MATH 323: DAY 17
NOVEMBER 2 THU

COVERING:

- Examples of questions involving CDF
    - How to find a constant
    - How to find probabilty within a CDF range
    - How to find Expected Value and Variance

Named Distributions:

- Uniform Distributions.
    - Defination
    - PDF and CDF
    - E(X) and Var(X)
- Exponential Distribution.
    - Defination
    - PDF and CDF
    - E(X) and Var(X)
    - Paramaterized form
    - Idea of memoryless property

2 November 2017

likelyhood of a value

Example

0 Suppose that a R.V X has a pdf $\left[f_X(x) = Cx^2 \text{ for } 0 < x < 1 \atop = 0 \text{ else} \right]$ ~~for $0 \le x < 1$~~

where C is some constant

i) Find C

ii) Find $P[1/4 \le X < 1/2]$

iii) Find $F_X(x)$ for all $-\infty < x < \infty$

iv) Find $E(X)$ and $Var(X)$

{Integrate over appropriate range}

Solution

i) We know $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$\therefore \int_{-\infty}^{0} 0\,dx + \int_0^1 cx^2\,dx + \int_1^{\infty} 0\,dx = 1$$

$$\therefore \left. C\frac{x^3}{3} \right]_0^1 = 1 \Rightarrow C = 3 \quad \text{Ans}$$

ii) $$P[1/4 \le X < 1/2] = \int_{1/4}^{1/2} 3x^2\,dx = \left. 3\frac{x^3}{3} \right]_{1/4}^{1/2}$$

$$= 7/64$$

iii) We have

thus
gets
u
sy Points

$$F_x(x) = \int_{-\infty}^{x} 0 \cdot dx \qquad \text{for all } x \leq 0$$

$$= 0$$

$$F_x(x) = \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{x} 3x^2 \, dx \qquad \text{for } 0 < x < 1$$

$$= x^3 \qquad \text{for } 0 < x < 1$$

So both cases are handled

You can write a third case for $x > 1$ too

$$F_x(x) = \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{1} 3x^2 \, dx + \int_{1}^{x} 0 \cdot dx$$

all $= 1$

iv) $E(x) = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$

$$\Rightarrow \boxed{\int_{-\infty}^{0} \underline{x \cdot 0} \, dx + \int_{0}^{1} x \cdot 3x^2 \, dx + \int_{1}^{\infty} \underline{x \cdot 0} \, dx}$$

$= 3/4$ Ans

$E(x^2) = \int_{-\infty}^{\infty} x^2 f_X(x) \, dx$

$= \int_{0}^{1} x^2 \, 3x^2 \, dx + \underline{0} + \underline{0}$

$= 3/5$

$\therefore$ $Var = \sigma^2 = 3/5 - (3/4)^2 = 3/80$

so $\sigma = \sqrt{3/80}$ Ans

Note: Watch out for a P.D.F whose ~~same~~ form changes over different regions of the real line. here P.D.F changes from 0 to $3x^2$ and back to 0.

Some Imp Named Distributions

The Uniform distribution:

Def: The r.v X is said to have a uniform distribution on the interval (a,b) if

$$f_X(x) = \frac{1}{b-a} \quad \text{for } a < x < b$$

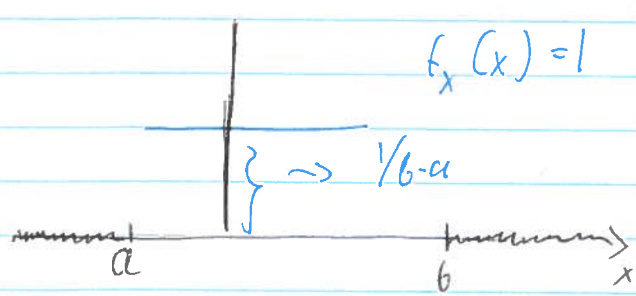
small f

$$= 0 \quad \text{for elsewhere.}$$

uniform = {for intervals of same length, prob of x belongs is same}

its pdf is of the form

$f_X(x) = 1$



Note: i) Reason for name: $P\left[X \in \text{any interval of Length } l \text{ inside } (a,b) \text{ is just } l/b-a\right]$

on that the probabilities are equally spread out over the interval (a, b)

2) The uniform distribution is used to model complete randomness in a continuous setting.

3) Notation $\qquad X \sim U(a, b)$

4 The $U(0,1)$ distribution is sometimes called the standard uniform distribution

5) CDF

We have

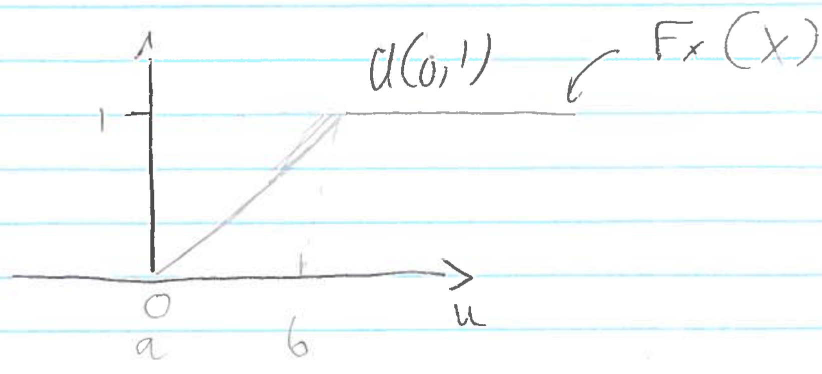
converting from $f(x)$ to $F(x)$

1) $F_x(x) = 0 \quad$ for $\quad x \le a$

2) $\displaystyle = \int_a^x \frac{1}{b-a}\, dx \quad$ for $\quad a < x < b$

$\displaystyle = \frac{x-a}{b-a} \qquad a < x < b$

3) $F_x(x) = 1 \quad$ for $\quad x \ge b$



6) $\mathcal{E}(x)$, well as Prob is uniformly spread

then $\mathcal{E}(x) = \int_a^b x \cdot \frac{1}{b-a}\, dx$

$= \frac{a+b}{2}$

$\mathcal{E}(x^2) = \int_a^b x^2 \cdot \frac{1}{b-a}\, dx$ from which it follows

that $Var(x) = \left\{\frac{(b-a)^2}{12}\right\}$ Ans

Note that PDF of a $U(a,b)$ R.V is discontinuous at a & b, although the C.D.F is continous.

The Exponential distribution

The R.V X has Exponential distribution with parameter $\beta > 0$ if

$$\begin{cases} f_X(x) = \dfrac{1}{\beta} e^{-x/\beta} & \text{for } x \geq 0 \\ \quad\quad\;\; = 0 & \text{for } x < 0 \end{cases}$$

Notes

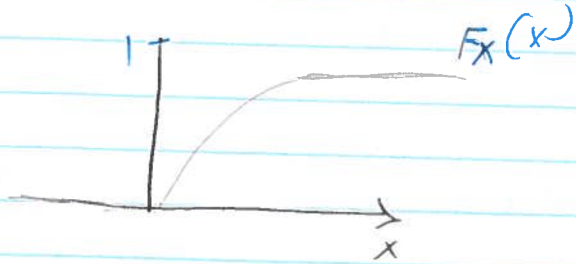
1) Clearly $f_X(x)$ is a P.D.F since a) $1/\beta \cdot e^{-x/\beta} \geq 0$
b) $\int_0^\infty \frac{1}{\beta} \cdot e^{-x/\beta} \, dx = 1$

2) $F_X(x) = 0$ for $x < 0$

$$= \int_0^x \frac{1}{\beta} \cdot e^{-y/\beta} \, dy \quad \text{[y is dummy variable]} \quad \text{for } x \geq 0$$

$$= 1 - e^{-x/\beta} \qquad x \geq 0$$

So CDF :



We have $P(X > x)$
$= 1 - P(X \leq x)$
$= X e^{-x/\beta}$ $\quad x \geq 0$
or 1 $\quad x < 0$

3) $\mathcal{E}(x) = \int_0^{\infty} x \frac{1}{\beta} e^{-x/\beta} \, dx$

$= \beta$

This is gotten using integration by parts or later seen moment generating function (which Prof prefers over intigration by parts)

Also $Var(x) = \mathcal{E}(x^2) - [\mathcal{E}(x)]^2$

$= \beta^2$

4) Sometimes the exp distributio is written in form

$f_x(x) = \lambda e^{-\lambda x}$ $\quad x \geq 0$

$= 0$ $\quad x < 0$

We say that the distribution has been PARAMETERIZED

o In this form the $\mathcal{E}(x) = 1/\lambda$ and $Var(x) = 1/\lambda^2$

Notice how $1/\lambda = \beta$ just parameterized

5 We write $X \sim Exp(\beta)$

6) Memoryless property of the exponential distribution:-

The exponential dist has following interesting property

$P(a \le X < a+h \mid X \ge a)$ → Given that $X \ge a$ or condition

$= P(a \le X < a+h \cap X \ge a) / P(X \ge a)$

$= P(a \le X < a+h) / P(X \ge a)$

$= [F_X(a+h) - F_X(a)] / 1 - F_X(a)$

Plugging Formula { $= \dfrac{1 - e^{-(a+h)/\beta} - (1 - e^{-a/\beta})}{e^{-a/\beta}}$

$= [1 - e^{-h/\beta}]$ Ans

↳ $P(0 < X < h)$

OR If you waited for a bus for 10 hours (given) what is the prob that bus will arrive in next 10 minutes

* its the probability that bus arrives in just 10 minutes (any 10 minutes)

↳ hence memoryless.

Note the memoryless property doesn't assert that

(*) $P(a \le X < a+h) = P(0 < X \le h)$

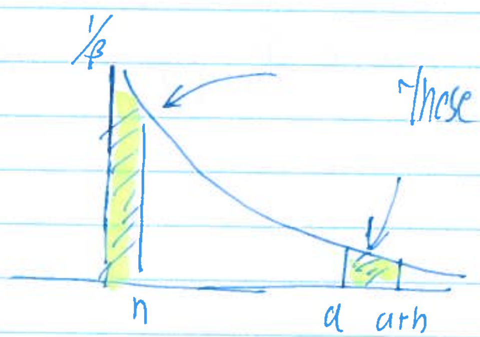
OR if $X \ge a$ the prob that X will be b/w a+h and a, is same as prob that X is b/w 0 & h.

There is no memory of how long you waited to observe the outcome X.

The Given component of condition disappears.

Now it can be shown that the exp distribution is the Unique Continuous distribution with such a property.

(*)



These two are not the same.