq \mu^2$
- $E(x)^2 \leq (E(x))^2$

4) $\mathcal{E}(x^2) \neq (\mathcal{E}(x))^2$ so in general

$\mathcal{E}(x^2) \geq (\mathcal{E}(x))^2$

Also $\mathcal{E}(xy) \neq \mathcal{E}(x) \cdot \mathcal{E}(y)$ for R-V's x, y