MATH 323: DAY 15
OCTOBER 24 TUE

COVERING:

- Continuing Mean and Variance
    - $E(X^2) \neq E(X)^2$ except in certain conditions
    - Example of Variance and Mean

Mean and Variance for Named Distributions:

Each of these have their own derivations:

- Discrete Uniform.
- Bernolli RV.
- Binomial RV.
- Poisson
- Geometric

Math 323

Oct 24^{th}, 2017

Assignment:

Last time $\underline{Var(X) = E(X^2) - \mu^2}$, $E(X^2) \neq (E(X))^2$

Except under certain conditions, as we shall see.

Eg) Suppose that a botanist knows that the leaf length of a certain plant has a prob. distri. (in cm)

Let X be the (pre-observation) r.v that describes the leaf length. We are told

$P[X=1.2] = .3$, $P[X=2.2] = .20$, $P[X=3.7] = .5$

Find the mean and standard deviation of the leaf length.

Sol: Let $x_1 = 1.2$, $x_2 = 2.2$ and $x_3 = 3.7$

$$\therefore E(X) = \sum_{i=1}^{3} x_i \, P[X = x_i] = 1.2 \cdot .3 + 2.2 \cdot .20 + 3.7 \cdot 0.5 = \boxed{2.65}$$

Next $\sigma^2 = E(X^2) - \mu^2 = \sum_{i=1}^{3} x_i^2 \, P[X = x_i] - (2.65)^2 = (1.2)^2 \cdot .3 + (2.2)^2 \cdot .20 + (3.7)^2 \cdot .5 - (2.65)^2$
$\boxed{= 1.2225}$

$$\therefore \sigma = \sqrt{\sigma^2} = \sqrt{1.2225} = 1.1057$$

The Means and Variances of some of the named distributions:

1. Discrete Uniform: $P(X = a_i) = \frac{1}{N}$ $\forall i = 1 \dots N$

We have $E(X) = \sum_{i=1}^{N} a_i \frac{1}{N} = \bar{a}$ (average of the values of X)

it is easy to see $Var(X) = \frac{1}{N} \sum_{i=1}^{N} (a_i - \bar{a})^2$ (check this)

2. Bernoulli r.v.: $E(X) = 1\cdot p + 0\cdot(1-p) = p$ $\qquad Var(X) = E(X^2) - p^2 = 1^2 p + 0(1-p) - p^2$
$= p(1-p)$

3. Binomial r.v.: $X \sim Bin(n,p)$

* $E(X) = np$

Proof: $E(X) = \sum_{x=1}^{n} x\binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=1}^{n} x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$

$$= \sum_{x=1}^{n} \frac{n(n-1)!}{(x-1)!(n-1-(x-1))!} p^x p^{x-1} (1-p)^{n-1-(x-1)}$$

$$= np \sum_{x-1=0}^{n-1} \binom{n-1}{x-1} p^{x-1} (1-p)^{n-1-(x-1)}$$

(let $y = x-1 \Rightarrow$) $= np \underbrace{\sum_{y=0}^{n-1} \binom{n-1}{y} p^y (1-p)^{n-1-y}}_{\overset{\shortparallel}{1} \text{ since it is sum of } n-1 \text{ : } Bin(n-1,p) \text{ prob.}} \boxed{= np = E(X)}$

To find σ^2, use $\sigma^2 = E(X^2) - \mu^2 = E(X^2) - (np)^2$

Recall: $E(X^2) = \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x} \xrightarrow{\text{first}} E[X(X-1)]$ *

We find $E(X^2)$ by first finding what it's called 1st factorial moment, $E(X(X-1))$.

We have $E[X(X-1)] = E(X^2) - E(X)$ $\quad \therefore E(X^2) = E[X(X-1)] + E(X)$

Now $E[X(X-1)] = \sum_{X=2}^{n} X(X-1)\binom{n}{X} p^X (1-p)^{n-X}$

$$= \sum_{x-2=0}^{n} \frac{n(n-1)(n-2)!\, p^{x-2} (1-p)^{n-2-(x-2)}}{(x-2)!\,\underbrace{(n-2-(x-2))!}_{(n-x)}}$$

$$= (n^2p^2 - np^2) \underbrace{\sum \text{stuff}}_{1 \text{ since sum of } n-2 \text{ } Bin(n-2,p) \text{ probs.}} = \boxed{n^2p^2 - np^2}$$

$E(X^2) = n^2p^2 - np^2 + np \qquad \therefore \sigma^2 = E(X^2) - n^2p^2 = n^2p^2 - np^2 + np - \overbrace{n^2p^2}^{\mu^2} = \boxed{np(1-p)}$

Poisson Distribution:

$$X = P_0(\lambda) \Rightarrow P[X=x] = \frac{\lambda^x e^{-\lambda}}{x!} \quad \forall x = 0,1,\ldots$$

$$E(X) = \sum_{x=1}^{\infty} \frac{x\lambda^x e^{-\lambda}}{x!} = e^{-\lambda}\lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

Let $y = x-1 \Rightarrow$ $\quad e^{-\lambda}\lambda \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} = e^{-\lambda}\lambda . e^{\lambda} = \lambda \,]$

ie the parameter λ is the mean of X.

To find $Var(X) = \sigma^2$, again we need to work with $E(X(X-1))$. Same idea as Binomial. *

We find $Var(X) = \sigma^2 = \lambda$ (The same as the mean)

Geometric r.v:

We have $P[X=x] = (1-p)^{x-1}p$ and $x = 1,2,\ldots$

$\boxed{\ast\ (1-p)^x p \quad \text{and } x = 0,1,\ldots \quad \#\ \text{failure before the } 1^{st} \text{ success}}$

So $E(X) = \sum_{x=1}^{\infty} x(1-p)^{x-1}p = p\sum_{x=1}^{\infty} x(1-p)^{x-1} = p\sum_{x=1}^{\infty} -\frac{d}{dp}(1-p)^{x-1}$

$$= -p\frac{d}{dp}\sum_{x=1}^{\infty}(1-p)^x = -p\frac{d}{dp}\frac{(1-p)}{1-(1-p)} = -p\frac{d}{dp}\left(\frac{1}{p}-1\right) = -p\cdot\left(-\frac{1}{p^2}\right) = \frac{1}{p}$$

* important *

Q) Does it make sense? yes it does

To find σ^2 once again we need to work with $E(X(X-1))$ Q) Why?

and differentiate twice; in the end we find $Var(X) = \frac{q}{p^2} = \frac{1-p}{p^2}$