Lecture Notes:
General Relativity and Cosmology

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Preface

The notes the former part mainly based on [0], the flow from math to perturbation theory
[0] mathematics and black hole
[0] talks on perturbation and inflation theory
[0] is observational cosmology

[0] and [0] is a
Currently, these are just drafts of these notes. There can be typos and mistakes anywhere. So, if you find anything that needs to be corrected or improved, please inform at yingqiu@post.kek.jp.
I am also planning to do think work on Group theory and qft. and after learning them, I plan learn geometry and algebra for physician more carefully. (beyond my research)

Contents

Lecture 1.

Special Relativity and Flat Spacetime

Referring to the book [0] of Chapter 1. Considering inertial Cartesian-like coordinates..

1.1 Spacetime

- events: interval between neighbring points of spacetime
- worldline: the path of a particle is a curve through spacetime, a parameterized one-dimensional set of events
- space interval between two events: $(\Delta s)^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu, \eta_{\mu\nu} = diag(-1,1,1,1)$
  - summuation convention - dummy indices
- lightcone: timelike $(\Delta s)^2 < 0$; spacetime $(\Delta s)^2 > 0$; lightlike/null $(\Delta s)^2 = 0$
- propertime: $(\Delta\tau)^2 = -(\Delta s)^2$

infinitesimal interval / line element:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{1.1}$$

spacelike curve:

$$\Delta s = \int \sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,d\lambda$$

timelike path:

$$\Delta \tau = \int \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,d\lambda$$

1.2 Lorentz transformation

coordiante systems that leave the interval invariant

$$x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^\nu \tag{1.2}$$

interval invariant: $\Delta s^2 = (\Delta x)^\intercal \eta (\Delta x) = (\Delta x')^\intercal \eta (\Delta x') = (\Delta x)^\intercal \Lambda^\intercal \eta \Lambda (\Delta x) \Rightarrow \eta = \Lambda^\intercal \eta \Lambda$ or

$$\eta_{\rho\sigma} = \Lambda^{\mu'}{}_{\rho}\eta_{\mu'\nu'}\Lambda^{\nu'}{}_{\sigma} = \Lambda^{\mu'}{}_{\rho}\Lambda^{\nu'}{}_{\sigma}\eta_{\mu'\nu'} \tag{1.3}$$

Lorentz group: the set of them forms a group under matrix multiplication

Spatial rotation / boosts: (in x-direction)

boost rotation angle of $\phi \in (-\infty, +\infty)$,

$$\Lambda^{\mu'}{}_{\nu} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then $t' = t\cosh\phi - x\sin\phi, x' = -t\sinh\phi + x\cosh\phi$. when $x' = 0 \Rightarrow v = x/t = \tanh\phi \Rightarrow$

$$t' = \gamma(t - vx)$$
$$x' = \gamma(x - vt)$$

where $\gamma = 1/\sqrt{1-v^2} = \cosh\phi$ $(\cosh\phi^2 - \sinh\phi^2)$. boosts correspond to changing coordinates by moving to a frame that travels at a constant velocity.

1.3 Vector and dual vector

- (real) vector space: a collection of vectors that can be added together and multiplied by real numbers in a linear way
- tangent space at p (T_p): the space of all vectors at the point p
- dimension of the space: infinit number of possible bases

any vector A can be written as $A = A^\mu \hat{e}_{(\mu)}$ ($\mu \in \{0,1,2,3\}$), where $\hat{e}_{(\mu)}$ basis vector and A^μ is components tangent vector:

$$V(\lambda) = V^\mu \hat{e}_{(\mu)}, \qquad V^\mu = \frac{dx^\mu}{d\lambda} \tag{1.4}$$

consider Lorentz transformation: $V^\mu \to V^{\mu'} = \Lambda^{\mu'}{}_\nu V^\nu$ (invariant), then

$$V = V^\mu \hat{e}_{(\mu)} = \Lambda^{\nu'}{}_\mu V^\mu \hat{e}_{(\nu')}$$

$\hat{e}_{(\nu')}$ is transformed coordinates system $\hat{e}_{(\nu')} = \Lambda^\mu_{\nu'} \hat{e}_{(\mu)}$, therefore we need

$$\Lambda^\mu_{\nu'} \Lambda^{\nu'}_\rho = \delta^\mu_\rho \tag{1.5}$$

- dual vector space: all linear maps from the original vector space to the real number
- cotangent space (T^*_p)

if $\omega \in T^*_p$, then $V, W \to$vecetor, $a, b \to$real number

$$\omega(aV + bW) = a\omega(V) + b(W) \in \mathbb{R}$$

similarly, $\omega = \omega_\mu \hat{\theta}^{(\mu)}$, and $\hat{\theta}^{(\nu)}(\hat{e}_{(\mu)}) = \delta^\nu_\mu$

Proof. $V(\omega) \equiv \omega(V) = \omega_\mu \hat{\theta}^{(\mu)}(V^\nu \hat{e}_{(\nu)}) = \omega_\mu V^\nu \hat{\theta}^{(\mu)} \hat{e}_{(\nu)} = \omega_\mu V^\nu \delta^\mu_\nu = \omega_\mu V^\mu \in \mathbb{R}$ □

$$\boxed{\omega_{\mu'} = \Lambda^\nu{}_{\mu'} \omega_\nu \qquad \hat{\theta}^{(\rho')} = \Lambda^{\rho'}{}_\sigma \hat{\theta}^{(\sigma)}} \tag{1.6}$$

e.g. the gradient of a scalar is a dual vector $d\phi = \frac{\partial\phi}{\partial x^\mu}\hat{\theta}^{(\mu)}$, then $\partial_{\mu'}\phi = \frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu\phi$

Remark 1. $\frac{\partial\phi}{\partial x^\mu} = \partial_\mu\phi = \phi_{,\mu}$

1.4 Tensor

a tensor T of type (k,l) is a multilinear map from T_p^* and T_p to $\mathbb{R}$: $T: T_p^* \times \cdots \times T_p^* \times T_p \cdots \times T_p \to \mathbb{R}$

- $\times$: Cartesian product
- $\otimes$: Tensor prodect

if $T(k,l)$ and $S(m,n)$, we define a $(k+m, l+n)$ tensor

$$\begin{aligned} &T \otimes S(\omega^{(1)}, \cdots, \omega^{(k)}, \cdots, \omega^{(k+m)}, V^{(1)}, \cdots, V^{(l)}, \cdots, V^{(l+n)}) \\ &\quad = T(\omega^{(1)}, \cdots, \omega^{(k)}, V^{(1)}, \cdots, V^{(l)}) \times S(\omega^{(k+1)}, \cdots, \omega^{(k+m)}, V^{(l+1)}, \cdots, V^{(l+n)}) \end{aligned}$$

In components notation,

$$\begin{aligned} &T = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \hat{e}_{(\mu_1)} \otimes \cdots \otimes \hat{e}_{(\mu_k)} \otimes \hat{\theta}^{(\nu_1)} \otimes \cdots \otimes \hat{\theta}^{(\nu_l)} \qquad \text{or} \\ &T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} = T(\hat{\theta}^{(\nu_1)}, \cdots, \hat{\theta}^{(\nu_l)}, \hat{e}_{(\mu_1)}, \cdots, \hat{e}_{(\mu_k)}) \end{aligned}$$

usually denote T as $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}$, the action on vectors and dual vectors to $\mathbb{R}$

$$T(\omega^{(1)}, \cdots, \omega^{(k)}, V^{(1)}, \cdots, V^{(l)}) = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \omega^{(1)}_{\mu_1} \cdots \omega^{(k)}_{\mu_k} V^{(1)\nu_1} \cdots V^{(l)\nu_l}$$

Tensor Lorentz transformation law:

$$T^{\mu'_1 \cdots \mu'_k}{}_{\nu'_1 \cdots \nu'_l} = \Lambda^{\mu'_1}{}_{\mu_1} \cdots \Lambda^{\mu'_k}{}_{\mu_k} \Lambda^{\nu'_1}{}_{\nu_1} \cdots \Lambda^{\nu'_l}{}_{\nu_l} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}$$

some examples:

1. Minkowski metric $\eta_{\mu\nu}$ $(0,2)$, inner product: $\eta(V,W) = \eta_{\mu\nu} V^\mu W^\nu = V \cdot W$
2. Kronecker delta δ^μ_ρ $(1,1)$
3. inverse metric $\eta^{\mu\nu}$ $(2,0)$, defined as $\eta^\mu \eta_{\nu\rho} = \eta_{\rho\nu} \eta^{\nu\mu} = \delta^\mu_\rho$
4. Levi-Civita symbel $\tilde{\varepsilon}_{\mu\nu\rho\sigma}$ $(0,4)$
5. Electromagnetic field strength tensor $F_{\mu\nu} = -F_{\nu\mu}$

well-defined tensor:

- transforming according to the tensor transformation law
- defining a unique multilinear map from a set of vector and dual vector to the real number

manipulation tensors

operation of contraction: $S^{\mu\rho}{}_\sigma = T^{\mu\nu\rho}{}_{\sigma\nu}$, $T^{\mu\nu\rho}{}_{\sigma\nu} \neq T^{\mu\rho\nu}{}_{\sigma\nu}$
raise and lower indices: $T^{\alpha\beta\mu}{}_\delta = \eta^\mu{}_\gamma T^{\alpha\beta}{}_{\gamma\sigma}$, $T_\mu{}^\beta{}_{\gamma\delta} = \eta_{\mu\alpha} T^{\alpha\beta}{}_{\gamma\sigma}$, $T_{\mu\nu}{}^{\rho\sigma} = \eta_{\mu\alpha}\eta_{\nu\beta}\eta^{\rho\gamma}\eta^{\sigma\delta} T^{\alpha\beta}{}_{\gamma\sigma}$
symmetric / antisymmetric: $S_{\mu\nu\rho} = S_{\nu\mu\rho}$ / $A_{\mu\rho} = -A_{\nu\mu\rho}$

1.5 Maxwell's equations

1.6 Energy and momentum

1.7 Classical field theory

"principle of least action"

- $\Phi^i(x^\mu)$: a set of spacetime-dependent fields
- S: functional/integral of fields
- $\mathscr{L}$: Langrange density

$$L = \int d^3x\, \mathscr{L}(\Phi^i, \partial_\mu \Phi^i) \tag{1.7}$$

$$S = \int dt\, L = \int d^4 \mathscr{L}(\Phi^i, \partial_\mu \Phi^i) \tag{1.8}$$

- natural unit: $[energy] = [mass] = [(length)^{-1}] = [(time)^{-1}]$, $[S] = [E][M] = M^0, [\mathscr{L}] = M^4$

consider $\Phi^i \to \Phi^i + \delta\Phi^i$, then $\partial_\mu \Phi^i \to \partial\Phi^i + \delta(\partial_\mu\Phi^i) = \partial\Phi^i + \partial\delta(\Phi^i)$ and $S \to S + \delta S$

$$\begin{aligned}\delta S &= \int d^4x\, \frac{\partial \mathscr{L}}{\partial \Phi^i}\delta\Phi^i + \frac{\partial \mathscr{L}}{\partial(\partial_\mu\Phi^i)}\partial_\mu(\delta\Phi^i) \\ &= \int d^4x \left[\frac{\partial \mathscr{L}}{\partial \Phi^i} - \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu\Phi^i)}\right)\right]\delta\Phi^i\end{aligned}$$

$\Rightarrow \dfrac{\delta S}{\delta \Phi^i} = \dfrac{\partial \mathscr{L}}{\partial \Phi^i} - \partial_\mu\left(\dfrac{\partial \mathscr{L}}{\partial(\partial_\mu\Phi^i)}\right) = 0$, which is the Euler-Langrange equation for a field theory in flat spacetime.

e.g.1 real scalar field

$\phi(x^\mu) \to \mathbb{R}$, $[\phi] = M^1$ (spin-less case)

$$\mathscr{L} = -\frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi) = \underbrace{\frac{1}{2}\dot{\phi}^2}_{\text{kinetic}} - \underbrace{\frac{1}{2}(\Delta\phi)^2}_{\text{gradient}} - \underbrace{V(\phi)}_{\text{potential}}$$

since $\dfrac{\partial \mathscr{L}}{\partial\phi} = -\dfrac{dV}{d\phi}$, $\dfrac{\partial \mathscr{L}}{\partial(\partial_\mu\phi)} = -\eta^{\mu\nu}\partial_\nu\phi$ and

$$\frac{\partial}{\partial(\partial_\mu\phi)}\left[\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi)\right] = \eta^{\rho\sigma}[\delta^\mu_\rho(\partial_\sigma\phi) + (\partial_\rho\phi)\delta^\mu_\sigma] = \eta^{\mu\sigma}(\partial_\sigma\phi) + \eta^{\rho\mu}(\partial_\rho\phi) = 2\eta^{\mu\nu}\partial_\nu\phi$$

therefore, $\Box\phi - \dfrac{dV}{d\phi} = 0 \quad (\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu)$. if $V = m^2\phi^2/2$, then we have Klein-Gordon equation as

$$\Box\phi - m^2\phi = 0 \tag{1.9}$$

e.g.2 vector potial field

$A_\mu = (\Phi, \vec{A})$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (gauge invariance)
consider gauge transformation: $A_\mu \to A_\mu + \partial_\mu\lambda(x)$, $F_{\mu\nu} \to F_{\mu\nu} + \partial_\mu\partial_\nu\lambda - \partial_\nu\partial_\mu\lambda = F_{\mu\nu}$,

Lecture 2.

Manifolds

Referring to the book [0] of Chapter 1.

2.1 Topological space

2.2 Manifold

Given two sets M and N, a map $\phi : M \to N$

- M: domain of ϕ; N: image of ϕ
- composition $\psi \circ \phi : A \to C$, by $(\psi \circ \phi)(a) = \psi(\phi(a))$
- one to one (injective): if each element N has at most on element of M mapped into it
- onto (surjective): ~ at least ~
- subset $U \subset N$, the element of M get mapped to U is called the preimage of U under ϕ, and $\phi^{-1}(U) \subset M$ (ϕ^{-1}: inverse map)

consider $\phi : \mathbb{R}^m \to \mathbb{R}^n$, $\quad y^1 = \phi^1(x^1, \cdots, x^m), \cdots, y^n = \phi^n(x^1, \cdots, x^m)$

- C^p: p_{th} derivative exists and is continuous
- $\phi : \mathbb{R}^m \to \mathbb{R}^n$ as C^p means at least C^p
- C^0 map: continuous but not necessariyly differentiable; C^∞ map: continuous, smooth

2.3 Maps between manifolds

consider a map $\phi : M \to N$, a function $f : N \to \mathbb{R}$, then $(f \circ \phi) : M \to \mathbb{R}$
define pull back ϕ^* as $\phi^* f = (f \circ \phi)$

2.4 Vectors, dual vectors, and tensors

an arbitary manifold, make things independent of coordinates

Claim 1. the tangent space T_p can be identified with the space of directional derivative operator along curves through p

$\{\partial_\mu\}$ at p form a basis for T_p, $\hat{e}_{(\mu)} = \partial_\mu$
consider an n-manifold M, a coordinate chart $\phi : M \to \mathbb{R}^n$, a curve $\gamma : \mathbb{R} \to M$, a function $f : M \to \mathbb{R}$,

$$\begin{aligned} \frac{d}{d\lambda} f = \frac{d}{d\lambda}(f \circ \gamma) &= \frac{d}{d\lambda}\left[(f \circ \phi^{-1})(\phi \circ \gamma)\right] \\ &= \frac{d(\phi \circ \gamma)^\mu}{d\lambda} \frac{\partial (f \circ \phi^{-1})}{\partial x^\mu} = \frac{dx^\mu}{d\lambda} \partial_\mu f \end{aligned} \quad \Rightarrow \quad \frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda} \partial_\mu$$

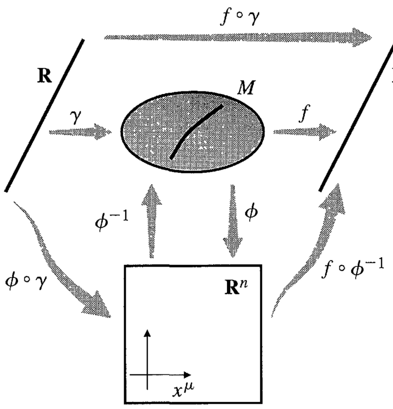

Figure 2.1: Tangle of maps

commutator: $[X,Y]^\mu = X^\lambda \partial_\lambda Y\mu - Y^\lambda \partial_\lambda X^\mu$
basis one form: $dx^{\mu'} = \dfrac{\partial x^{\mu'}}{\partial x^\mu} dx^\mu$, $\omega^{\mu'} = \dfrac{\partial x^{\mu'}}{\partial x^\mu}\omega^\mu$
(k,l) form tensor is a multilinear map form k dual vector and l vector to $\mathbb{R}$,

$$T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} = T(dx^{\mu_1},\cdots,dx^{\mu_k},\partial_{\nu_1},\cdots,\partial_{\nu_l}) = T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}\partial_{\mu_1}\otimes\cdots\otimes\partial_{\mu_k}\otimes dx^{\nu_1}\otimes\cdots\otimes dx^{\nu_l}$$

4-vector: $A'^\mu = \dfrac{\partial x'^\mu}{\partial x^\nu}A^\nu$; 4-tensor: $C'_{\mu\nu} = \dfrac{\partial x^\alpha}{\partial x'^\mu}\dfrac{\partial x^\beta}{\partial x'^\nu}C_{\alpha\beta}$
transformation law:

$$T^{\mu'_1\cdots\mu'_k}{}_{\nu'_1\cdots\nu'_l} = \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}}\cdots\frac{\partial x^{\mu'_k}}{\partial x^{\mu_k}}\frac{\partial x^{\nu'_1}}{\partial x^{\nu_1}}\cdots\frac{\partial x^{\nu'_l}}{\partial x^{\nu_l}}T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} \tag{2.1}$$

e.g. consider a symmetric (0,2) tensor S on a two-dim manifold, $(x^1 = x, x^2 = y)$ and $S_{\mu\nu} = \begin{pmatrix}1 & 0\\ 0 & x^2\end{pmatrix}$,

$$S = S_{\mu\nu}(dx^\mu\otimes dx^\nu) = dx\otimes dx + x^2 dy\otimes dy = (dx)^2 + x^2(dy)^2$$

then introduce a new coordinate $\begin{cases} x' = 2x/y \\ y' = y/2 \end{cases} \Rightarrow \begin{cases} x = x'y' \\ y = 2y' \end{cases} \Rightarrow \begin{cases} dx = y'dx' + x'dy' \\ dy = 2dy' \end{cases}$, plug this into up equation, we got

$$S_{\mu'\nu'} = \begin{pmatrix}(y')^2 & x'y' \\ x'y' & (x')^2 + 4(x'y')^2\end{pmatrix}$$

is still symmetric.
consider flat to general spaces, three things change

1. partial derivatives
2. metric
3. Levi-Civita

2.5 The metric

in a curved space,

- metric: $g_{\mu\nu}$, (0,2)type tensor

- inverse metric: $g^{\mu\nu}$, defined as

$$g^{\mu\nu} g_{\nu\sigma} = g_{\lambda\sigma} g^{\lambda\mu} = \delta^{\mu}_{\sigma} \tag{2.2}$$

restrictions on $g_{\mu\nu}$:

1. symmetric $(0,2)$ tensor
2. be nondegenerate, determinant $g = |g_{\mu\nu}| \neq 0$

consider locally inertial coordinates: $g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$, $\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}(p) = 0$

2.6 Differential forms

2.7 Integration

Lecture 3.

Curvature

Referring to the book [0] of Chapter 1 and [0] of Chapter 1.

3.1 Covariant differentiation

Parallel transport

partial derivative ∂ is not tensor;

d: an operator that reduceds to the partial derivative in flat space with inertial coordinates, but transforms as a tensor on an arbitary manifold

Intruduce connection

∇: $(k,l) \to (k,l+1)$

Properties:

1. linearity:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \qquad \nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu}\omega_\lambda \tag{3.1}$$

more general expression:

$$\begin{aligned}\nabla_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} =& \partial_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} \\ &+ \Gamma\end{aligned} \tag{3.2}$$

Define connection

Christoffel connection / Levi-Civita connection / Riemannian connection

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$$

Divergence of a vector

from $\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda$, $\Gamma^\mu_{\mu\lambda} = \frac{1}{\sqrt{|g|}}\partial_\lambda \sqrt{|g|}$ (eq:..)

3.2 Geodesics

$$\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = 0 \qquad \Leftrightarrow \qquad \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \tag{3.3}$$

Short-distance diefinition

Properties of geodesics

3.3 The Riemann curvature tensor

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \tag{3.4}$$

Properties of the Riemann tensor

1. $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$
2. $R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}$
3. $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$
4. $R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = R_{\rho[\sigma\mu\nu]} = 0$
5. $R_{[\rho\sigma\mu\nu]} = 0$

Proof. df □

Ricci tensor and Ricci scalar

Einstein tensor

3.4 Metric determinant

$$\Gamma^{\mu}_{\mu\lambda} \tag{3.5}$$

3.5 Levi-Civita tensor

3.6 Geodesic deviation

geometrical meaning of Riemann tensor

- γ_0, γ_1: geodesics
- $u^{\mu} = \dfrac{\partial x^{\mu}}{\partial t}$: vector field, tangent to geodesics, satisfy $u^{\nu}\nabla_{\nu}u^{\mu} = 0$
- $\xi^{\mu} = \dfrac{\partial x^{\mu}}{\partial s}$: tangent vector field

"relative velocity of geodesics":

$$V^{\mu} = (\nabla_u \xi)^{\mu} = u^{\rho}\nabla_{\rho}\xi^{\mu}$$

"relative accelaration of geodesics":

$$A^{\mu} = (\nabla_u V)^{\mu} = u^{\rho}\nabla_{\rho}V^{\mu} = u^{\rho}\nabla_{\rho}(u^{\sigma}\nabla_{\sigma}S^{\mu})$$

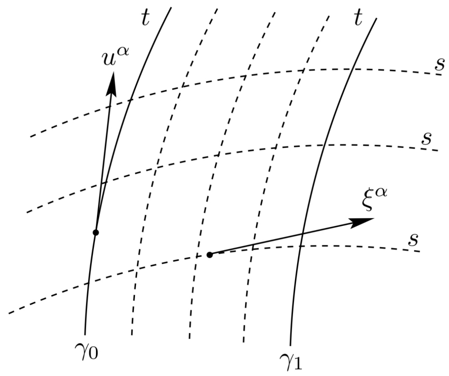

Figure 3.1: Deviation vector

Geodesic deviation equation

$$\frac{D^2\xi^\mu}{dt^2} = -R^\mu{}_{\nu\rho\sigma} u^\nu u^\rho \xi^\sigma \tag{3.6}$$

it shows curvature produces a relative acceleration between two neighbouring geodesics

3.7 Local flatness

3.7.1 Local flatness theorem (single point)

consider a single point

free-falling observers see no effect of gravity in their immediate vicinity.

3.7.2 Fermi normal coordinates (entire geodesics)

consider an entire geodesics

geometric construction

Lecture 4.

Lie Derivatives and Killing fields

Referring to the book [0] of Chapter 1 and [0] of Chapter 1.

4.1 Maps of manifolds

a smooth map $\phi : M \to N$, function $f : N \to \mathbb{R}$, all smooth tensor field $\mathscr{F}_N(k,l)$

Definition 1. The pullback map $\phi^*: \mathscr{F}_N \to \mathscr{F}_M$ is defined as

$$(\phi^* f)\Big|_p := f\Big|_{\phi(p)}, \qquad \forall f \in \mathscr{F}_N, p \in M$$

i.e. $\phi^* f = f \circ \phi$, see Fig 4.1.

Definition 2. The pushforward map $\phi_* : V_p \to V_{\phi(p)}$ is defined as

$$(\phi_* v)(f) := v(\phi^* f), \qquad \forall f \in \mathscr{F}_N, \forall v^a \in V_p, \phi_* v^a \in V_{\phi(p)}$$

i.e. the action of $\phi_* v$ on any function is the action of v on the pull back of that function.

4.2 Lie differentiation

$\mathfrak{L}$

4.3 Killing vector field

4.3.1 Symmetries

isometries: symmetries of the metric

e.g. $\partial_{\sigma_*} g_{\mu\nu} = 0 \quad \Rightarrow$

4.3.2 Killing equation

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0, \quad \text{or} \quad \nabla_{(\mu} K_{\nu)} = 0, \quad \text{or} \quad \nabla_\mu K_\nu = \nabla_{[\mu} K_{\nu]} \tag{4.1}$$

Conserved energy

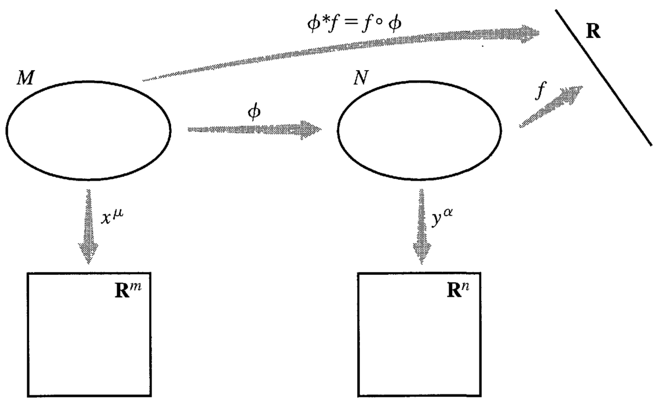

Figure 4.1: $\phi^* f = f \circ \phi$

Lecture 5.

Geodesic Congruences

5.1 Energy condition

5.2 Kinematics of a deformable medium

5.3 Congruence of timelike geodesics

5.4 Congruence of null geodesics

Lecture 6.

Einstein's Equation

Referring to the book [0] of Chapter 4.

6.1 Newtonian limit

satisfy

1. weakness of gravitational field $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $|h_{\mu\nu}| \ll 1$. since $g^{\mu\nu} g_{\nu\sigma} = \delta^{\mu}_{\sigma} \Rightarrow g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$
2. $\dfrac{dx^i}{d\tau} \ll \dfrac{dt}{d\tau} \quad \Rightarrow \quad \dfrac{d^2 x^\mu}{d\tau^2} + \Gamma^{\mu}_{00} \left(\dfrac{dt}{d\tau} \right)^2 = 0$
3. $\partial_0 g_{\mu\nu} = 0 \quad \Rightarrow \quad \Gamma^{\mu}_{00} = -\dfrac{1}{2} g^{\mu\lambda} \partial_\lambda g_{00} = -frac12(\eta^{\mu\lambda} - h^{\mu\lambda}) \partial_\lambda (\eta_{00} + h_{00}) = -\dfrac{1}{2} \eta^{\mu\lambda} \partial_\lambda h_{00}$

Therefore,

6.2 The Einstein field equation

By Bianchi identies $\nabla^\mu G_{\mu\nu} = 0$, assume $G_{\mu\nu} = \kappa_{\mu\nu}$, since $\nabla_\mu T = 0$, then we have

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{6.1}$$

Properties of Einstein's equation

6.3 Langrangian formulation

action: $S = \int \mathscr{L}(\Phi^i, \nabla_\mu \Phi^i) d^n x$

6.4

Lecture 7.

Hypersurfaces

Referring to the book [0] of Chapter 3. From intrinsic and extrinsic geometry of a hypersurfae, to discontinuities of the metric.

7.1 Description of hypersurfaces

- Hypersurface (Σ): a 3-dim submanifold in a 4-dim spacetime manifold (either timelike, spacelike or null).
- Σ like restriction $\Phi(x^\alpha) = 0$ or parametric equation $x^\alpha = x^\alpha(y^a)$, where y^a ($a = 1, 2, 3$) are coordinates intrinsics to Σ.

Normal vector

7.2 Integration on hypersurfaces

Surface element (non-null case)

Surface element (null case)

7.3 Gauss-Stokes theorem

First version: Gauss' theorem

$$\int_{\mathcal{V}} A^\alpha{}_{;\alpha} \sqrt{-g} d^4x = \oint_{\partial\mathcal{V}} A^\alpha d\Sigma_\alpha \tag{7.1}$$

Second version: Stokes' theorem

$$\int_{\Sigma} B^{\alpha\beta}{}_{:\beta} d\Sigma_\alpha = \frac{1}{2} \oint_{\partial\Sigma} B^{\alpha\beta} dS_{\alpha\beta} \tag{7.2}$$

7.4 Differentiation of tangent vector fields

Lecture 8.

Perturbation Theory and Gravitational Radiation

Refering to

8.1 Perturbation

Lecture 9.

Lagrangian and Hamiltonian Formulation of GR

Referring to the book [0] of Chapter 4.

9.1 Lagrangian formulation

9.2 Hamiltonian formulation

9.3 Mass and angular momentum

Lecture 10.

The Schwarzschild Solution

Referring to the book [0] of Chapter 4.

10.1 Lagrangian formulation

10.2 Hamiltonian formulation

10.3 Mass and angular momentum

Lecture 11.

More General Black Holes

Lecture 12.

The Expansion of the Universe

Lecture 13.

The Early Universe

Lecture 14.

Growth of Structure I: Linear Theory

Lecture 15.

The Cosmic Microwave Background

Lecture 16.

The Polarized CMB

Lecture 17.

Growth of Structure II: Beyond Linear Theory

Lecture 18.

Probes of Structure: Gravitational Lensing

Lecture 19.

Analysis and Inference

Lecture 20.

Field theory

Lecture 21.

QFT in Curved Spacetime

Lecture 22.

The Standard Inflationary Universe

Lecture 23.

Inflation and the Cosmological Perturbation

.Recommended Resources

Books

[0] Sean Carroll. *Spacetime and geometry: an introduction to general relativity*. eng. San Francisco: Addison Wesley, 2004. ISBN: 978-0-8053-8732-2 (pp. i, 1, 5, 8, 11, 14)

[0] Eric Poisson. *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics*. en. Cambridge: Cambridge University Press, 2004. ISBN: 978-0-511-60660-1. DOI: 10.1017/CBO9780511606601. URL: https://www.cambridge.org/core/product/identifier/9780511606601/type/book (visited on 10/18/2024) (pp. i, 8, 11, 15, 17, 18)

[0] David Hilary Lyth and Andrew R. Liddle. *The primordial density perturbation: cosmology, inflation and the origin of structure*. eng. Cambridge, UK New York: Cambridge University Press, 2009. ISBN: 978-0-511-53985-5 978-0-511-65060-4 978-0-511-81920-9 (p. i)

[0] Scott Dodelson and Fabian Schmidt. *Modern cosmology*. eng. Second edition. London San Diego, CA Cambridge, MA Oxford: Academic Press, an imprint of Elsevier, 2021. ISBN: 978-0-12-815948-4 (p. i)

[0] Robert M. Wald. *General relativity*. Chicago: University of Chicago Press, 1984. ISBN: 978-0-226-87032-8 978-0-226-87033-5 (p. i)

[0] Canbin Liang and Bin Zhou. *Differential Geometry and General Relativity: Volume 1*. en. Graduate Texts in Physics. Singapore: Springer Nature Singapore, 2023. ISBN: 978-981-9900-21-3 978-981-9900-22-0. DOI: 10.1007/978-981-99-0022-0. URL: https://link.springer.com/10.1007/978-981-99-0022-0 (visited on 11/02/2024) (p. i)