22 Sept Lecture 6: Cont: Conditional Probability.

- $P[B|A] = \frac{P[B \cap A]}{P(A)}$, $P(A) > 0$

Imp Result of Def of Cond. Prob:-

- As a consquence of its defination, we have the multiplication rule of probabilities.

Multiplication rule:- Given events A, B. $P(A \cap B)$
$= P(B|A)\, P(A)$
$= P(A|B)\, P(B)$

Reason for importance: well if you want to find $P(A \cap B)$ might be difficult to do directly

- If you know $P(A), P(B)$ you dont always know $P(A \cap B)$

- But if you know $P(A)$ and $P(B|A)$ or $P(A|B)$. you can use multiplication formula, you can find. $P(A \cap B)$

Example:- Suppose in the marble exp you draw two marb
1) what is the probility of obtaining the outcome R_1 and G_2 (ie $R_1 \cap G_2$)

2) What is the prob of obtaining green on second Draw

Solution →

Solution:

1) By the mult, $P(R_1 \cap G_2) = P(G_2|R_1)\,P(R_1)$
or $P(R_1|G_2)\,P(G_2)$

Both are equally correct, but natural order of question is just more convinent.

So $P(G_2|R_1) = 4/9 \qquad P(R_1) = 6/10$

$= 24/90$

- Can be done by counting and even by permutation. But complex Q's are harder to do with alternative options.

2) Want $P(G_2)$. Idea: write G_2 as the union of two disjoint 2-draw outcomes.

$G_2 = (G_2 \cap R_1) \cup (G_2 \cap G_1)$

Now observe $G_2 \cap R_1$ and $G_2 \cap G_1$ are disjoint

$\therefore$ by Axiom 3 $P(G_2) = P(G_2 \cap R_1) + P(G_2 \cap G_1)$

// Think condition $\begin{cases} P(G_2|R_1)P(R_1) + P(G_2|G_1)P(G_1) \\ = 4/9 \cdot 6/10 + 3/9 \cdot 4/10 \end{cases}$

Note: conditions ~~[illegible]~~ infere that there is disjoint

Note:

Full power of mult. Rule can be seen in following Question

- In marble problem, what prob of getting sequence: Seq
  Seq $\{R_1, R_2, G_3, G_4, R_5\}$
- Or Find $= P(R_1 \cap R_2 \cap G_3 \cap G_4 \cap R_5)$

Idea: Condition backwards making it trivial

- $P(\underbrace{R_1 \cap R_2 \cap G_3 \cap G_4}_{B} \cap \underbrace{R_5}_{A})$

- We have $P(A \cap B) = P(A \mid B)\, P(B)$
  ie $P(R_1 \cap R_2 \cap G_3 \cap G_4 \cap R_5)$
  $\equiv P(R_5 \mid R_1 \cap R_2 \cap G_3 \cap G_4) \cdot P(R_1 \cap R_2 \cap G_3 \cap G_4)$

- You can further break it down as well

  (1) $P(R_5 \mid R_1 \cap R_2 \cap G_3 \cap G_4)$
  $\cdot\, P(G_4 \mid R_1 \cap R_2 \cap G_3)$ (2)
  $\cdot\, P(G_3 \mid R_1 \cap R_2)$ (3) $P(R_2 \mid R_1)$ (4) $\cdot\, P(R_1)$ (5)

  each component is easy to find.

1) $4/6$ • 2) $3/7$ • 3) $4/8$ • 4) $5/9$ • 5) $6/10$

or $\left\{\frac{4}{6} \cdot \frac{3}{7} \cdot \frac{4}{8} \cdot \frac{5}{9} \cdot \frac{6}{10}\right\}$ = Ans

Same thing in counting is too complex

Complex Question : Test for Brest Cancer

Pos = positive Bc = has Breast cancer

Given data

$P(Pos \mid Bc) = .95$

$P(Neg \mid Bc^c) = .95$

$P(BC) = 5/1000$

$P(BC \mid Pos) =$ .85

.56

.605 } guess

What porestion of women have brest cancer and will test positive

0 Note seeing having BC is 5/1000 this forces the BC | Pos to be small as probability of having it in first place is small

$P(BC \mid Pos) = 0.0857$

Is Part of Bayes Theorem

Law of total Probability

let A be any event. Let $B_1, B_2 \cdots\cdots B_m$ be events that satisfy

up: i) $B_i \cap B_j = \phi$ For $i \neq j$ (B_i's are Disjoint)
ii) $B_1 \cup B_2 \cdots \cup B_m = S$ (make all of Sample Space)

[such set of B's are called a Partition of S]

Theorem:-
$$P(A) = \sum \underbrace{P(A \mid B_i)}\ \underbrace{P(B_i)}$$

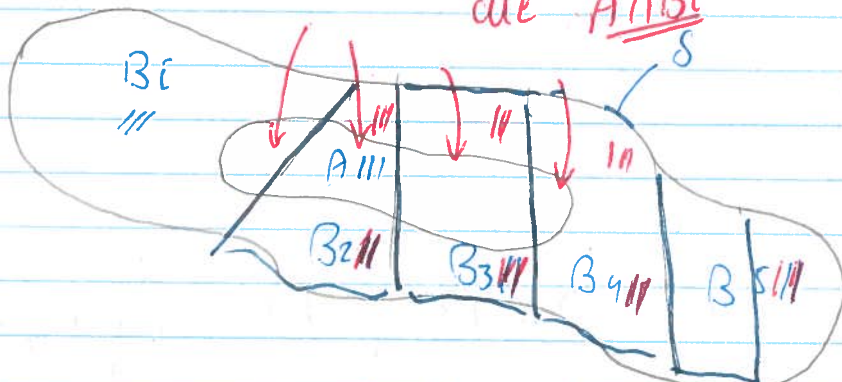
Sum of cond. Prob and all B_i's

Note: [A could be complicated to find its probability directly. However we may know the conditional Probs on the R.H.S or able to find them easily, in addition to $P(B_i)$'s]

Proof

the small area here are $A \cap B_i$

[B_i's are whole section]



All these are Independently dis joint. Since they are Disjoint,

$$P(A) = P\left(\bigcup_{i=1}^{m} (A \cap B_i)\right)$$

Note:- why are the RHS, LHS equal, cause union of B_i's = Sample space

so $P(A) = P(\bigcup_{i=1}^{m} (A \cap B_i))$

$$= \sum_{i=1}^{m} P(A \cap B_i)$$

$$= \sum_{i=1}^{m} P(A \mid B_i) \cdot P(B_i).$$

Note that it is possible that for some i, $A \cap B_i$ is ϕ. No Problem. Cause its probility or $P(A \cap B_i) = P(\phi) = 0$

- Message if any Questions ☺

Important related Theorem — Bayes theorem.

Let A be any event. Let $B_1, B_2 \dots\dots$ satisfy

i) $B_i \cap B_j = \phi$

ii) $\bigcup_{i=1} B_i = S$

$\Bigg\}$ assumptions

Theorem:- $$P(B_k \mid A) = \frac{P(A \mid B_k)\, P(B_k)}{\sum_{j=1}^{m} P(B_j \mid A) \cdot \cancel{} \cdot P(B_j)}$$

Every $k = 1, 2 \dots j$

Law of Prob vs Bayes Theorem.

You are given conditional and marginal Propenty, you are asked to reverse it, then it is Bayes Theorem

Prof made a mistake; correction of Theorem:

$$P(B_k \mid A) = \frac{P(A \mid B_k) \cdot P(B_m)}{\sum_{j=1}^{n} P(A \mid B_j) \cdot P(B_j)}$$

This is the corrected version, mistake was here

Futher:. Proof $P(B_k \mid A) = \frac{P(B_k \cap A)}{P(A)}, P(A) > 0$

$$= \frac{P(A \mid B_k) \cdot P(B_k)}{\sum_{j=1}^{m} P(A \mid B_j) \cdot P(B_j)}$$

look how denominator is law of prob

Example : Formal version of B. cancer diagnosis

Suppose diagnostic test for Breast cancer has the following property

1) if a woman has breast cancer, the probability she'll be tested positive (+ve) is .95

2) " " " doesn't have B-cancer, Prob she'll be tested negative (-ve) is .95

3) Finally suppose that 5/1000 women have Breast cancer (BC)

Question

i) What proportion of women will test positive for BC

ii) if a women test positive, what's the prob that she has BC

$\longleftrightarrow$

Advice: Keep your events as simple as possible to start with.

- Pos or (+ve) = event of positive outcome
- Pos^c or (-ve) or Neg = event of negative
- BC = event woman got Breast cancer
- BC^c = Not Breast cancer

Solution Note : The sample Space can be written as $\{S = BC \cup BC^c\}$ => are Disjoint

therefore by law of total Prob,

$$P(Pos) = P(Pos \mid BC) \cdot P(BC) + P(Pos \mid BC^c) \cdot P(BC^c)$$

- We can always get BC^c as $1 - BC$

TBe Continued.