AMS 513 Financial Derivatives and Stochastic Calculus Mar 18, 2018

Homework Assignment 1

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Compile Environment

Trinomial pricing code is built on Ubuntu 16.04 LTS(one of the most popular GNU/Linux distribution). The code can be complied by g++, a language-specific driver program of GCC(GNU Complier Collection). I strongly recommend you compile source code under the same system environment. This source code may not be compiled correctly by compiler of Visual Studio, Visual C++. If it is really difficult to install another operating system on one computer, Windows Subsystem for Linux(https://docs.microsoft.com/en-us/windows/wsl/about) is also acceptable. Please make sure run below bash commands in Terminal before compile source code.

Listing 1: bash commands

```
sudo apt update
sudo apt upgrade
sudo apt-get install -y --no-install-recommends \
build-essential \
gcc -7 \
g++ -7 \
sudo apt-get clean && \
sudo  rm -rf /var/lib/apt/lists/* /tmp/* /var/tmp/*
```

Trinomial Tree Model

The stochastic differential equation for risk-neutral geometric Brownian motion model of an asset price paying a continuous yield is

$$dS = (r - \delta)Sdt + \sigma Sdz$$

As in the case the binomial model, it is convenient to use $x = ln(S)$

$$dx = vdt + \sigma dz, v = r - \delta - \frac{1}{2}\sigma^2$$

For a trinomial model of asset price, over a small time interval Δt, the asset price can go up by Δx, stay same or go down by Δx, with probabilities p_u, p_m, p_d.

The drift and volatility parameters of the continuous time process are captured by

$\Delta x, p_u, p_m and p_d$. It is a good choice the $\Delta x = \sigma\sqrt{3\Delta t}$. The relationship of parameters of the continuous, time process is obtained by equating the mean and variance over time interval Δt and requiring that the probabilities sum to one:

$$E[\Delta x] = p_u\Delta x + p_m * 0 + p_d\Delta x = v\Delta t$$
$$E[\Delta x^2] = p_u\Delta x^2 + p_m * 0 + p_d\Delta x^2 = \sigma^2\Delta t + v^2\Delta t^2$$
$$p_u + p_m + p_d = 1$$

Then we solve these equations and get

$$p_u = \frac{1}{2}(\frac{\sigma^2\Delta t + v^2\Delta t^2}{\Delta x^2} + \frac{v\Delta t}{\Delta x})$$
$$p_m = 1 - \frac{\sigma^2\Delta t + v^2\Delta t^2}{\Delta x^2}$$
$$p_d = \frac{1}{2}(\frac{\sigma^2\Delta t + v^2\Delta t^2}{\Delta x^2} + \frac{v\Delta t}{\Delta x})$$

The result on time step i and level of asset price j is

$$C_{i,j} = e^{-r\Delta t}(p_u C_{i+1,j+1} + pmC_{i+1,j} + p_d C_{i+1,j_1})$$

The Explicit Finite Difference Method

The idea of finite difference methods(Schwartz, 1977) is to simply the PDE by replacing the partial differentials with finite differences. The explicit finite difference method to the Black-Scholes PDE

$$-\frac{\partial C}{\partial t} = \frac{1}{2}S^2\sigma^2\frac{\partial^2 C}{\partial S^2} + (r-\delta)S\frac{\partial C}{\partial S} - rC$$

As $x = ln(S)$, we get

$$-\frac{\partial C}{\partial t} = \frac{1}{2}\sigma^2\frac{\partial^2 C}{\partial x^2} + v\frac{\partial C}{\partial x} - rC$$

This equation is a PDE with constant coefficients, the do not depend on x or t, which make the application of finite difference methods much easier.

When implementing the finite differences methods, we assume that time and space can be divided up to discrete intervals (Δt and Δx), this is the finite difference grid or lattice.

We use a forward difference for $\frac{\partial C}{\partial t}$ and central differences for $\frac{\partial^2 C}{\partial x^2}$s to obtain

$$-\frac{C_{i+1,j}-C_{i,j}}{\Delta t}=\frac{1}{2}\sigma^2\frac{C_{i+1,j+1}-2C_{i+1,j}+C_{i+1,j-1}}{\Delta x^2}+v\frac{C_{i+1,j+1}-C_{i+1,j-1}}{2\Delta x}-rC_{i,j}$$

which can be written as

$$\begin{aligned}C_{i,j}&=p_uC_{i+1,j+1}+p_mC_{i+1,j}+p_dC_{i+1,j_1}\\p_u&=\frac{1}{2}\Delta t(\frac{\sigma^2}{\Delta x^2}+\frac{v}{\Delta x})\\p_m&=1-\Delta t\frac{\sigma^2}{\Delta x^2}\\p_d&=\frac{1}{2}(\frac{\sigma^2}{\Delta x^2}+\frac{v}{\Delta x})\end{aligned}$$

If I don't mention on purpose, these figures are plotted by R x64 3.43. **Convergence** Denote the current asset price S is 100 and volatility σ is 20%, the continuously compounded interest rate r is 6% per year. The asset pays a continuous dividend yield *div* of 3% per year. And the strike price K of 4 kinds of options is 100 and the maturity T is 1 year.

According to Black-Scholes Formula, we get the price of European Call Option is 9.1352 and the price European Put Option is of 6.2671.

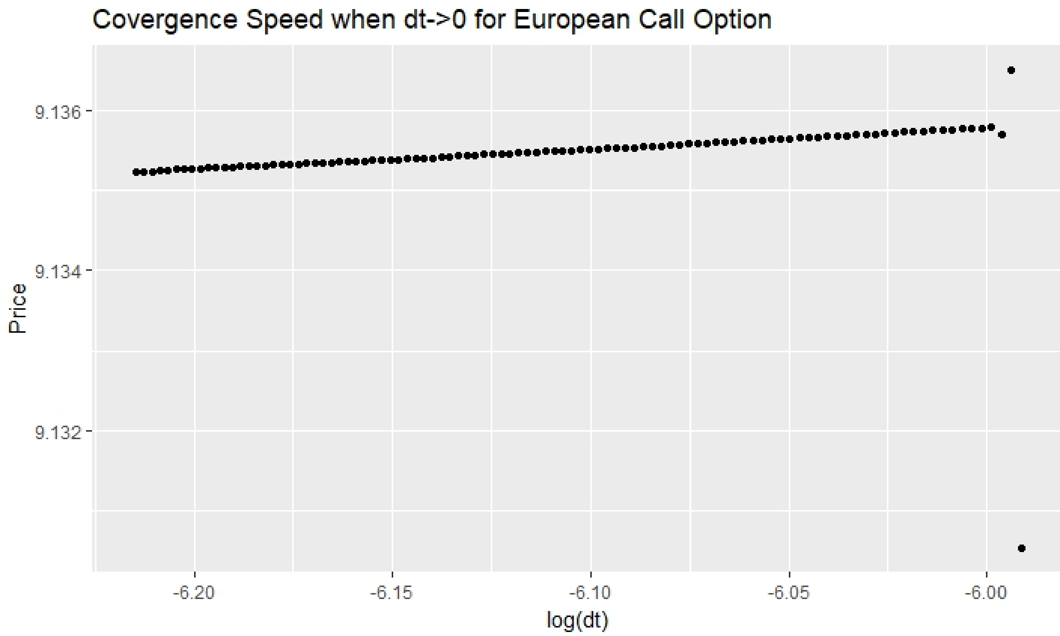
As for $\Delta t \to 0$, we can get $N \to \infty$ because $\Delta t = \frac{T}{N}$. We plot the figure from $N = 5$ to $N = 500$.

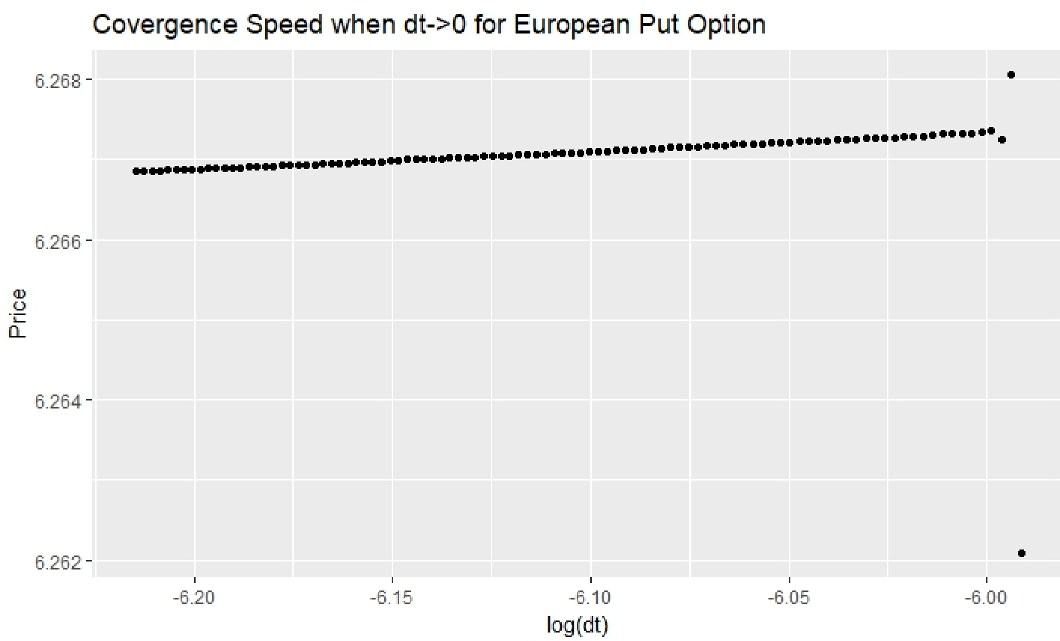
The price of European Call Option is 9.13523, which is converge to the result of Black-Scholes Formula, 9.1352.

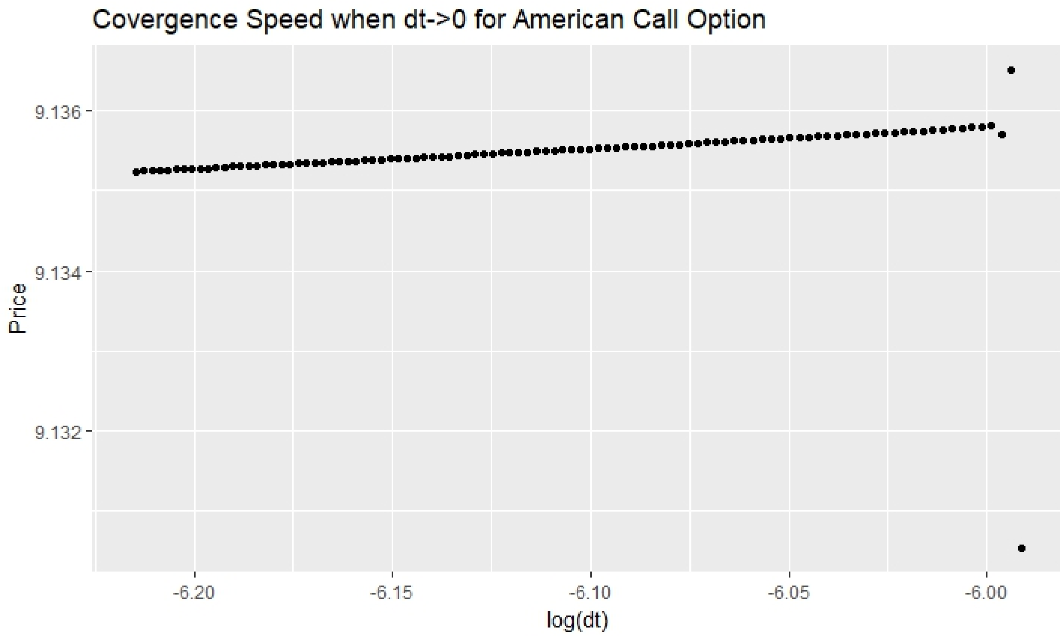
The price of European Call Option is 6.26685, which is converge to the result of Black-Scholes Formula, 6.2671.

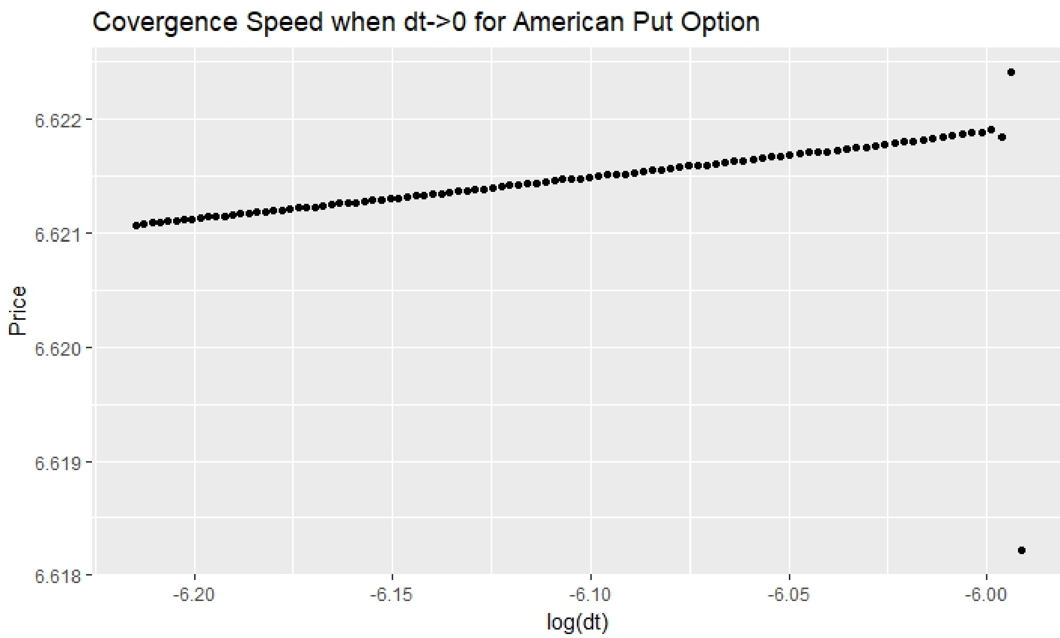
The price of American Call Option is 9.13524. And the price of American Put Option is 6.62107.

We plot the figure from $N = 5$ to $N = 500$.The convergence speed of these 4 options is as below.


Covergence Speed when dt->0 for European Call Option
9.136
9.134
9.132
Price
-6.20
-6.15
-6.10
-6.05
-6.00
log(dt)


Covergence Speed when dt->0 for European Put Option
6.268
6.266
6.264
6.262
Price
-6.20
-6.15
-6.10
-6.05
-6.00
log(dt)





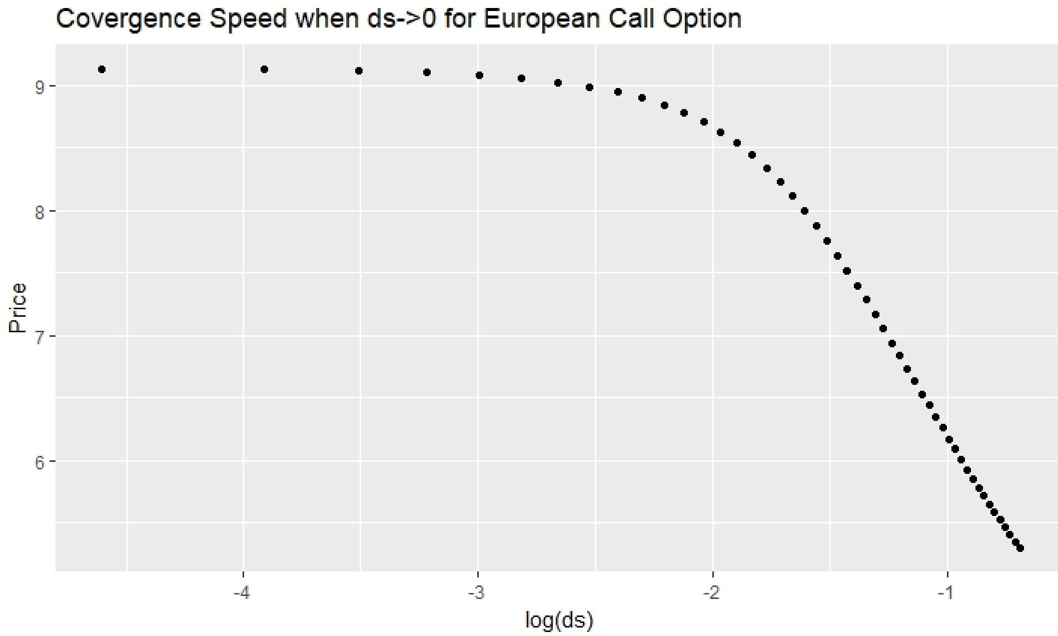
As for $\Delta s \to 0$, The price of European Call Option is 9.13523, which is converge to the result of Black-Scholes Formula, 9.1352.

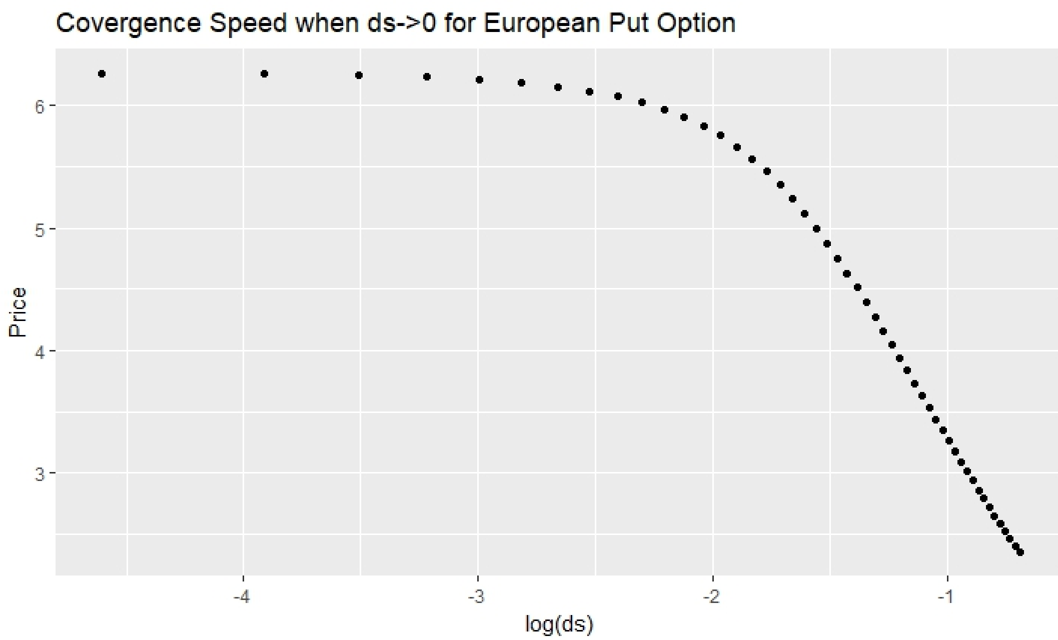
The price of European Call Option is 6.26685, which is converge to the result of Black-Scholes Formula, 6.2671.

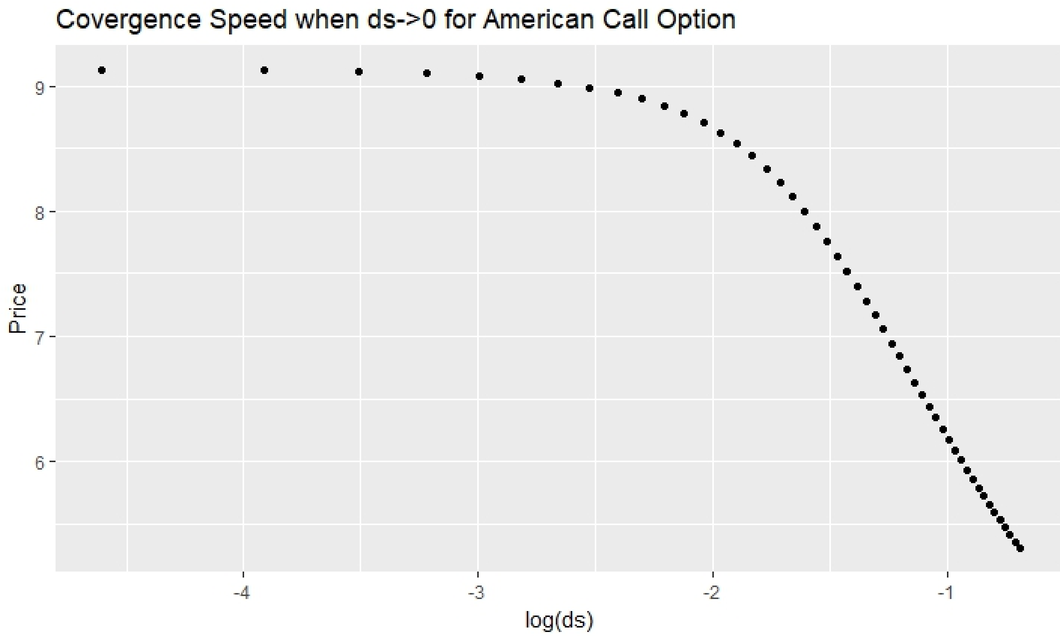
The price of American Call Option is 9.13524. And the price of American Put Option is

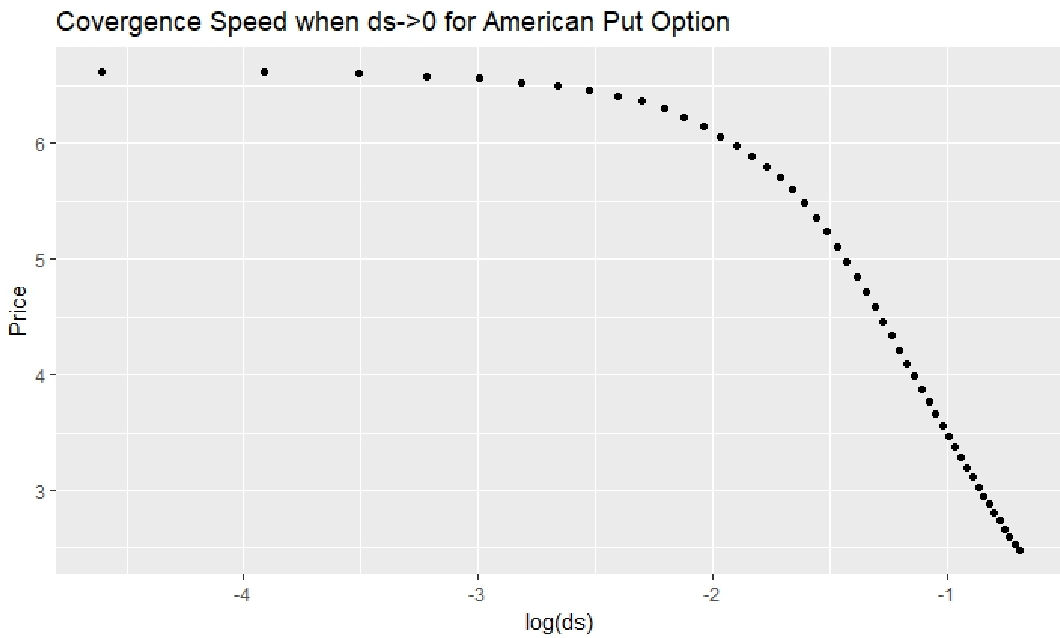
6.62107.

We plot the figure from $ds = 0.5$ to $ds = 0.01$.The convergence speed of these 4 options is as below.





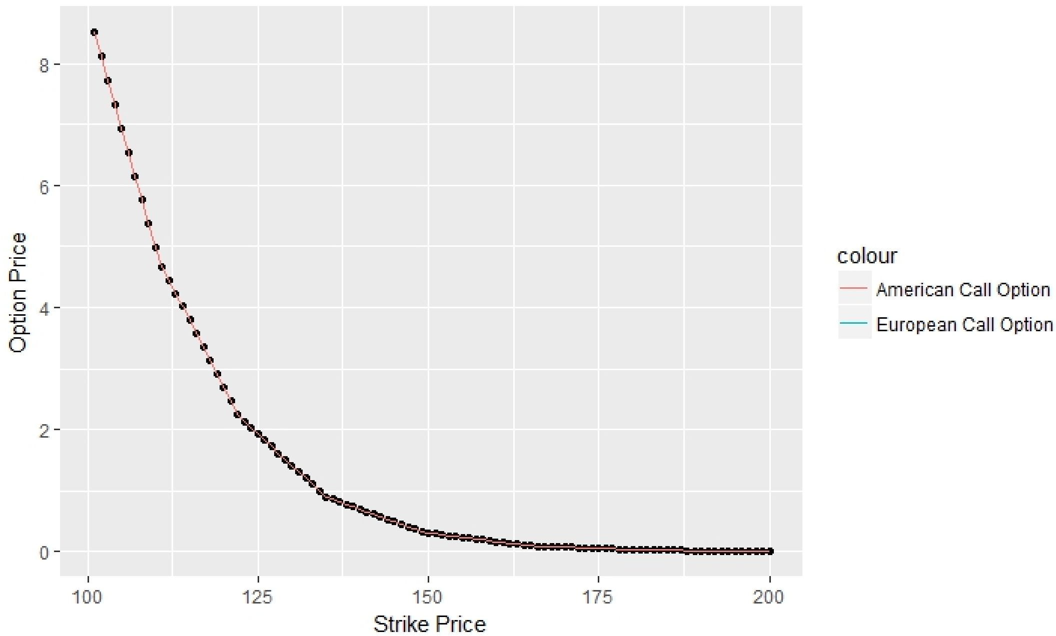


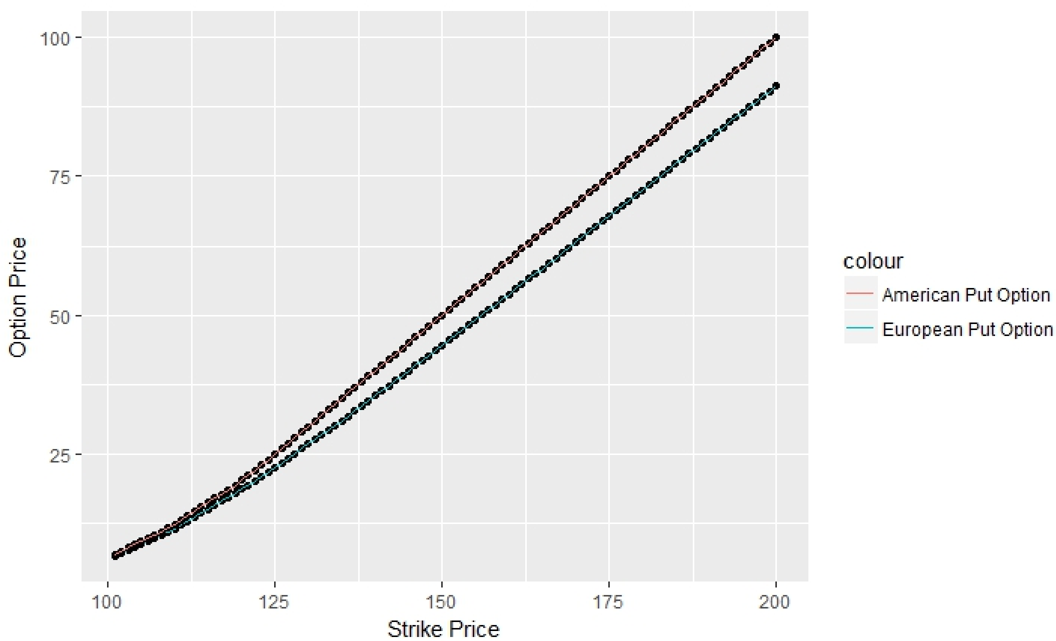


Europeans $\leqslant$ Americans

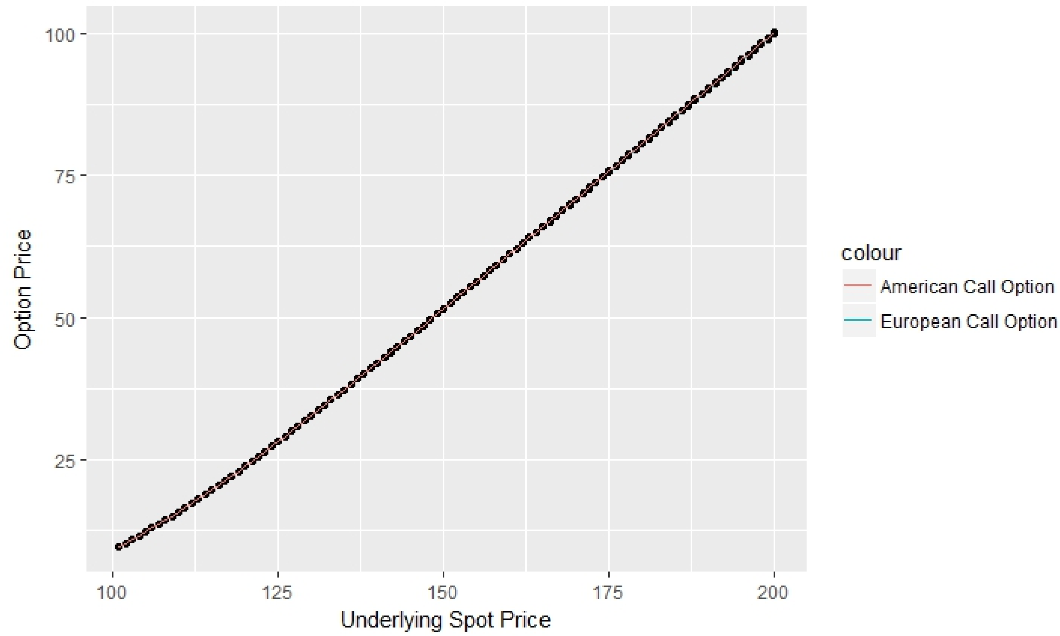
As for the relationship of European option and American option, we can plot their prices in different condition in below figures. **Note:** There seems like only one line in some figures, but these two groups of data are too close actually. The line of American options overlapped the European one. I tried to add some random jitter into those figures by ggplot2 package of R, however, it caused that some points of European options price are higher than American ones. In face, the vertical coordinates of all American option price are no less than the Europeans. I will attached my raw data to the email.

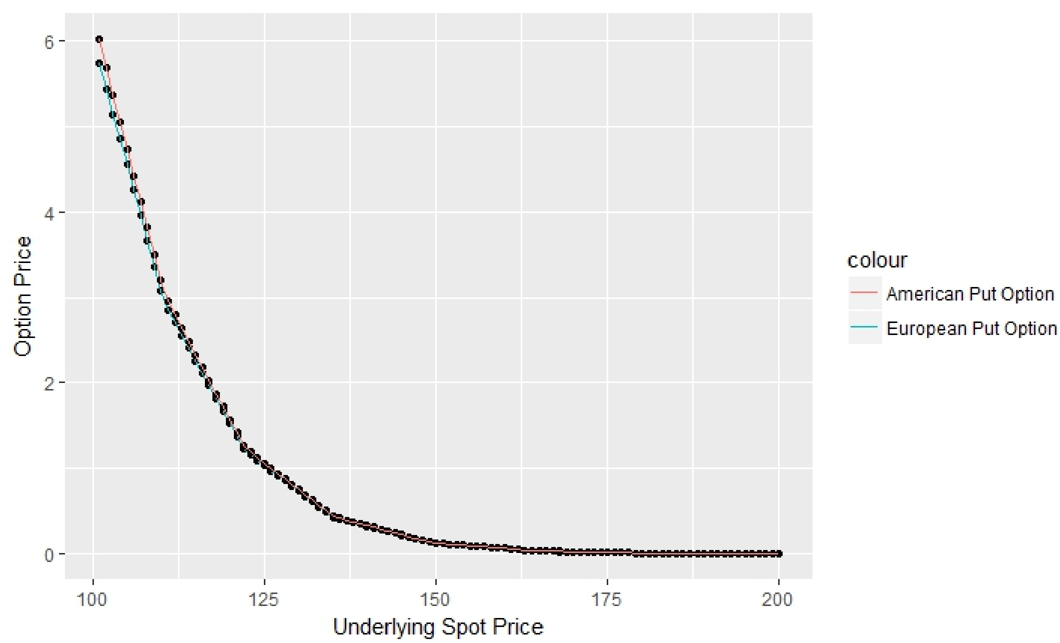
When the strike price of option changed, we get



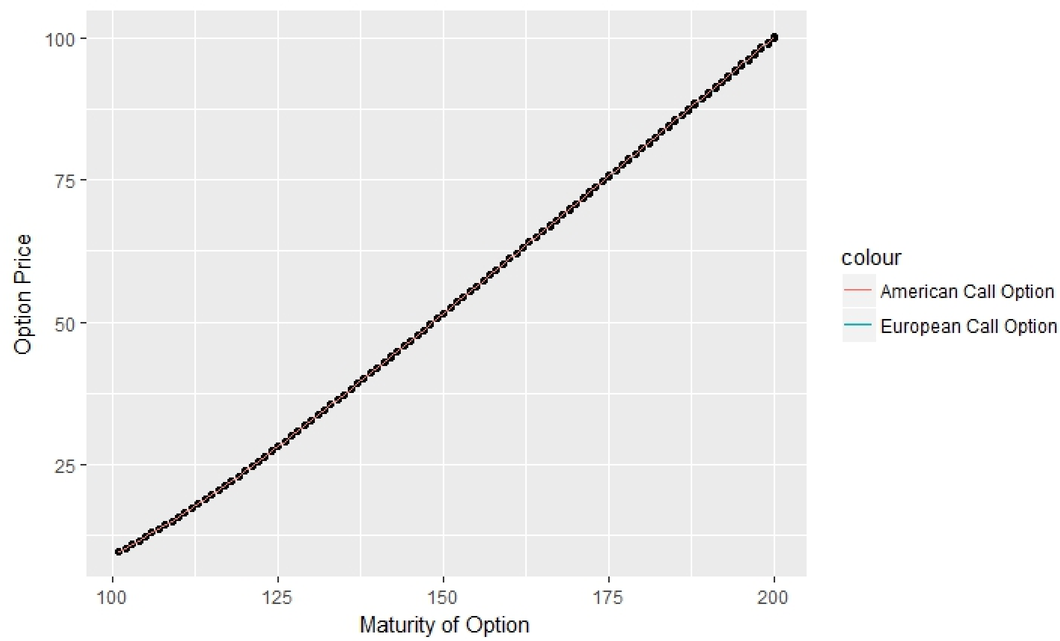


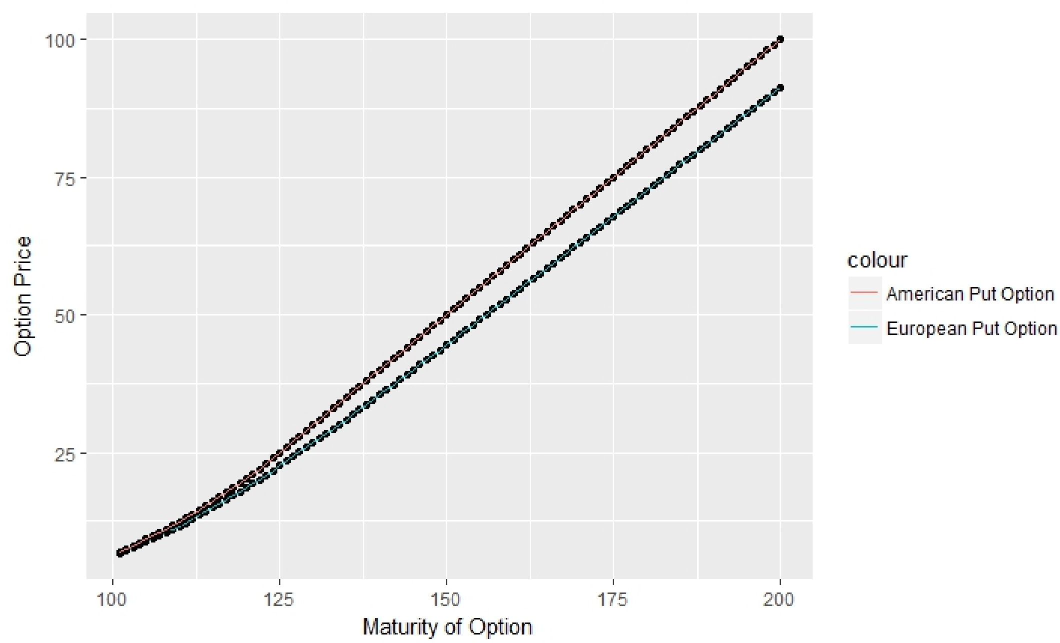
From these 2 figures, we conclude that price of American option are always higher than Europeans when their strike price changed.
When the spot price of underlying asset changed, we get



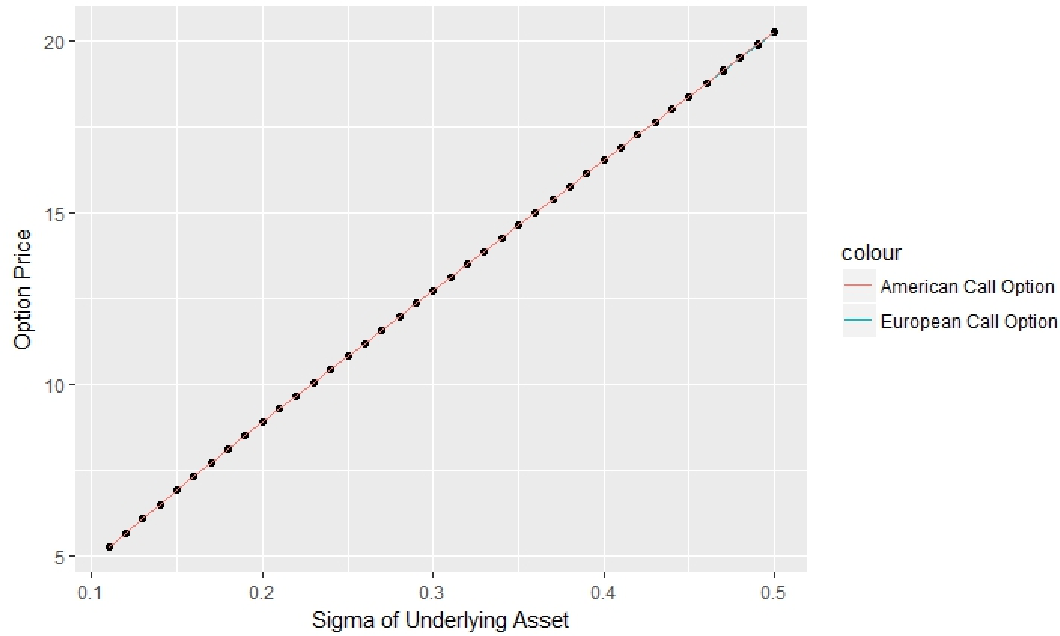


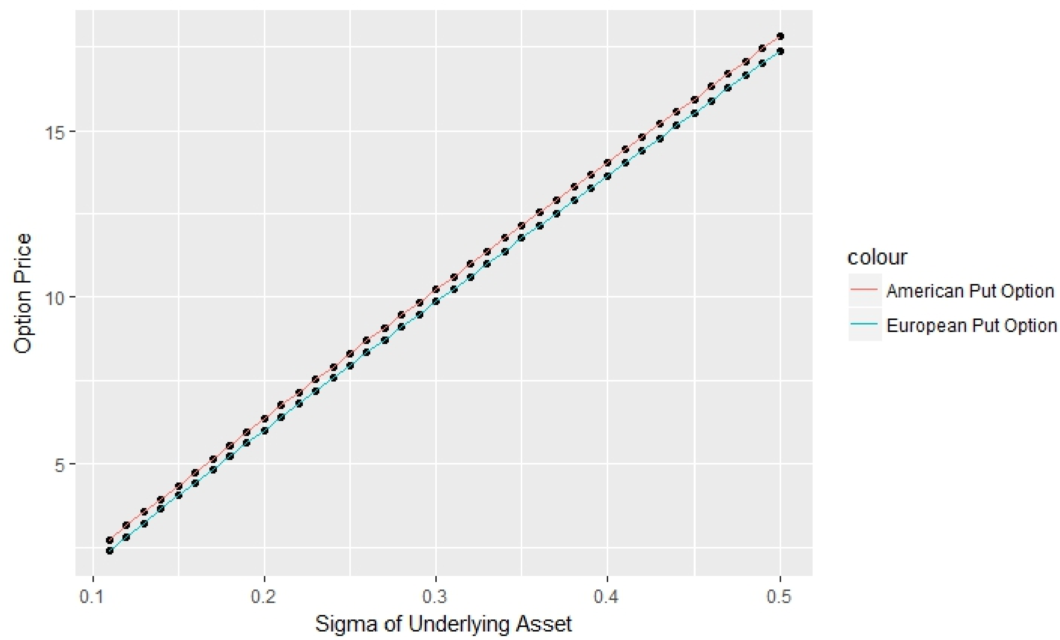
From these 2 figures, we conclude that price of American option are always higher than Europeans when price of their underlying asset changed.
When the maturity of options changed, we get



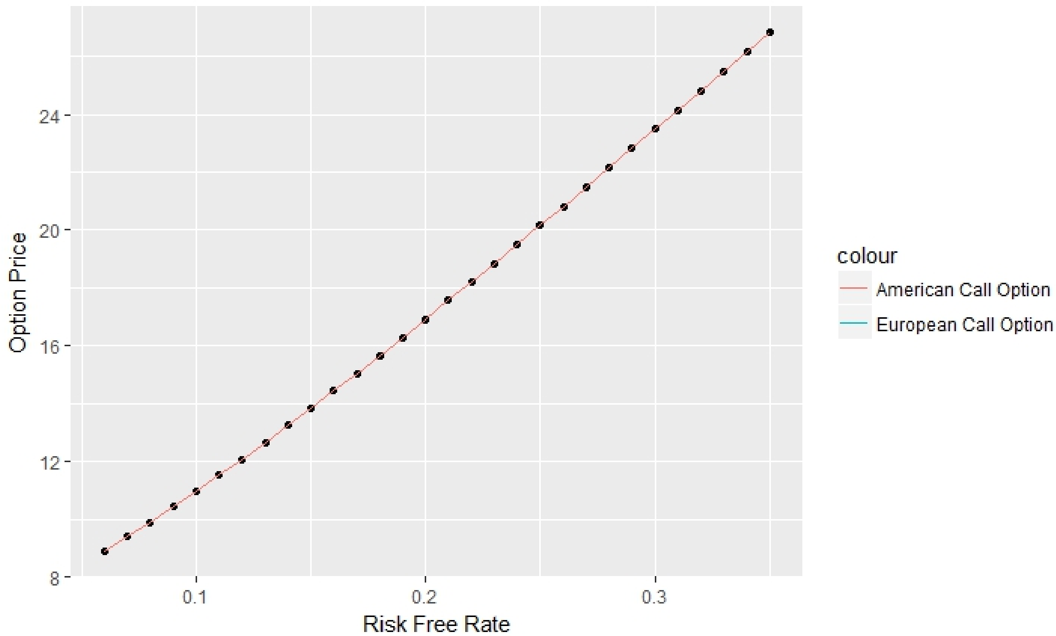


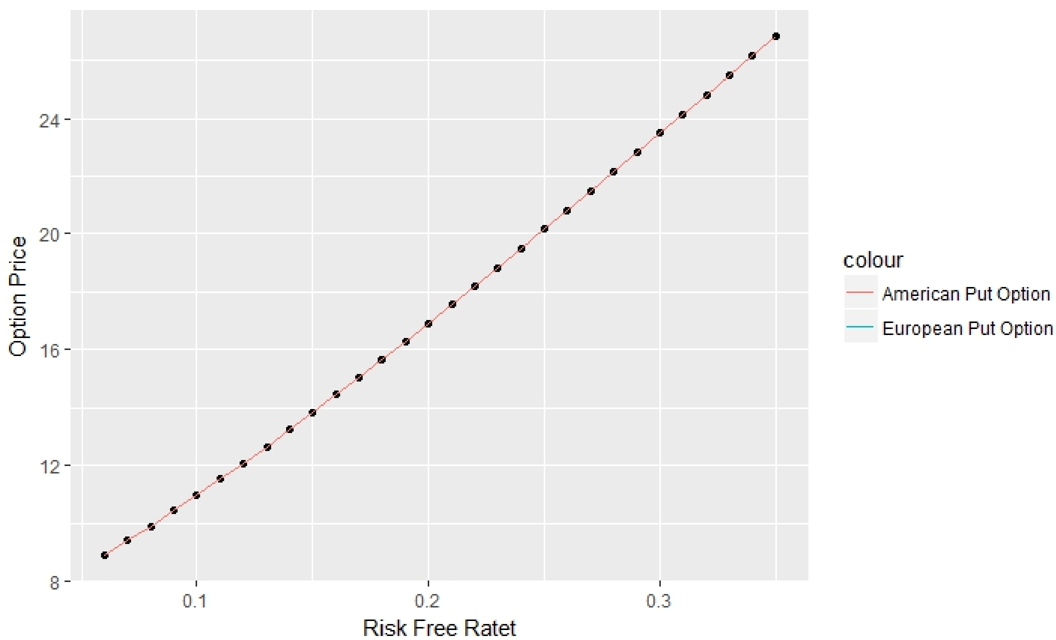
From these 2 figures, we conclude that price of American option are always higher than Europeans when price of their maturity changed.
When the σ of underlying asset changed, we get



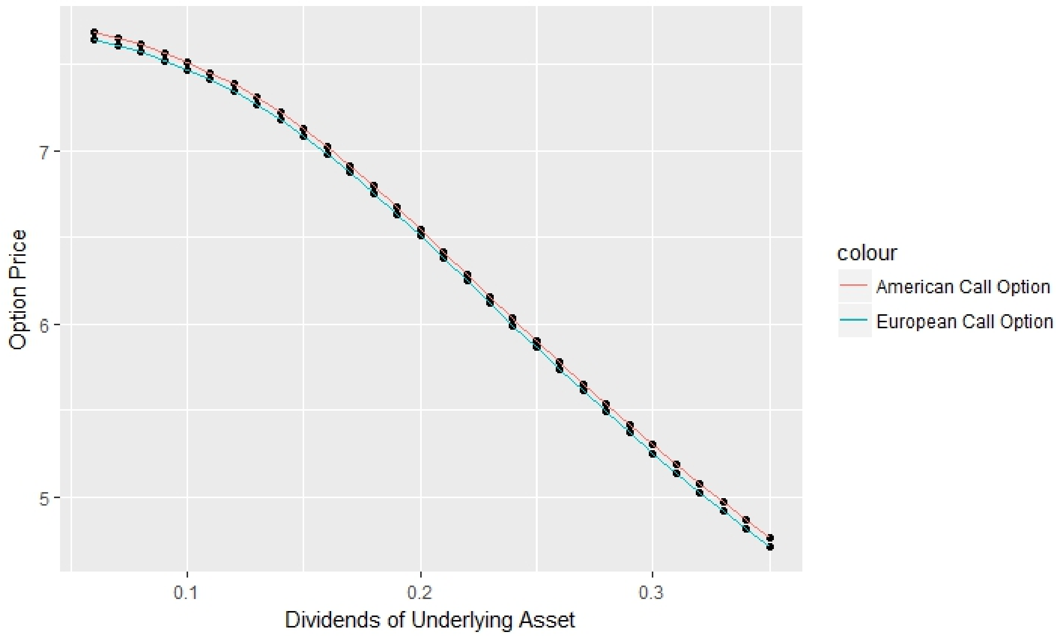


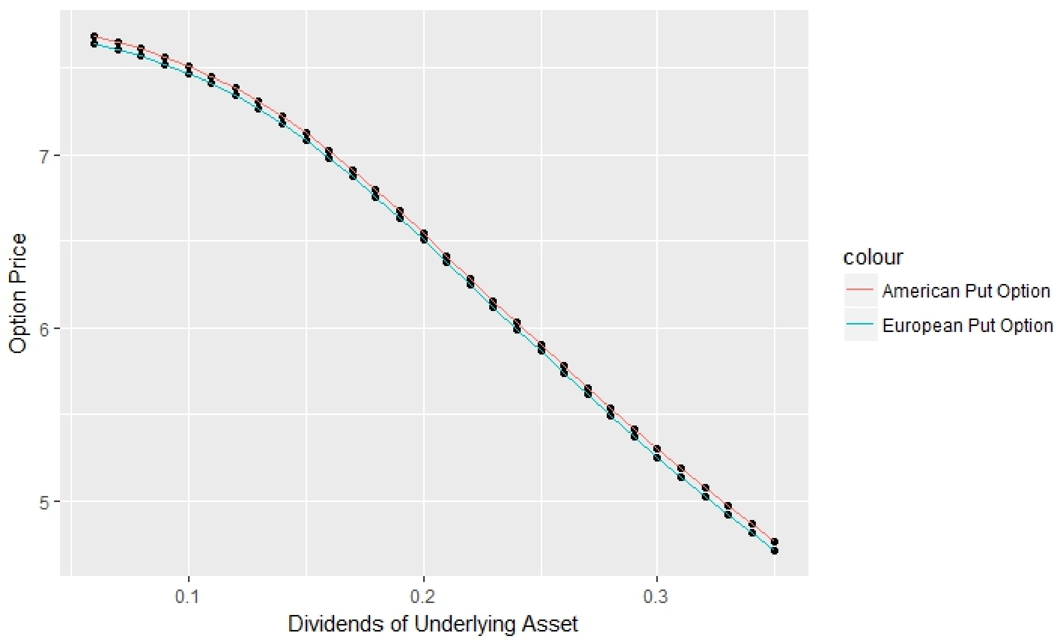
From these 2 figures, we conclude that price of American option are always higher than Europeans when σ of their underlying asset changed.
When risk-free rate changed, we get





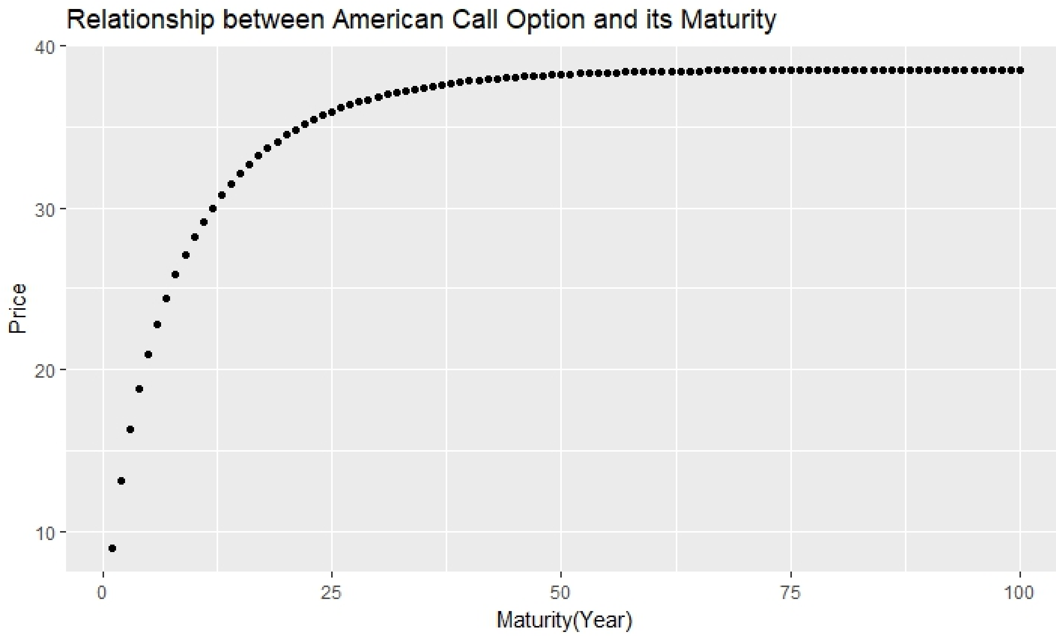
From these 2 figures, we conclude that price of American option are always higher than Europeans when risk-free rate changed.
When dividends rate of their underlying changed, we get

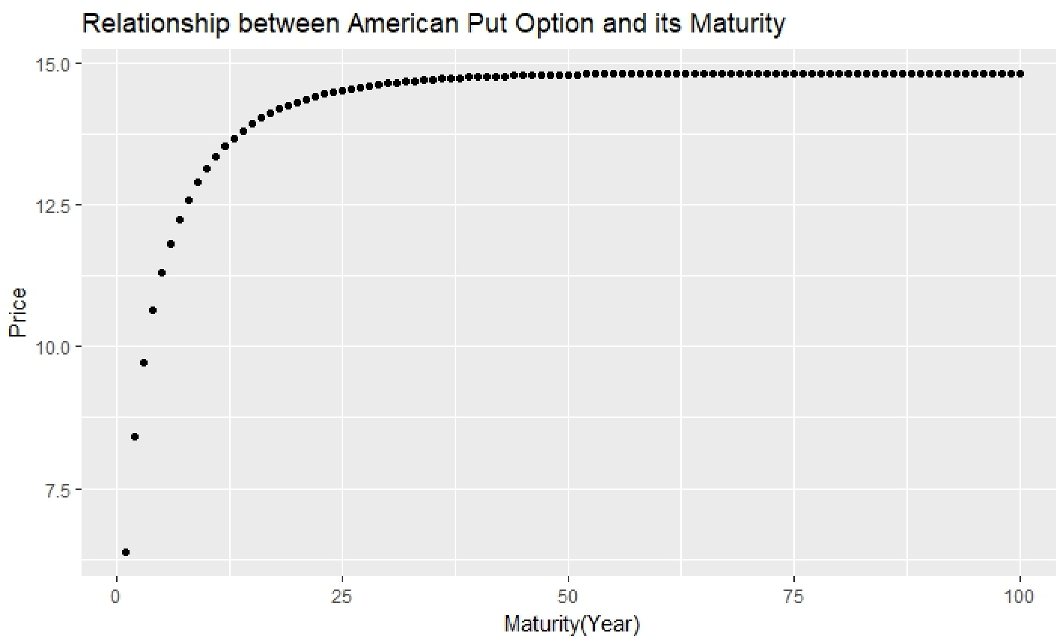




From these 2 figures, we conclude that price of American option are always higher than Europeans when dividends rate of their underlying changed.

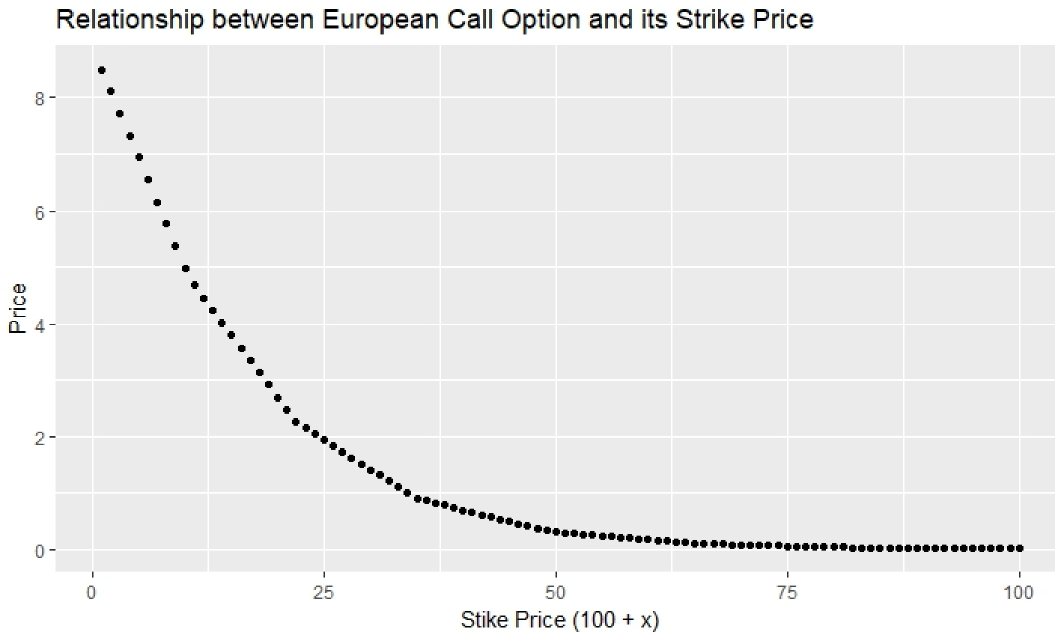
American Options and its Maturity

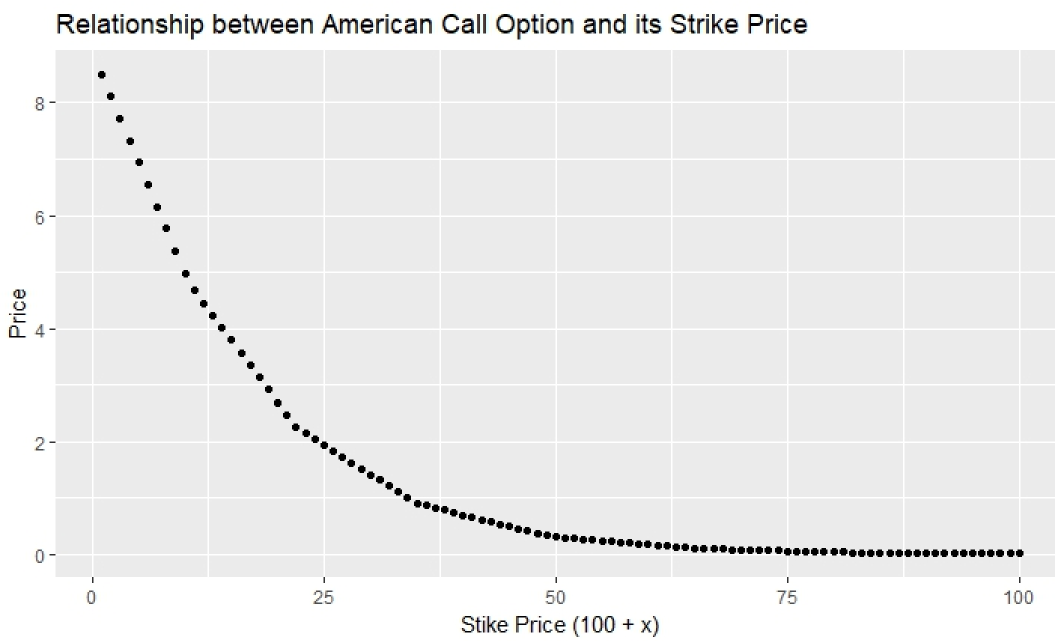




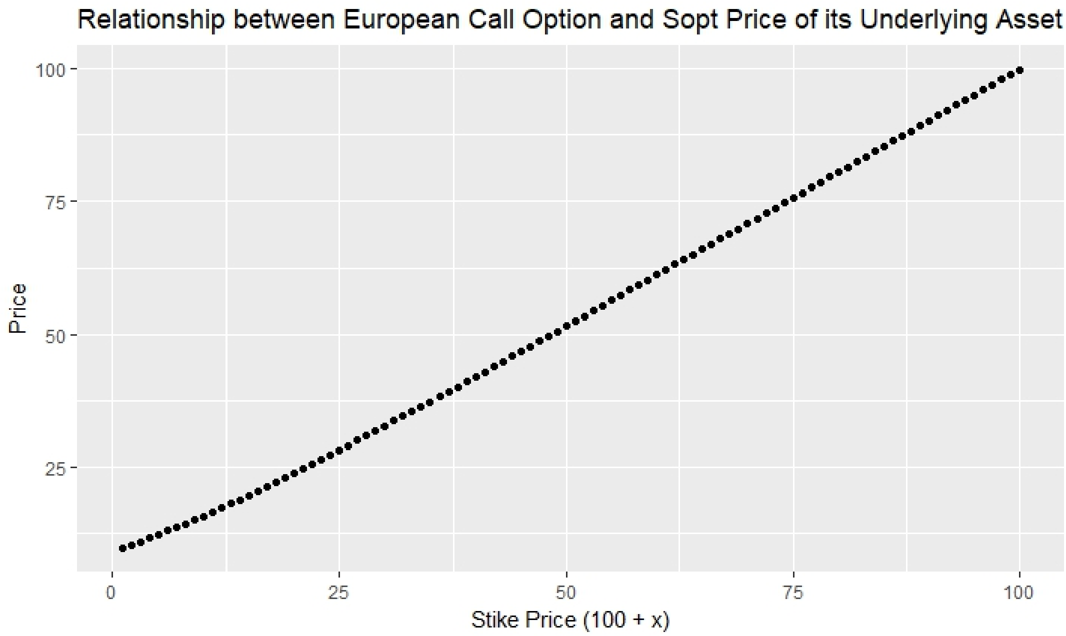
According to these two figures, we can get the price of American Options will increase if their maturity increase form 1 year to 100 years. Thus, the θ of American Options is larger than 0.

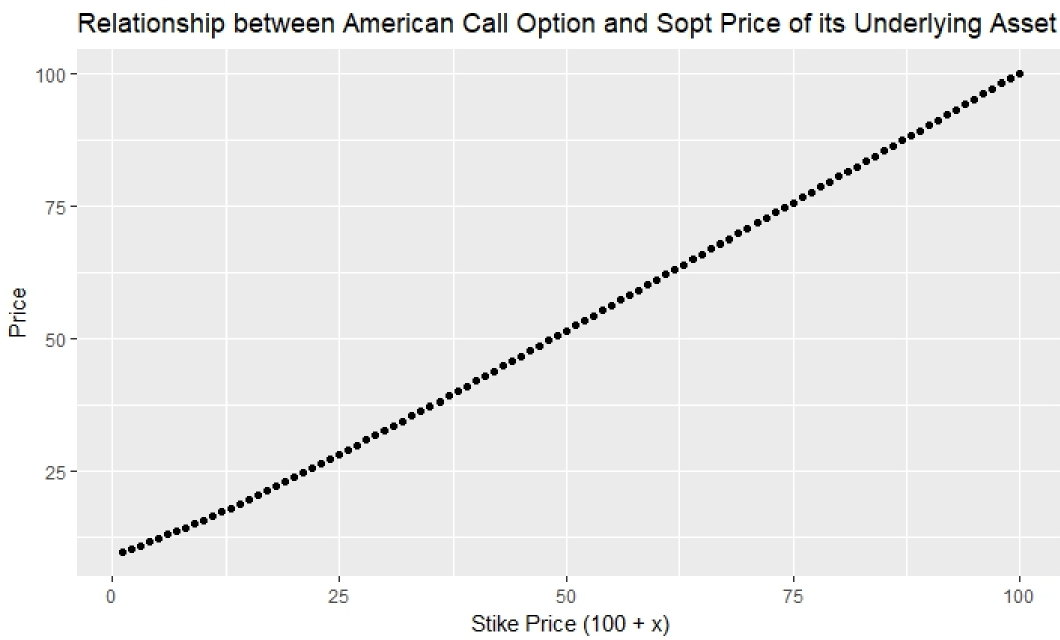
Options with $\delta > 0$





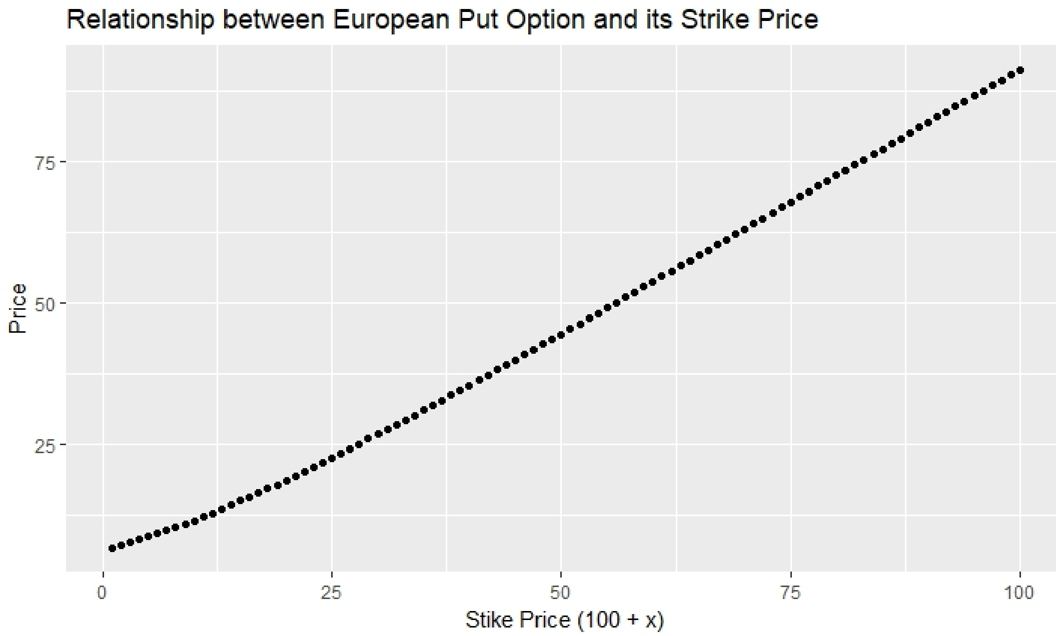
According to these two figures, we can get the price of 2 types of Call Options will decrease if their strike price increase form 100.0 to 200.0.



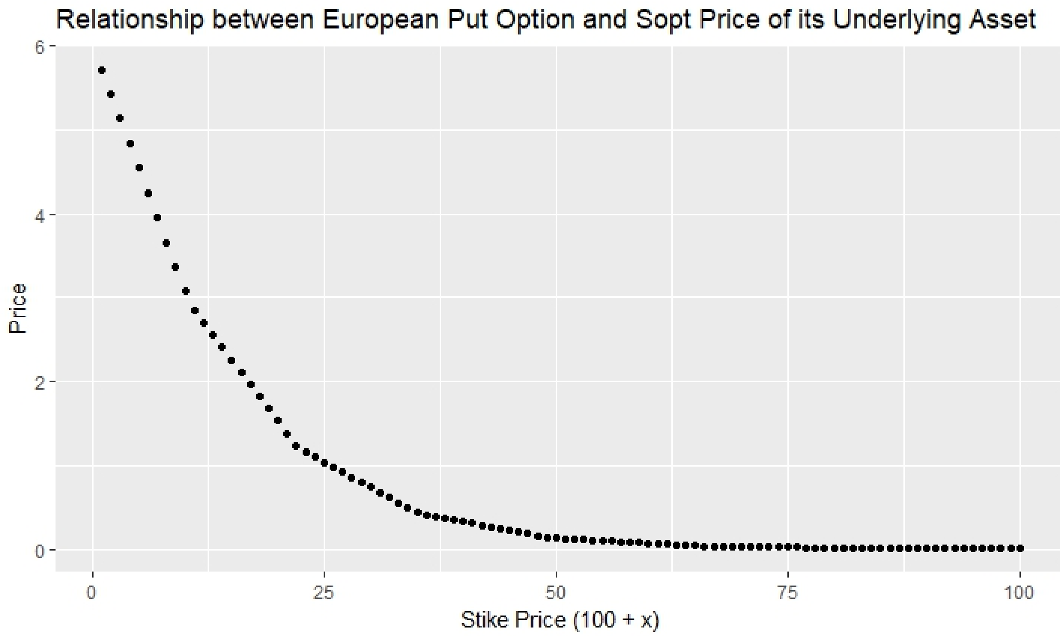


According to these two figures, we can get the price of 2 types of Call Options will increase if spot price of their underlying asset increase form 100.0 to 200.0.

Options with $\delta < 0$

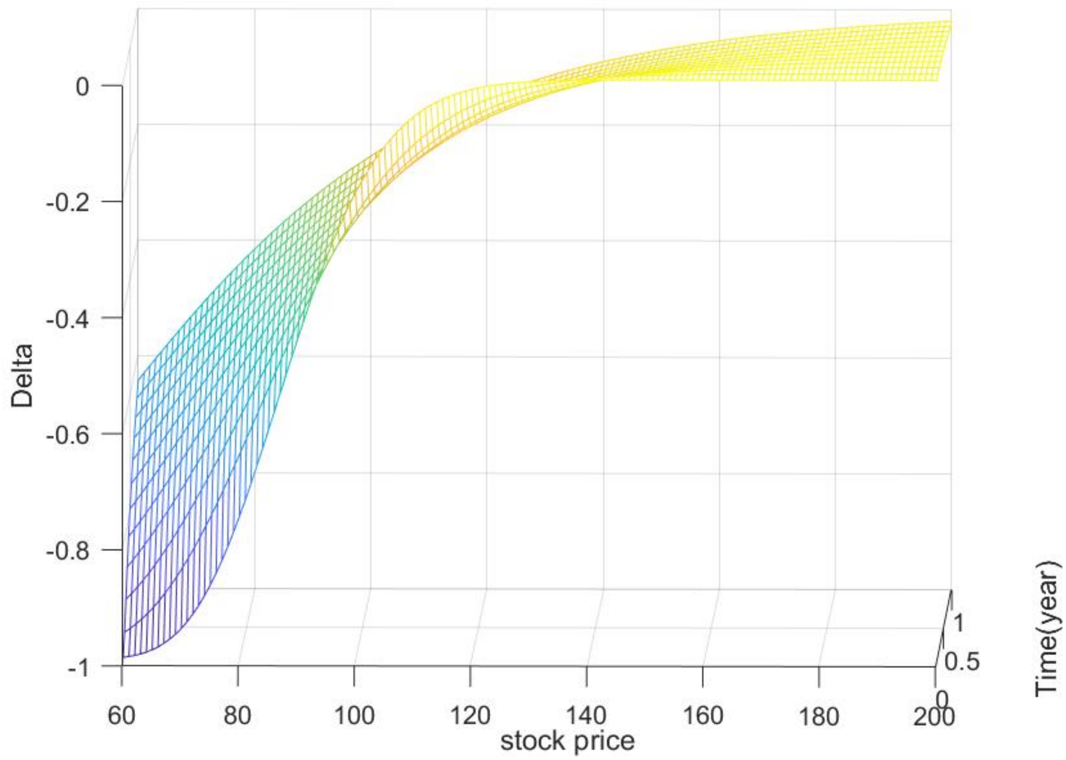


According to these two figures, we can get the price of 2 types of Put Options will increase if their strike price increase form 100.0 to 200.0.

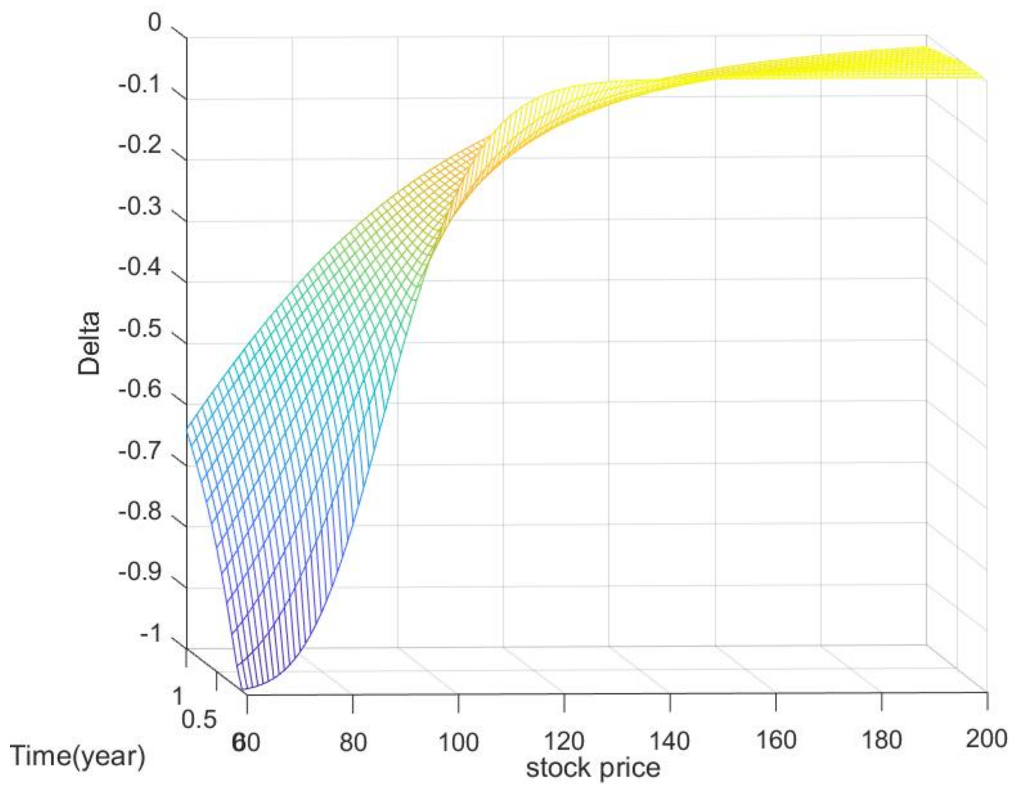


According to these two figures, we can get the price of 2 types of Call Options will decrease if spot price of their underlying asset increase form 100.0 to 200.0.

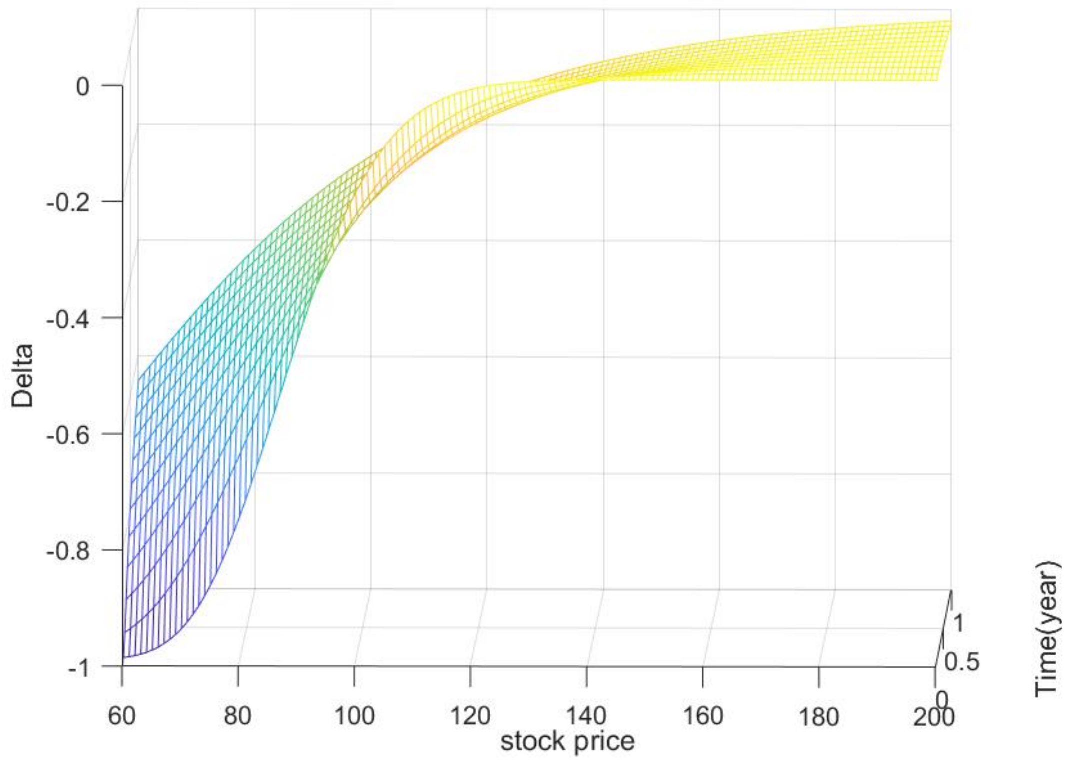
Γ of Options



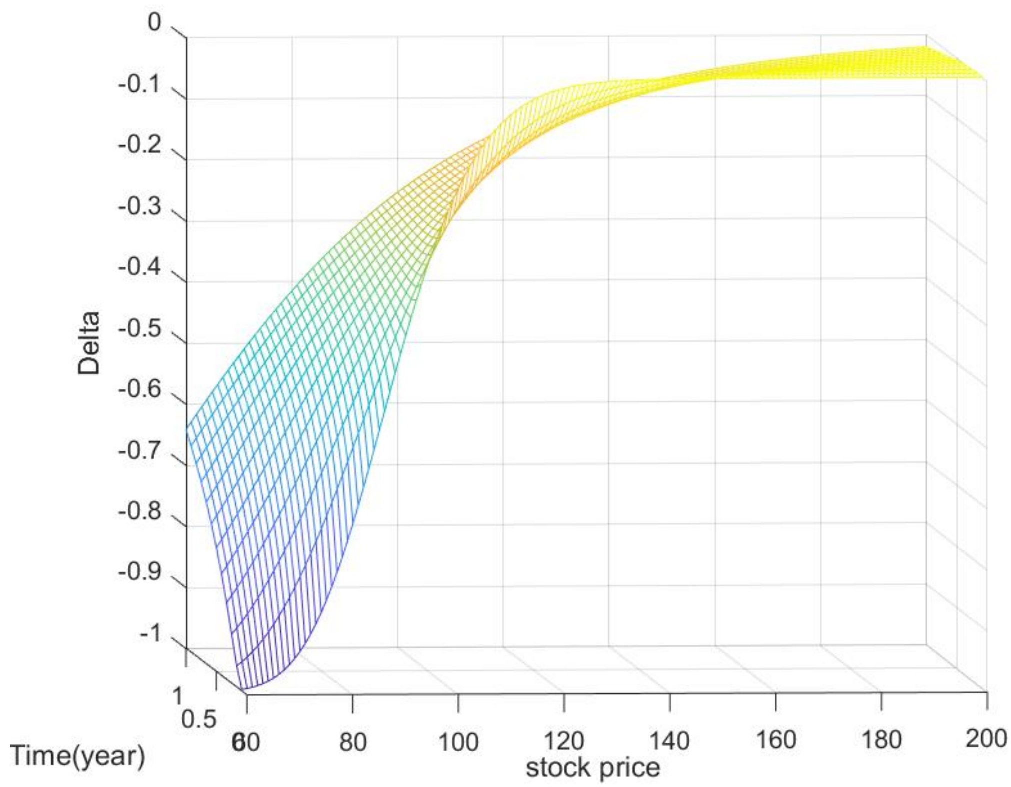
Delta
0
-0.2
-0.4
-0.6
-0.8
-1
60
80
100
120
140
160
180
200
stock price
Time(year)
0
0.5
1



Delta
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
-1
60
80
100
120
140
160
180
200
stock price
Time(year)
1
0.5
0



Delta
0
-0.2
-0.4
-0.6
-0.8
-1
60
80
100
120
140
160
180
200
stock price
Time(year)
0
0.5
1



Delta
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
-1
60
80
100
120
140
160
180
200
stock price
Time(year)
0
0.5
1

These 4 3-D figures were plotted by MATLAB R2017b, the x-axis is underlying stock price, the y -axis is maturity and the z-axis is Delta of options. These 4 figures are European Call Option, European Put Option, American Call Option and American Put Option respectively. The strike price is 90.0 maturity is 3 month, risk-free rate is 10 and the volatility of underlying asset is 50%.

From these figures, we can get the δ will increase if the spot price of underlying asset. Thus, $\frac{\partial \delta}{\partial S} > 0$. As $\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\partial \delta}{\partial S}$. Therefore, we can get $\Gamma > 0$