Homework 1 (CS 401)
Deadline: **M. Feb. 7, 11:59 pm CST**

1- (30 pts)

Consider ***m*** hospitals, each with a certain number of available positions for hiring residents. There are ***n*** medical students graduating in a given year, each interested in joining one of the hospitals.

Assume each hospital has a ranking of the students in order of preference, and each student has a ranking of the hospitals in order of preference. Also assume there are more students graduating than the available slots in the ***m*** hospitals.

The goal is to assign each student to at most one hospital, in such a way that all available positions in all hospitals are filled. (Since there is a surplus of students, there will be some students who do not get assigned to any hospital).

We say that an assignment of students to hospitals is stable if neither of the following situations arises.

- First type of instability: There are students ***s*** and ***s'*** and a hospital ***h*** so that:
  - ***s*** is assigned to ***h***, and
  - ***s'*** is assigned to no hospital, and
  - ***h*** prefers ***s'*** to ***s***.
- Second type of instability: There are students ***s*** and ***s'***, and hospitals ***h*** and ***h***, so that
  - ***s*** is assigned to ***h***, and
  - ***s'*** is assigned to ***h'***, and
  - ***h*** prefers ***s'*** to ***s***, and
  - ***s'*** prefers ***h*** to ***h'***.

So, we basically have the stable marriage problem, except that (i) hospitals generally want more than one resident, and (ii) there is a surplus of medical students.

Show that there is always a stable assignment of students to hospitals and give an efficient algorithm to find one.

2- (40 pts – 10 pts each)

i- Show that for any $b > 0$, $(n + a)^b = \theta(n^b)$.
ii-Solve the recurrence $T(n) = 2T(\sqrt{n}) + \log n$. Your solution should be asymptotically tight. Hint: defining the variable m such that $n = 2^m$, set $S(m) = T(2^m)$.
iii- Without using the master theorem, derive the complexity of $T(n) = T\left(\frac{n}{2}\right) + \log n$
iv- Using the master theorem, derive the complexity of $T(n) = 7T\left(\frac{n}{3}\right) + 5\sqrt{n} + \log^3 n$

3- (30 pts)

Given a directed graph, develop an efficient algorithm to determine whether it is connected (AKA semi-connected). A directed graph G is connected, if for every pair of nodes u and v: u is reachable from v or v is reachable from u. Prove that your algorithm is correct and analyze its running-time.

Bonus (20 pts)

Given a directed graph, provide an efficient algorithm (faster than cubic time) that finds the number of paths between all pairs of nodes.

- Abolfazl Asudeh