Name Solution

Midterm Exam

MEAM 520, Introduction to Robotics
University of Pennsylvania
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You must take this exam independently, without assistance from anyone else. You may bring in a calculator and two 8.5" ×11" sheets of notes for reference. Aside from these two pages of notes, you may not consult any outside references, such as the textbook or the Internet. Any suspected violations of Penn's Code of Academic Integrity will be reported to the Office of Student Conduct for investigation.

This exam consists of several problems. We recommend you look at all of the problems before starting to work. If you need clarification on any question, please ask a member of the teaching team. When you work out each problem, please show all steps and $\boxed{\text{box your answer}}$. On problems involving actual numbers, please keep your solution symbolic for as long as possible; this will make your work easier to follow and easier to grade. The exam is worth a total of 100 points, and partial credit will be awarded for the correct approach even when you do not arrive at the correct answer.

	Points	Score
Problem 1	20	
Problem 2	20	
Problem 3	15	
Problem 4	20	
Problem 5	25	
Total	100	

I agree to abide by the University of Pennsylvania Code of Academic Integrity during this exam. I pledge that all work is my own and has been completed without the use of unauthorized aid or materials.

Signature ______________________

Date ______________________

Problem 1: Short Answer *(20 points)*

a. Why is it important to indicate the **coordinate frame** in which a vector is expressed, e.g., p^0? *(2 points)*

Vectors must be expressed in the same frame to be added or subtracted. The superscript lowers the likelihood of mistakes and confusion.

b. Compared to other methods such as Euler Angles, explain one **advantage** of using a 3×3 **rotation matrix** for representing the orientation of a three-dimensional object. *(2 points)*

- It is easy to look at a rotation matrix and visualize the orientation.
- A rotation matrix is directly useful for calculations.

c. Compared to other methods such as Euler Angles, explain one **disadvantage** of using a 3×3 **rotation matrix** for representing the orientation of a three-dimensional object. *(2 points)*

- Requires more storage space on your hard drive (9 floats rather than 3).
- Susceptible to numerical rounding, which may make its determinant not precisely 1.

d. Serial robotic manipulators are composed of revolute and prismatic joints. Explain one advantage of using a **revolute** joint instead of a prismatic joint in a robot. *(2 points)*

- More compact.
- Easier to manufacture.
- Easier to actuate with a rotating actuator
- Can change orientation of end-effector.

e. Explain one advantage of using a **prismatic** joint instead of a revolute joint. *(2 points)*

- The forward and inverse kinematics are much easier to calculate.
- Less susceptible to singularities
- Workspace can be rectangular

f. Describing a rigid-body transformation in three dimensions generally requires six numbers. Why then are only four DH parameters (a, α, d, θ) needed to describe link i's pose relative to link $i-1$ in a serial manipulator? *(4 points)*

The DH convention includes two constraint we must follow when placing our frames: ① The x_i axis is perpendicular to the z_{i-1} axis
② The x_i axis intersects the z_{i-1} axis.

These 2 constraints reduce the degrees of freedom from 6 to 4.

g. In the DH convention, the homogeneous transformation representing one step in a serial linkage chain can be represented as a product of four basic transformations, as follows:

$$A_i = \mathrm{Rot}_{z,\theta_i}\mathrm{Trans}_{z,d_i}\mathrm{Trans}_{x,a_i}\mathrm{Rot}_{x,\alpha_i}$$

Here, Rot indicates a rotation around the subscripted axis by the noted angle, and Trans indicates a translation along the specified axis by the noted distance.

Which pairs of the four matrices on the right-hand side commute? Explain why these pairs commute. Find all permutations of these four matrices that yield the same homogeneous transformation matrix, A_i. You do not need to calculate the product. *(6 points)*

Rotation about and translation along the same axis commute because the axis stays the same. Successive translations commute because the frame does not rotate.

All permutations: $\mathrm{Rot}_{z,\theta}\,\mathrm{Trans}_{z,d}\,\mathrm{Trans}_{x,a}\,\mathrm{Rot}_{x,\alpha}$ (orig

$\mathrm{Trans}_{z,d}\,\mathrm{Rot}_{z,\theta}\,\mathrm{Trans}_{x,a}\,\mathrm{Rot}_{x,\alpha}$

$\mathrm{Rot}_{z,\theta}\,\mathrm{Trans}_{z,d}\,\mathrm{Rot}_{x,\alpha}\,\mathrm{Trans}_{x,a}$

$\mathrm{Trans}_{z,d}\,\mathrm{Rot}_{z,\theta}\,\mathrm{Rot}_{x,\alpha}\,\mathrm{Trans}_{x,a}$

$\mathrm{Rot}_{z,\theta}\,\mathrm{Trans}_{x,a}\,\mathrm{Trans}_{z,d}\,\mathrm{Rot}_{x,\alpha}$

Extra explanation using matrices:

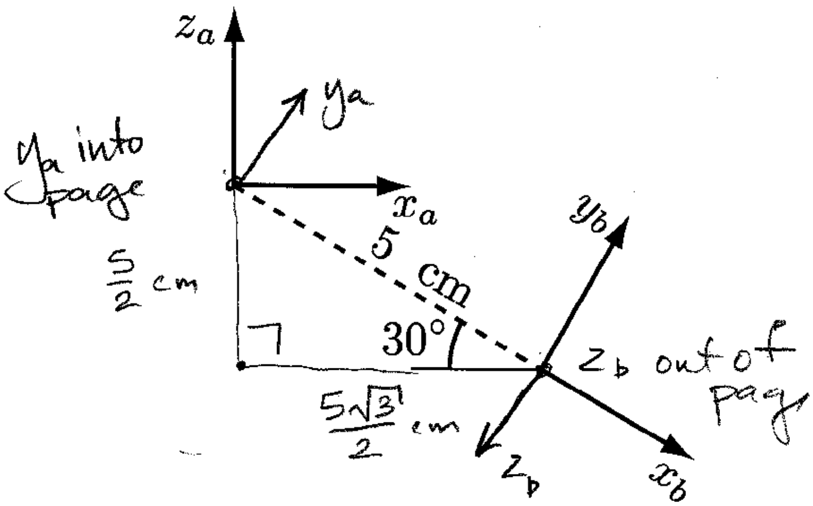
$$\begin{bmatrix} R_z & \begin{matrix}0\\0\\0\end{matrix} \\ 0\ 0\ 0 & 1 \end{bmatrix}\begin{bmatrix} I & d_z \\ 0\ 0\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_z & R_z d_z \\ 0\ 0\ 0 & 1 \end{bmatrix}$$

$$R_z = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad d_z = \begin{bmatrix} 0 \\ 0 \\ d_z \end{bmatrix}$$

$R_z d_z = d_z$ b/c rotating around z does not change z coordinate

Problem 2: *(20 points)*

The diagram below depicts two coordinate frames in three dimensions. Note that all of the unit vectors shown and the dashed line segment labeled with a length are in the same plane.

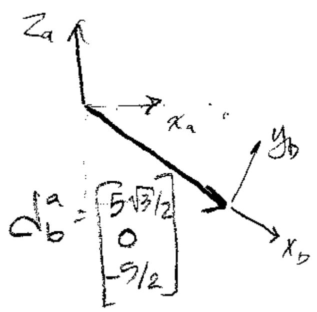


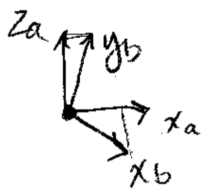
location of origin of frame b in frame a

a. What is H_b^a, the homogeneous transformation representing the position and orientation of frame b in frame a? *(10 points)*

$$H_b^a = \begin{bmatrix} & R_b^a & & d_b^a \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 & 5\sqrt{3}/2\text{ cm} \\ 0 & 0 & -1 & 0\text{ cm} \\ -1/2 & \sqrt{3}/2 & 0 & -5/2\text{ cm} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

x_b expressed in frame a — y_b in frame a — z_b expressed in frame a





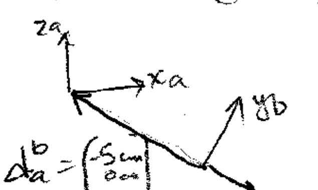
x_a in frame b — y_a in frame b — z_a in frame b — location of origin in frame b

b. What is H_a^b? *(10 points)*

$$H_a^b = \begin{bmatrix} & R_a^b & & d_a^b \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 0 & -1/2 & -5\text{cm} \\ 1/2 & 0 & \sqrt{3}/2 & 0\text{cm} \\ 0 & -1 & 0 & 0\text{cm} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_a^b = (R_b^a)^T \qquad d_a^b = -R_a^b d_b^a$$

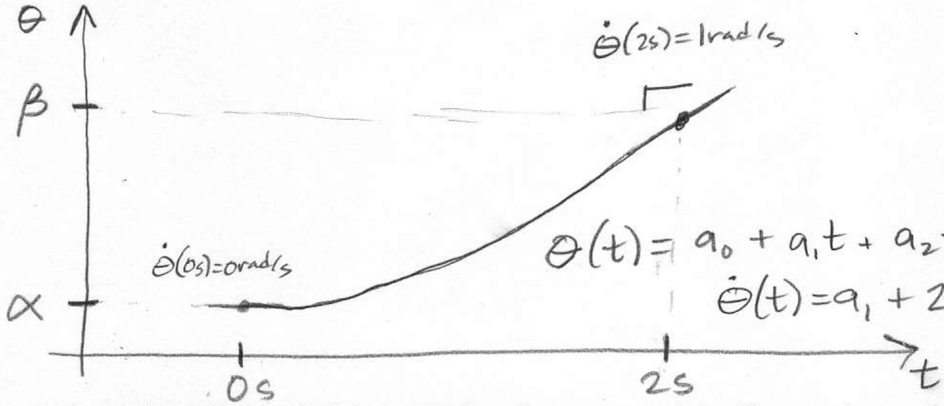
$$= \begin{bmatrix} \sqrt{3}/2 & 0 & -1/2 \\ 1/2 & 0 & \sqrt{3}/2 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -5\sqrt{3}/2\text{ cm} \\ 0 \\ 5/2\text{ cm} \end{bmatrix} = \begin{bmatrix} -\frac{15}{4}\text{cm} - \frac{5}{4}\text{cm} \\ -\frac{5\sqrt{3}}{4}\text{cm} + \frac{5\sqrt{3}}{4}\text{cm} \\ 0\text{cm} \end{bmatrix} = \begin{bmatrix} -\frac{20}{4}\text{cm} \\ 0\text{cm} \\ 0\text{cm} \end{bmatrix} = \begin{bmatrix} -5\text{cm} \\ 0\text{cm} \\ 0\text{cm} \end{bmatrix}$$



Problem 3: Calculating a Trajectory *(15 points)*

We seek a trajectory $\theta(t)$ that will move a revolute robot joint from rest (not moving) at position α at time $t = 0$ s to be at position β in 2 seconds with a final velocity of 1 rad/s. Assume $\alpha < \beta$.

a. **Roughly sketch the trajectory** as a function of time, labeling the constraints. *(3 points)*



$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$
$$\dot\theta(t) = a_1 + 2a_2 t + 3a_3 t^2$$

b. Compute a **cubic polynomial** that satisfies these constraints. Be sure to provide the correct units for all coefficients in the equation. *(12 points)*

$$\theta(0s) = \alpha = a_0 + \cancel{a_1(0s)} + \cancel{a_2(0s)^2} + \cancel{a_3(0s)^3}$$
$$\boxed{a_0 = \alpha}$$

$$\dot\theta(0s) = 0\,\text{rad/s} = a_1 + \cancel{2a_2(0s)} + \cancel{3a_3(0s)^2}$$
$$\boxed{a_1 = 0\,\text{rad/s}}$$

$$\theta(2s) = \beta = \alpha + \cancel{0\,\text{rad/s}(2s)} + a_2(2s)^2 + a_3(2s)^3$$
$$\beta - \alpha = 4s^2 \cdot a_2 + 8s^3 \cdot a_3 \quad (*)$$

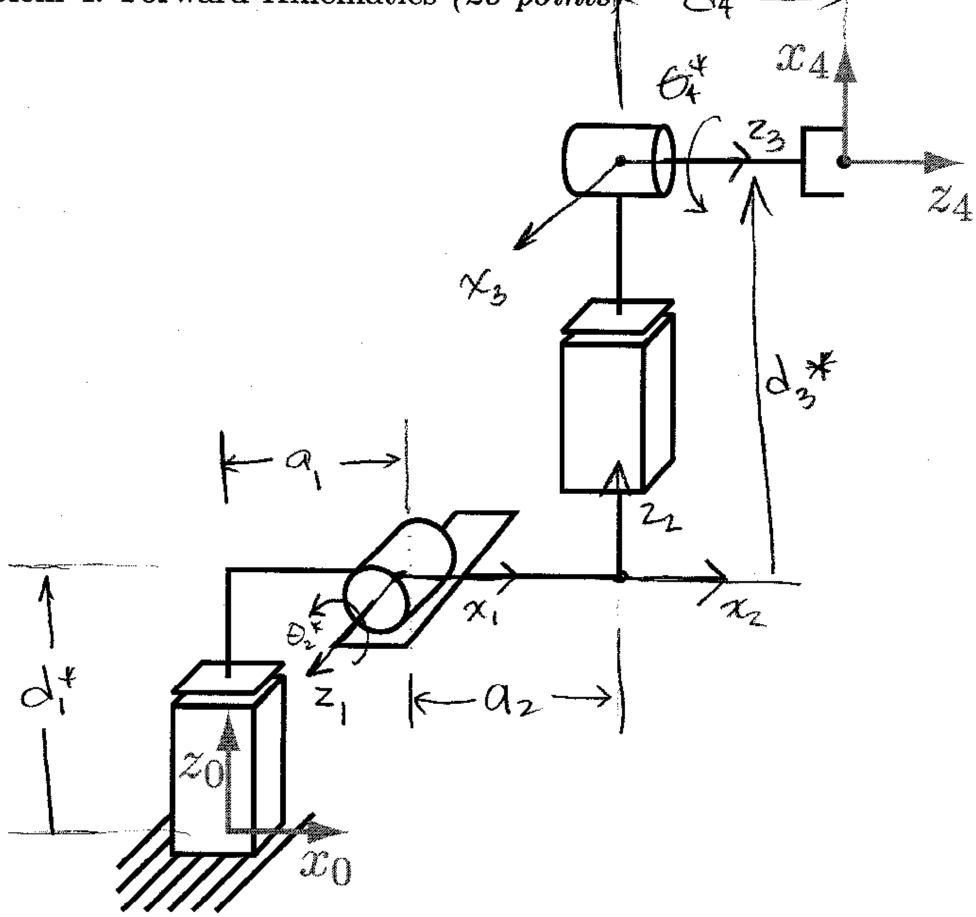
$$\dot\theta(2s) = 1\,\text{rad/s} = \cancel{0\,\text{rad/s}} + 2a_2(2s) + 3a_3(2s)^2$$
$$1\,\text{rad/s} = 4s \cdot a_2 + 12s^2 \cdot a_3 \quad (*)$$ 2 eqns, 2 unknowns (a_2, a_3)
$$4s \cdot a_2 = 1\,\text{rad/s} - 12s^2 \cdot a_3$$
$$a_2 = 0.25\,\text{rad/s}^2 - 3s \cdot a_3$$

$(*)$
$$\beta - \alpha = 4s^2(0.25\,\text{rad/s}^2 - 3s \cdot a_3) + 8s^3 \cdot a_3$$
$$\beta - \alpha = 1\,\text{rad} - 12s^3 \cdot a_3 + 8s^3 \cdot a_3 = 1\,\text{rad} - 4s^3 a_3$$
$$4s^3 a_3 = 1\,\text{rad} + \alpha - \beta$$
$$\boxed{a_3 = \frac{(1\,\text{rad} + \alpha - \beta)}{4s^3} = 0.25\,\frac{\text{rad}}{s^3} + 0.25\,\frac{\text{rad}}{s^3}(\alpha - \beta)}$$

$$a_2 = 0.25\,\frac{\text{rad}}{s^2} - 3s \cdot a_3$$
$$a_2 = 0.25\,\frac{\text{rad}}{s^2} - 3s\left(\frac{1+\alpha-\beta}{4s^3}\right)$$
$$= 0.25\,\frac{\text{rad}}{s^2} - 0.75\,\frac{\text{rad}}{s^2} + \frac{3(\beta-\alpha)}{4s^2}$$
$$= -0.5\,\frac{\text{rad}}{s^2} + 0.75\,s^{-2}(\beta-\alpha)$$
$$\boxed{a_2 = -0.5\,\frac{\text{rad}}{s^2} + 0.75\,s^{-2}(\beta-\alpha)}$$

$$\boxed{\theta(t) = \alpha + 0\,\frac{\text{rad}}{s}t + a_2 t^2 + a_3 t^3}$$
cubic polynomial!

Problem 4: Forward Kinematics *(20 points)*



a. Draw **frames 1 through 3** on the above diagram, following the DH convention. *(4 points)*

b. The diagram shows both prismatic joints extended to a positive displacement, while both revolute joints are shown at zero. Fill in the table of **DH parameters** below. Use a superscript star to indicate joint variables, e.g., d_1^*. On the figure, label any DH parameters that you introduce and also mark the positive direction for all joint variables. *(16 points)*

i	a	α	d	θ
1	a_1	$90°$	d_1^*	0
2	a_2	$-90°$	0	θ_2^*
3	0	$-90°$	d_3^*	$-90°$
4	0	0	d_4^*	$\theta_4^* - 90°$

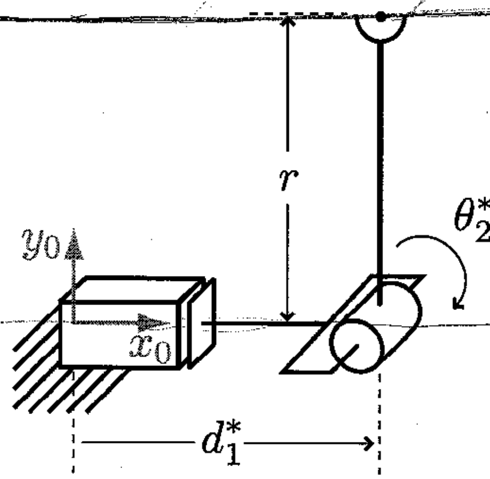
Other solutions are possible, since positive directions were not specified for the revolute joints.

No solutions

Problem 5: Inverse Position Kinematics and Jacobian *(25 points)*

one solution on line $y=r$

2 solutions.



The diagram above shows a PR manipulator. The prismatic joint is drawn at a positive displacement, and the revolute joint is drawn at zero rotation. Positive joint directions are marked with arrows. The center of the gripper (black dot) has the following position with respect to frame 0:

one solution on $y=-r$

$$p^0 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} d_1^* + r\sin\theta_2^* \\ r\cos\theta_2^* \end{bmatrix}$$

No solutions

a. Assume this robot has no joint limits and will not collide with itself in any configuration. Given a desired position of the end effector $[x \quad y]^T$, find **all possible solutions** to this robot's planar inverse position kinematics. In addition to providing equations for d_1^* and θ_2^*, please state **how many solutions there are** to the inverse kinematics problem, and explain how the number of solutions depends on the desired position. *(10 points)*

Algebraic:

$x = d_1^* + r\sin\theta_2^*$

$\boxed{d_1^* = x - r\sin\theta_2^*}$

plug in the angle chosen for θ_2^* (2 options)

$y = r\cos\theta_2^*$ ← this one is simpler, so solve first.

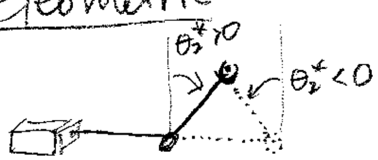
$\cos\theta_2^* = \dfrac{y}{r}$

$\sin\theta_2^* = \pm\sqrt{1 - y^2/r^2}$

option ① $\theta_2^* = \mathrm{atan2}\left(\dfrac{+\sqrt{1-y^2/r^2}}{y/r}\right)$

option ② $\theta_2^* = \mathrm{atan2}\left(\dfrac{-\sqrt{1-y^2/r^2}}{y/r}\right)$

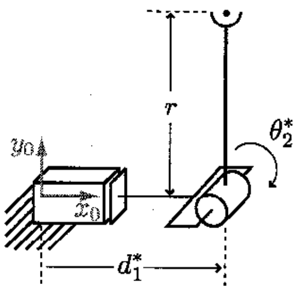
Geometric:



There are 2 solutions if the $|y|<r$. Only one Solution if $|y|=r$. And zero solutions if $|y|>r$

Could also use $\mathrm{acos}(y/r)$ but we need to recognize there are 2 solutions

s | +solution
-solution



$$p^0 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} d_1^* + r\sin\theta_2^* \\ r\cos\theta_2^* \end{bmatrix}$$

b. Calculate the **linear velocity Jacobian** J_v for this robot. *(7 points)*

$$J_v = \begin{bmatrix} \partial x/\partial d_1^* & \partial x/\partial \theta_2^* \\ \partial y/\partial d_1^* & \partial y/\partial \theta_2^* \end{bmatrix} = \boxed{\begin{bmatrix} 1 & r\cos\theta_2^* \\ 0 & -r\sin\theta_2^* \end{bmatrix}}$$

c. Use your answers from above to derive the **singular configurations** of the arm, if any. **Sketch** the manipulator in each singular configuration that you found, and explain what effect the singularity has on the robot's motion in that configuration. *(8 points)*

Look for where $\det(J_v) = 0$

$$\det(J_v) = (1)(-r\sin\theta_2^*) - (0)(r\cos\theta_2^*)$$

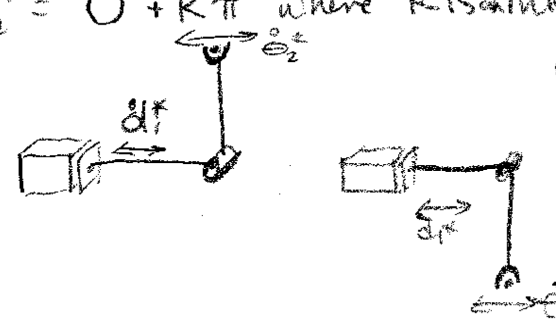
$$= -r\sin\theta_2^* = 0$$

$r = 0$

θ_2^* has no effect on end-effector position if $r=0$.
Lose ability to move
in y direction

$\sin\theta_2^* = 0$

$\theta_2^* = 0 + k\pi$ where k is an integer



Robot cannot move in y direction at these singularities.