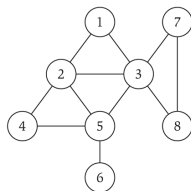


### 3.1 Basic Definitions and Applications

#### Undirected Graphs

Undirected graph.  $G = (V, E)$

- $V$  = nodes.
- $E$  = edges between pairs of nodes.
- Captures pairwise relationship between objects.
- Graph size parameters:  $n = |V|$ ,  $m = |E|$ .



$V = \{1, 2, 3, 4, 5, 6, 7, 8\}$

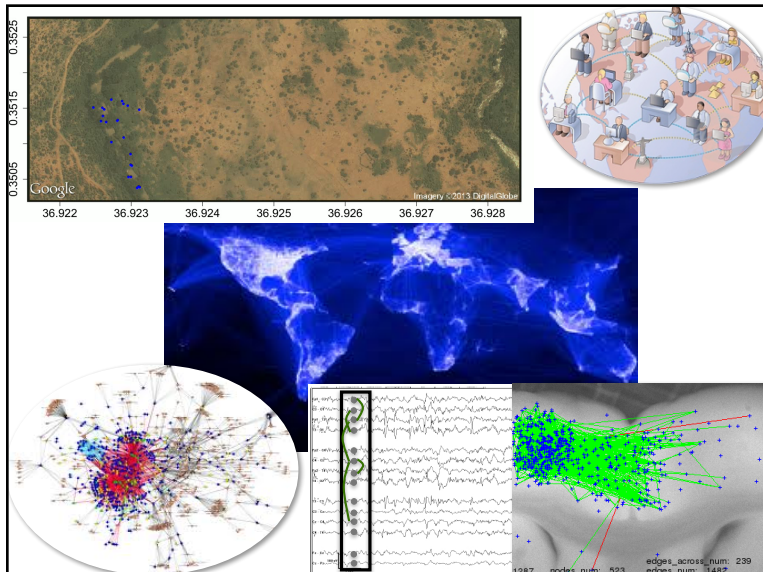
$E = \{1-2, 1-3, 2-3, 2-4, 2-5, 3-5, 3-7, 3-8, 4-5, 5-6, 7-8\}$

$n = 8$

$m = 11$

#### Some Graph Applications

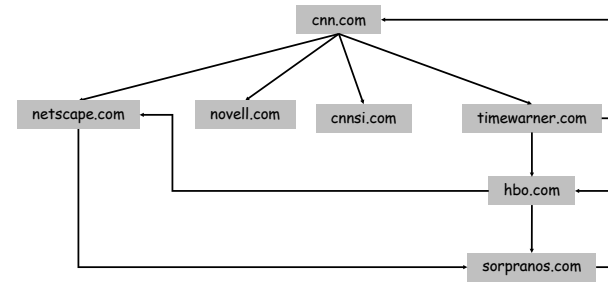
Graph	Nodes	Edges
transportation	street intersections	highways
communication	computers	fiber optic cables
World Wide Web	web pages	hyperlinks
social	people/animals	relationships
food web	species	predator-prey
software systems	functions	function calls
scheduling	tasks	precedence constraints
circuits	gates	wires
gene-regulatory	genes/proteins	regulation



## World Wide Web

### Web graph.

- Node: web page.
- Edge: hyperlink from one page to another.

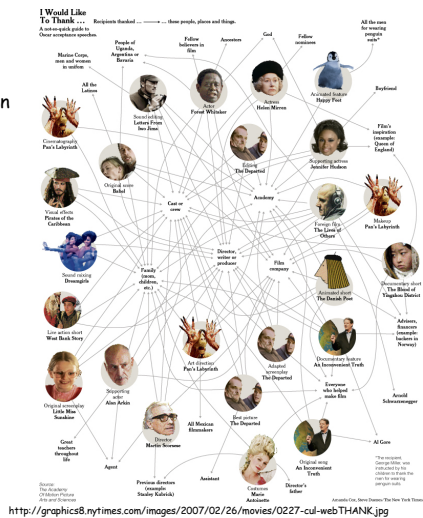


6

## Academy Awards Thanks

### Social network graph.

- Node: people.
- Edge: relationship between two people.

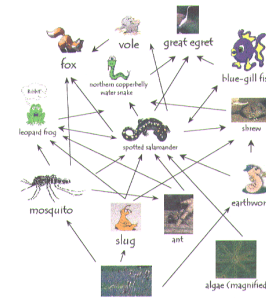


7

## Ecological Food Web

### Food web graph.

- Node = species.
- Edge = from prey to predator.



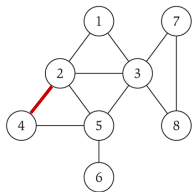
Reference: <http://www.twingroves.district196.k12.il.us/Wetlands/Salamander/SalGraphics/salfoodweb.gif>

8

### Graph Representation: Adjacency Matrix

**Adjacency matrix.**  $n$ -by- $n$  matrix with  $A_{uv} = 1$  if  $(u, v)$  is an edge.

- Two representations of each edge.
- Space proportional to  $n^2$ .
- Checking if  $(u, v)$  is an edge takes  $\Theta(1)$  time.
- Identifying all edges takes  $\Theta(n^2)$  time.



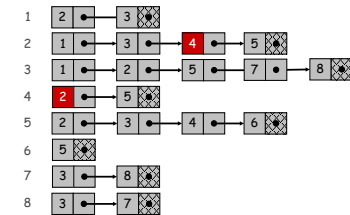
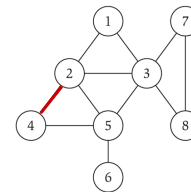
	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

9

### Graph Representation: Adjacency List

**Adjacency list.** Node indexed array of lists.

- Two representations of each edge.
- Space proportional to  $m + n$ .
- Checking if  $(u, v)$  is an edge takes  $O(\deg(u))$  time. degree = number of neighbors of  $u$
- Identifying all edges takes  $\Theta(m + n)$  time.



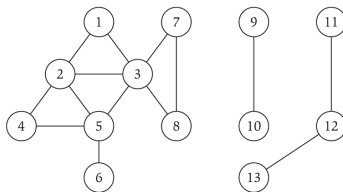
10

### Paths and Connectivity

**Def.** A **path** in an undirected graph  $G = (V, E)$  is a sequence  $P$  of nodes  $v_1, v_2, \dots, v_{k-1}, v_k$  with the property that each consecutive pair  $v_i, v_{i+1}$  is joined by an edge in  $E$ .

**Def.** A path is **simple** if all nodes are distinct.

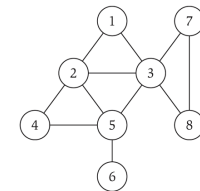
**Def.** An undirected graph is **connected** if for every pair of nodes  $u$  and  $v$ , there is a path between  $u$  and  $v$ .



11

### Cycles

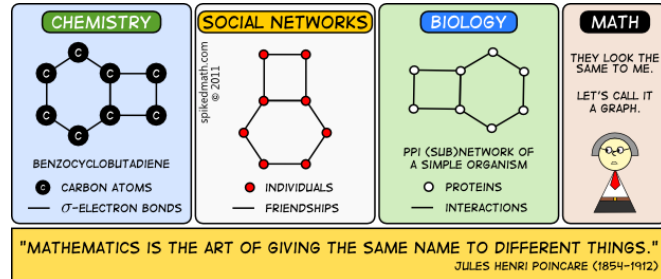
**Def.** A **cycle** is a path  $v_1, v_2, \dots, v_{k-1}, v_k$  in which  $v_1 = v_k$ ,  $k > 3$ , and the first  $k-1$  nodes are all distinct.



cycle  $C = 1-2-4-5-3-1$

12

### Graphs, continuing



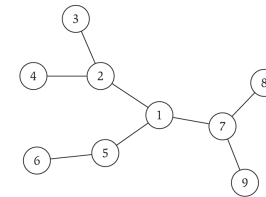
13

### Trees

**Def.** An undirected graph is a **tree** if it is connected and does not contain a cycle.

**Theorem.** Let  $G$  be an undirected graph on  $n$  nodes. Any two of the following statements imply the third.

- $G$  is connected.
- $G$  does not contain a cycle.
- $G$  has  $n-1$  edges.

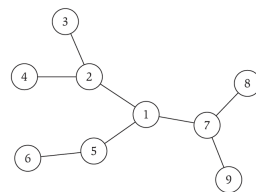


14

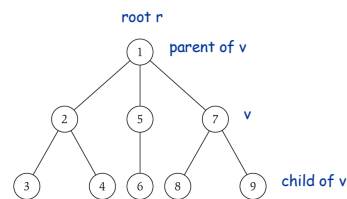
### Rooted Trees

**Rooted tree.** Given a tree  $T$ , choose a root node  $r$  and orient each edge away from  $r$ .

**Importance.** Models hierarchical structure.



a tree

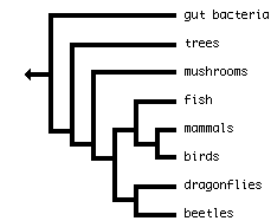


the same tree, rooted at 1

15

### Phylogeny Trees

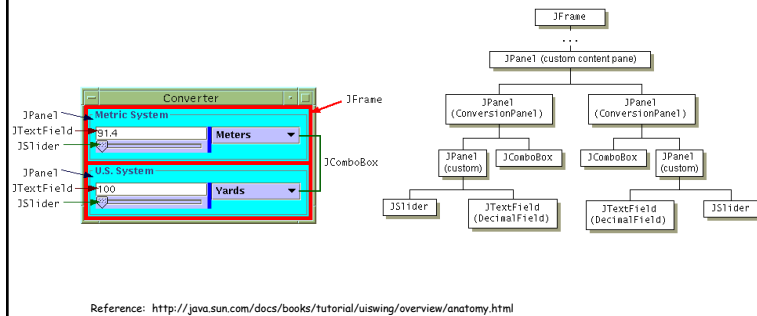
**Phylogeny trees.** Describe evolutionary history of species.



16

## GUI Containment Hierarchy

**GUI containment hierarchy.** Describe organization of GUI widgets.



Reference: <http://java.sun.com/docs/books/tutorial/uiswing/overview/anatomy.html>

17

### 3.2 Graph Traversal

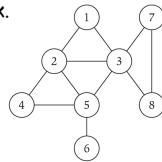
## Connectivity

**s-t connectivity problem.** Given two node  $s$  and  $t$ , is there a path between  $s$  and  $t$ ?

**s-t shortest path problem.** Given two node  $s$  and  $t$ , what is the length of the shortest path between  $s$  and  $t$ ?

### Applications.

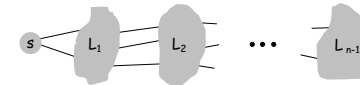
- LinkedIn contacts.
- Maze traversal.
- Travel planning.
- Fewest number of hops in a communication network.



19

## Breadth First Search

**BFS intuition.** Explore outward from  $s$  in all possible directions, adding nodes one "layer" at a time.



BFS algorithm.

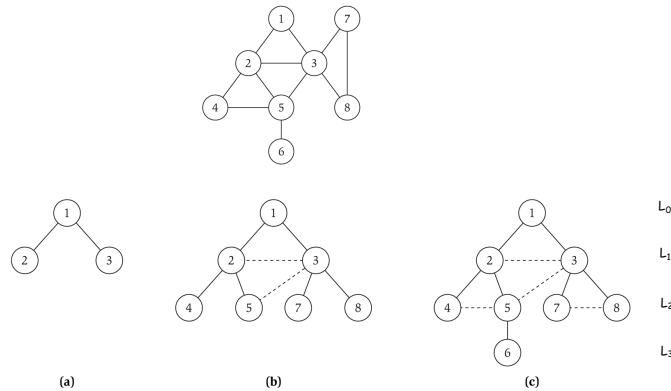
- $L_0 = \{s\}$ .
- $L_1 =$  all neighbors of  $L_0$ .
- $L_2 =$  all nodes that do not belong to  $L_0$  or  $L_1$ , and that have an edge to a node in  $L_1$ .
- $L_{i+1} =$  all nodes that do not belong to an earlier layer, and that have an edge to a node in  $L_i$ .

**Theorem.** For each  $i$ ,  $L_i$  consists of all nodes at distance exactly  $i$  from  $s$ . There is a path from  $s$  to  $t$  iff  $t$  appears in some layer.

20

### Breadth First Search

**Property.** Let  $T$  be a BFS tree of  $G = (V, E)$ , and let  $(x, y)$  be an edge of  $G$ . Then the level of  $x$  and  $y$  differ by at most 1.



21

### Breadth First Search: Analysis

**Theorem.** The above implementation of BFS runs in  $O(m + n)$  time if the graph is given by its adjacency list representation.

**Pf.**

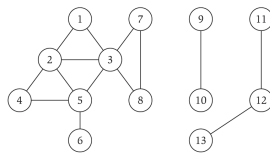
- Easy to prove  $O(n^2)$  running time:
  - at most  $n$  lists  $L[i]$
  - each node occurs on at most one list; for loop runs  $\leq n$  times
  - when we consider node  $u$ , there are  $\leq n$  incident edges  $(u, v)$ , and we spend  $O(1)$  processing each edge
- Actually runs in  $O(m + n)$  time:
  - when we consider node  $u$ , there are  $\deg(u)$  incident edges  $(u, v)$
  - total time processing edges is  $\sum_{u \in V} \deg(u) = 2m$

↑  
each edge  $(u, v)$  is counted exactly twice  
in sum: once in  $\deg(u)$  and once in  $\deg(v)$

22

### Connected Component

**Connected component.** Find all nodes reachable from  $s$ .



Connected component containing node 1 =  $\{1, 2, 3, 4, 5, 6, 7, 8\}$ .

23

### Flood Fill

**Flood fill.** Given lime green pixel in an image, change color of entire blob of neighboring lime pixels to blue.

- Node: pixel.
- Edge: two neighboring lime pixels.
- Blob: connected component of lime pixels.

recolor lime green blob to blue



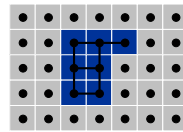
24

### Flood Fill

**Flood fill.** Given lime green pixel in an image, change color of entire blob of neighboring lime pixels to blue.

- Node: pixel.
- Edge: two neighboring lime pixels.
- Blob: connected component of lime pixels.

recolor lime green blob to blue



25

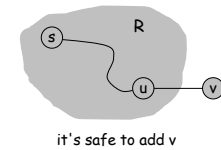
### Connected Component

**Connected component.** Find all nodes reachable from  $s$ .

---

$R$  will consist of nodes to which  $s$  has a path  
 Initially  $R = \{s\}$   
 While there is an edge  $(u, v)$  where  $u \in R$  and  $v \notin R$   
     Add  $v$  to  $R$   
 Endwhile

---



**Theorem.** Upon termination,  $R$  is the connected component containing  $s$ .

- BFS = explore in order of distance from  $s$ .
- DFS = explore in a different way.

26

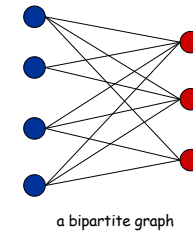
## 3.4 Testing Bipartiteness

### Bipartite Graphs

**Def.** An undirected graph  $G = (V, E)$  is **bipartite** if the nodes can be colored red or blue such that every edge has one red and one blue end.

**Applications.**

- Stable marriage: men = red, women = blue.
- Scheduling: machines = red, jobs = blue.



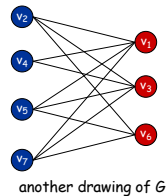
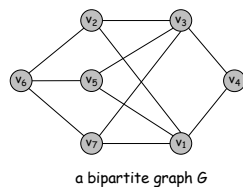
a bipartite graph

28

### Testing Bipartiteness

**Testing bipartiteness.** Given a graph  $G$ , is it bipartite?

- Many graph problems become:
  - easier if the underlying graph is bipartite (matching)
  - tractable if the underlying graph is bipartite (independent set)
- Before attempting to design an algorithm, we need to understand structure of bipartite graphs.

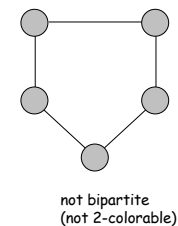
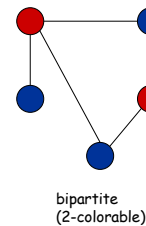


29

### An Obstruction to Bipartiteness

**Lemma.** If a graph  $G$  is bipartite, it cannot contain an odd length cycle.

**Pf.** Not possible to 2-color the odd cycle, let alone  $G$ .

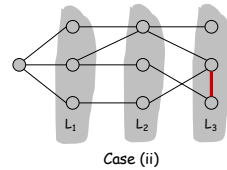
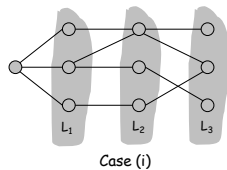


30

### Bipartite Graphs

**Lemma.** Let  $G$  be a connected graph, and let  $L_0, \dots, L_k$  be the layers produced by BFS starting at node  $s$ . Exactly one of the following holds.

- No edge of  $G$  joins two nodes of the same layer, and  $G$  is bipartite.
- An edge of  $G$  joins two nodes of the same layer, and  $G$  contains an odd-length cycle (and hence is not bipartite).



31

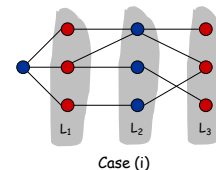
### Bipartite Graphs

**Lemma.** Let  $G$  be a connected graph, and let  $L_0, \dots, L_k$  be the layers produced by BFS starting at node  $s$ . Exactly one of the following holds.

- No edge of  $G$  joins two nodes of the same layer, and  $G$  is bipartite.
- An edge of  $G$  joins two nodes of the same layer, and  $G$  contains an odd-length cycle (and hence is not bipartite).

**Pf.** (i)

- Suppose no edge joins two nodes in the same layer.
- By previous lemma, this implies all edges join nodes on same level.
- Bipartition: red = nodes on odd levels, blue = nodes on even levels.



32



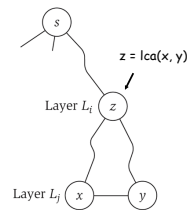
### Bipartite Graphs

**Lemma.** Let  $G$  be a connected graph, and let  $L_0, \dots, L_k$  be the layers produced by BFS starting at node  $s$ . Exactly one of the following holds.

- (i) No edge of  $G$  joins two nodes of the same layer, and  $G$  is bipartite.
- (ii) An edge of  $G$  joins two nodes of the same layer, and  $G$  contains an odd-length cycle (and hence is not bipartite).

**Pf.** (ii)

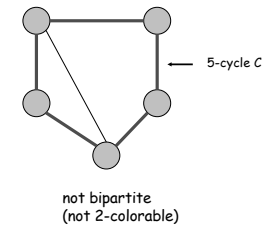
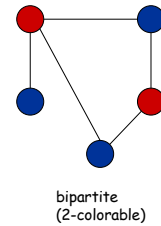
- Suppose  $(x, y)$  is an edge with  $x, y$  in same level  $L_j$ .
- Let  $z = \text{lca}(x, y)$  = lowest common ancestor.
- Let  $L_i$  be level containing  $z$ .
- Consider cycle that takes edge from  $x$  to  $y$ , then path from  $y$  to  $z$ , then path from  $z$  to  $x$ .
- Its length is  $1 + \underbrace{(j-i)}_{\text{path from } y \text{ to } z} + \underbrace{(j-i)}_{\text{path from } z \text{ to } x}$ , which is odd.



33

### Obstruction to Bipartiteness

**Corollary.** A graph  $G$  is bipartite iff it contains no odd length cycle.



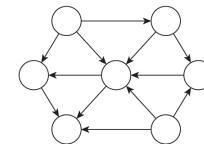
34

## 3.5 Connectivity in Directed Graphs

### Directed Graphs

**Directed graph.**  $G = (V, E)$

- Edge  $(u, v)$  goes from node  $u$  to node  $v$ .



**Ex.** Web graph - hyperlink points from one web page to another.

- Directedness of graph is crucial.
- Modern web search engines exploit hyperlink structure to rank web pages by importance.

36

### Graph Search

**Directed reachability.** Given a node  $s$ , find all nodes reachable from  $s$ .

**Directed  $s$ - $t$  shortest path problem.** Given two nodes  $s$  and  $t$ , what is the length of the shortest path between  $s$  and  $t$ ?

**Graph search.** BFS extends naturally to directed graphs.

**Web crawler.** Start from web page  $s$ . Find all web pages linked from  $s$ , either directly or indirectly.

37

### Strong Connectivity

**Def.** Node  $u$  and  $v$  are **mutually reachable** if there is a path from  $u$  to  $v$  and also a path from  $v$  to  $u$ .

**Def.** A graph is **strongly connected** if every pair of nodes is mutually reachable.

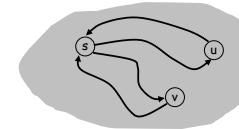
**Lemma.** Let  $s$  be any node.  $G$  is strongly connected iff every node is reachable from  $s$ , and  $s$  is reachable from every node.

**Pf.**  $\Rightarrow$  Follows from definition.

**Pf.**  $\Leftarrow$  Path from  $u$  to  $v$ : concatenate  $u$ - $s$  path with  $s$ - $v$  path.

Path from  $v$  to  $u$ : concatenate  $v$ - $s$  path with  $s$ - $u$  path.  $\cdot$

$\swarrow$   
ok if paths overlap

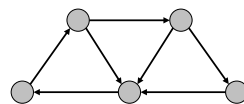


38

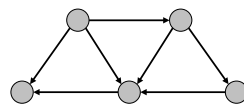
### Strong Connectivity: Algorithm

**Theorem.** Can determine if  $G$  is strongly connected in  $O(m + n)$  time.  
**Pf.**

- Pick any node  $s$ .
- Run BFS from  $s$  in  $G$ .
- Run BFS from  $s$  in  $G^{rev}$ .  $\swarrow$  reverse orientation of every edge in  $G$
- Return true iff all nodes reached in both BFS executions.
- Correctness follows immediately from previous lemma.  $\cdot$



strongly connected



not strongly connected

39

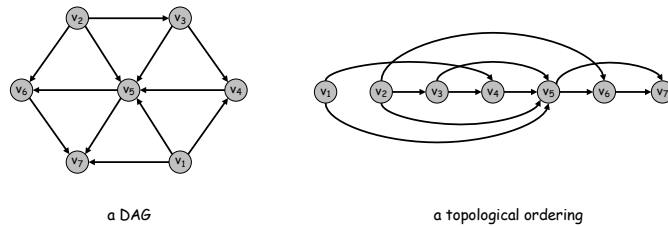
## 3.6 DAGs and Topological Ordering

### Directed Acyclic Graphs

**Def.** An **DAG** is a directed graph that contains no directed cycles.

**Ex.** Precedence constraints: edge  $(v_i, v_j)$  means  $v_i$  must precede  $v_j$ .

**Def.** A **topological order** of a directed graph  $G = (V, E)$  is an ordering of its nodes as  $v_1, v_2, \dots, v_n$  so that for every edge  $(v_i, v_j)$  we have  $i < j$ .



41

### Precedence Constraints

**Precedence constraints.** Edge  $(v_i, v_j)$  means task  $v_i$  must occur before  $v_j$ .

**Applications.**

- Course prerequisite graph: course  $v_i$  must be taken before  $v_j$ .
- Compilation: module  $v_i$  must be compiled before  $v_j$ .
- Pipeline of computing jobs: output of job  $v_i$  needed to determine input of job  $v_j$ .

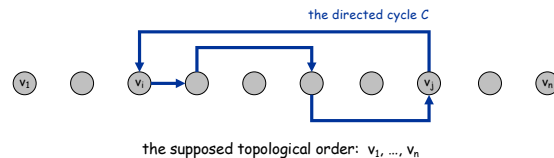
42

### Directed Acyclic Graphs

**Lemma.** If  $G$  has a topological order, then  $G$  is a DAG.

**Pf.** (by contradiction)

- Suppose that  $G$  has a topological order  $v_1, \dots, v_n$  and that  $G$  also has a directed cycle  $C$ . Let's see what happens.
- Let  $v_i$  be the lowest-indexed node in  $C$ , and let  $v_j$  be the node just before  $v_i$ ; thus  $(v_j, v_i)$  is an edge.
- By our choice of  $i$ , we have  $i < j$ .
- On the other hand, since  $(v_j, v_i)$  is an edge and  $v_1, \dots, v_n$  is a topological order, we must have  $j < i$ , a contradiction.



43

### Directed Acyclic Graphs

**Lemma.** If  $G$  has a topological order, then  $G$  is a DAG.

**Q.** Does every DAG have a topological ordering?

**Q.** If so, how do we compute one?

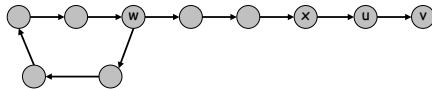
44

## Directed Acyclic Graphs

**Lemma.** If  $G$  is a DAG, then  $G$  has a node with no incoming edges.

**Pf.** (by contradiction)

- Suppose that  $G$  is a DAG and every node has at least one incoming edge. Let's see what happens.
- Pick any node  $v$ , and begin following edges backward from  $v$ . Since  $v$  has at least one incoming edge  $(u, v)$  we can walk backward to  $u$ .
- Then, since  $u$  has at least one incoming edge  $(x, u)$ , we can walk backward to  $x$ .
- Repeat until we visit a node, say  $w$ , twice.
- Let  $C$  denote the sequence of nodes encountered between successive visits to  $w$ .  $C$  is a cycle. ·



45

## Directed Acyclic Graphs

**Lemma.** If  $G$  is a DAG, then  $G$  has a topological ordering.

**Pf.** (by induction on  $n$ )

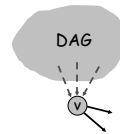


- Base case: true if  $n = 1$ .
- Given DAG on  $n > 1$  nodes, find a node  $v$  with no incoming edges.
- $G - \{v\}$  is a DAG, since deleting  $v$  cannot create cycles.
- By inductive hypothesis,  $G - \{v\}$  has a topological ordering.
- Place  $v$  first in topological ordering; then append nodes of  $G - \{v\}$  in topological order. This is valid since  $v$  has no incoming edges. ·

To compute a topological ordering of  $G$ :

Find a node  $v$  with no incoming edges and order it first  
Delete  $v$  from  $G$

Recursively compute a topological ordering of  $G - \{v\}$   
and append this order after  $v$



46

## Topological Sorting Algorithm: Running Time

**Theorem.** Algorithm finds a topological order in  $O(m + n)$  time.

**Pf.**

- Maintain the following information:
  - `count[w]` = remaining number of incoming edges
  - $S$  = set of remaining nodes with no incoming edges
- Initialization:  $O(m + n)$  via single scan through graph.
- Update: to delete  $v$ 
  - remove  $v$  from  $S$
  - decrement `count[w]` for all edges from  $v$  to  $w$ , and add  $w$  to  $S$  if `count[w]` hits 0
  - this is  $O(1)$  per edge ·

47