MEAM 520

More Trajectory Planning

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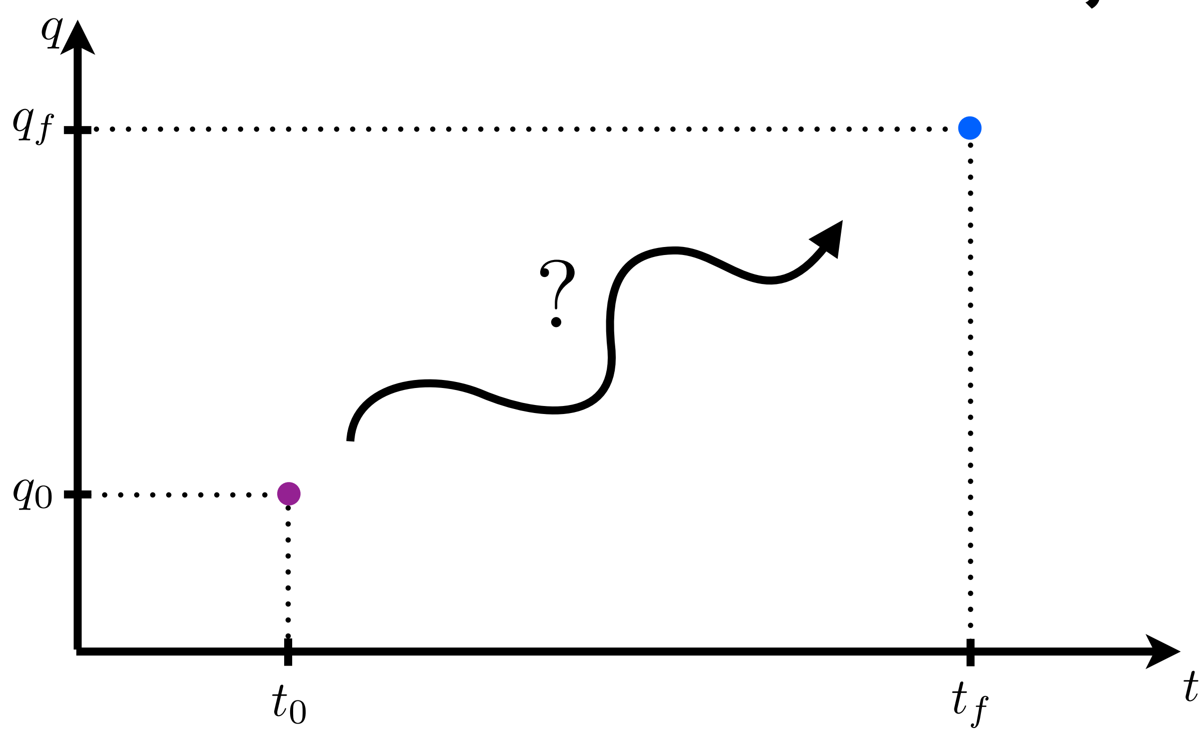
GRASP LABORATORY

Lecture 11: October 3, 2013

Trajectory Planning



	Initial Conditions	Final Conditions
Position	$q(t_0) = q_0$	$q(t_f) = q_f$
Velocity	$\dot{q}(t_0) = v_0$	$\dot{q}(t_f) = v_f$
Acceleration	$\ddot{q}(t_0) = \alpha_0$	$\ddot{q}(t_f) = \alpha_f$
Jerk	$\dddot{q}(t_0) \neq \infty$	$\dddot{q}(t_f) \neq \infty$

First-Order Polynomial (Line)

$$q(t) = a_0 + a_1 t$$

Third-Order Polynomial (Cubic)

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Solving for Coefficients

$$\begin{bmatrix} q_0 \\ v_0 \\ q_f \\ v_f \end{bmatrix} = \begin{bmatrix} 1 & t_0 & {t_0}^2 & {t_0}^3 \\ 0 & 1 & 2t_0 & 3{t_0}^2 \\ 1 & t_f & {t_f}^2 & {t_f}^3 \\ 0 & 1 & 2t_f & 3{t_f}^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Fifth-Order Polynomial (Quintic)

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

Sequencing Successive Low-Order Polynomials Through Multiple Via Points

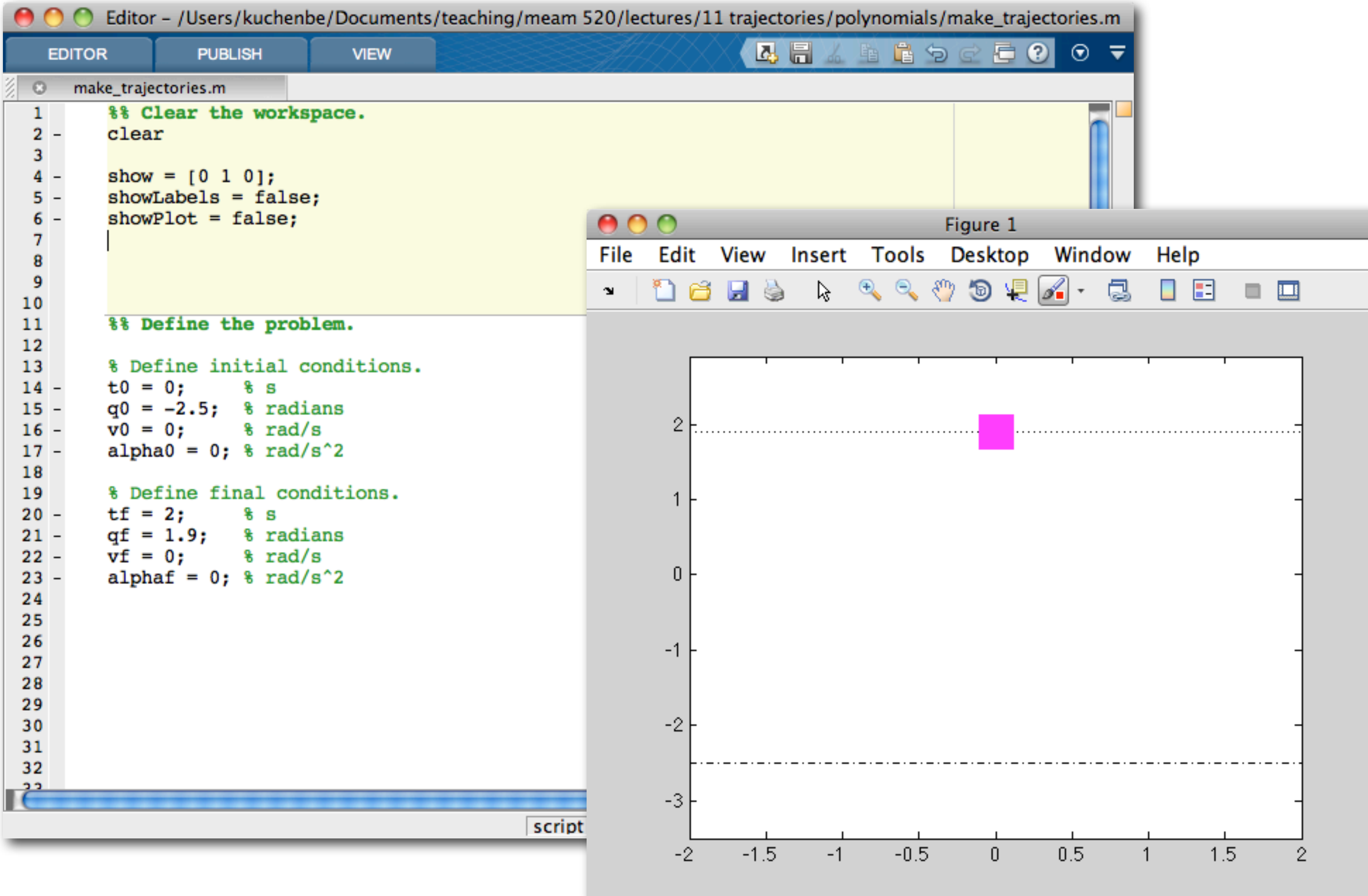
Linear Segment with Parabolic Blends (LSPB, 1 Line + 2 Quadratics)

$$q(t) = b_0 + b_1 t + b_2 t^2 \quad q(t) = a_0 + a_1 t \quad q(t) = c_0 + c_1 t + c_2 t^2$$

Minimum Time Trajectory (Bang-Bang, 2 Quadratics)

$$q(t) = b_0 + b_1 t + b_2 t^2 \quad q(t) = c_0 + c_1 t + c_2 t^2$$

Activity 1


Editor – /Users/kuchenbe/Documents/teaching/meam 520/lectures/11 trajectories/polynomials/make_trajectories.m
EDITOR
PUBLISH
VIEW
make_trajectories.m
%% Clear the workspace.
clear
show = [0 1 0];
showLabels = false;
showPlot = false;
%% Define the problem.
% Define initial conditions.
t0 = 0;   % s
q0 = -2.5;   % radians
v0 = 0;   % rad/s
alpha0 = 0; % rad/s^2
% Define final conditions.
tf = 2;   % s
qf = 1.9;   % radians
vf = 0;   % rad/s
alphaf = 0; % rad/s^2
script
Figure 1
File
Edit
View
Insert
Tools
Desktop
Window
Help
2
1
0
-1
-2
-3
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

Activity 2

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Trajectory Planning Questions

1. The equation $q(t) = a_0 + a_1 t$ defines a line. Solve for the coefficients a_0 and a_1 that satisfy the initial and final position constraints of $q(t_0) = q_0$ and $q(t_f) = q_f$.

2. We discussed using linear algebra to solve for the coefficients of the cubic polynomial that satisfies the specified conditions. Will there always be a solution? If no, when does it fail?

$$\begin{bmatrix} q_0 \\ v_0 \\ q_f \\ v_f \end{bmatrix} = \begin{bmatrix} 1 & t_0 & {t_0}^2 & {t_0}^3 \\ 0 & 1 & 2t_0 & 3{t_0}^2 \\ 1 & t_f & {t_f}^2 & {t_f}^3 \\ 0 & 1 & 2t_f & 3{t_f}^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

3. For which of the five trajectory types can q leave the interval between q_0 and q_f for the time span $t_0 \leq t \leq t_f$? Explain.

Work with one or two partners to answer the first three questions.

1. The equation $q(t) = a_0 + a_1 t$ defines a line. Solve for the coefficients a_0 and a_1 that satisfy the initial and final position constraints of $q(t_0) = q_0$ and $q(t_f) = q_f$.

$$q(t) = a_0 + a_1 t$$

$$q_0 = a_0 + a_1 t_0$$

$$q_f = a_0 + a_1 t_f$$

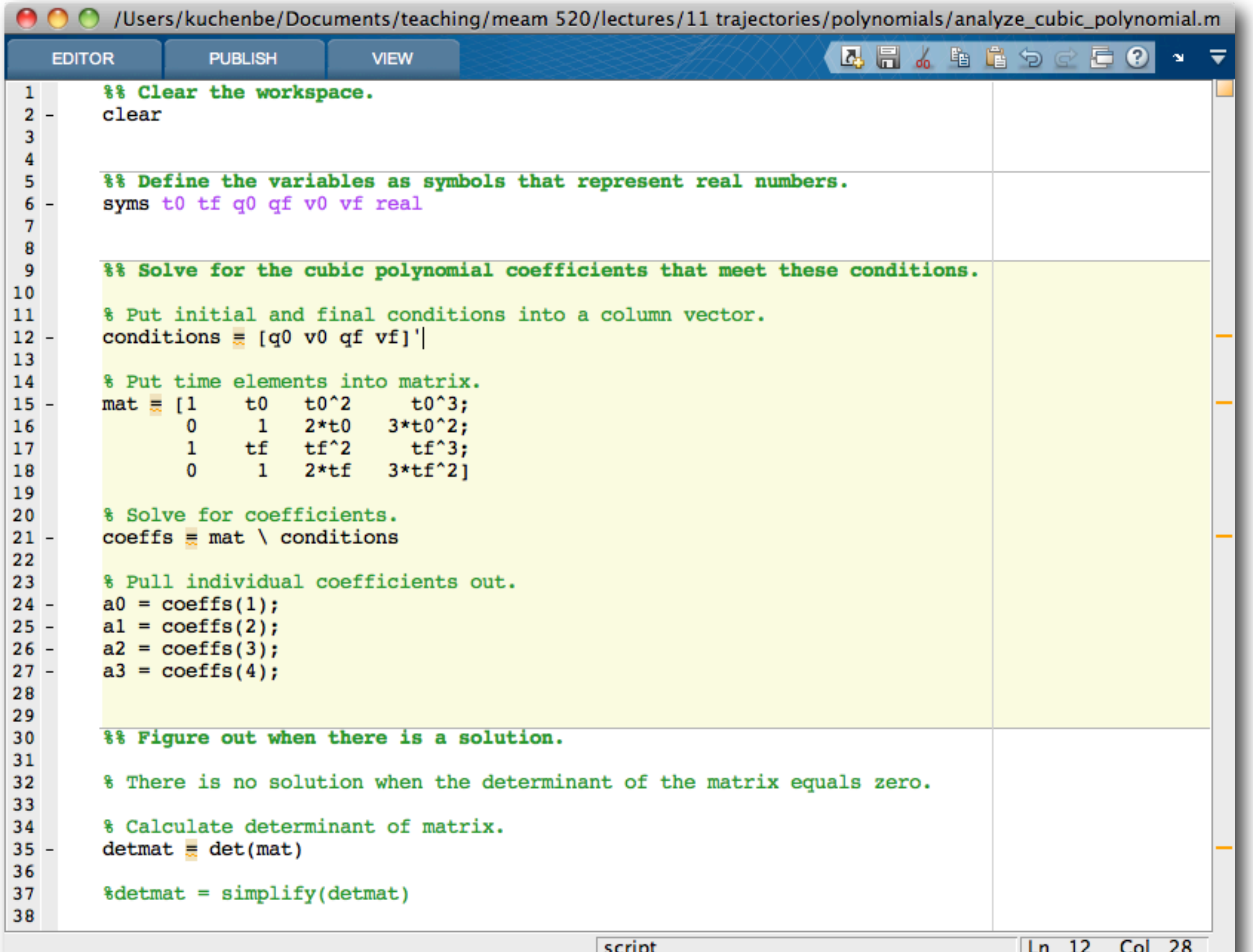
$$\boxed{a_0 = q_0 - \frac{q_f - q_0}{t_f - t_0} \cdot t_0}$$

$$\boxed{a_1 = \frac{q_f - q_0}{t_f - t_0}}$$

2. We discussed using linear algebra to solve for the coefficients of the cubic polynomial that satisfies the specified conditions. Will there always be a solution? If no, when does it fail?

$$\begin{bmatrix} q_0 \\ v_0 \\ q_f \\ v_f \end{bmatrix} = \begin{bmatrix} 1 & t_0 & {t_0}^2 & {t_0}^3 \\ 0 & 1 & 2t_0 & 3{t_0}^2 \\ 1 & t_f & {t_f}^2 & {t_f}^3 \\ 0 & 1 & 2t_f & 3{t_f}^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

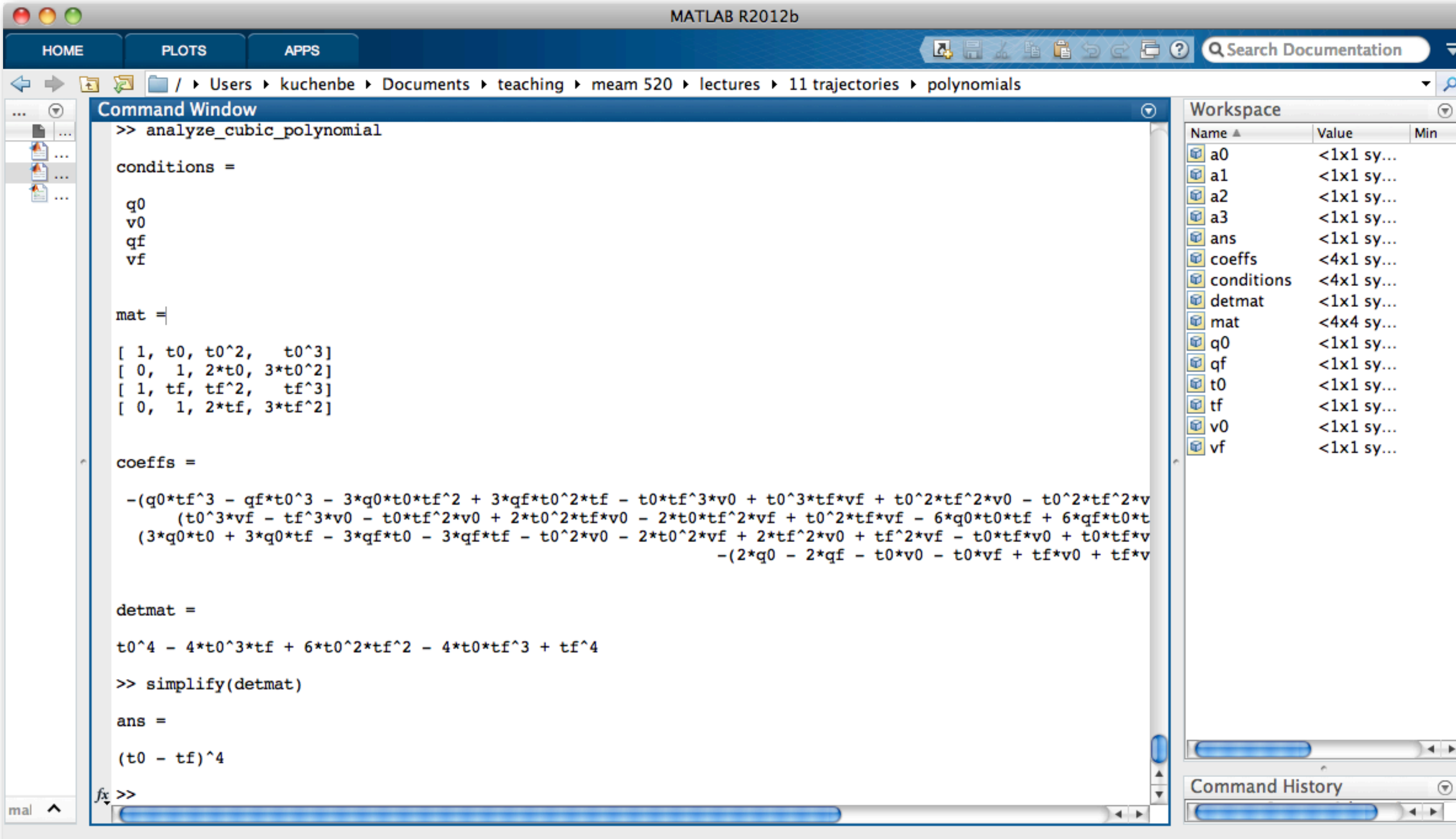
fails when $t_f = t_0$


/Users/kuchenbe/Documents/teaching/meam 520/lectures/11 trajectories/polynomials/analyze_cubic_polynomial.m

EDITOR PUBLISH VIEW

```
%% Clear the workspace.
clear


%% Define the variables as symbols that represent real numbers.
syms t0 tf q0 qf v0 vf real


%% Solve for the cubic polynomial coefficients that meet these conditions.

% Put initial and final conditions into a column vector.
conditions = [q0 v0 qf vf]'

% Put time elements into matrix.
mat = [1     t0    t0^2      t0^3;
       0      1    2*t0    3*t0^2;
       1     tf    tf^2      tf^3;
       0      1    2*tf    3*tf^2]

% Solve for coefficients.
coeffs = mat \ conditions

% Pull individual coefficients out.
a0 = coeffs(1);
a1 = coeffs(2);
a2 = coeffs(3);
a3 = coeffs(4);


%% Figure out when there is a solution.

% There is no solution when the determinant of the matrix equals zero.

% Calculate determinant of matrix.
detmat = det(mat)

%detmat = simplify(detmat)
```

script Ln 12 Col 28


MATLAB R2012b
HOME
PLOTS
APPS
Search Documentation
/ Users kuchenbe Documents teaching meam 520 lectures 11 trajectories polynomials
Command Window
>> analyze_cubic_polynomial
conditions =
q0
v0
qf
vf
mat =
[1, t0, t0^2, t0^3]
[0, 1, 2*t0, 3*t0^2]
[1, tf, tf^2, tf^3]
[0, 1, 2*tf, 3*tf^2]
coeffs =
-(q0*tf^3 - qf*t0^3 - 3*q0*t0*tf^2 + 3*qf*t0^2*tf - t0*tf^3*v0 + t0^3*tf*vf + t0^2*tf^2*v0 - t0^2*tf^2*v
(t0^3*vf - tf^3*v0 - t0*tf^2*v0 + 2*t0^2*tf*v0 - 2*t0*tf^2*vf + t0^2*tf*vf - 6*q0*t0*tf + 6*qf*t0*t
(3*q0*t0 + 3*q0*tf - 3*qf*t0 - 3*qf*tf - t0^2*v0 - 2*t0^2*vf + 2*tf^2*v0 + tf^2*vf - t0*tf*v0 + t0*tf*v
-(2*q0 - 2*qf - t0*v0 - t0*vf + tf*v0 + tf*v
detmat =
t0^4 - 4*t0^3*tf + 6*t0^2*tf^2 - 4*t0*tf^3 + tf^4
>> simplify(detmat)
ans =
(t0 - tf)^4
>>
Workspace
Name
Value
Min
a0 <1x1 sy...
a1 <1x1 sy...
a2 <1x1 sy...
a3 <1x1 sy...
ans <1x1 sy...
coeffs <4x1 sy...
conditions <4x1 sy...
detmat <1x1 sy...
mat <4x4 sy...
q0 <1x1 sy...
qf <1x1 sy...
t0 <1x1 sy...
tf <1x1 sy...
v0 <1x1 sy...
vf <1x1 sy...
Command History

3. For which of the five trajectory types can q leave the interval between q_0 and q_f for the time span $t_0 \leq t \leq t_f$? Explain.

First-Order Polynomial (Line) Does not leave interval.

$$q(t) = a_0 + a_1 t$$

Third-Order Polynomial (Cubic) Could leave interval*.

*Depends on initial and final velocities. When both are zero, does not leave interval.

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Fifth-Order Polynomial (Quintic) Could leave interval**.

**Depends on initial and final velocities and accelerations. When all are zero, does not leave interval.

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

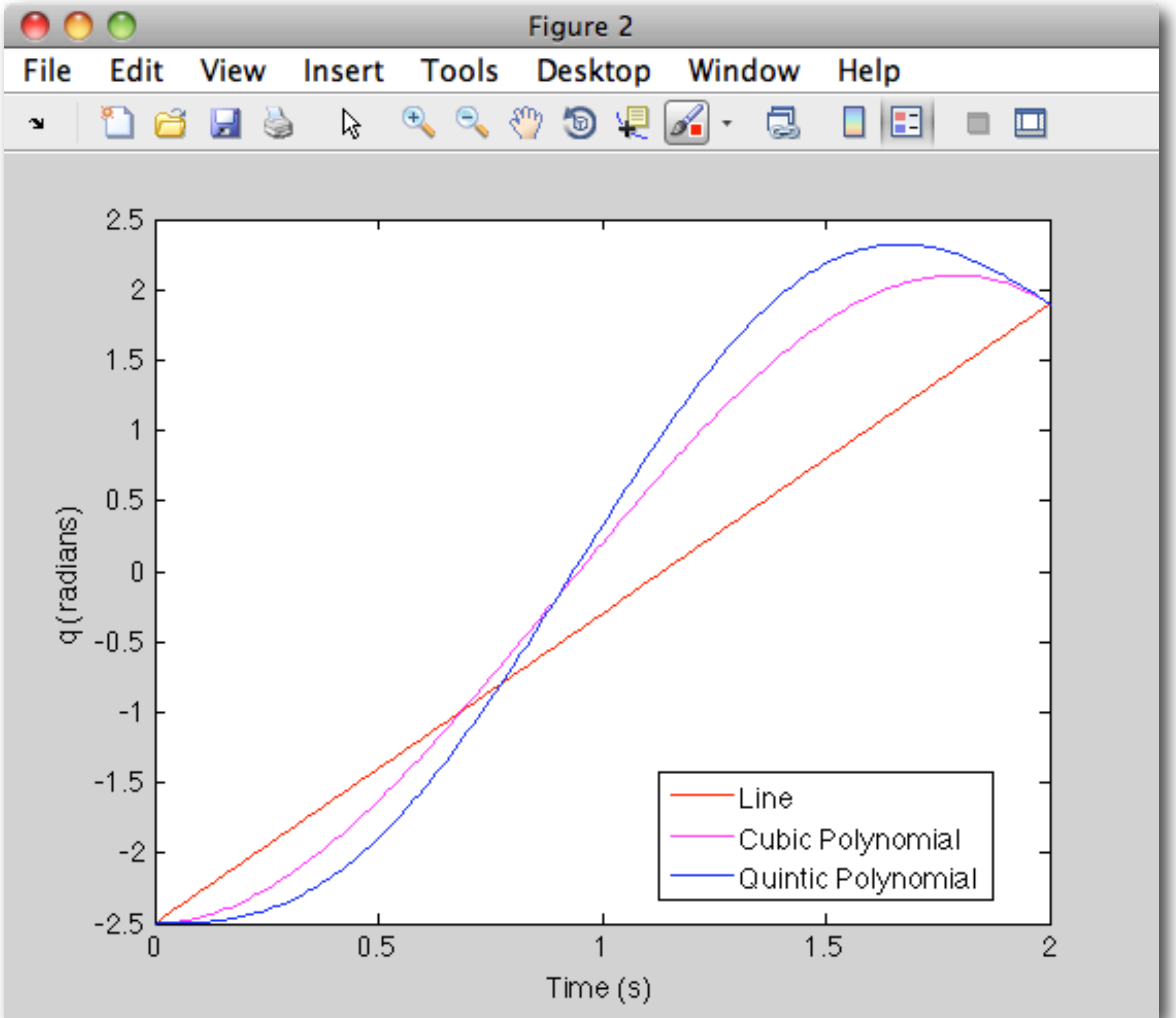
Linear Segment with Parabolic Blends (LSPB, 1 Line + 2 Quadratics) Could leave interval*.

$$q(t) = b_0 + b_1 t + b_2 t^2 \quad q(t) = a_0 + a_1 t \quad q(t) = c_0 + c_1 t + c_2 t^2$$

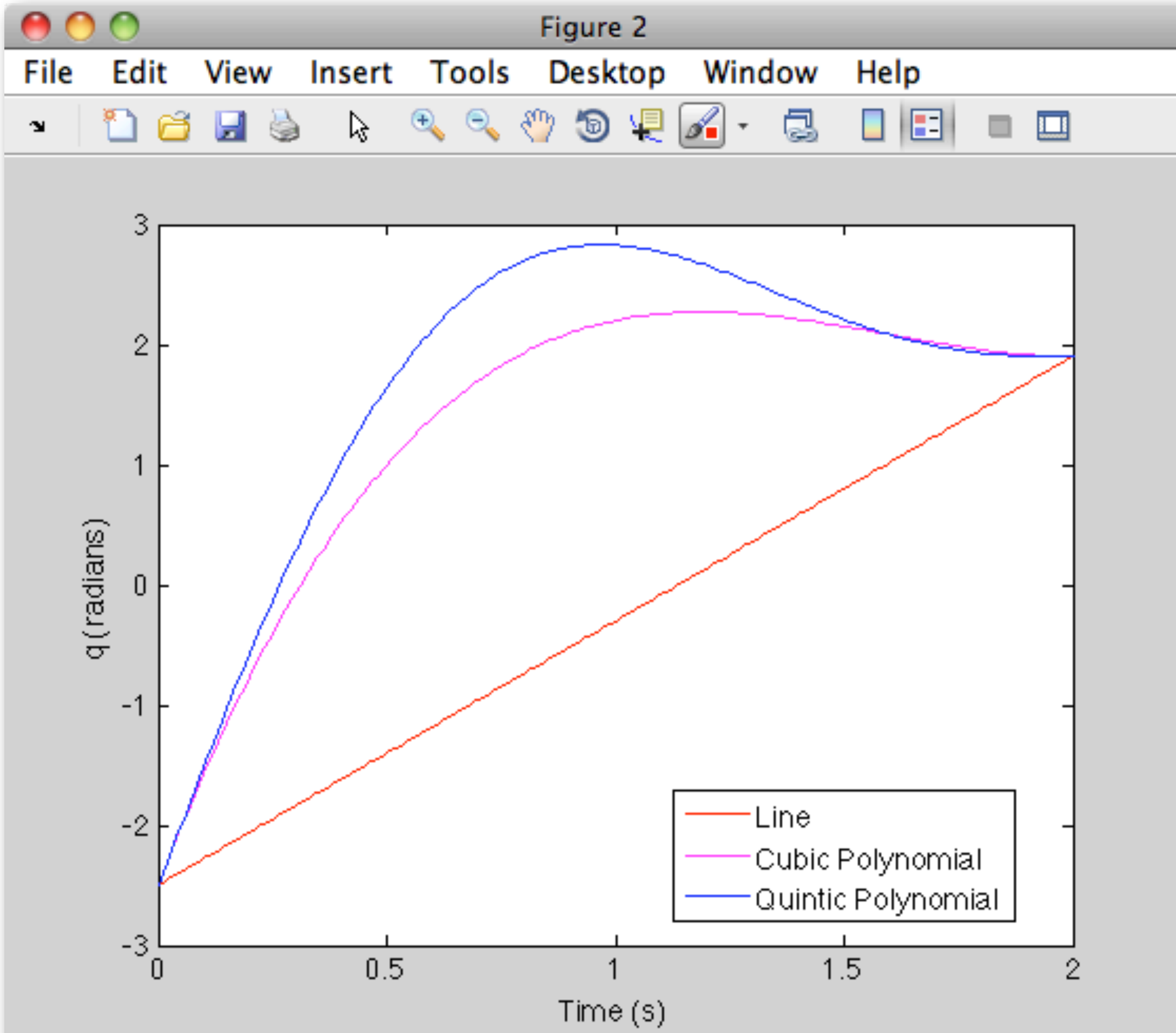
Minimum Time Trajectory (Bang-Bang, 2 Quadratics) Could leave interval*.

$$q(t) = b_0 + b_1 t + b_2 t^2 \quad q(t) = c_0 + c_1 t + c_2 t^2$$

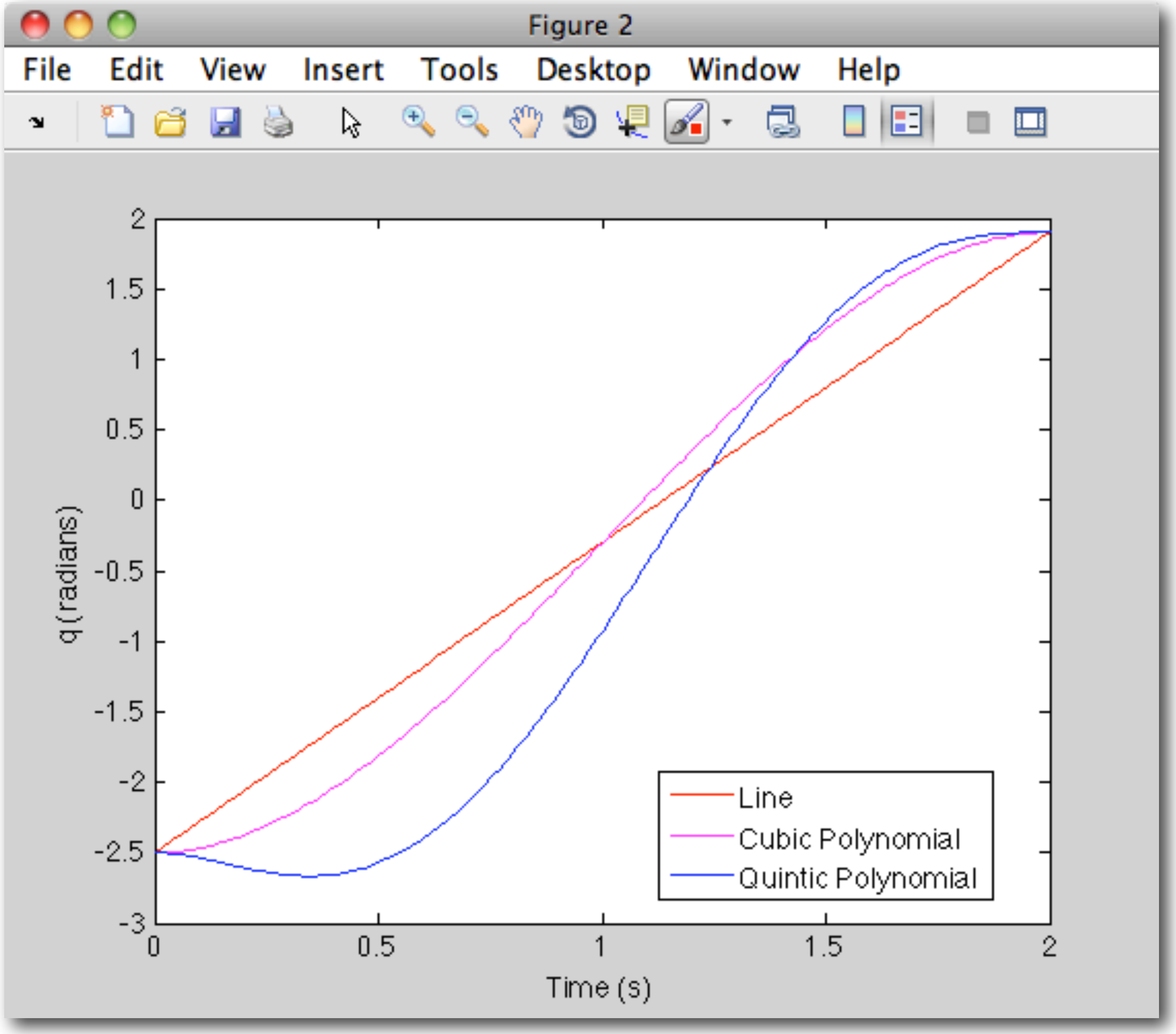
Final velocity less than zero



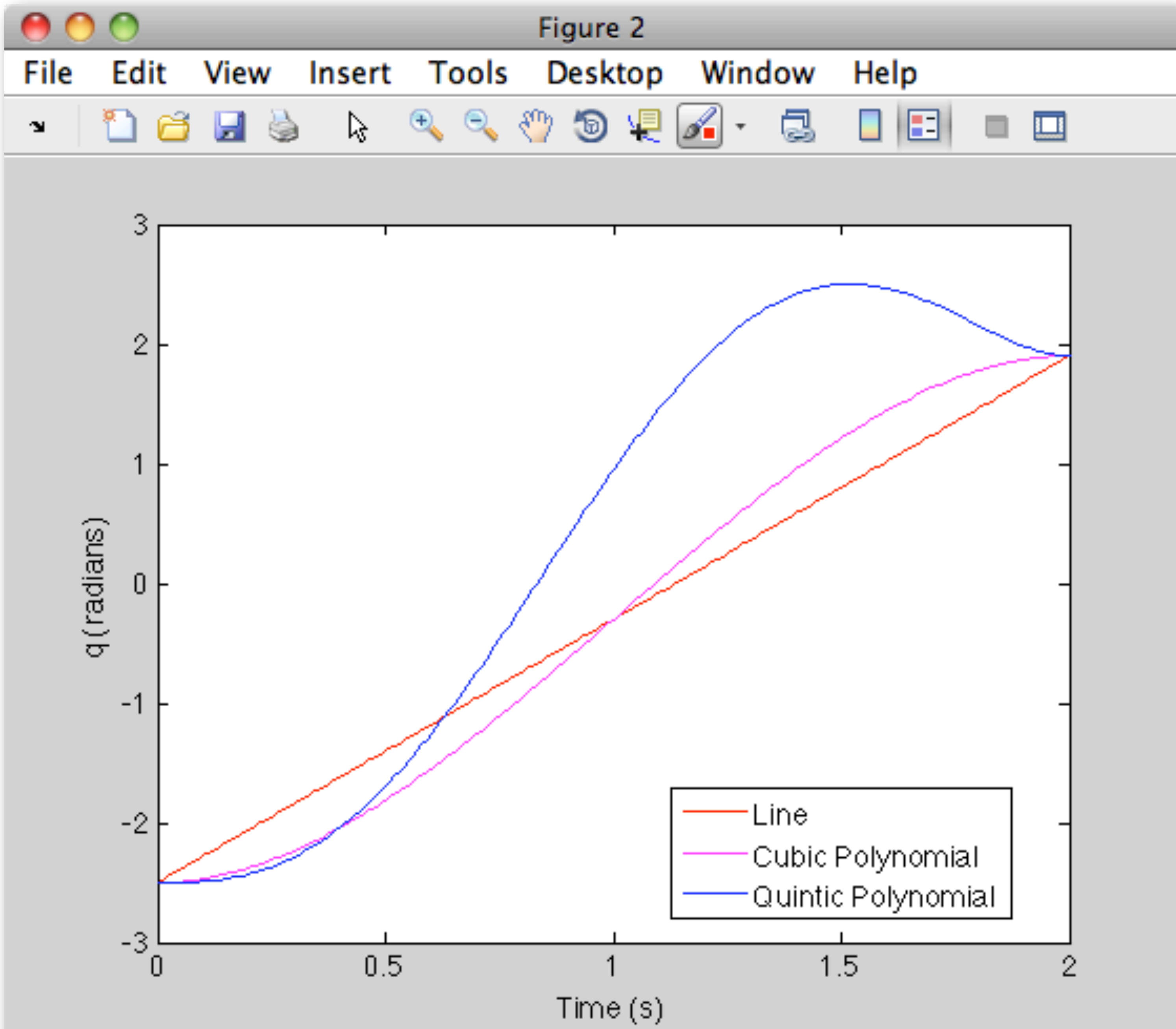
Initial velocity greater than zero and large



Initial acceleration less than zero



Final acceleration greater than zero



Activity 3

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4. Why would one ever use a line or a cubic polynomial instead of a quintic polynomial?

5. How does the idea of sequencing low-order polynomials such as cubics through multiple via points relate to LSPB and Bang-Bang trajectories?

6. Set up the equations to solve for all the coefficients of a general LSPB given initial time t_0, final time t_f, initial position q_0, final position q_f, initial velocity v_0, final velocity v_f, and blend duration t_b.

Work with one or two partners to answer the last three questions.

$$q(t) = b_0 + b_1 t + b_2 t^2 \quad q(t) = a_0 + a_1 t \quad q(t) = c_0 + c_1 t + c_2 t^2$$

4. Why would one ever use a line or a cubic polynomial instead of a quintic polynomial?

- Want constant velocity (line).
- Your robot is sufficiently rigid, so you don't care about minimal jerk.
- Need lower computational complexity, e.g., real-time calculations on a microcontroller.
- Need lower memory usage, e.g., implementation on a microcontroller.
- Want to limit maximum speed.
- More ideas from class?

5. How does the idea of sequencing low-order polynomials such as cubics through multiple via points relate to LSPB and Bang-Bang trajectories?

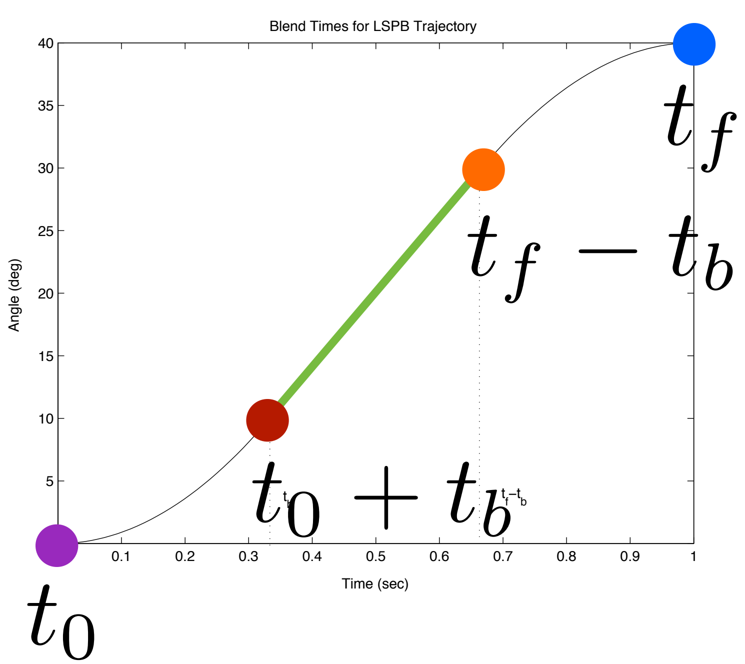
- A linear segment with parabolic blends *is* a sequence of low-order polynomials: quadratic + line + quadratic.
- A bang-bang trajectory *is* a sequence of low-order polynomials: quadratic + quadratic.
- But, for LSPB and BB we don't care about the particular position or velocity of the robot at the switching times. We just require position and velocity to be continuous at these points.

6. Set up the equations to solve for all the coefficients of a general LSPB given initial time t_0, final time t_f, initial position q_0, final position q_f, initial velocity v_0, final velocity v_f, and blend duration t_b.

$$q(t) = b_0 + b_1 t + b_2 t^2 \qquad q(t) = a_0 + a_1 t \qquad q(t) = c_0 + c_1 t + c_2 t^2$$

$$\dot{q}(t) = b_1 + 2b_2 t \qquad \dot{q}(t) = a_1 \qquad \dot{q}(t) = c_1 + 2c_2 t$$

8 parameters – need 8 equations



Position and velocity at four points in time

$$q_0 = b_0 + b_1 t_0 + b_2 t_0^2$$

$$v_0 = b_1 + 2b_2 t_0$$

$$b_0 + b_1(t_0 + t_b) + b_2(t_0 + t_b)^2 = a_0 + a_1(t_0 + t_b)$$

$$b_1 + 2b_2(t_0 + t_b) = a_1$$

$$\ldots$$

What questions do you have?

Trajectory Planning Questions

1. The equation $q(t) = a_0 + a_1 t$ defines a line. Solve for the coefficients a_0 and a_1 that satisfy the initial and final position constraints of $q(t_0) = q_0$ and $q(t_f) = q_f$.

2. We discussed using linear algebra to solve for the coefficients of the cubic polynomial that satisfies the specified conditions. Will there always be a solution? If no, when does it fail?

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3. For which of the five trajectory types can q leave the interval between q_0 and q_f for the time span $t_0 \leq t \leq t_f$? Explain.

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These would have been good homework questions, but instead I want you to use these concepts in a project....