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Chapter 9

Number Systems

The Decimal System

- System based on decimal digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) to represent numbers
- For example the number 83 means eight tens plus three:

$$83 = (8 * 10) + 3$$

- The number 4728 means four thousands, seven hundreds, two tens, plus eight:

$$4728 = (4 * 1000) + (7 * 100) + (2 * 10) + 8$$

- The decimal system is said to have a ***base***, or ***radix***, of 10. This means that each digit in the number is multiplied by 10 raised to a power corresponding to that digit's position:

$$83 = (8 * 10^1) + (3 * 10^0)$$

$$4728 = (4 * 10^3) + (7 * 10^2) + (2 * 10^1) + (8 * 10^0)$$

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Decimal Fractions

- The same principle holds for decimal fractions, but negative powers of 10 are used. Thus, the decimal fraction 0.256 stands for 2 tenths plus 5 hundredths plus 6 thousandths:

$$0.256 = (2 * 10^{-1}) + (5 * 10^{-2}) + (6 * 10^{-3})$$

- A number with both an integer and fractional part has digits raised to both positive and negative powers of 10:

$$442.256 = (4 * 10^{2}) + (4 + 10^{1}) + (2 * 10^{0}) + (2 * 10^{-1}) + (5 * 10^{-2})$$
$$+ (6 * 10^{-3})$$

- ***Most significant digit***
  - The leftmost digit (carries the highest value)
- ***Least significant digit***
  - The rightmost digit

Table 9.1
Positional Interpretation of a Decimal Number

4	7	2	2	5	6
100s 10^2 position 2	10s 10^1 position 1	1s 10^0 position 0	tenths 10^{-1} position −1	hundredths 10^{-2} position −2	thousandths 10^{-3} position −3

Positional Number Systems

- Each number is represented by a string of digits in which each digit position *i* has an associated weight r^i, where *r* is the *radix*, or *base,* of the number system.

- The general form of a number in such a system with radix *r* is

$$(\ldots a_3 a_2 a_1 a_0 . a_{-1} a_{-2} a_{-3} \ldots)_r$$

where the value of any digit a_i *is an integer in the range* $0 \le a_i < r$. The dot between a_0 *and* a_{-1} is called the ***radix point.***

Table 9.2
Positional Interpretation of a Number in Base 7

Position	4	3	2	1	0	–1
Value in exponential form	7^4	7^3	7^2	7^1	7^0	7^{-1}
Decimal value	2401	343	49	7	1	1/7

The Binary System

- Only two digits, 1 and 0
- Represented to the base 2
- The digits 1 and 0 in binary notation have the same meaning as in decimal notation:

$$0_2 = 0_{10}$$

$$1_2 = 1_{10}$$

- To represent larger numbers each digit in a binary number has a value depending on its position:

$$10_2 = (1 * 2^1) + (0 * 2^0) = 2_{10}$$

$$11_2 = (1 * 2^1) + (1 * 2^0) = 3_{10}$$

$$100_2 = (1 * 2^2) + (0 * 2^1) + (0 * 2^0) = 4_{10}$$

and so on. Again, fractional values are represented with negative powers of the radix:

$$1001.101 = 2^3 + 2^0 + 2^{-1} + 2^{-3} = 9.625_{10}$$

Converting Between Binary and Decimal

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Binary notation to decimal notation:

- Multiply each binary digit by the appropriate power of 2 and add the results

Decimal notation to binary notation:

- Integer and fractional parts are handled separately

Integers

For the integer part, recall that in binary notation, an integer represented by

$$b_{m-1}b_{m-2}\ldots b_2b_1b_0 \qquad b_i = 0 \text{ or } 1$$

has the value

$$(b_{m-1} * 2^{m-1}) + (b_{m-2} * 2^{m-2}) + \ldots + (b_1 * 2^1) + b_0$$

Suppose it is required to convert a decimal integer N into binary form. If we divide N by 2, in the decimal system, and obtain a quotient N_1 and a remainder R_0, we may write

$$N = 2 * N_1 + R_0 \qquad R_0 = 0 \text{ or } 1$$

Next, we divide the quotient N_1 by 2. Assume that the new quotient is N_2 and the new remainder R_1. Then

$$N_1 = 2 * N_2 + R_1 \qquad R_1 = 0 \text{ or } 1$$

so that

$$N = 2(2N_2 + R_1) + R_0 = (N_2 * 2^2) + (R_1 * 2^1) + R_0$$

If next

$$N_2 = 2N_3 + R_2$$

we have

$$N = (N_3 * 2^3) + (R_2 * 2^2) + (R_1 * 2^1) + R_0$$

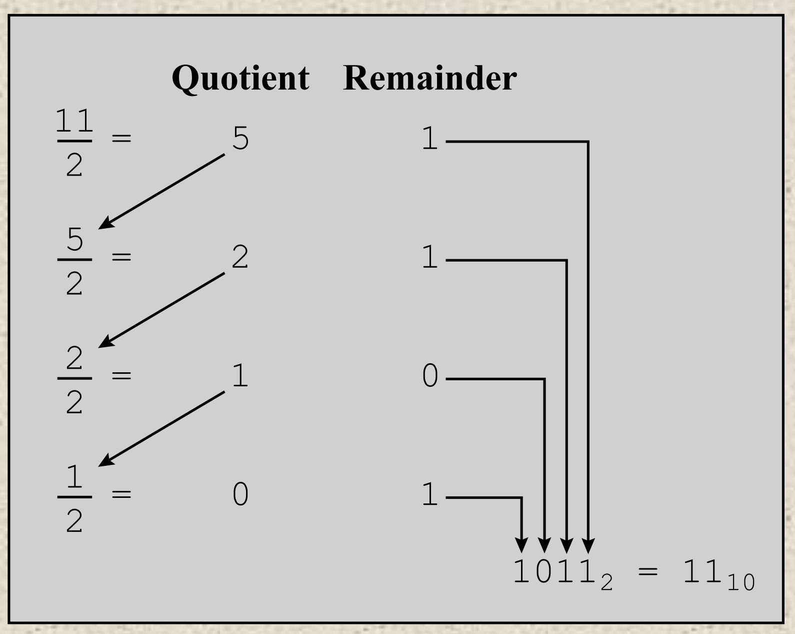
Continued . . .

Integers

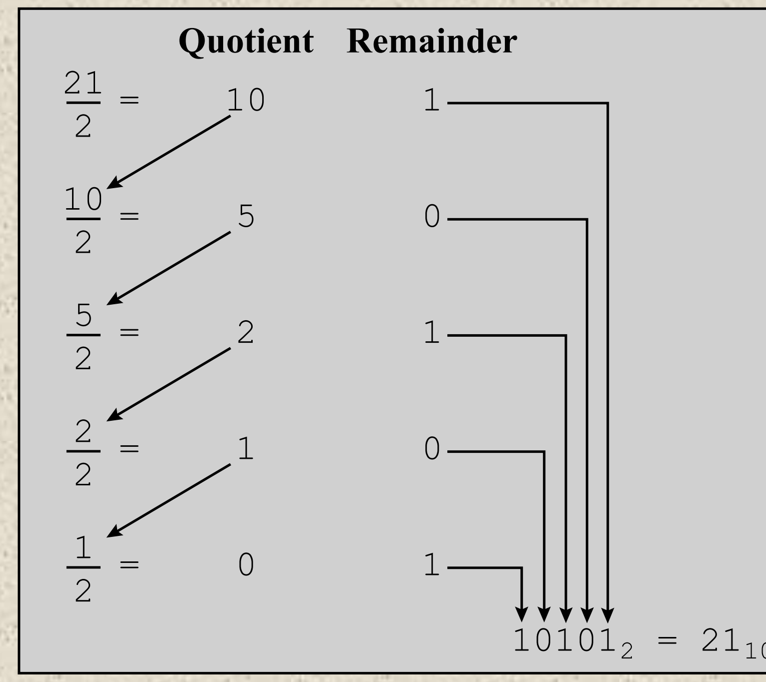
Because $N > N_1 > N_2 \ldots$, continuing this sequence will eventually produce a quotient $N_{m-1} = 1$ (except for the decimal integers 0 and 1, whose binary equivalents are 0 and 1, respectively) and a remainder R_{m-2}, which is 0 or 1. Then

$$N = (1 * 2^{m-1}) + (R_{m-2} * 2^{m-2}) + \ldots + (R_2 * 2^2) + (R_1 * 2^1) + R_0$$

which is the binary form of *N*. Hence, we convert from base 10 to base 2 by repeated divisions by 2. The remainders and the final quotient, 1, give us, in order of increasing significance, the binary digits of *N*.



(a) 11_{10}



(b) 21_{10}

Figure 9.1 Examples of Converting from Decimal Notation to Binary Notation for Integers

Fractions

For the fractional part, recall that in binary notation, a number with a value between 0 and 1 is represented by

$$0.b_{-1}b_{-2}b_{-3}\ldots \qquad b_i = 0 \text{ or } 1$$

and has the value

$$(b_{-1} * 2^{-1}) + (b_{-2} * 2^{-2}) + (b_{-3} * 2^{-3})\ldots$$

This can be rewritten as

$$2^{-1} * (b_{-1} + 2^{-1} * (b_{-2} + 2^{-1} * (b_{-3} + \ldots)\ldots))$$

Suppose we want to convert the number $F\ (0 < F < 1)$ from decimal to binary notation. We know that F can be expressed in the form

$$F = 2^{-1} * (b_{-1} + 2^{-1} * (b_{-2} + 2^{-1} * (b_{-3} + \ldots)\ldots))$$

If we multiply F by 2, we obtain,

$$2 * F = b_{-1} + 2^{-1} * (b_{-2} + 2^{-1} * (b_{-3} + \ldots)\ldots)$$

Continued . . .

Fractions

From this equation, we see that the integer part of $(2 * F)$, *which must be* either 0 or 1 because $0 < F < 1$, *is simply* b_{-1}. *So we can say* $(2 * F) = b_{-1} + F_1$, where $0 < F_1 < 1$ *and where*

$$F_1 = 2-1 * (b_{-2} + 2^{-1} * (b_{-3} + 2^{-1} * (b_{-4} + . . .) . . .))$$

To find b_{-2}, we repeat the process.
At each step, the fractional part of the number from the previous step is multiplied by 2. The digit to the left of the decimal point in the product will be 0 or 1 and contributes to the binary representation, starting with the most significant digit. The fractional part of the product is used as the multiplicand in the next step.

Figure 9.2

Examples of Converting from Decimal Notation To Binary Notation For Fractions

Product	Integer Part	0.110011_2
$0.81 \times 2 = 1.62$	1	
$0.62 \times 2 = 1.24$	1	
$0.24 \times 2 = 0.48$	0	
$0.48 \times 2 = 0.96$	0	
$0.96 \times 2 = 1.92$	1	
$0.92 \times 2 = 1.84$	1	

(a) $0.81_{10} = 0.110011_2$ (approximately)

Product	Integer Part	0.01_2
$0.25 \times 2 = 0.5$	0	
$0.5 \times 2 = 1.0$	1	

(b) $0.25_{10} = 0.01_2$ (exactly)

Hexadecimal Notation

- Binary digits are grouped into sets of four bits, called a *nibble*
- Each possible combination of four binary digits is given a symbol, as follows:

0000 = 0	0100 = 4	1000 = 8	1100 = C
0001 = 1	0101 = 5	1001 = 9	1101 = D
0010 = 2	0110 = 6	1010 = A	1110 = E
0011 = 3	0111 = 7	1011 = B	1111 = F

- Because 16 symbols are used, the notation is called *hexadecimal* and the 16 symbols are the *hexadecimal digits*
- Thus

$$2C_{16} = (2_{16} * 16^1) + (C_{16} * 16^0)$$

$$= (2_{10} * 16^1) + (12_{10} * 16^0) = 44$$

Table 9.3

Decimal, Binary, and Hexadecimal

Decimal (base 10)	Binary (base 2)	Hexadecimal (base 16)
0	0000	0
1	0001	1
2	0010	2
3	0011	3
4	0100	4
5	0101	5
6	0110	6
7	0111	7
8	1000	8
9	1001	9
10	1010	A
11	1011	B
12	1100	C
13	1101	D
14	1110	E
15	1111	F
16	0001 0000	10
17	0001 0001	11
18	0001 0010	12
31	0001 1111	1F
100	0110 0100	64
255	1111 1111	FF
256	0001 0000 0000	100

Hexadecimal Notation

Not only used for representing integers but also as a concise notation for representing any sequence of binary digits

Reasons for using hexadecimal notation are:

- It is more compact than binary notation
- In most computers, binary data occupy some multiple of 4 bits, and hence some multiple of a single hexadecimal digit
- It is extremely easy to convert between binary and hexadecimal notation

Summary

Chapter 9

Number Systems

- The decimal system
- Positional number systems
- The binary system
- Converting between binary and decimal
  - Integers
  - Fractions
- Hexadecimal notation