Problem 3: Finding Roll, Pitch, and Yaw Angles *(15 points)*

A rotation matrix R can be described as a product of successive rotations about the principal coordinate axes of the **fixed frame**. If we define the convention to be first a yaw about x_0 through an angle ψ, then pitch about the y_0 by an angle θ, and finally roll about the z_0 by an angle ϕ, we obtain the following transformation matrix, where s_ϕ signifies $\sin(\phi)$, c_ϕ signifies $\cos(\phi)$, etc.:

$$R = R_{z,\phi}R_{y,\theta}R_{x,\psi} = \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}$$

Imagine you have a numerical rotation matrix $\mathcal{R}$ that has no zero entries. We will represent it with the following 3×3 array of r_{ij} scalar values.

$$\mathcal{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Find **closed-form expressions** that would enable you to **solve for all possible sets of ψ, θ, and ϕ** that would produce a given numerical matrix $\mathcal{R}$ using the above formula for R. *(15 points)*

This problem is very similar to the steps needed to calculate Euler angles from a numerical rotation matrix, as with Puma.

Notice $r_{31} = -s\theta$

$s\theta^2 + c\theta^2 = 1$

$c\theta^2 = 1 - s\theta^2$

$c\theta = \pm\sqrt{1 - r_{31}^2}$

$$\theta = \begin{cases} \operatorname{atan2}\left(\frac{-r_{31}}{\sqrt{1-r_{31}^2}}\right) & \text{option ①} \\ \text{or} \\ \operatorname{atan2}\left(\frac{-r_{31}}{-\sqrt{1-r_{31}^2}}\right) & \text{option ②} \end{cases}$$

if using option ①, $c\theta > 0$

notice r_{11} and r_{21} involve $c\phi$ and $s\phi$

r_{32} and r_{33} involve $c\psi$ and $s\psi$

$$\phi = \operatorname{atan2}\left(\frac{r_{21}}{r_{11}}\right)$$

$$\psi = \operatorname{atan2}\left(\frac{r_{32}}{r_{33}}\right)$$

if option ②, $c\theta < 0$

$$\phi = \operatorname{atan2}\left(\frac{-r_{21}}{-r_{11}}\right)$$

$$\psi = \operatorname{atan2}\left(\frac{-r_{32}}{-r_{33}}\right)$$