MEAM 520

More Rotation Matrices

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GRASP
LABORATORY

Lecture 4: September 10, 2013

Homework 1:
MATLAB Programming and Reachable Workspace

MEAM 520, University of Pennsylvania
Katherine J. Kuchenbecker, Ph.D.

September 4, 2013

This assignment is due on Tuesday, September 10, by midnight (11:59:59 p.m.) Your code should be submitted via email according to the instructions at the end of this document. Late submissions will be accepted until Thursday, September 12, by noon (11:59:59 a.m.), but they will be penalized by 10% for each partial or full day late, up to 20%. After the late deadline, no further assignments may be submitted.

You may talk with other students about this assignment, ask the teaching team questions, use a calculator and other tools, and consult outside sources such as the Internet. To help you actually learn the material, what you write down should be your own work, not copied from any other individual or team. Any submissions suspected of violating Penn's Code of Academic Integrity will be reported to the Office of Student Conduct. If you get stuck, post a question on Piazza or go to office hours!

Individual vs. Pair Programming

This class will use the programming language MATLAB to analyze and simulate robotic systems and also to control real robots. Some students in the class have never used MATLAB before, and others are quite familiar with it. The goal of this assignment is to get everyone starting to use MATLAB to improve their understanding of robotic systems.

If you have not used MATLAB much before, you should do this assignment with another student in our class. If you are already pretty comfortable with MATLAB, you should do this assignment alone. Read the assignment to decide which option is right for you.

If you do this homework with a partner, you may work with anyone you choose; the only stipulation is that they also have only a little MATLAB experience. If you are looking for a partner, consider using the "Search for Teammates!" tool on Piazza.

If you are in a pair, you should work closely with your partner throughout this assignment, following the paradigm of pair programming. You will turn in one MATLAB script for which you are both jointly responsible, and you will both receive the same grade. Please follow these pair programming guidelines, which were adapted from "All I really need to know about pair programming I learned in kindergarten," by Williams and Kessler, *Communications of the ACM*, May 2000:

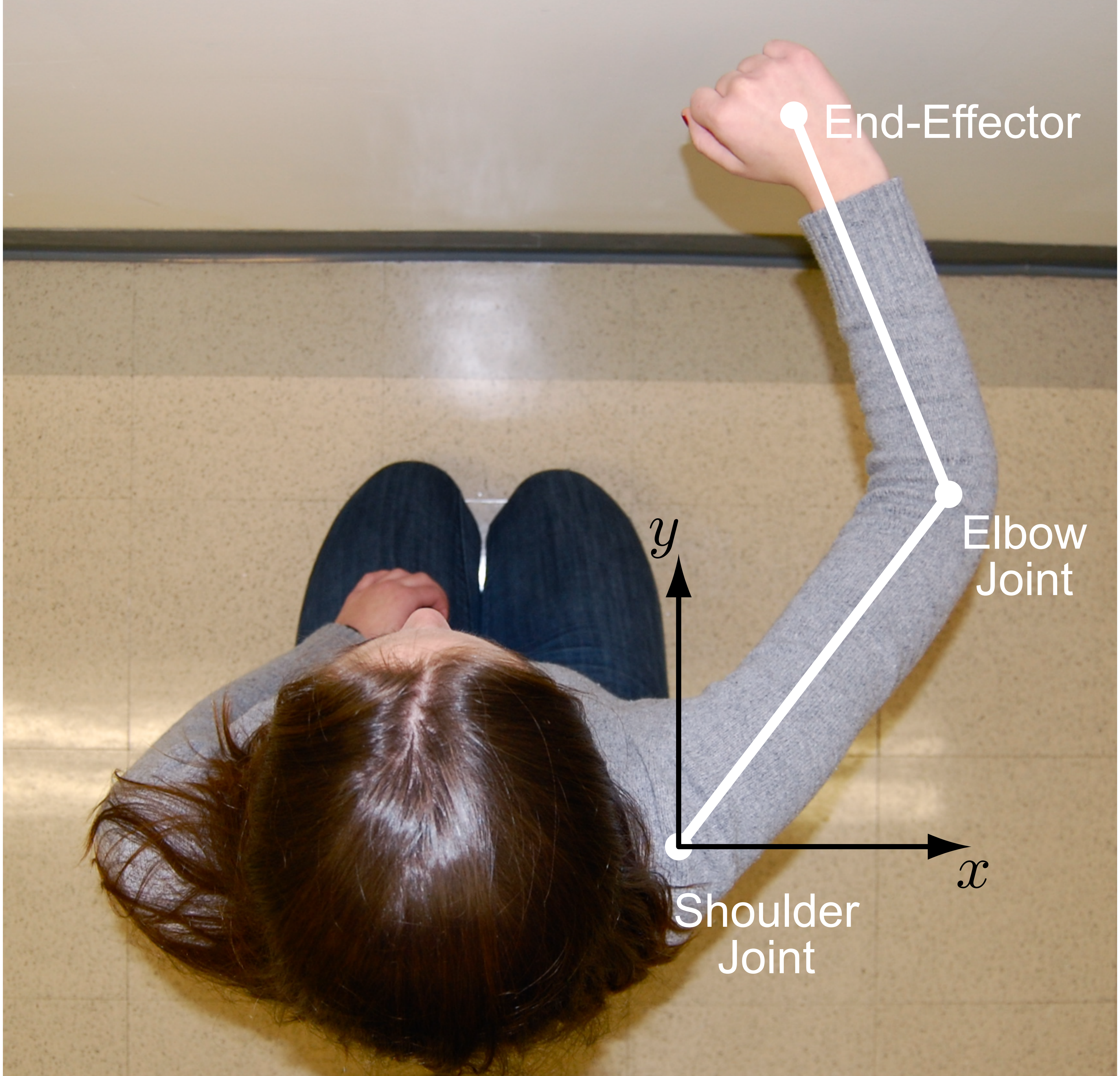
- Start with a good attitude, setting aside any skepticism and expecting to jell with your partner.
- Don't start writing code alone. Arrange a meeting with your partner as soon as you can.
- Use just one computer, and sit side by side; a desktop computer with a large monitor is better for this than a laptop. Make sure both partners can see the screen.
- At each instant, one partner should be driving (using the mouse and keyboard or recording design ideas) while the other is continuously reviewing the work (thinking and making suggestions).
- Change driving/reviewing roles at least every thirty minutes, *even if one partner is much more experienced than the other*. You may want to set a timer to help you remember to switch.
- If you notice a bug in the code your partner is typing, wait until they finish the line to correct them.
- Stay focused and on-task the whole time you are working together.

1

Homework 1

Due today by midnight.

Late deadline is Thursday at noon.


End-Effector
Elbow
Joint
y
x
Shoulder
Joint

Homework 2:
Rotation Matrices and Homogeneous Transformations

MEAM 520, University of Pennsylvania
Katherine J. Kuchenbecker, Ph.D.

September 10, 2013

This paper-based assignment is due on Tuesday, September 17, by midnight (11:59:59 p.m.) You should aim to turn it in during class that day. If you don't finish until later in the day, you can turn it in to Professor Kuchenbecker's office, Towne 224, in the bin or under the door. Late submissions will be accepted until Thursday, September 19, by noon (11:59:59 a.m.), but they will be penalized by 10% for each partial or full day late, up to 20%. After the late deadline, no further assignments may be submitted.

You may talk with other students about this assignment, ask the teaching team questions, use a calculator and other tools, and consult outside sources such as the Internet. To help you actually learn the material, what you write down should be your own work, not copied from any other individual or a solution manual. Any submissions suspected of violating Penn's Code of Academic Integrity will be reported to the Office of Student Conduct. If you get stuck, post a question on Piazza or go to office hours!

These problems are from the textbook, *Robot Modeling and Control* by Spong, Hutchinson, and Vidyasagar (SHV). Please follow the extra clarifications and instructions when provided. Write in pencil, show your work clearly, $\boxed{\text{box your answers}}$, and staple together all pages of your assignment. This assignment is worth a total of 20 points.

1. SHV 2-6, page 65 – Verifying Three Properties of $R_{z,\theta}$ *(2 points)*
2. SHV 2-10, page 66 – Sequence of Rotations *(2 points)*
   Please specify each element of each matrix in symbolic form and show the order in which the matrices should be multiplied; as stated in the problem, you do not need to perform the matrix multiplication.
3. SHV 2-14, page 67 – Rotating a Coordinate Frame *(4 points)*
   Sketch the initial, intermediate, and final frames by reading the text in the problem. Make your drawings big, and remember the right-hand rule. Then find R in two ways: by inspection of your sketch and by calculation. Check your solutions against one another.
4. SHV 2-23, page 68 – Axis/Angle Representation *(4 points)*
   Be careful with the sketch, and remember the right-hand rule.
5. SHV 2-39, page 70 – Homogeneous Transformations *(4 points)*
   Treat frame $o_2x_2y_2z_2$ as being located at the center of the cube's bottom surface (as drawn in Figure 2.14), not at the center of the cube (as stated in the problem). Be careful with notation; you are looking for H_1^0, H_2^0, H_3^0, and H_3^2.
6. SHV 2-43, page 71-72 – Commutativity of Homogeneous Transformations *(4 points)*

Homework 2

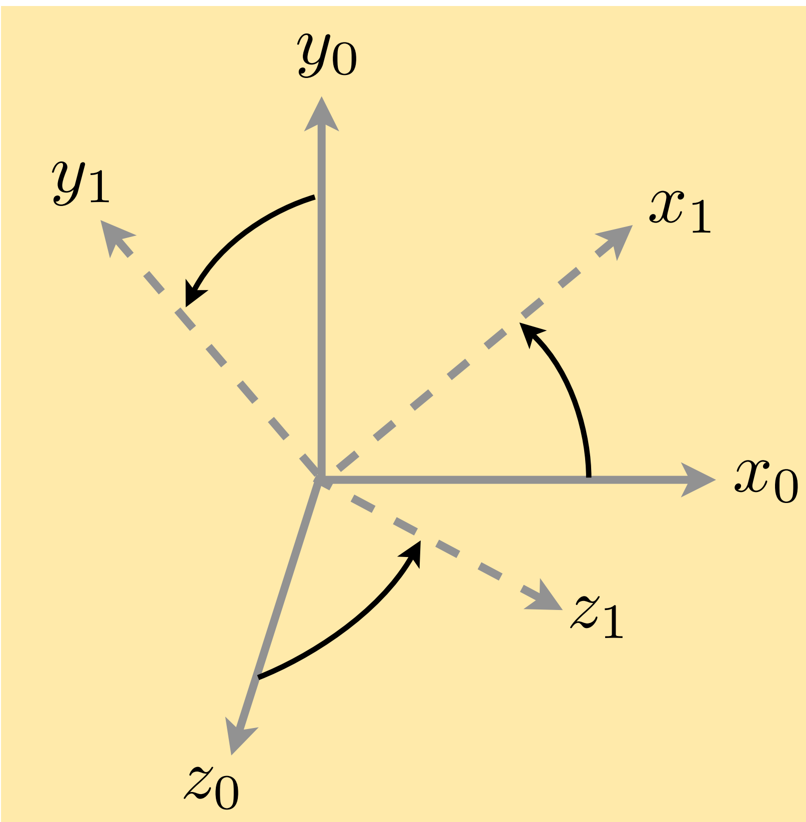
Due next Tuesday by midnight.

Late deadline is next Thursday at noon.

Rotation Matrices

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Rotation Matrices - Interpretation 1 of 3



Represents the orientation of one coordinate frame with respect to another frame

$$\mathbf{R}_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

Orientation of frame 1 w.r.t. frame 0

columns are of unit length

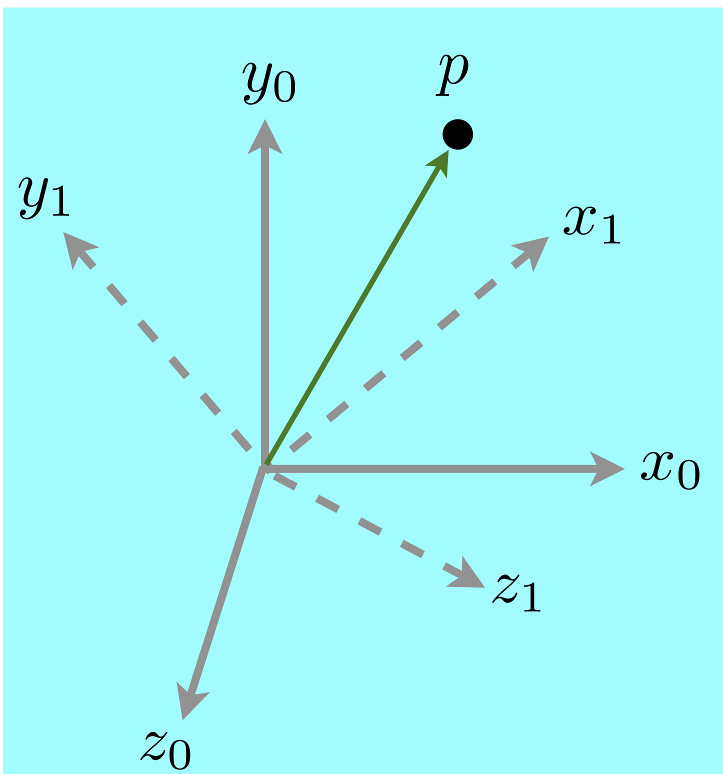
columns are mutually orthogonal

$$\det \mathbf{R}_1^0 = +1$$

$$\mathbf{R}_0^1 = ? \qquad \mathbf{R}_0^1 = \left(\mathbf{R}_1^0\right)^T$$

$$\left(\mathbf{R}_1^0\right)^T = \left(\mathbf{R}_1^0\right)^{-1}$$

Rotation Matrices - Interpretation 2 of 3



Coordinate transformation relating the coordinates of a point p in two different frames

$$\mathbf{R}_1^0 \qquad \mathbf{v}_p^1$$

$$\mathbf{v}_p^0 = ?$$

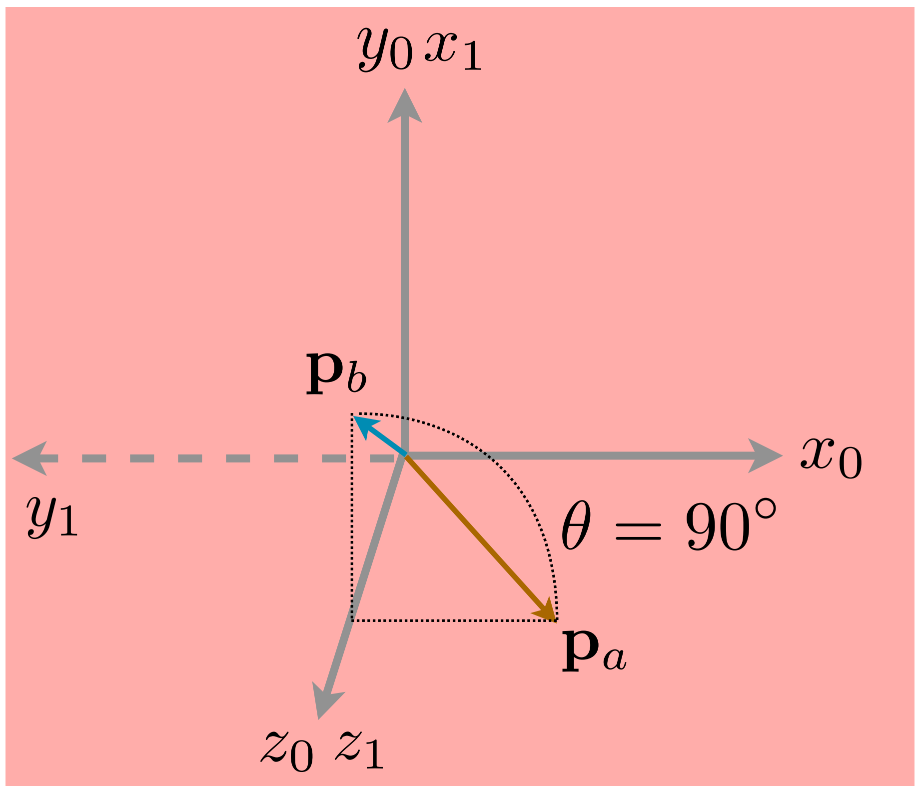
$$\boxed{\mathbf{v}_p^0 = \mathbf{R}_1^0 \mathbf{v}_p^1}$$

Subscript and superscript cancel

$$\mathbf{v}_p^1 = ? \qquad \left(\mathbf{R}_1^0\right)^{-1} \mathbf{v}_p^0 = \cancel{\left(\mathbf{R}_1^0\right)^{-1} \mathbf{R}_1^0} \mathbf{v}_p^1 \qquad \mathbf{v}_p^1 = \left(\mathbf{R}_1^0\right)^T \mathbf{v}_p^0$$

$$\boxed{\mathbf{v}_p^1 = \mathbf{R}_0^1 \mathbf{v}_p^0}$$

Rotation Matrices - Interpretation 3 of 3



Operator taking a vector and rotating it to yield a new vector in the same coordinate frame

$$\mathbf{p}_a^0 \qquad \mathbf{R}_1^0$$

$$\mathbf{p}_b^0 = ?$$

$$\mathbf{p}_b^0 = \mathbf{R}_1^0\, \mathbf{p}_b^1$$

$$\mathbf{p}_b^1 = \mathbf{p}_a^0$$

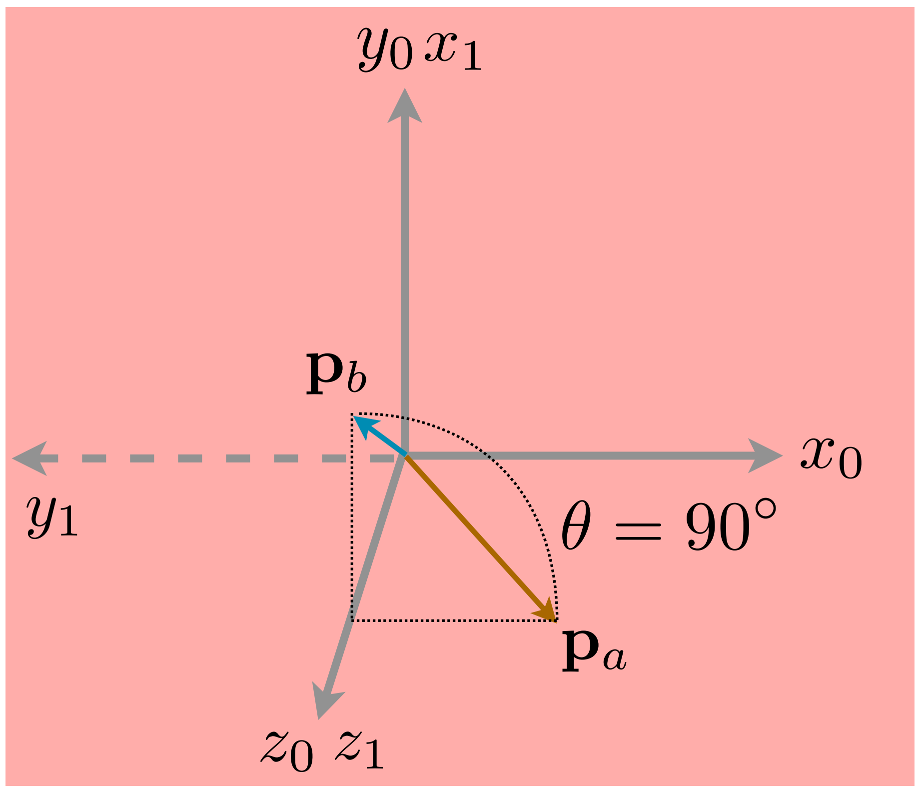
$$\mathbf{p}_b^0 = \mathbf{R}_1^0\, \mathbf{p}_a^0 \qquad \boxed{\mathbf{p}_b^0 = \mathbf{R}\, \mathbf{p}_a^0}$$

Composite Rotations

SHV 2.4

Rotation Matrices - Interpretation 3 of 3

What if I want to apply another rotation to this vector? Method depends on which axis I want to rotate around.



Operator taking a vector and rotating it to yield a new vector in the same coordinate frame

$$\mathbf{p}_a^0 \qquad \mathbf{R}_1^0$$

$$\mathbf{p}_b^0 = ?$$

For example:
rotate 45° around y_0
vs.
rotate 45° around y_1

$$\mathbf{p}_b^0 = \mathbf{R}_1^0\,\mathbf{p}_b^1$$

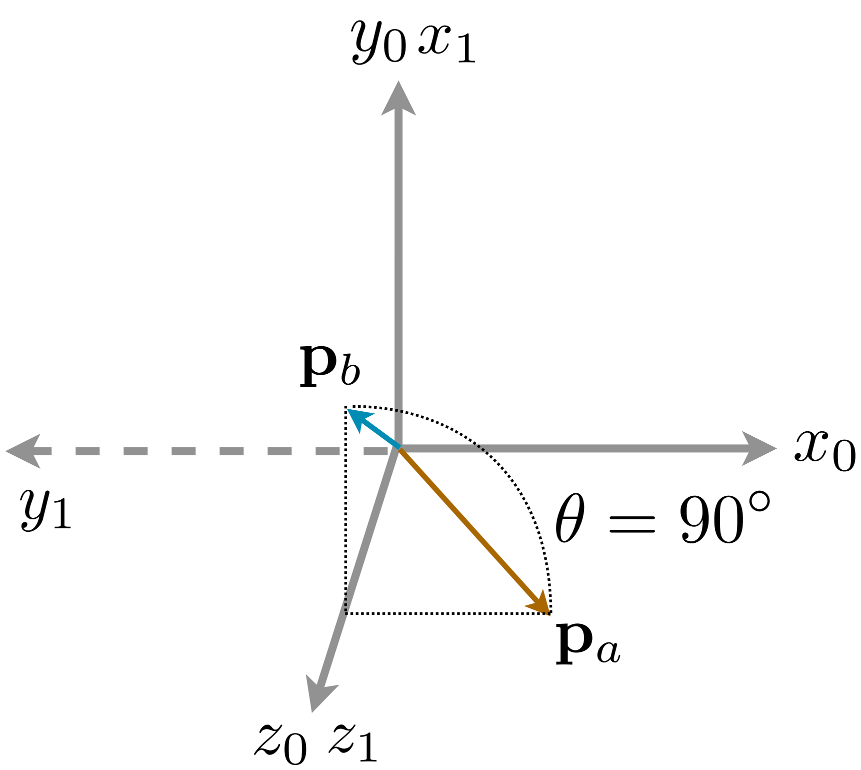
$$\mathbf{p}_b^1 = \mathbf{p}_a^0$$

$$\mathbf{p}_b^0 = \mathbf{R}_1^0\,\mathbf{p}_a^0 \qquad \boxed{\mathbf{p}_b^0 = \mathbf{R}\,\mathbf{p}_a^0}$$

The **order** in which a sequence of rotations is performed is crucial.

Thus, the **order** in which the rotation matrices are multiplied together is crucial.

What if I want to apply another rotation to this vector? Method depends on which axis I want to rotate around.



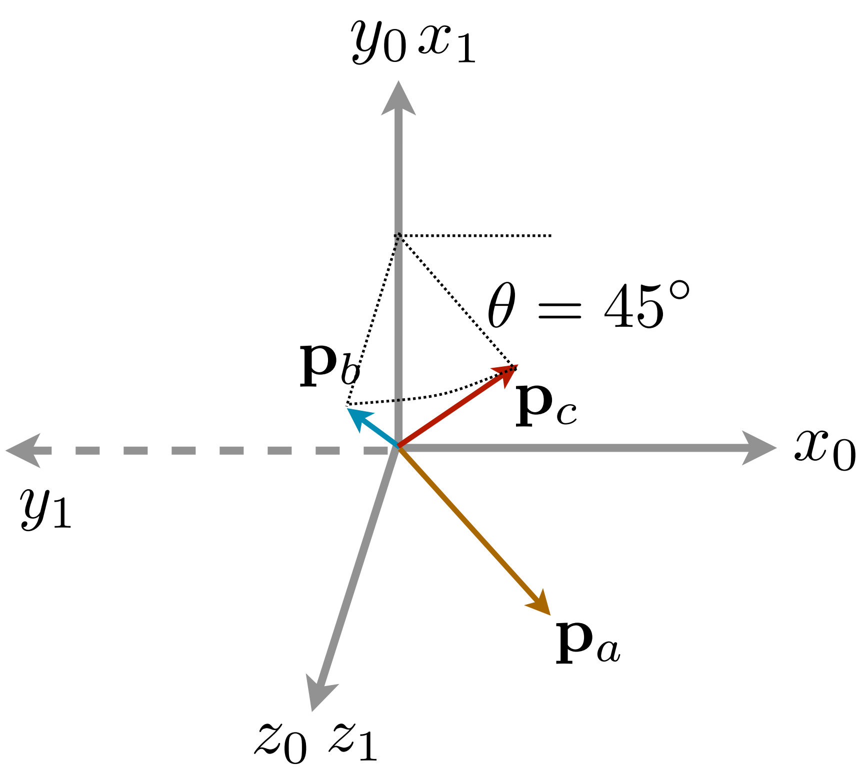
For example:

rotate 45° around y_0

vs.

rotate 45° around y_1

What if I want to apply another rotation to this vector?
Method depends on which axis I want to rotate around.



$$\mathbf{p}_b^0 = \mathbf{R}\,\mathbf{p}_a^0 \qquad \mathbf{R}'$$

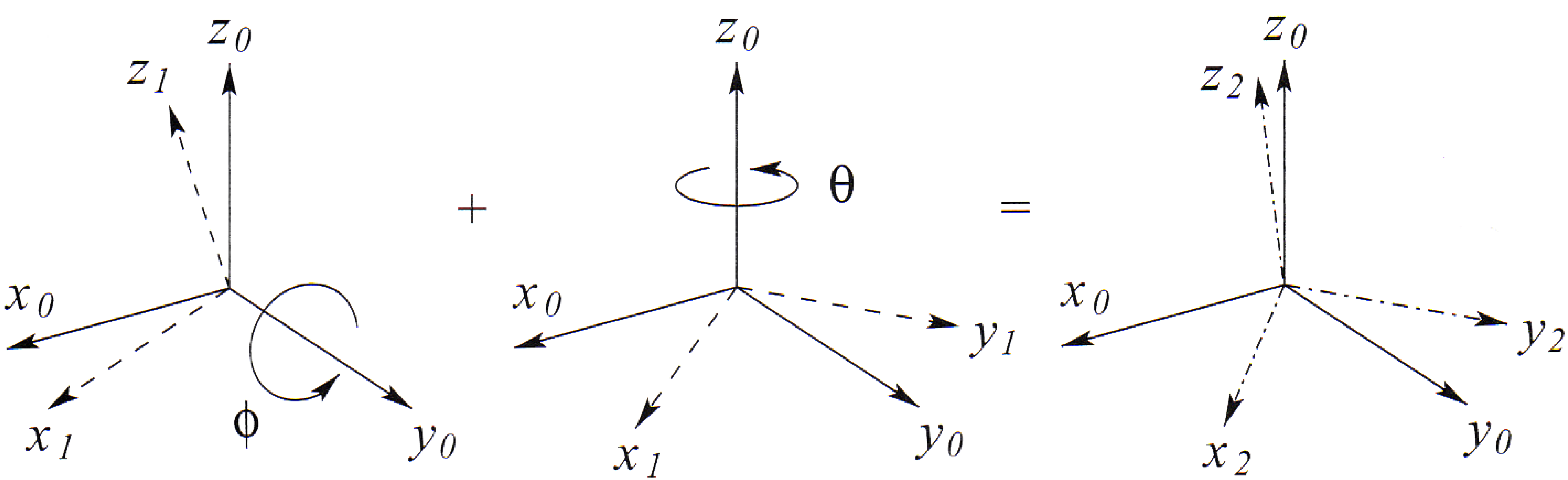
$$\mathbf{p}_c^0 = ?$$

$$\boxed{\mathbf{p}_c^0 = \mathbf{R}'\mathbf{p}_b^0}$$

$$\boxed{\mathbf{p}_c^0 = \mathbf{R}'\,\mathbf{R}\,\mathbf{p}_a^0}$$

For example:
rotate 45° around y_0
vs.
rotate 45° around y_1

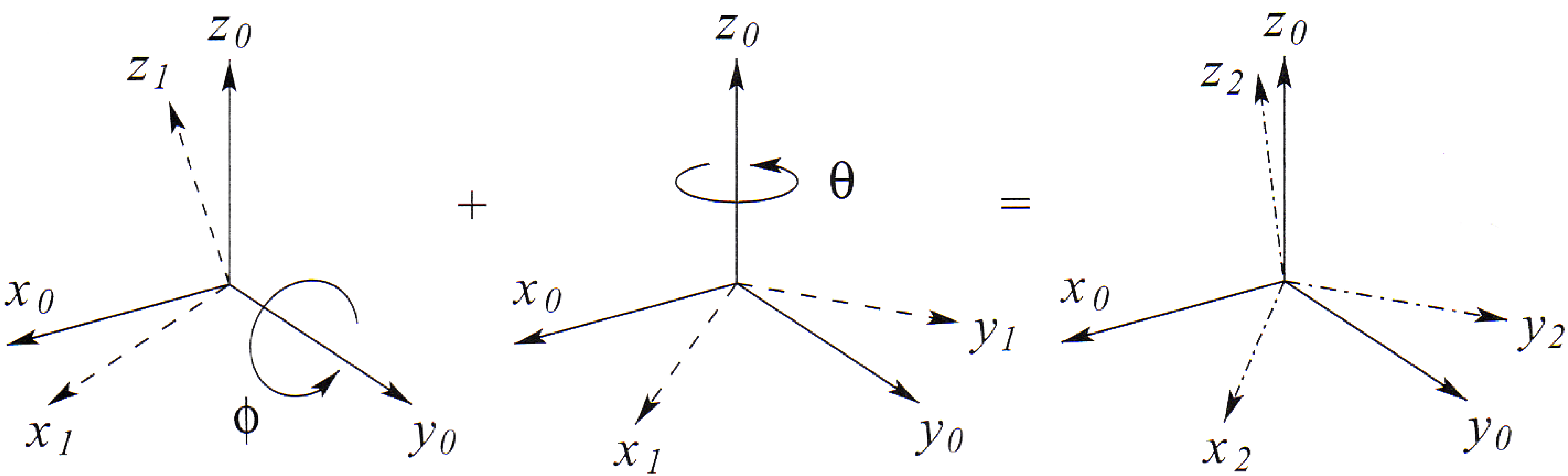
Composition of Rotations with Respect to a Fixed Frame



Composition of Rotations with Respect to a Fixed Frame

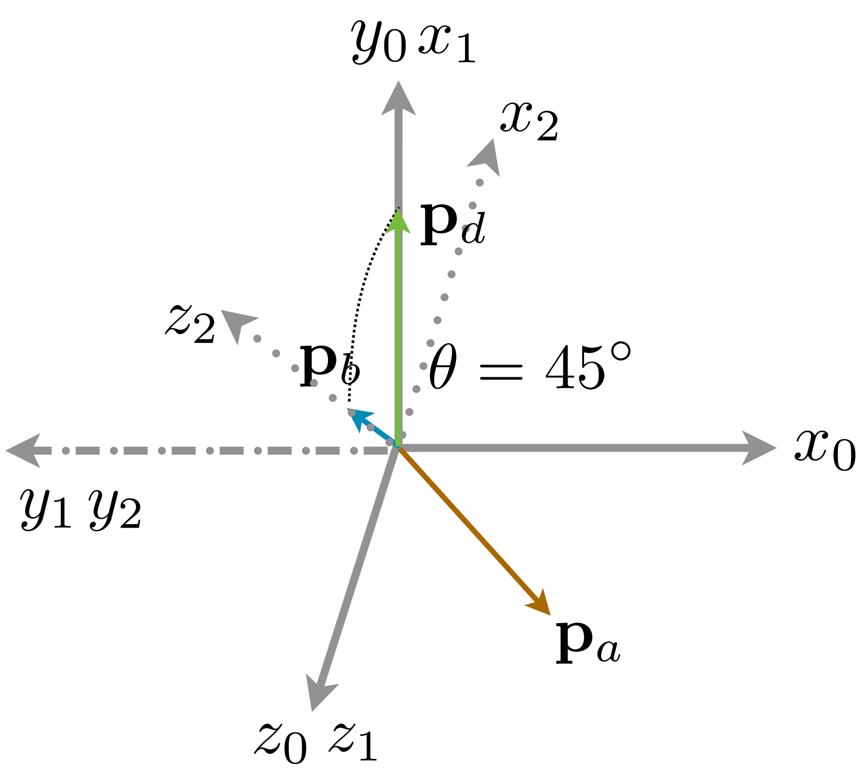
the result of a successive rotation about a fixed frame can be found by **pre-multiplying** by the corresponding rotation matrix

$$\mathbf{R}_2^0 = \mathbf{R}\,\mathbf{R}_1^0$$



Note that $\mathbf{R}$ is a rotation about the original frame

What if I want to apply another rotation to this vector?
Method depends on which axis I want to rotate around.

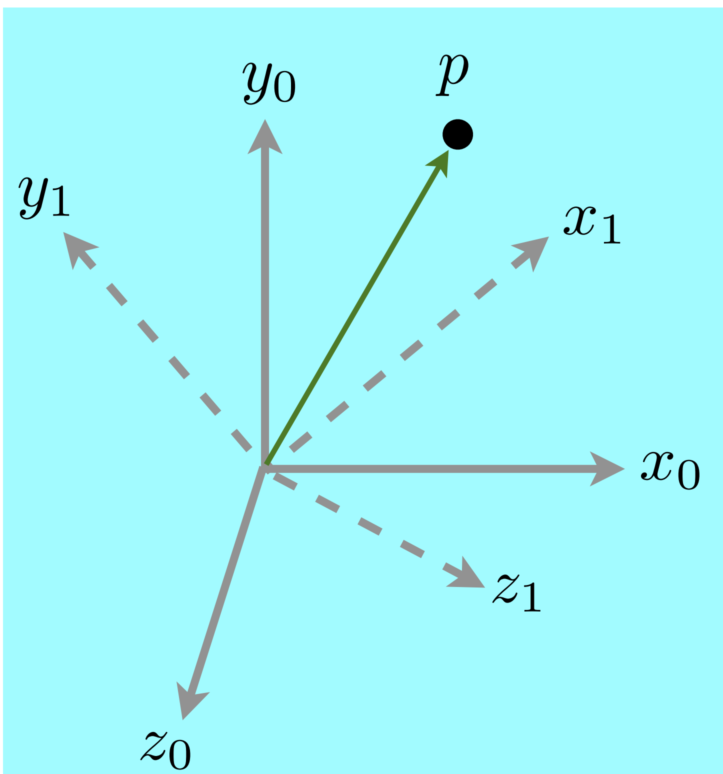


$$\mathbf{p}_d^0 = ?$$

For example:
rotate 45° around y_0
vs.
rotate 45° around y_1

Rotation Matrices - Interpretation 2 of 3

Remember...



Coordinate transformation relating the coordinates of a point p in two different frames

$$\mathbf{R}_1^0 \qquad \mathbf{v}_p^1$$

$$\mathbf{v}_p^0 = ?$$

$$\boxed{\mathbf{v}_p^0 = \mathbf{R}_1^0 \mathbf{v}_p^1}$$

Subscript and superscript cancel

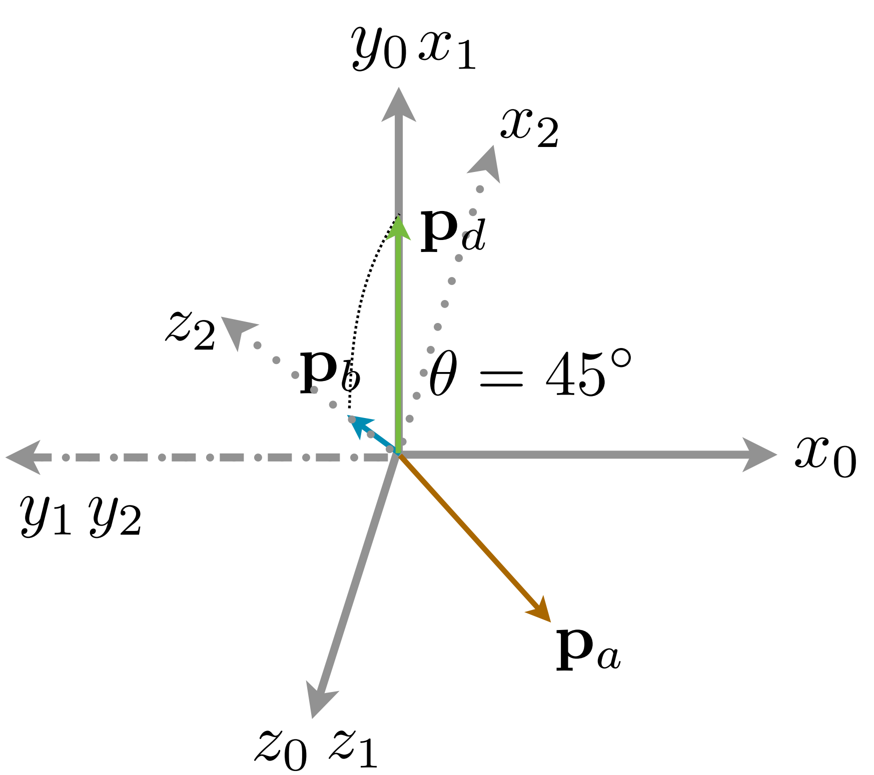
$$\mathbf{v}_p^1 = ?$$

$$\left(\mathbf{R}_1^0\right)^{-1} \mathbf{v}_p^0 = \cancel{\left(\mathbf{R}_1^0\right)^{-1} \mathbf{R}_1^0} \mathbf{v}_p^1$$

$$\mathbf{v}_p^1 = \left(\mathbf{R}_1^0\right)^T \mathbf{v}_p^0$$

$$\boxed{\mathbf{v}_p^1 = \mathbf{R}_0^1 \mathbf{v}_p^0}$$

What if I want to apply another rotation to this vector?
Method depends on which axis I want to rotate around.



For example:
rotate 45° around y_0
vs.
rotate 45° around y_1

$$\mathbf{p}_d^2 = \mathbf{p}_a^0$$

$$\mathbf{p}_d^0 = ?$$

$$\boxed{\mathbf{p}_d^0 = \mathbf{R}_1^0\, \mathbf{R}_2^1\, \mathbf{p}_a^0}$$

interp. 3 (operator)

$$\mathbf{p}_d^0 = \mathbf{R}_2^0\, \mathbf{p}_d^2$$

$$\mathbf{p}_d^0 = \mathbf{R}_1^0\, \mathbf{p}_d^1$$

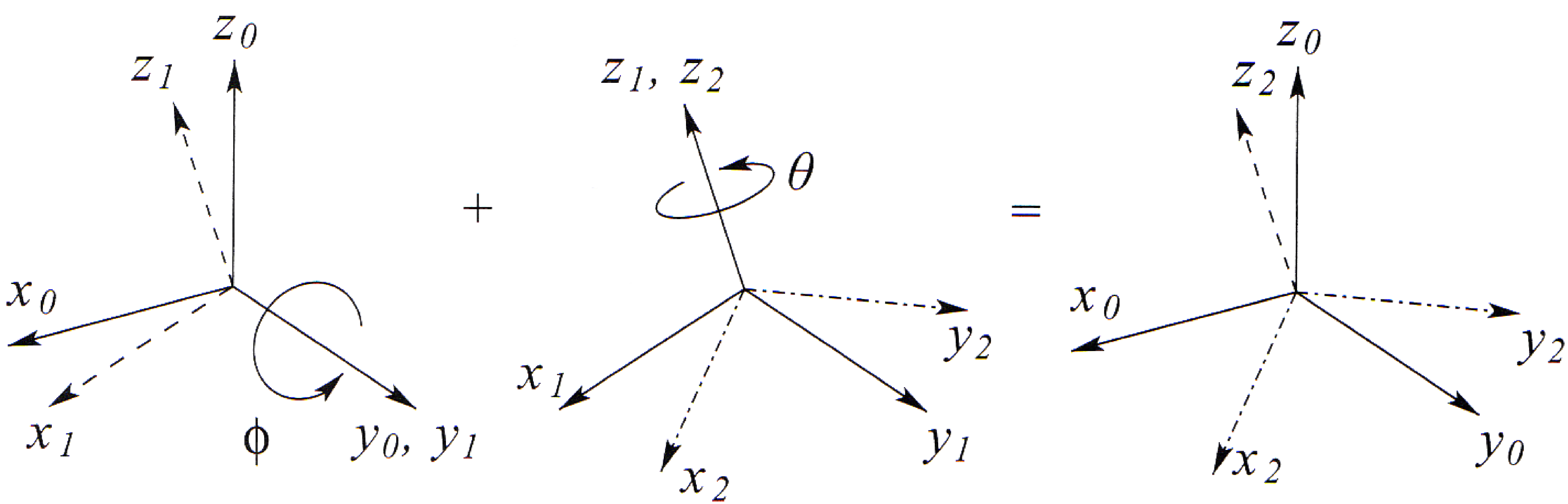
$$\mathbf{p}_d^1 = \mathbf{R}_2^1\, \mathbf{p}_d^2$$

$$\boxed{\mathbf{R}_2^0 = \mathbf{R}_1^0\, \mathbf{R}_2^1}$$

$$\boxed{\mathbf{p}_d^0 = \mathbf{R}_1^0\, \mathbf{R}_2^1\, \mathbf{p}_d^2}$$

interp. 2
(frames)

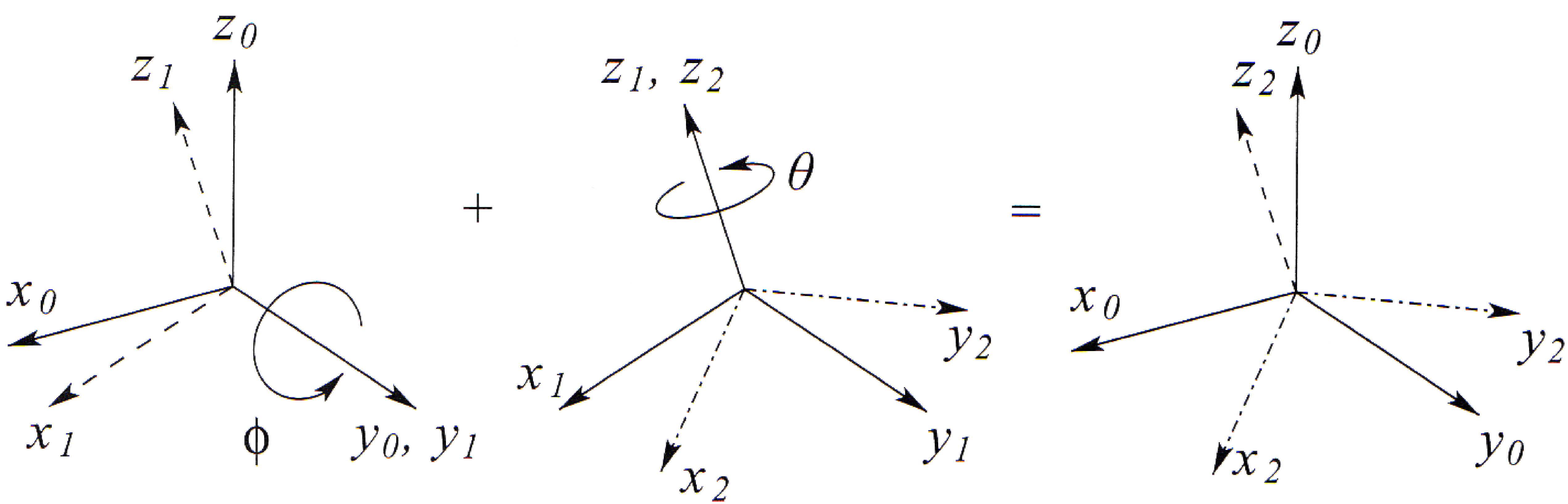
Composition of Rotations with Respect to the Current Frame



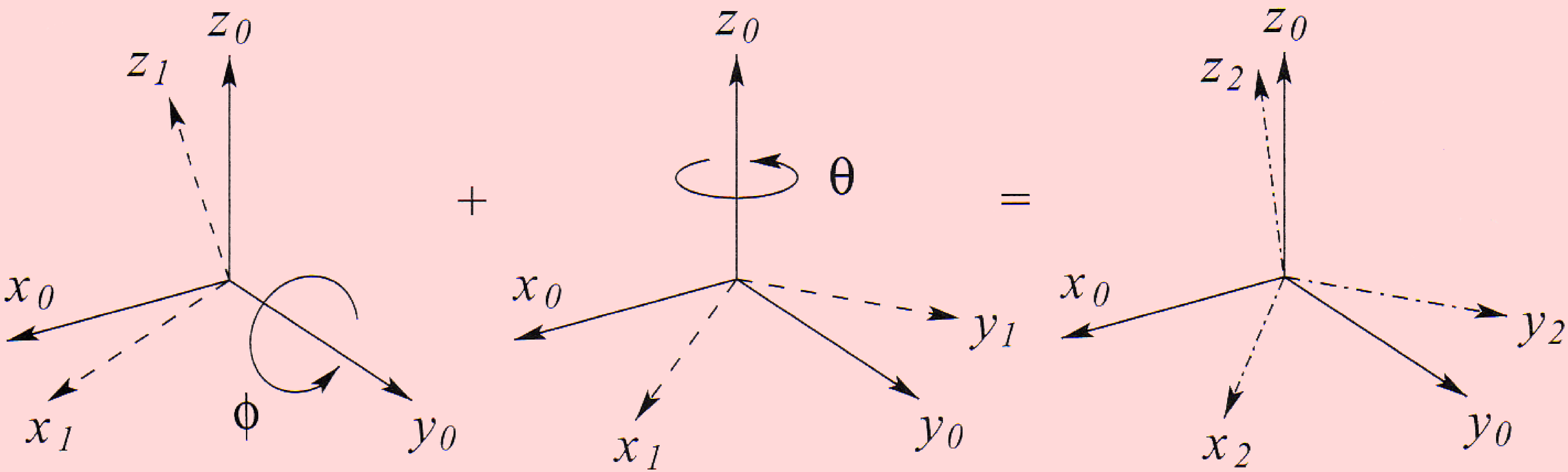
Composition of Rotations with Respect to the Current Frame

the result of a successive rotation about the current (intermediate) frame can be found by **post-multiplying** by the corresponding rotation matrix

$$\mathbf{R}_2^0 = \mathbf{R}_1^0\,\mathbf{R}_2^1$$



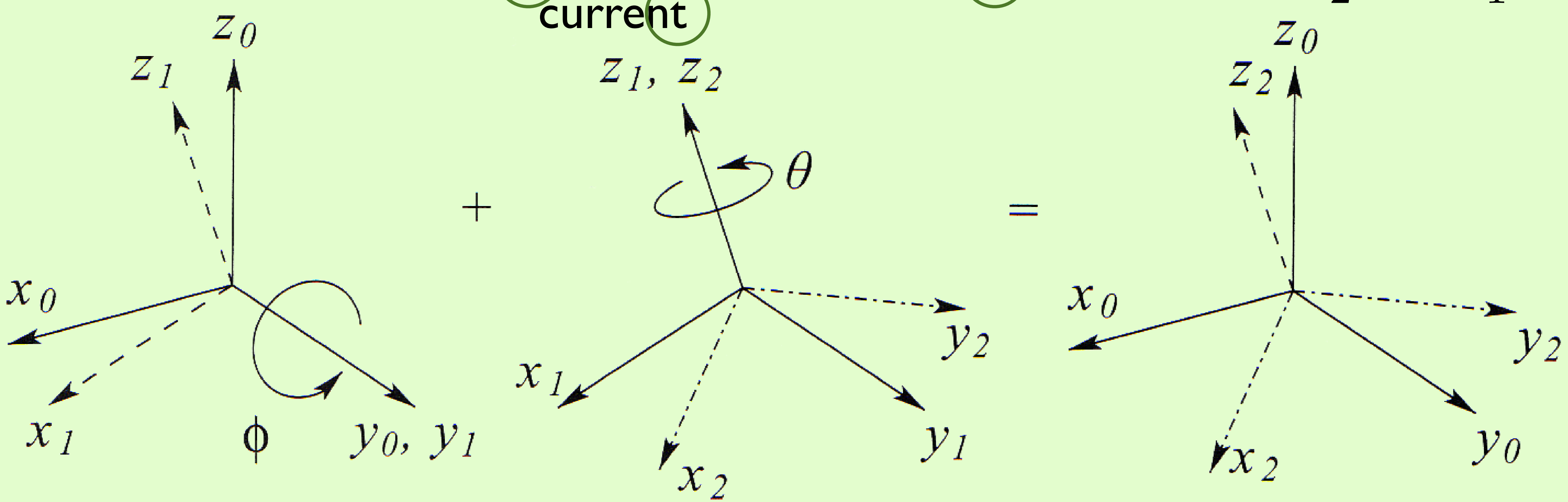
successive rotation about fixed frame? **pre-multiply** $\mathbf{R}_2^0 = \mathbf{R}\,\mathbf{R}_1^0$



Which of these is more commonly used in robotics?

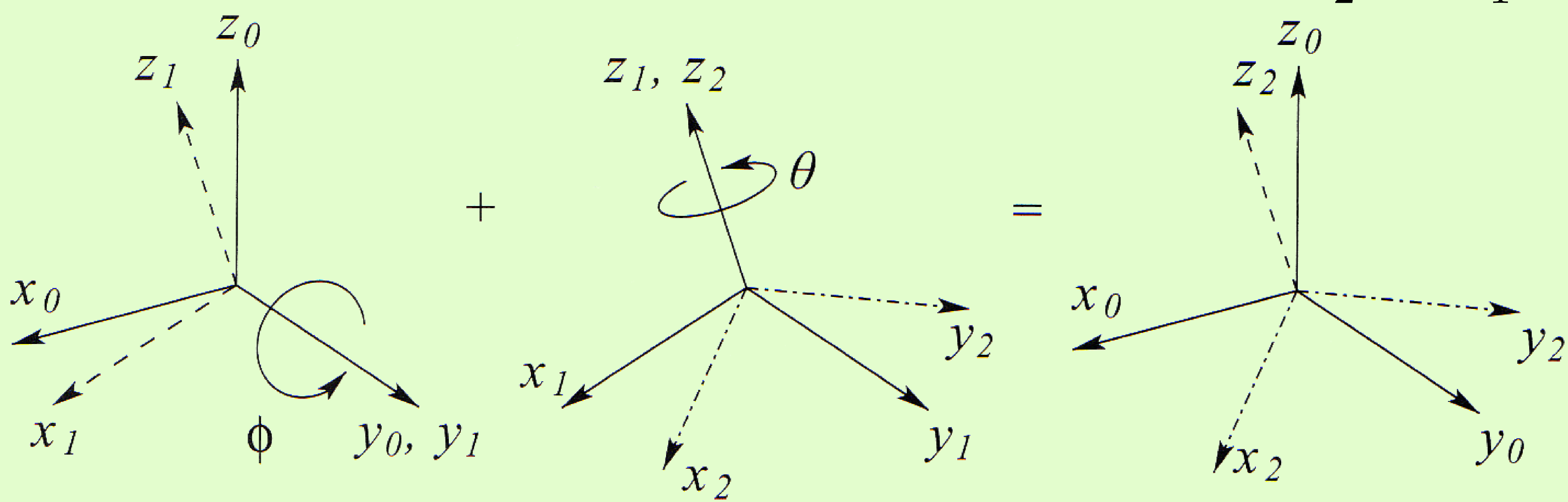
successive rotation about intermediate frame? **post-multiply** $\mathbf{R}_2^0 = \mathbf{R}_1^0\,\mathbf{R}_2^1$

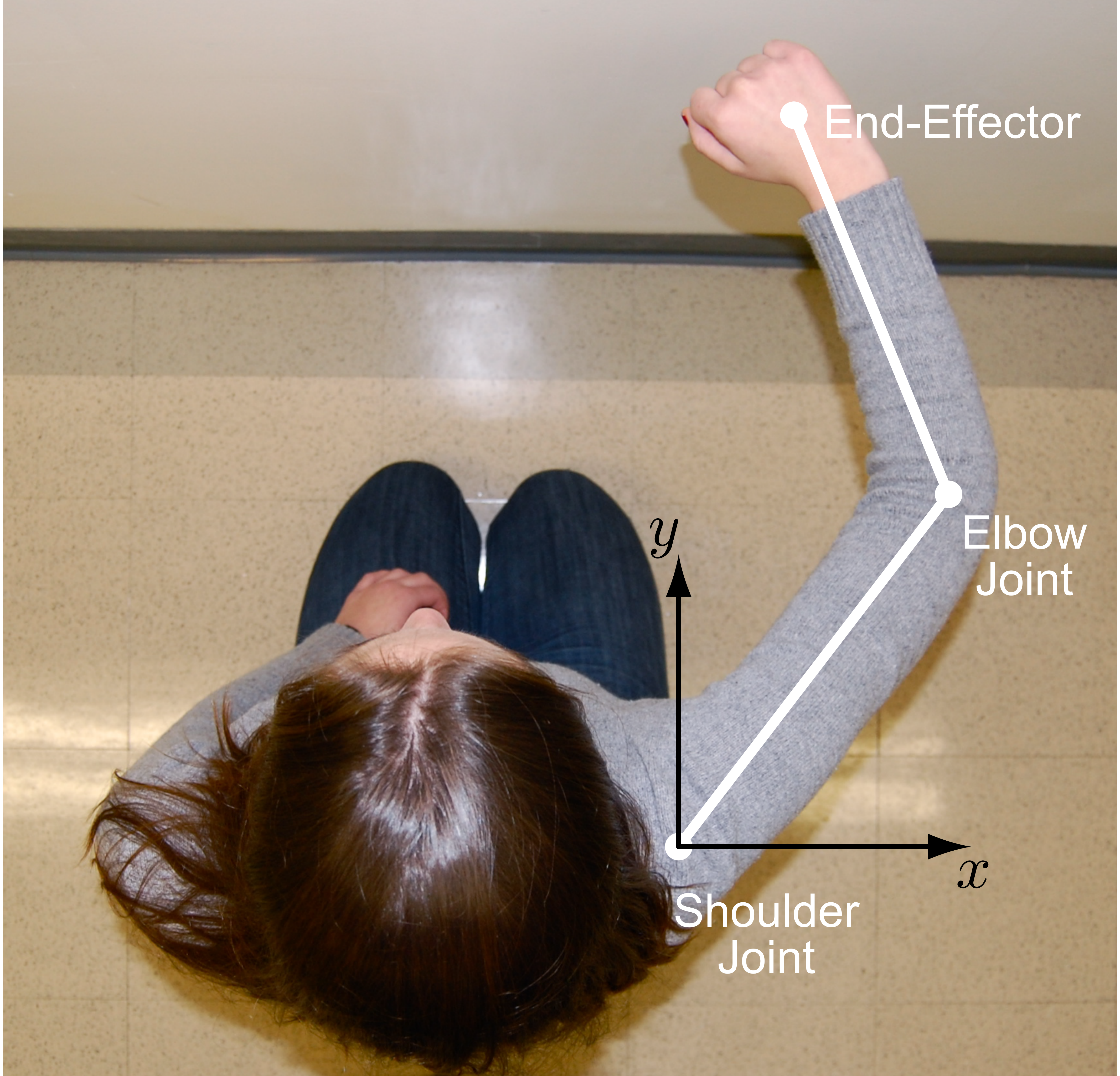
current



Which of these is more commonly used in robotics?

successive rotation about intermediate frame? **post-multiply** $\mathbf{R}_2^0 = \mathbf{R}_1^0 \, \mathbf{R}_2^1$

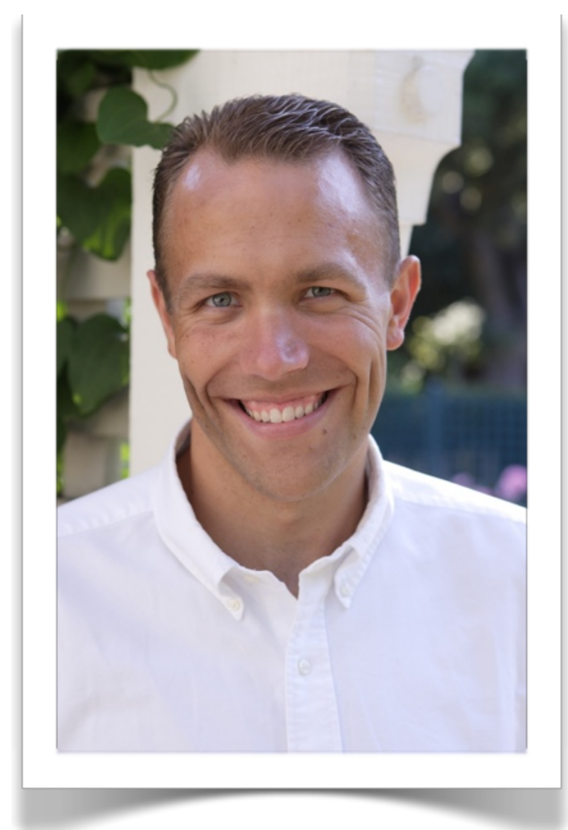



End-Effector
Elbow
Joint
y
x
Shoulder
Joint

You should read SHV 2.4 and think about this.

Parameterizing Rotations

These slides are adapted from ones created by Jonathan Fiene for MEAM 520 in Spring 2012.



SHV 2.5

Parameterization of Rotations

$$\mathbf{R}_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

in three dimensions, no more than 3 values are needed to specify an arbitrary rotation

which means that the 9-element rotation matrix has at least 6 redundancies

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numerous methods have been developed to represent rotation/orientation with less redundancy

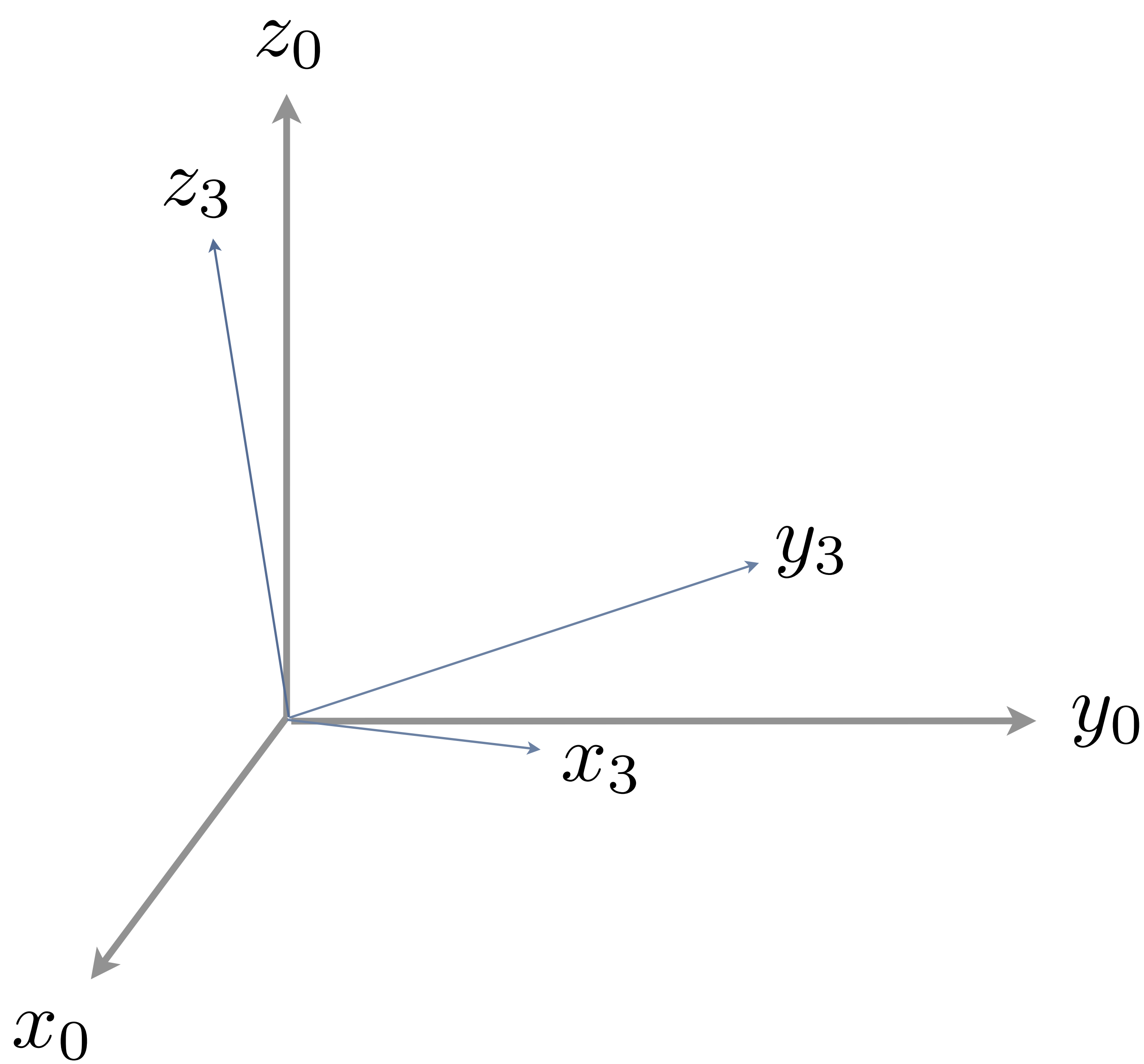
Euler Angles

Roll, Pitch, Yaw Angles

Axis/Angle Representation

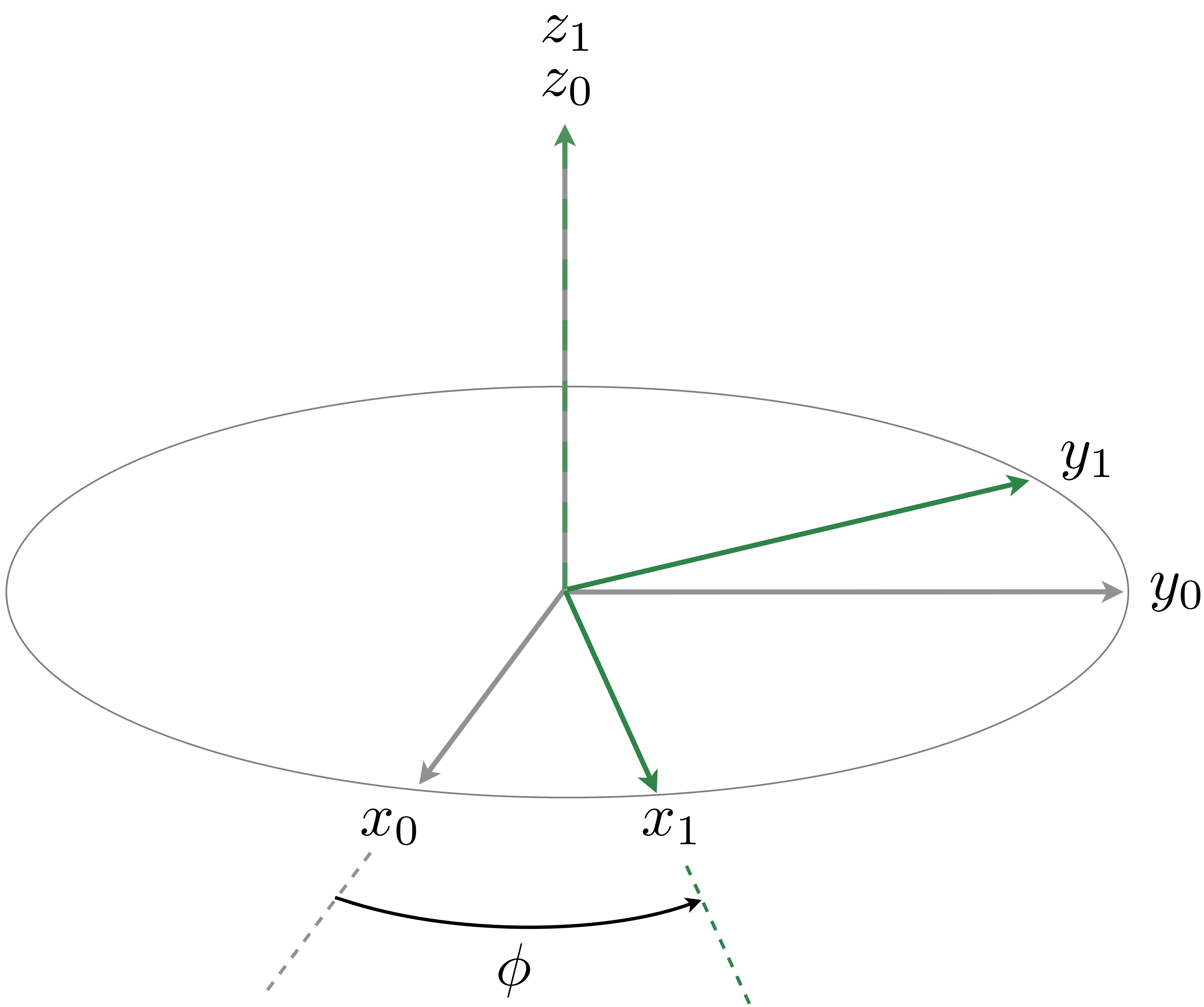
Conventions vary, so always check definitions!

Define a set of three **intermediate** angles, ϕ, θ, ψ , to go from $0 \rightarrow 3$

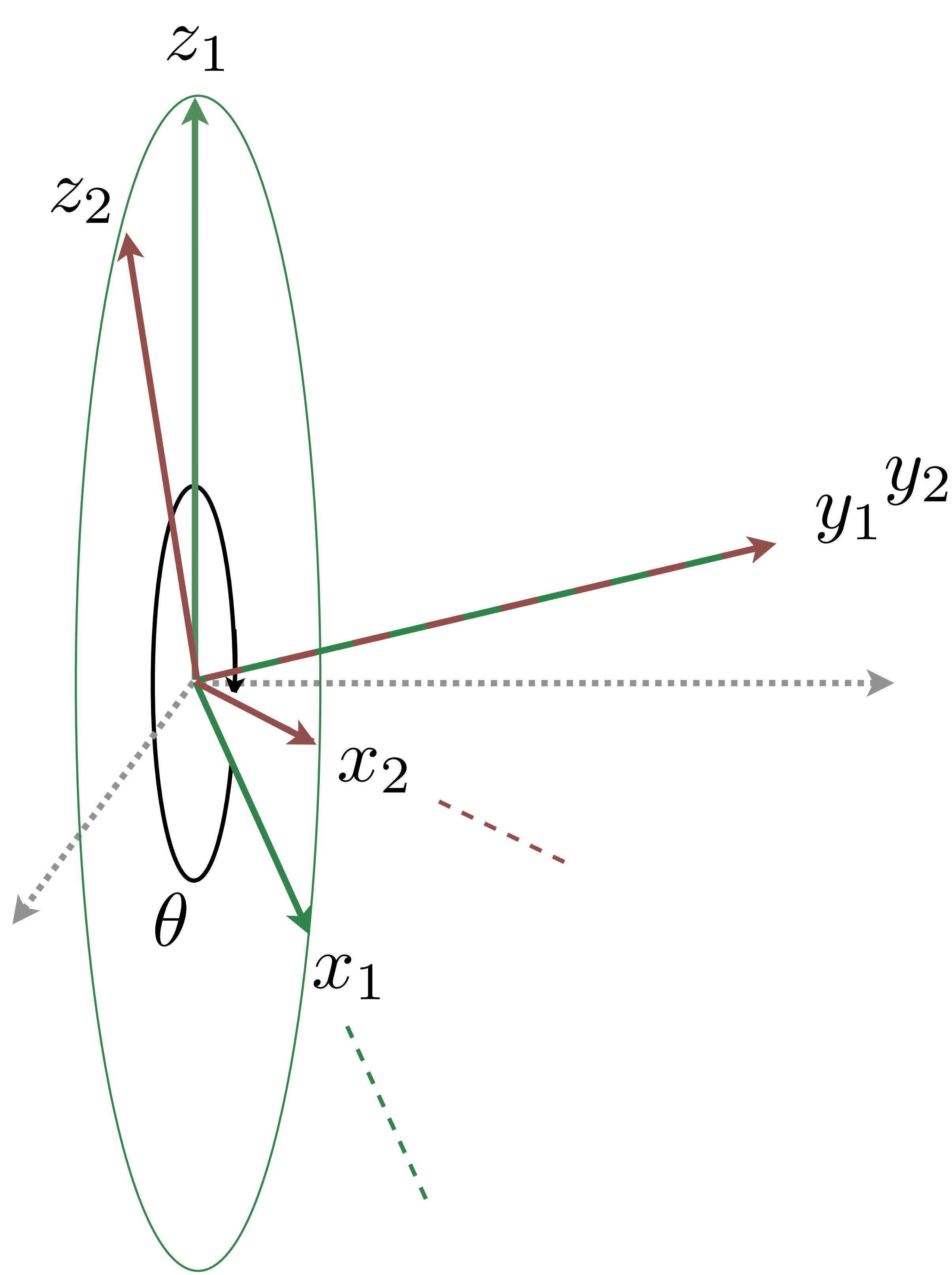


Our book uses a Z-Y-Z convention for Euler angles.

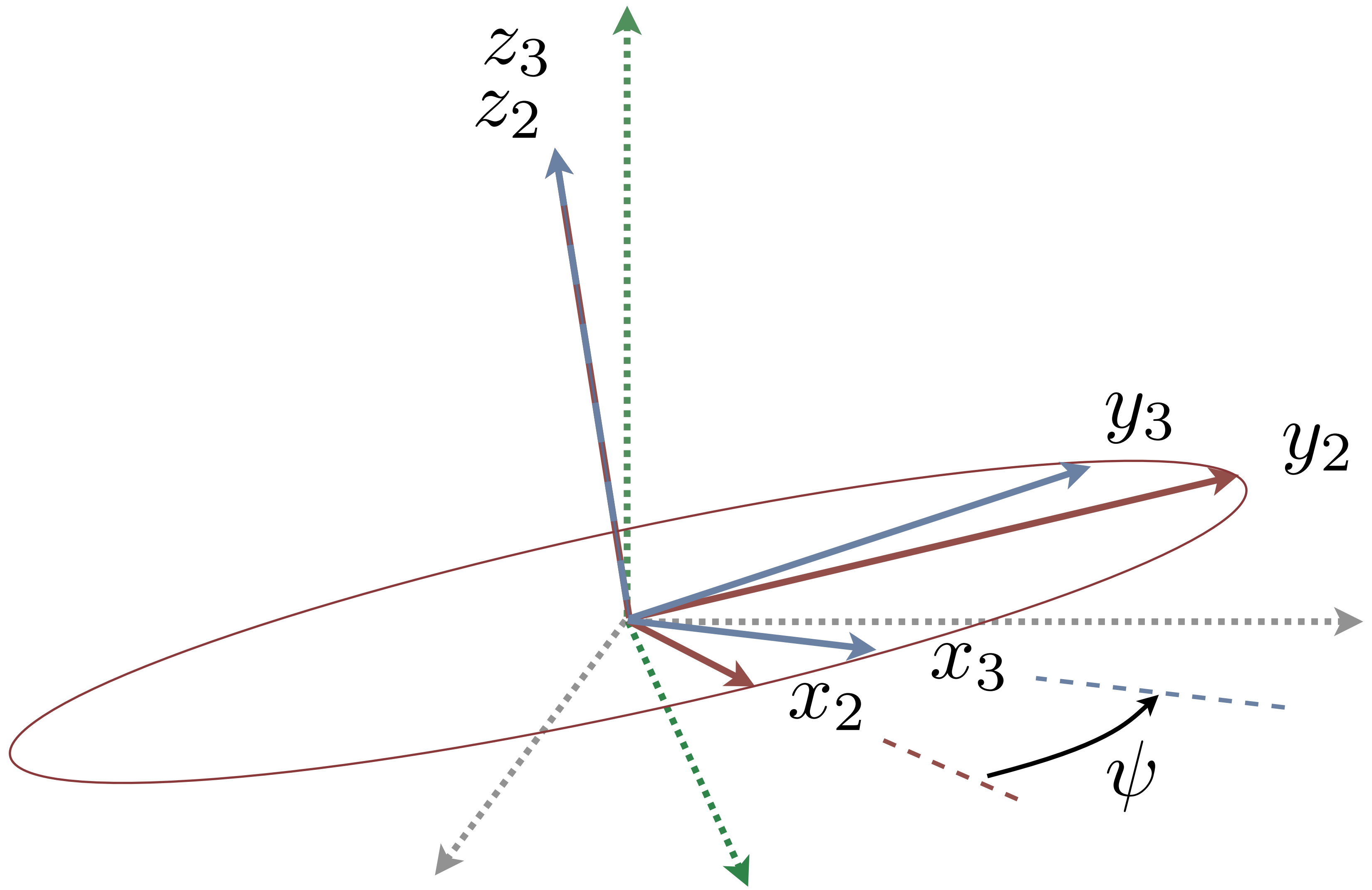
step 1: rotate by ϕ about z_0



step 2: rotate by θ about y_1

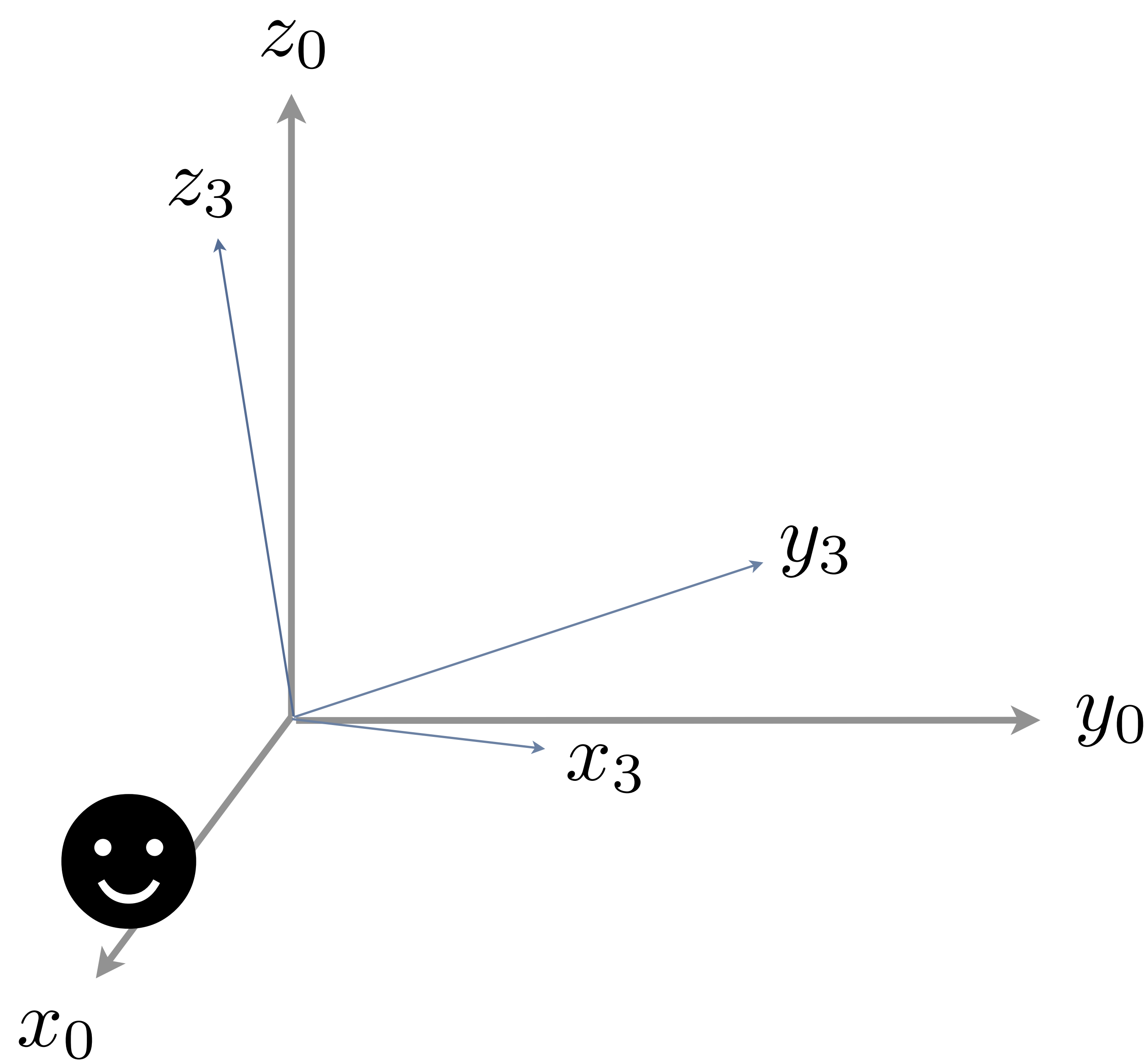


step 3: rotate by ψ about z_2



Euler Angles

Define a set of three **intermediate** angles, ϕ, θ, ψ , to go from $0 \rightarrow 3$



Think about looking out the x-axis with your head, looking left/right, then tilting up/down, then spinning your head around its long axis.

Should we pre- or post-multiply the successive rotations?

Euler Angles to Rotation Matrix

(**post**-multiply using the **basic rotation matrices**)

$$\mathbf{R} = \mathbf{R}_{z,\phi}\,\mathbf{R}_{y,\theta}\,\mathbf{R}_{z,\psi}$$

$$s_\theta = \sin\theta$$

$$c_\theta = \cos\theta$$

$$= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

Rotation Matrix to Euler Angles?

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Use atan2 to determine ϕ for both θ options

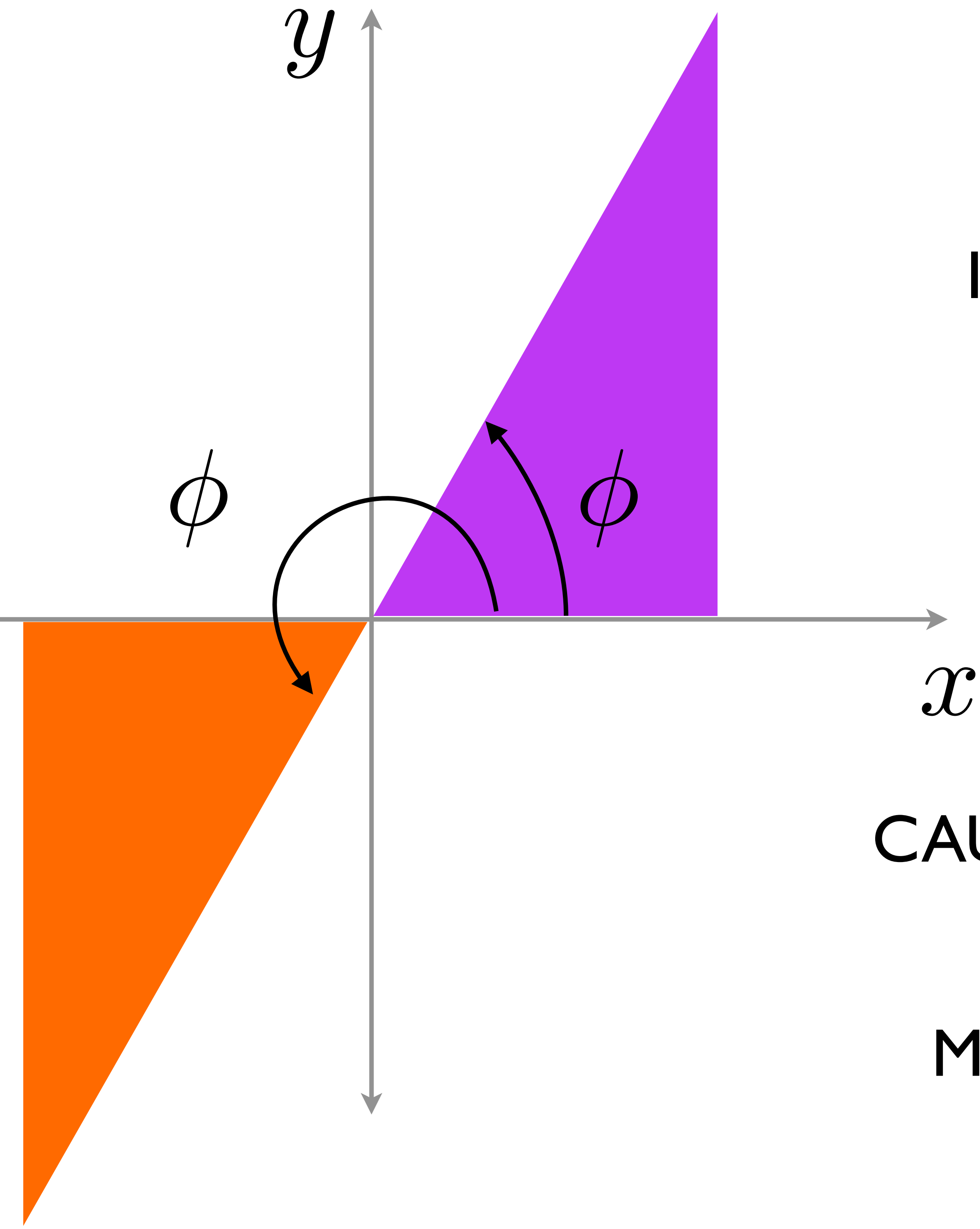
$$= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

Use atan2 to determine ψ for both θ options

Two solutions for θ because sign of s_θ is not known.

$$\phi = ? \quad \theta = ? \quad \psi = ?$$

atan2



atan2 is the two-argument inverse tangent function.

It preserves the signs of the numerator and denominator, returning angles in all four quadrants.

Read Appendix A.1 to learn more.

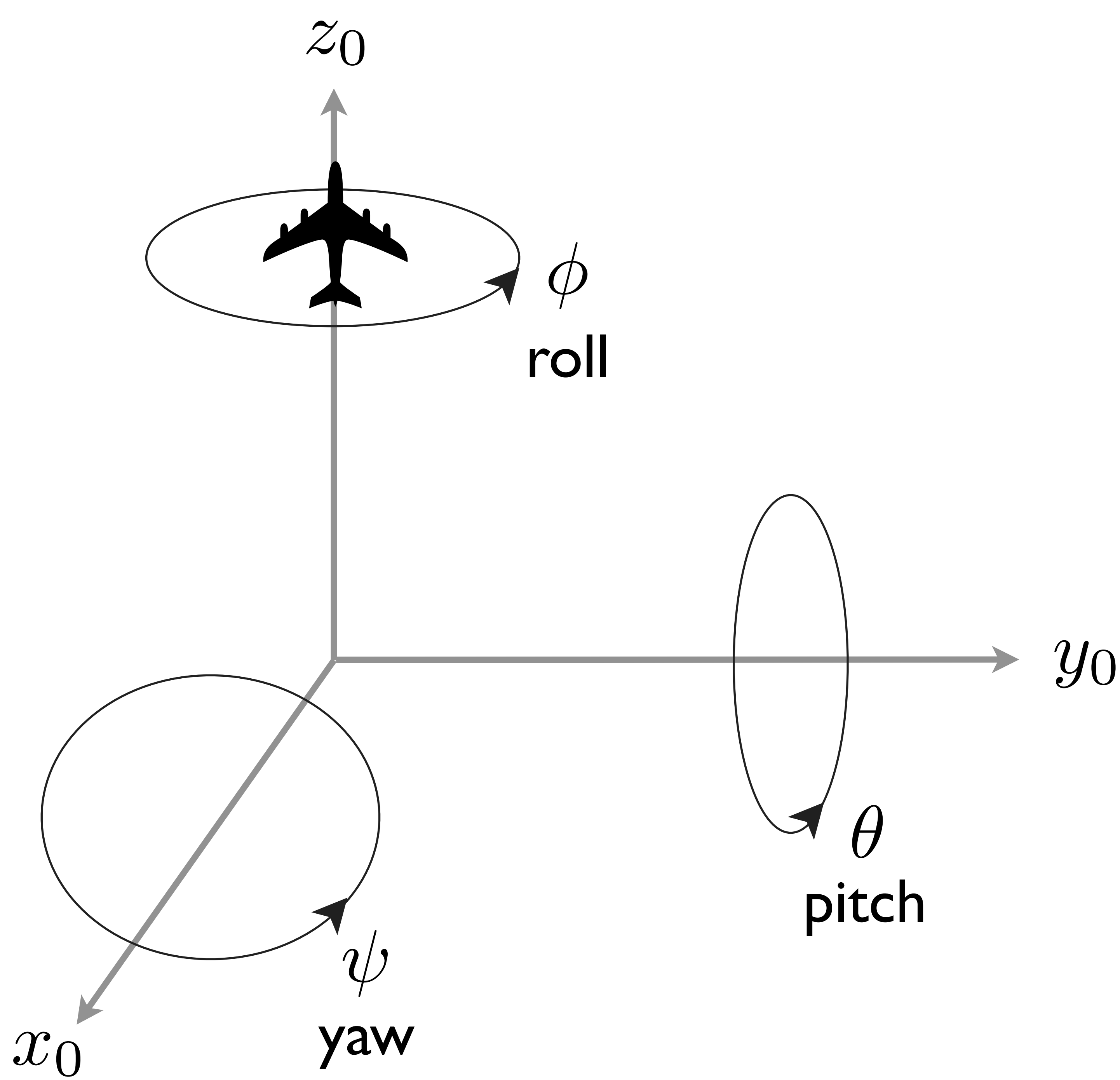
CAUTION: Our book lists atan2 arguments in the opposite order of MATLAB.

MATLAB expects atan2(num, den) or similarly atan2(y, x).

SHV uses atan2(den, num) or similarly atan2(x, y).

defined as a set of three angles about a **fixed** reference

Our book uses an X-Y-Z convention for Yaw, Pitch, Roll angles.



Think about being a plane flying in the z-axis direction. Yaw is turning left/right, pitch is tilting up/down, and roll rotates around travel direction.

Should we pre- or post-multiply the successive rotations?

Roll, Pitch, Yaw Angles to Rotation Matrices

(**pre**-multiply using the **basic rotation matrices**)

$$\mathbf{R} = \mathbf{R}_{z,\phi}\,\mathbf{R}_{y,\theta}\,\mathbf{R}_{x,\psi}$$

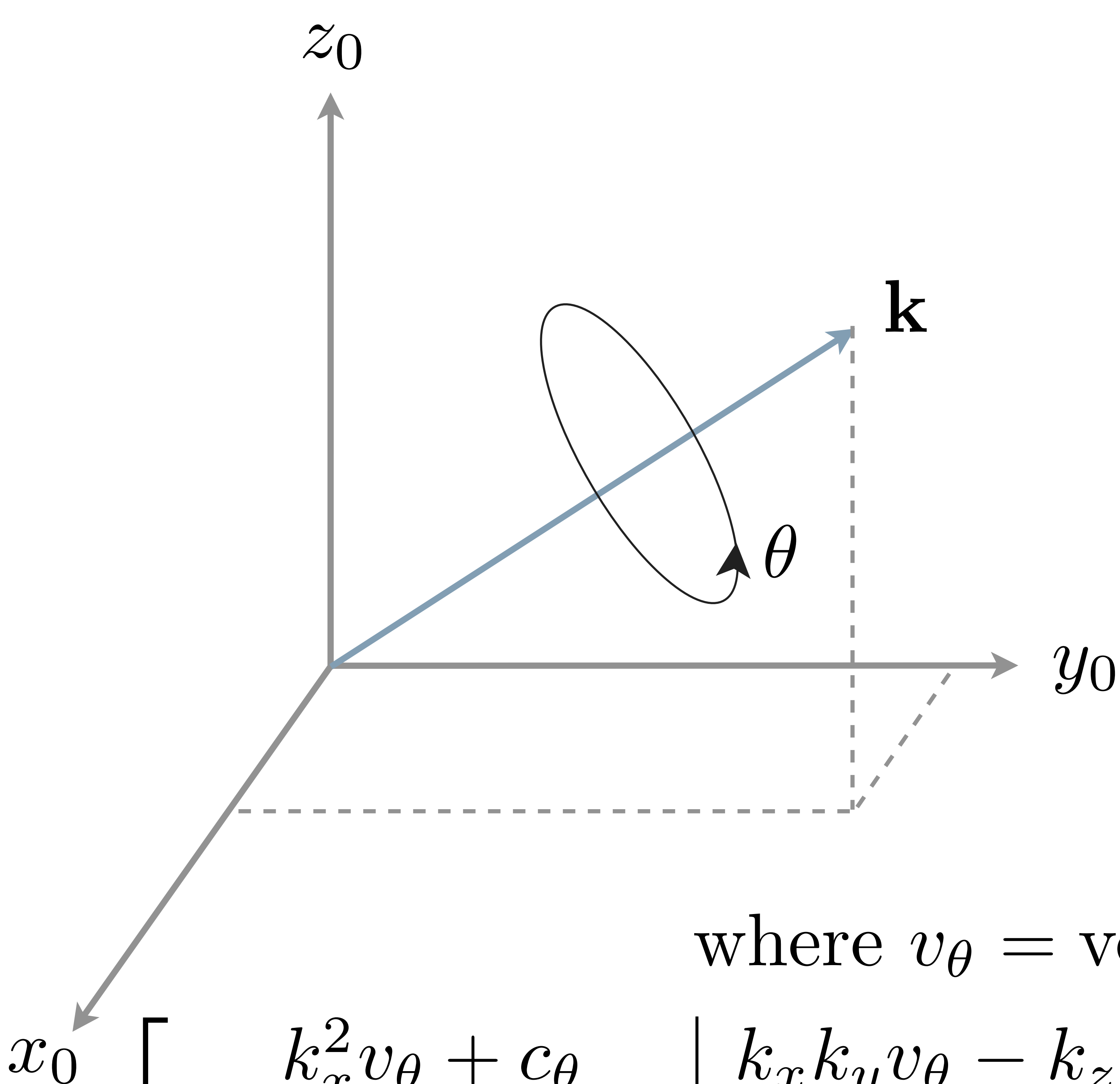
$$= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix}$$

$$= \begin{bmatrix} c_\phi c_\theta & c_\phi s_\theta s_\psi - s_\phi c_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}$$

You can convert from a rotation matrix to these angles in a manner similar to the procedure for Euler angles.

Axis/Angle Representation

rotation by an angle about an axis in space



$$\mathbf{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}$$

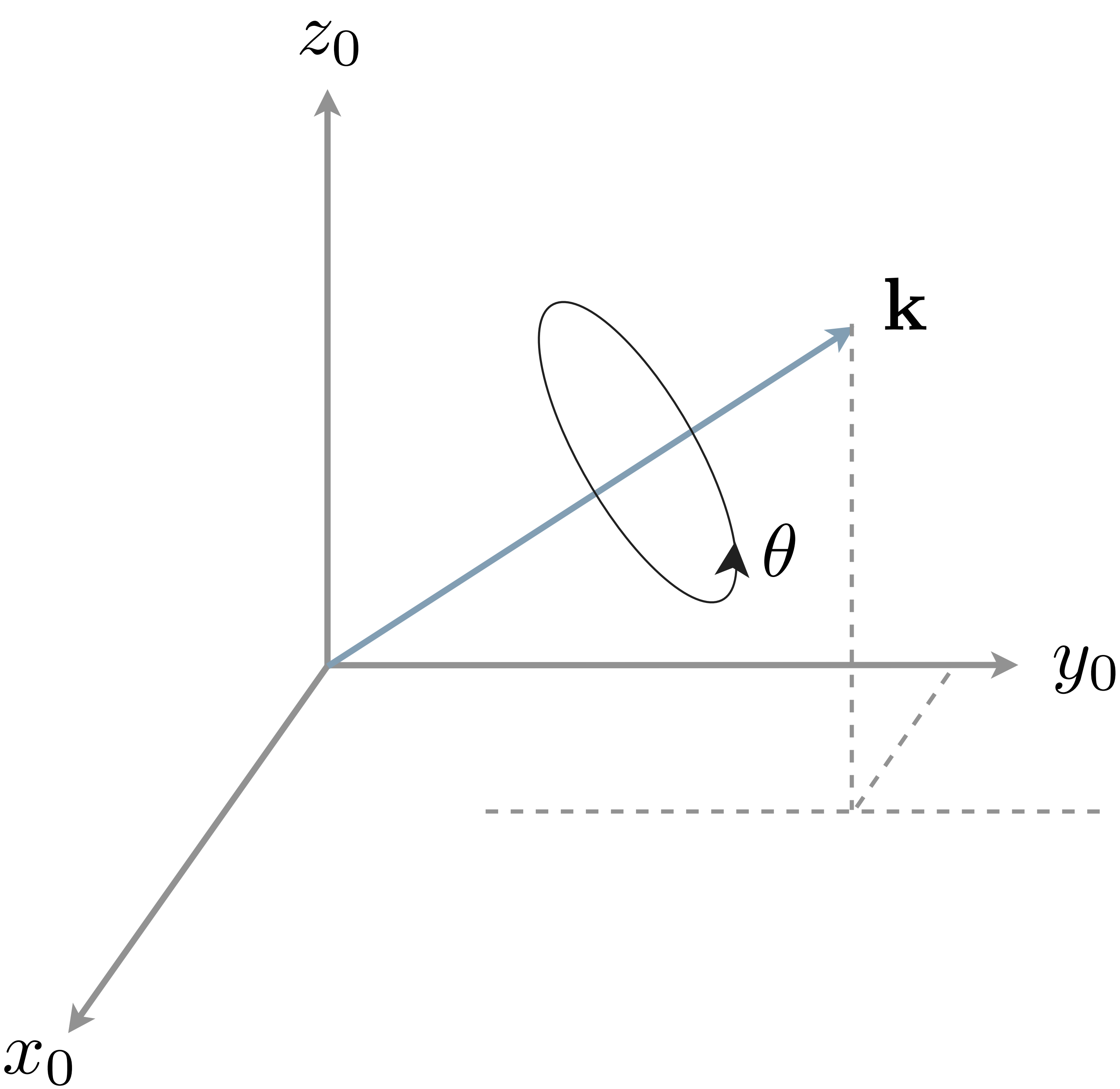
$$k_x^2 + k_y^2 + k_z^2 = 1$$

$$\text{where } v_\theta = \operatorname{vers}\theta = 1 - \cos\theta$$

$$R_{k,\theta} = \left[\begin{array}{c|c|c} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{array}\right]$$

Axis/Angle Representation

any rotation matrix can be represented this way!



$$R = R_{k,\theta}$$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Is axis/angle solution unique?

No. $R_{k,\theta} = R_{-k,-\theta}$

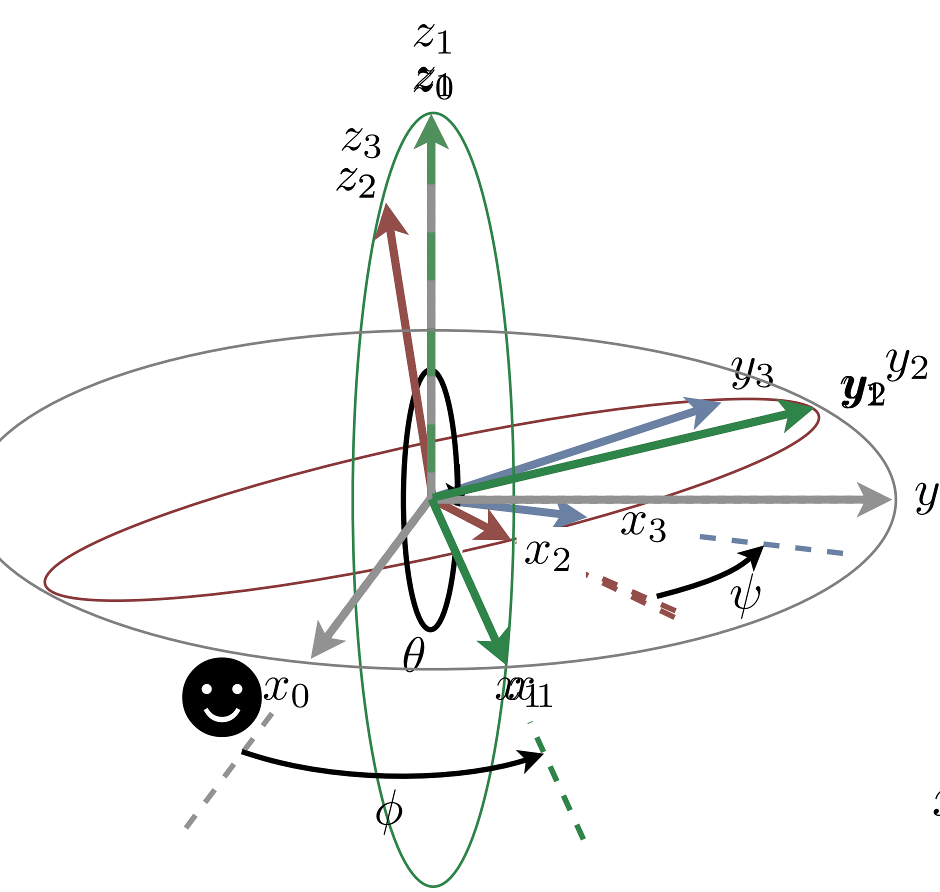
$$\theta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

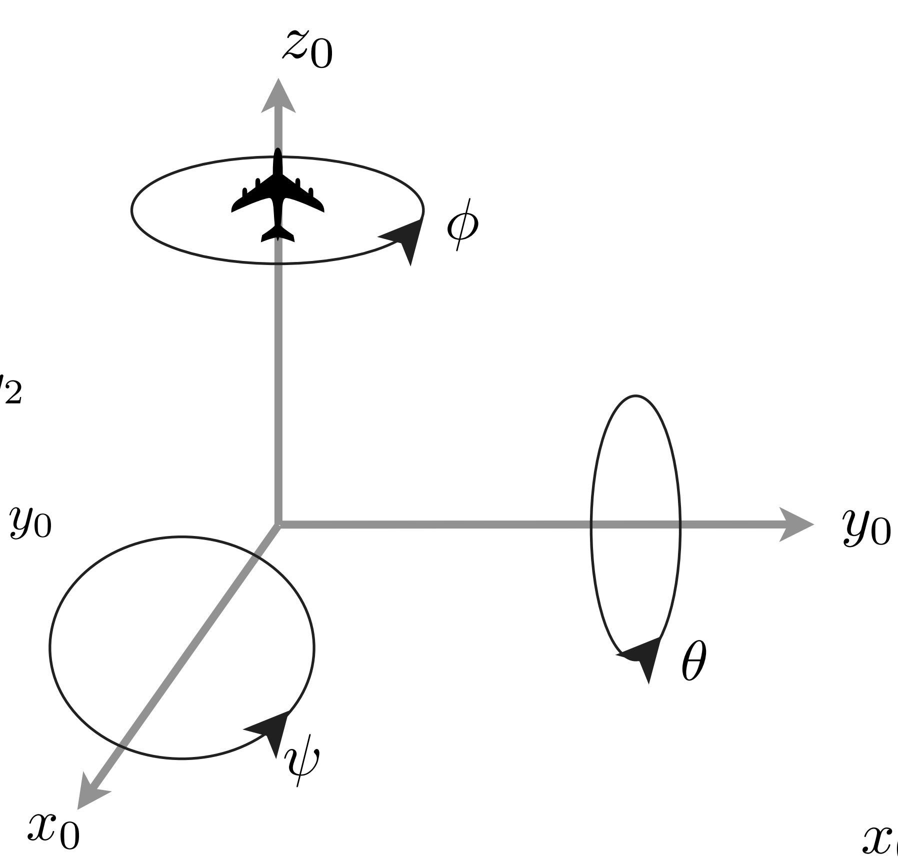
$$k = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

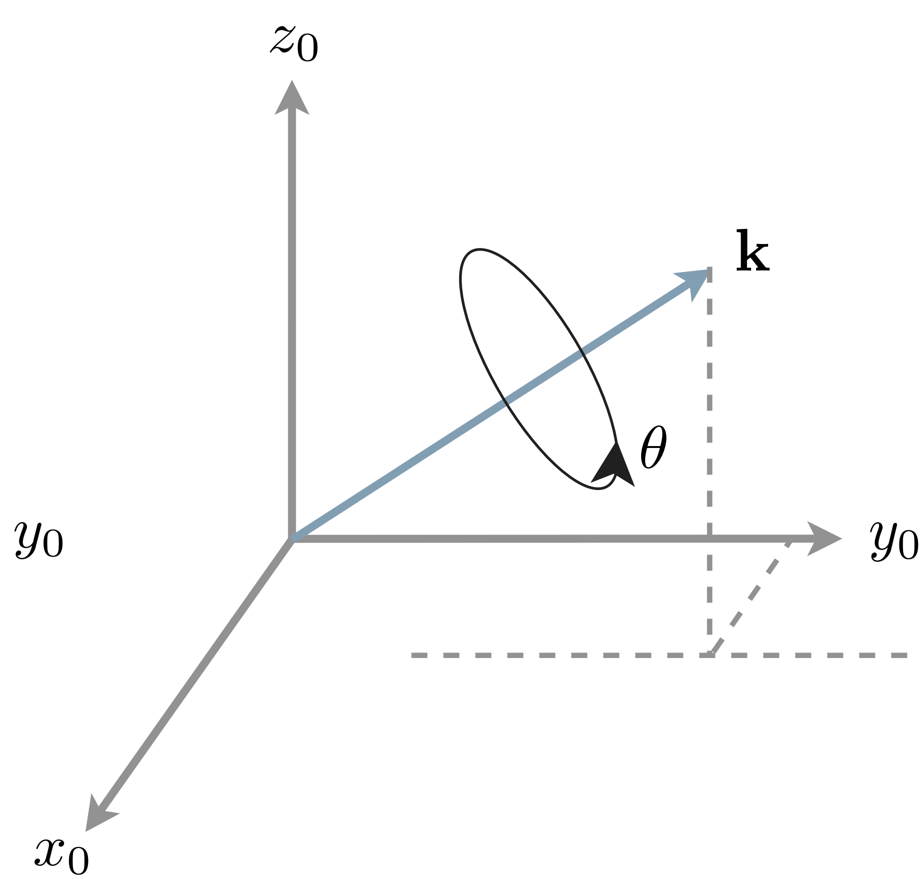
Talk to the person next to you.

Explain one of the three parameterization approaches to your partner, then switch.

Talk about the third one together.







Questions ?