Reading Notes: General Relativity and Cosmology

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Preface

The notes the former part mainly based on [0], the flow from math to perturbation theory
[0] mathematics and black hole
[0] talks on perturbation and inflation theory
[0] is observational cosmology

[0] and [0] is a
Currently, these are just drafts of these notes. There can be typos and mistakes anywhere. So, if you find anything that needs to be corrected or improved, please inform at yingqiu@post.kek.jp.
I am also planning to do think work on Group theory and qft. and after learning them, I plan learn geometry and algebra for physician more carefully. (beyond my research)

Notation and conventions

Greek letter indices $(\mu, \nu, \cdots)$ run from 0 to 3, lower-case Latin indices $(i, j, \cdots)$ run from 1 to 3,

Contents

PART I

DIFFERENTIAL GEOMETRY

This part, the note contains from mathematics to a application of perturbation and radiation.
the mathematics part are little difficult df

Special Relativity and Flat Spacetime

Referring to the book [0] of Chapter 1. Considering inertial Cartesian-like coordinates..

1.1 Spacetime

- events: interval between neighbring points of spacetime
- worldline: the path of a particle is a curve through spacetime, a parameterized one-dimensional set of events
- space interval between two events: $(\Delta s)^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu, \eta_{\mu\nu} = diag(-1,1,1,1)$
  - summuation convention - dummy indices
- lightcone: timelike $(\Delta s)^2 < 0$; spacetime $(\Delta s)^2 > 0$; lightlike/null $(\Delta s)^2 = 0$
- propertime: $(\Delta\tau)^2 = -(\Delta s)^2$

infinitesimal interval / line element:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{1.1}$$

spacelike curve:

$$\Delta s = \int \sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda$$

timelike path:

$$\Delta\tau = \int \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda$$

1.2 Lorentz transformation

coordiante systems that leave the interval invariant

$$x^{\mu'} = \lambda^{\mu'}{}_\nu x^\nu \tag{1.2}$$

interval invariant: $\Delta s^2 = (\Delta x)^\intercal \eta(\Delta x) = (\Delta x')^\intercal \eta(\Delta x') = (\Delta x)^\intercal \lambda^\intercal \eta \lambda(\Delta x) \Rightarrow \eta = \lambda^\intercal \eta \lambda$ or

$$\eta_{\rho\sigma} = \lambda^{\mu'}{}_\rho \eta_{\mu'\nu'} \lambda^{\nu'}{}_\sigma = \lambda^{\mu'}{}_\rho \lambda^{\nu'}{}_\sigma \eta_{\mu'\nu'} \tag{1.3}$$

Lorentz group: the set of them forms a group under matrix multiplication

Spatial rotation / boosts: (in x-direction)

boost rotation angle of $\phi \in (-\infty, +\infty)$,

$$\lambda^{\mu'}{}_\nu = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then $t' = t\cosh\phi - x\sin\phi, x' = -t\sinh\phi + x\cosh\phi$. when $x' = 0 \Rightarrow v = x/t = \tanh\phi \Rightarrow$

$$t' = \gamma(t - vx)$$
$$x' = \gamma(x - vt)$$

where $\gamma = 1/\sqrt{1-v^2} = \cosh\phi\ (\cosh\phi^2 - \sinh\phi^2)$. boosts correspond to changing coordinates by moving to a frame that travels at a constant velocity.

1.3 Vector and dual vector

- (real) vector space: a collection of vectors that can be added together and multiplied by real numbers in a linear way
- tangent space at p (T_p): the space of all vectors at the point p
- dimension of the space: infinit number of possible bases

any vector A can be written as $A = A^\mu \hat{e}_{(\mu)}$ ($\mu \in \{0,1,2,3\}$), where $\hat{e}_{(\mu)}$ basis vector and A^μ is components
tangent vector:

$$V(\lambda) = V^\mu \hat{e}_{(\mu)}, \qquad V^\mu = \frac{dx^\mu}{d\lambda} \tag{1.4}$$

consider Lorentz transformation: $V^\mu \to V^{\mu'} = \lambda^{\mu'}{}_\nu V^\nu$ (invariant), then

$$V = V^\mu \hat{e}_{(\mu)} = \lambda^{\nu'}{}_\mu V^\mu \hat{e}_{(\nu')}$$

$\hat{e}_{(\nu')}$ is transformed coordinates system $\hat{e}_{(\nu')} = \lambda^\mu_{\nu'} \hat{e}_{(\mu)}$, therefore we need

$$\lambda^\mu_{\nu'} \lambda^{\nu'}_\rho = \delta^\mu_\rho \tag{1.5}$$

- dual vector space: all linear maps from the original vector space to the real number
- cotangent space (T^*_p)

if $\omega \in T^*_p$, then $V, W \to$vecetor, $a, b \to$real number

$$\omega(aV + bW) = a\omega(V) + b(W) \in \mathbb{R}$$

similarly, $\omega = \omega_\mu \hat{\theta}^{(\mu)}$, and $\hat{\theta}^{(\nu)}(\hat{e}_{(\mu)}) = \delta^\nu_\mu$

Proof. $V(\omega) \equiv \omega(V) = \omega_\mu \hat{\theta}^{(\mu)}(V^\nu \hat{e}_{(\nu)}) = \omega_\mu V^\nu \hat{\theta}^{(\mu)} \hat{e}_{(\nu)} = \omega_\mu V^\nu \delta^\mu_\nu = \omega_\mu V^\mu \in \mathbb{R}$ □

$$\boxed{\omega_{\mu'} = \lambda^\nu{}_{\mu'} \omega_\nu \qquad \hat{\theta}^{(\rho')} = \lambda^{\rho'}{}_\sigma \hat{\theta}^{(\sigma)}} \tag{1.6}$$

e.g. the gradient of a scalar is a dual vector $d\phi = \frac{\partial \phi}{\partial x^\mu} \hat{\theta}^{(\mu)}$, then $\partial_{\mu'}\phi = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \phi$

Note. $\frac{\partial \phi}{\partial x^\mu} = \partial_\mu \phi = \phi_{,\mu}$

1.4 Tensor

a tensor T of type (k,l) is a multilinear map from T^*_p and T_p to $\mathbb{R}$: $T : T^*_p \times \cdots \times T^*_p \times T_p \cdots \times T_p \to \mathbb{R}$

- $\times$: Cartesian product
- $\otimes$: Tensor prodect

if $T(k,l)$ and $S(m,n)$, we define a $(k+m, l+n)$ tensor

$$\begin{aligned} &T \otimes S(\omega^{(1)}, \cdots, \omega^{(k)}, \cdots, \omega^{(k+m)}, V^{(1)}, \cdots, V^{(l)}, \cdots, V^{(l+n)}) \\ &= T(\omega^{(1)}, \cdots, \omega^{(k)}, V^{(1)}, \cdots, V^{(l)}) \times S(\omega^{(k+1)}, \cdots, \omega^{(k+m)}, V^{(l+1)}, \cdots, V^{(l+n)}) \end{aligned}$$

In components notation,

$$
\begin{aligned}
&T = T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}\hat{e}_{(\mu_1)} \otimes \cdots \otimes \hat{e}_{(\mu_k)} \otimes \hat{\theta}^{(\nu_1)} \otimes \cdots \otimes \hat{\theta}^{(\nu_l)} \qquad \text{or} \\
&T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} = T(\hat{\theta}^{(\nu_1)}, \cdots, \hat{\theta}^{(\nu_l)}, \hat{e}_{(\mu_1)}, \cdots, \hat{e}_{(\mu_k)})
\end{aligned}
$$

usually denote T as $T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}$, the action on vectors and dual vectors to $\mathbb{R}$

$$T(\omega^{(1)}, \cdots, \omega^{(k)}, V^{(1)}, \cdots, V^{(l)}) = T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}\omega^{(1)}_{\mu_1}\cdots\omega^{(k)}_{\mu_k}V^{(1)\nu_1}\cdots V^{(l)\nu_l}$$

Tensor Lorentz transformation law:

$$T^{\mu_1'\cdots\mu_k'}{}_{\nu_1'\cdots\nu_l'} = \lambda^{\mu_1'}{}_{\mu_1}\cdots\lambda^{\mu_k'}{}_{\mu_k}\lambda^{\nu_1'}{}_{\nu_1}\cdots\lambda^{\nu_l'}{}_{\nu_l}T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}$$

some examples:

1. Minkowski metric $\eta_{\mu\nu}$ $(0,2)$, inner product: $\eta(V,W) = \eta_{\mu\nu}V^\mu W^\nu = V\cdot W$
2. Kronecker delta δ^μ_ρ $(1,1)$
3. inverse metric $\eta^{\mu\nu}$ $(2,0)$, defined as $\eta^\mu\eta_{\nu\rho} = \eta_{\rho\nu}\eta^{\nu\mu} = \delta^\mu_\rho$
4. Levi-Civita symbol $\tilde{\epsilon}_{\mu\nu\rho\sigma}$ $(0,4)$
5. Electromagnetic field strength tensor $F_{\mu\nu} = -F_{\nu\mu}$

> **Remark.** well-defined tensor:
>
> - transforming according to the tensor transformation law
> - defining a unique multilinear map from a set of vector and dual vector to the real number

manipulation tensors

operation of contraction: $S^{\mu\rho}{}_\sigma = T^{\mu\nu\rho}{}_{\sigma\nu}$, $T^{\mu\nu\rho}{}_{\sigma\nu} \neq T^{\mu\rho\nu}{}_{\sigma\nu}$
raise and lower indices: $T^{\alpha\beta\mu}{}_\delta = \eta^\mu{}_\gamma T^{\alpha\beta}{}_{\gamma\sigma}$, $T_\mu{}^\beta{}_{\gamma\delta} = \eta_{\mu\alpha}T^{\alpha\beta}{}_{\gamma\sigma}$, $T_{\mu\nu}{}^{\rho\sigma} = \eta_{\mu\alpha}\eta_{\nu\beta}\eta^{\rho\gamma}\eta^{\sigma\delta}T^{\alpha\beta}{}_{\gamma\sigma}$
symmetric / antisymmetric: $S_{\mu\nu\rho} = S_{\nu\mu\rho}$ / $A_{\mu\rho} = -A_{\nu\mu\rho}$

1.5 Maxwell's equations

written as

$$
\begin{aligned}
\tilde{\epsilon}^{ijk}\partial_j B_k - \partial_0 E^i &= J^i \\
\partial_i E^i &= J^0 \\
\tilde{\epsilon}^{ijk}\partial_j E_k + \partial_0 B^i &= 0 \\
\partial_i B^i &= 0
\end{aligned}
\quad\Rightarrow\quad
\begin{aligned}
\partial_i F^{iji} - \partial_0 F^{0i} &= J^i \\
\partial_i F^{0i} &= J^0 \\
\partial &= \\
-\frac{1}{2}\tilde{\epsilon}^{ijk}\partial_i F^{jk} &= 0
\end{aligned}
\quad\Rightarrow\quad
\begin{aligned}
\partial_\mu F^{\nu\mu} &= J^\nu \\
\partial_{[\mu}F_{\nu\lambda]} &= 0
\end{aligned}
\tag{1.7}
$$

where $F^{0i} = E^i, F^{ij} = \tilde{\epsilon}^{ijk}B_k$, i.e. $F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$ and its inverse is

$F^{0i} = B^i$

1.6 Energy and momentum

consider massive particle path $x^\mu(\tau)$, four velocity $U^\mu = \frac{dx^\mu}{d\tau}$,
from $d\tau^2 = -\eta_{\mu\nu}dx^\mu dx^\nu$, $\eta_{\mu\nu}U^\mu U^\nu = -1$ (automatically normalized)

- momentum four-vector: $p^\mu = mU^\mu$, energy: $E = p^0$
- energy momentrum tensor: $T^{\mu\nu}$ (the flux of p^μ across a surface of constant x^ν)
  - T^{00}: the flux of p^0 in x^0 direction;
  - $T^{0i} = T^{i0}$: momentum density;
  - T^{ij}: momentum flux/stress

for a perfect fluid, we have $T = \mathrm{diag}(\rho, p, p, p)$, i.e. $T^{\mu\nu} = (\rho + p)U^\mu U^\nu + p\eta^{\mu\nu}$, ρ: energy density, p: pressure
satisfy relativistic eqn of enerfy conservation $\partial_\mu T^{\mu\nu} = 0$,

$$\begin{aligned}
U_\nu \partial_\mu T^{\mu\nu} &= U_\nu \left(\partial_\mu(\rho + p)U^\mu U^\nu + (\rho + p)(U^\nu \partial_\mu U^\mu + U^\mu \partial_\mu U^\nu) + \partial_\mu \eta^{\mu\nu} p\right) \\
&= -\partial_\mu(\rho + p)U^\mu + (\rho + p)(-\partial_\mu U^\mu) + U_\nu \partial^\nu p \\
&= -\partial_\mu(\rho U^\mu) - p\partial_\mu U^\mu = 0
\end{aligned}$$

here use $U_\nu \partial_\mu U^\nu = \frac{1}{2}\partial_\mu(U_\nu U^\nu) = 0, \partial_\mu \eta^{\mu\nu} = \partial^\nu$
if look into a ordinary non-relativistic $U^\mu = (1, v^i)$, $|v^i| \ll 1$, $p \ll \rho$, then got a continuity eqn for the energy density

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0$$

1.7 Classical field theory

"principle of least action"

- $\Phi^i(x^\mu)$: a set of spacetime-dependent fields
- S: functional/integral of fields
- $\mathscr{L}$: Langrange density

$$L = \int d^3x \mathscr{L}(\Phi^i, \partial_\mu \Phi^i) \tag{1.8}$$

$$S = \int dt L = \int d^4 \mathscr{L}(\Phi^i, \partial_\mu \Phi^i) \tag{1.9}$$

- natural unit: $[energy] = [mass] = [(length)^{-1}] = [(time)^{-1}]$, $[S] = [E][M] = M^0, [\mathscr{L}] = M^4$

consider $\Phi^i \to \Phi^i + \delta\Phi^i$, then $\partial_\mu \Phi^i \to \partial\Phi^i + \delta(\partial_\mu \Phi^i) = \partial\Phi^i + \partial\delta(\Phi^i)$ and $S \to S + \delta S$

$$\begin{aligned}
\delta S &= \int d^4x \frac{\partial \mathscr{L}}{\partial \Phi^i}\delta\Phi^i + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi^i)}\partial_\mu(\delta\Phi^i) \\
&= \int d^4x \left[\frac{\partial \mathscr{L}}{\partial \Phi^i} - \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi^i)}\right)\right]\delta\Phi^i
\end{aligned}$$

$\Rightarrow \dfrac{\delta S}{\delta \Phi^i} = \dfrac{\partial \mathscr{L}}{\partial \Phi^i} - \partial_\mu\left(\dfrac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi^i)}\right) = 0$, which is the Euler-Langrange equation for a field theory in flat space-time.

e.g.1 real scalar field

$\phi(x^\mu) \to \mathbb{R}$, $[\phi] = M^1$ (spin-less case)

$$\mathscr{L} = -\frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi) = \underbrace{\frac{1}{2}\dot{\phi}^2}_{\text{kinetic}} - \underbrace{\frac{1}{2}(\Delta\phi)^2}_{\text{gradient}} - \underbrace{V(\phi)}_{\text{potential}}$$

since $\frac{\partial \mathscr{L}}{\partial \phi} = -\frac{dV}{d\phi}$, $\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} = -\eta^{\mu\nu}\partial_\nu \phi$ and

$$\frac{\partial}{\partial(\partial_\mu \phi)}\left[\eta^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi)\right] = \eta^{\rho\sigma}[\delta^\mu_\rho(\partial_\sigma \phi) + (\partial_\rho \phi)\delta^\mu_\sigma] = \eta^{\mu\sigma}(\partial_\sigma \phi) + \eta^{\rho\mu}(\partial_\rho \phi) = 2\eta^{\mu\nu}\partial_\nu \phi$$

therefore, $\Box\phi - \frac{dV}{d\phi} = 0$ [1]. if $V = m^2\phi^2/2$, then we have Klein-Gordon equation as

$$\Box\phi - m^2\phi = 0 \tag{1.10}$$

e.g.2 vector potial field

$A_\mu = (\Phi, \vec{A})$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (gauge invariance)
consider gauge transformation: $A_\mu \to A_\mu + \partial_\mu \lambda(x)$, $F_{\mu\nu} \to F_{\mu\nu} + \partial_\mu\partial_\nu\lambda - \partial_\nu\partial_\mu\lambda = F_{\mu\nu}$,
Review Maxwell's eqns: $\partial_{[\mu}F_{\nu\sigma]} = \partial_{[\mu}\partial_\nu A_{\sigma]} - \partial_{[\mu}\partial_\sigma A_{\nu]}$,
$\partial_\mu F^{\nu\mu} = J^\nu \Leftrightarrow \frac{\partial \mathscr{L}}{\partial A_\nu} - \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu)}\right) = 0$, where $\mathscr{L} = -\frac{1}{4}F_\mu F^\mu + A_\mu J^\mu$

Proof. $\frac{\partial \mathscr{L}}{\partial A_\nu} = J^\mu$, $\quad F_\mu F^\mu = F_{\alpha\beta}F^{\alpha\beta} = \eta^{\alpha\rho}\eta^{\beta\sigma}F_{\alpha\beta}F_{\rho\sigma}$, $\quad \frac{\partial F^\beta_\alpha}{\partial(\partial_\mu A_\nu)} = \delta^\mu_\alpha\delta^\nu_\beta - \delta^\beta_\mu\delta^\nu_\alpha$

$$\begin{aligned}\frac{\partial(F_{\alpha\beta}F^{\alpha\beta})}{\partial(\partial_\mu A_\nu)} =& \eta^{\alpha\rho}\eta^{\beta\sigma}\left[\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}F_{\rho\sigma} + \frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_\nu)}F_{\alpha\beta}\right] = \left[\left(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\beta_\mu\delta^\nu_\alpha\right)F_{\rho\sigma} + \left(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho\right)F_{\alpha\beta}\right] \\ =& (\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\nu\rho}\eta^{\mu\sigma})F_{\rho\sigma} + (\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\nu}\eta^{\beta\mu})F_{\alpha\beta} = F^{\mu\nu} - F^{\nu\mu} + F^{\mu\nu} - F^{\nu\mu\nu} = 4F^{\mu\nu}\end{aligned}$$

□

Gauge field, Higgs field, fermions fields...

[1] $\Box \equiv \partial^\mu\partial_\mu = \eta^{\mu\nu}\partial_\mu\partial_\nu$

Manifolds

Referring to the book [0] of Chapter 1.

2.1 Introduction to topological space

2.2 Manifold

Given two sets M and N, a map $\phi : M \to N$

- M: domain of ϕ; N: image of ϕ
- composition $\psi \circ \phi : A \to C$, by $(\psi \circ \phi)(a) = \psi(\phi(a))$
- one to one (injective): if each element N has <u>at most</u> on element of M mapped into it
- onto (surjective): ~ at least ~
- subset $U \subset N$, the element of M get mapped to U is called the preimage of U under ϕ, and $\phi^{-1}(U) \subset M$ (ϕ^{-1}: inverse map)

consider $\phi : \mathbb{R}^m \to \mathbb{R}^n, \quad y^1 = \phi^1(x^1, \cdots, x^m), \cdots, y^n = \phi^n(x^1, \cdots, x^m)$

- C^p: p_{th} derivative exists and is continuous
- $\phi : \mathbb{R}^m \to \mathbb{R}^n$ as C^p means at least C^p
- C^0 map: continuous but not necessariyly differentiable; C^∞ map: continuous, smooth

call two sets M and N are diffeomorphic if C^∞ map $\phi : M \to N$ with a C^∞ inverse map $\phi^{-1} : N \to M$, and ϕ is diffeomorphism

definition of minifold

- open ball: $|x-y| < r$, where $|x-y| = \left(\sum_i (x^i - y^i)^2\right)^{1/2}$
- open set: in $\mathbb{R}^n$, is a set constructed from an arbitary union of open ball
- $V \subset \mathbb{R}^n$ is open if for any $y \in V$, there is an open ball centered at y that is completely inside y

consider a char/coordinate system, U is an open set in M, $\phi(U)$ is open in $\mathbb{R}^n$. A C^∞ atlas is an indexed collection of chart (U_α, ϕ_α) and (U_β, ϕ_β) that satisfy, see Fig (2.1)

1. U_α cover M
2. $U_\alpha \cap U_\beta \neq \emptyset$
   i.e. $(\phi_\alpha \circ \phi_\beta^{-1})$ take points in $\phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ onto an open set $\phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$, and all of these maps must be C^∞

> **Definition 1.** a $C^\infty n-$dimensional manifold: a set M along with a maximal atlas, one that contains every possible compatible chart.

Chain rule: ($f : \mathbb{R}^m \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^l$, $x^a \in \mathbb{R}^m$, $y^b \in \mathbb{R}^n$)

$$\frac{\partial}{\partial x^a}(g \circ g)^c = \sum_b \frac{\partial f^b}{\partial x^a}\frac{\partial g^c}{\partial y^b}, \qquad \frac{\partial}{\partial x^a} = \sum_b \frac{\partial y^b}{\partial x^a}\frac{\partial}{\partial y^b}$$

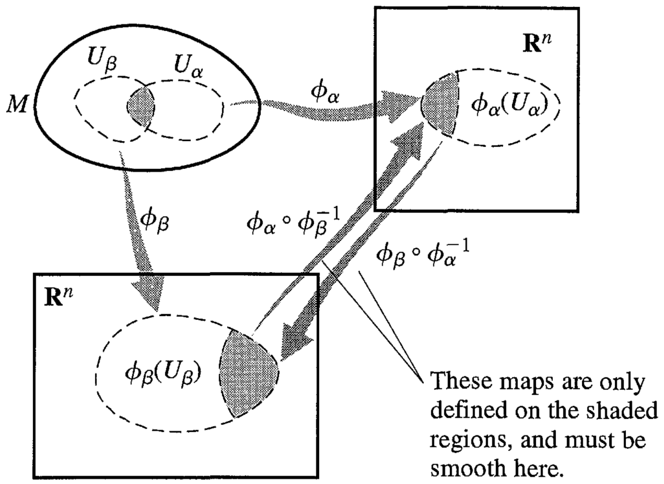

Figure 2.1: Overlapping coordinate chart

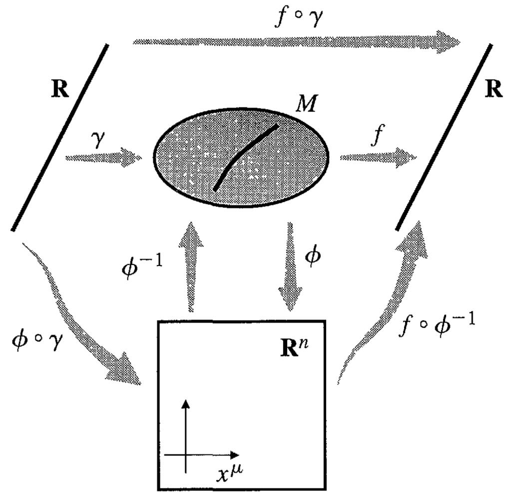

Figure 2.2: Tangle of maps

2.3 Maps between manifolds

consider a map $\phi : M \to N$, a function $f : N \to \mathbb{R}$, then $(f \circ \phi) : M \to \mathbb{R}$
define pull back ϕ^* as $\phi^* f = (f \circ \phi)$

2.4 Vectors, dual vectors, and tensors

an arbitary manifold, make things independent of coordinates
the tangent space T_p can be identified with the space of directional derivative operator along curves through p
$\{\partial_\mu\}$ at p form a basis for T_p, $\hat{e}_{(\mu)} = \partial_\mu$
consider an n-manifold M, a coordinate chart $\phi : M \to \mathbb{R}^n$, a curve $\gamma : \mathbb{R} \to M$, a function $f : M \to \mathbb{R}$,

$$\begin{aligned} \frac{d}{d\lambda} f = \frac{d}{d\lambda}(f \circ \gamma) &= \frac{d}{d\lambda}\left[(f \circ \phi^{-1})(\phi \circ \gamma)\right] \\ &= \frac{d(\phi \circ \gamma)^\mu}{d\lambda} \frac{\partial (f \circ \phi^{-1})}{\partial x^\mu} = \frac{dx^\mu}{d\lambda} \partial_\mu f \end{aligned} \quad \Rightarrow \quad \frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda} \partial_\mu$$

commutator: $[X, Y]^\mu = X^\lambda \partial_\lambda Y\mu - Y^\lambda \partial_\lambda X^\mu$
basis one form: $dx^{\mu'} = \dfrac{\partial x^{\mu'}}{\partial x^\mu} dx^\mu$, $\omega^{\mu'} = \dfrac{\partial x^{\mu'}}{\partial x^\mu} \omega^\mu$
(k, l) form tensor is a multilinear map form k dual vector and l vector to $\mathbb{R}$,

$$T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} = T(dx^{\mu_1}, \cdots, dx^{\mu_k}, \partial_{\nu_1}, \cdots, \partial_{\nu_l}) = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_k} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_l}$$

4-vector: $A'^\mu = \dfrac{\partial x'^\mu}{\partial x^\nu} A^\nu$; 4-tensor: $C'_{\mu\nu} = \dfrac{\partial x^\alpha}{\partial x'^\mu} \dfrac{\partial x^\beta}{\partial x'^\nu} C_{\alpha\beta}$
transformation law:

$$T^{\mu'_1 \cdots \mu'_k}{}_{\nu'_1 \cdots \nu'_l} = \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu'_k}}{\partial x^{\mu_k}} \frac{\partial x^{\nu'_1}}{\partial x^{\nu_1}} \cdots \frac{\partial x^{\nu'_l}}{\partial x^{\nu_l}} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \tag{2.1}$$

e.g. consider a symmetric (0,2) tensor S on a two-dim manifold, $(x^1 = x, x^2 = y)$ and $S_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & x^2 \end{pmatrix}$,

$$S = S_{\mu\nu}(dx^\mu \otimes dx^\nu) = dx \otimes dx + x^2 dy \otimes dy = (dx)^2 + x^2 (dy)^2$$

then introduce a new coordinate $\begin{cases} x' = 2x/y \\ y' = y/2 \end{cases} \Rightarrow \begin{cases} x = x'y' \\ y = 2y' \end{cases} \Rightarrow \begin{cases} dx = y'dx' + x'dy' \\ dy = 2dy' \end{cases}$, plug this into up equation, we got

$$S_{\mu'\nu'} = \begin{pmatrix} (y')^2 & x'y' \\ x'y' & (x')^2 + 4(x'y')^2 \end{pmatrix}$$

is still symmetric.
consider flat to general spaces, three things change

1. partial derivatives
2. metric
3. Levi-Civita

2.5 The metric

metric: returens the actual physical distance between two infinitesimally close points in spacetime defined in some arbitrary coordinate system.

- metric: $g_{\mu\nu}$, (0,2)type tensor
- inverse metric: $g^{\mu\nu}$, defined as

$$g^{\mu\nu} g_{\nu\sigma} = g_{\lambda\sigma} g^{\lambda\mu} = \delta^{\mu}_{\sigma} \tag{2.2}$$

restrictions on $g_{\mu\nu}$:

1. symmetric $(0,2)$ tensor
2. be nondegenerate, determinant $g = |g_{\mu\nu}| \neq 0$

consider locally inertial coordinates: $g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$, $\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}(p) = 0$

2.6 Differential forms

2.7 Integration

Curvature

Referring to the book [0] of Chapter 1 and [0] of Chapter 1.

3.1 Covariant differentiation

Parallel transport

partial derivative ∂ is not tensor; <—- need to ..
d: an operator that reduceds to the partial derivative in flat space with inertial coordinates, but transforms as a tensor on an arbitary manifold

Intruduce connection

- covariant derivaties ∇: $(k,l) \to (k,l+1)$
- connection coeffeicient: $\Gamma^{\nu}_{\mu\lambda}$

covariant derivative of vector and dual vector:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^{\nu}_{\mu\lambda} V^\lambda \qquad \nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^{\lambda}_{\mu\nu}\omega_\lambda$$

Proof. since $\omega_\lambda V^\lambda \in \mathbb{R}$, and $\partial_\mu(\omega_\lambda V^\lambda) = \nabla_\mu(\omega_\lambda V^\lambda)$, if $\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \widetilde{\Gamma}^{\lambda}_{\mu\nu}\omega_\lambda$
l.h.s $= (\partial_\mu \omega_\lambda) V^\lambda + \omega_\lambda(\partial_\mu V^\lambda)$
r.h.s $= (\nabla_\mu \omega_\lambda) V^\lambda + \omega_\lambda(\nabla_\mu V^\lambda) = (\partial_\mu \omega_\lambda) V^\lambda + \widetilde{\Gamma}^{\sigma}_{\mu\lambda}\omega_\sigma V^\lambda + \omega_\lambda \partial_\mu V^\lambda + \omega_\lambda \Gamma^{\lambda}_{\mu\rho} V^\rho$

$$\Rightarrow \qquad 0 = \widetilde{\Gamma}^{\sigma}_{\mu\lambda}\omega_\sigma V^\lambda + \omega_\lambda \Gamma^{\lambda}_{\mu\rho} V^\rho \quad \xRightarrow[\lambda\to\sigma]{\rho\to\lambda} \quad \widetilde{\Gamma}^{\sigma}_{\mu\lambda} = -\Gamma^{\sigma}_{\mu\lambda}$$

□

it satisfy $\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu$, therefore we can have the transformation of connection is

$$\Gamma^{\nu'}{}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^{\nu}{}_{\mu\lambda} + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial^\mu \partial^\lambda}$$

Proof. extend the eqn
l.h.s $= \partial_{\mu'} V^{\nu'} + \Gamma^{\nu'}{}_{\mu'\lambda'} V^{\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} V^\nu \frac{\partial}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu} + \Gamma^{\nu'}{}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda$
r.h.s $= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^{\nu}{}_{\mu\lambda} V^\lambda \quad \Rightarrow \quad \Gamma^{\nu'}{}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda + \frac{\partial x^\mu}{\partial x^{\mu'}} V^\lambda \frac{\partial}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\lambda} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^{\nu}{}_{\mu\lambda} V^\lambda$

□

Properties:

1. linearity: $\nabla(T+S) = \nabla T + \nabla S$
2. Leibiniz (product) rule: $\nabla(\otimes S) = (\nabla T)\otimes S + T \otimes (\nabla S)$
3. commutes with contractions: $\nabla_\mu (T^{\lambda}{}_{\lambda\rho}) = (\nabla T)_\mu{}^{\lambda}{}_{\lambda\rho}$
4. reduced to partial derivative on scalars: $\nabla_\mu \phi = \partial_\mu \phi$

more general expression:

$$\begin{aligned}\nabla_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} =& \partial_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} \\ &+ \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1\lambda\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_l} + \cdots \\ &- \Gamma^{\lambda}_{\sigma\nu_1} T^{\mu_1\mu_2\cdots\mu_k}{}_{\lambda\nu_2\cdots\nu_l} - \Gamma^{\lambda}_{\sigma\nu_2} T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\lambda\cdots\nu_l} - \cdots\end{aligned} \tag{3.1}$$

Define connection

consider a large connection on manifold

- $\Gamma^\lambda_{\mu\nu} = \widehat{\Gamma}^\lambda_{\mu\nu} + S^\lambda_{\mu\nu}$
- torsion tensor: $T^\lambda{}_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = 2\Gamma^\lambda_{[\mu\nu]}$ (antisymmetric)

  torsion free: $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{(\mu\nu)}$ (symmetric)
- metric compatibility: $\nabla_\rho g_{\mu\nu} = 0, \quad \nabla_\lambda \epsilon_{\mu\nu\rho\sigma} = 0$

  $\Rightarrow \quad g_{\mu\lambda}\nabla_\rho V^\lambda = \nabla + \rho(g_{\mu\nu}V^\lambda) = \nabla_\rho V_\mu$

Gamma connection / Levi-Civita connection / Riemannian connection

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$$

Proof. metric compatible, torsion-free and symmetry of connection

$\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda} = 0, \quad \nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\lambda_{\mu\rho} g_{\lambda\rho} - \Gamma^\lambda_{\mu\rho} g_{\nu\lambda} = 0, \quad \nabla_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - \Gamma^\lambda_{\nu\rho} g_{\lambda\mu} - \Gamma^\lambda_{\nu\mu} g_{\rho\lambda} = 0$

$\Rightarrow \quad \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2\Gamma^\lambda_{\mu\nu} g_{\lambda\rho} = 0 \quad \Rightarrow \quad (g_{\lambda\rho} g^{\sigma\rho} = \delta^\sigma_\lambda)$

□

Divergence of a vector

from $\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda$, $\Gamma^\mu_{\mu\lambda} = \frac{1}{\sqrt{|g|}} \partial_\lambda \sqrt{|g|}$ [1], then

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} V^\mu\right)$$

3.2 Geodesics

$$\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = 0 \qquad \Leftrightarrow \qquad \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \tag{3.2}$$

Short-distance diefinition

Properties of geodesics

3.3 The Riemann curvature tensor

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{3.3}$$

[1] Eqn:.. prove it

Properties of the Riemann tensor

1. $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$
2. $R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}$
3. $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$
4. $R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = R_{\rho[\sigma\mu\nu]} = 0$
5. $R_{[\rho\sigma\mu\nu]} = 0$

Proof. df □

Ricci tensor and Ricci scalar

- Ricci tensor: $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = R_{\nu\mu}$
- Ricci scalar: $R = R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu}$

Einstein tensor

3.4 Metric determinant

$$\Gamma^{\mu}_{\mu\lambda} \tag{3.4}$$

3.5 Levi-Civita tensor

3.6 Maximally symmetric spaces

$$R_{\rho\sigma\mu\nu} = \kappa(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}) \tag{3.5}$$

where

$$\kappa = \frac{R}{n(n-1)}$$

- $\kappa > 0$: de Sitter space

3.7 Geodesic deviation

geometrical meaning of Riemann tensor

- Γ_0, Γ_1: geodesics
- $u^{\mu} = \dfrac{\partial x^{\mu}}{\partial t}$: vector field, tangent to geodesics, satisfy $u^{\nu}\nabla_{\nu}u^{\mu} = 0$
- $\xi^{\mu} = \dfrac{\partial x^{\mu}}{\partial s}$: tangent vector field

"relative velocity of geodesics":

$$V^{\mu} = (\nabla_u \xi)^{\mu} = u^{\rho}\nabla_{\rho}\xi^{\mu}$$

"relative accelaration of geodesics":

$$A^{\mu} = (\nabla_u V)^{\mu} = u^{\rho}\nabla_{\rho}V^{\mu} = u^{\rho}\nabla_{\rho}(u^{\sigma}\nabla_{\sigma}S^{\mu})$$

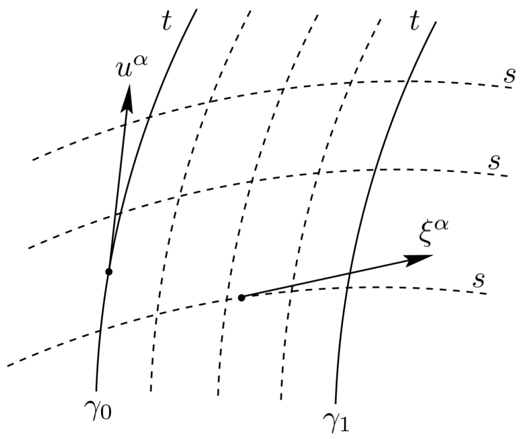

Figure 3.1: Deviation vector

Geodesic deviation equation

$$\frac{D^2\xi^\mu}{dt^2} = -R^\mu{}_{\nu\rho\sigma} u^\nu u^\rho \xi^\sigma \tag{3.6}$$

it shows curvature produces a relative acceleration between two neighbouring geodesics

3.8 Local flatness

3.8.1 Local flatness theorem (single point)

consider a single point

free-falling observers see no effect of gravity in their immediate vicinity.

3.8.2 Fermi normal coordinates (entire geodesics)

consider an entire geodesics

geometric construction

Chapter 4.

Lie Derivatives and Killing Fields

Referring to the book [0] of Chapter 1 and [0] of Chapter 1.

4.1 Maps of manifolds

a smooth map $\phi : M \to N$, function $f : N \to \mathbb{R}$, all smooth tensor field $\mathscr{F}_N(k,l)$

Definition 2. The pullback map $\phi^*: \mathscr{F}_N \to \mathscr{F}_M$ is defined as

$$(\phi^* f)\Big|_p := f\Big|_{\phi(p)}, \qquad \forall f \in \mathscr{F}_N,\, p \in M$$

i.e. $\phi^* f = f \circ \phi$, see Fig 8.3.

Definition 3. The pushforward map $\phi_* : V_p \to V_{\phi(p)}$ is defined as

$$(\phi_* v)(f) := v(\phi^* f), \qquad \forall f \in \mathscr{F}_N, \forall v^a \in V_p, \phi_* v^a \in V_{\phi(p)}$$

i.e. the action of $\phi_* v$ on any function is the action of v on the pull back of that function.

4.2 Lie differentiation

$\mathfrak{L}$

4.3 Killing vector field

4.3.1 Symmetries

isometries: symmetries of the metric

e.g. $\partial_{\sigma_*} g_{\mu\nu} = 0 \quad \Rightarrow$

4.3.2 Killing equation

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0, \quad \text{or} \quad \nabla_{(\mu} K_{\nu)} = 0, \quad \text{or} \quad \nabla_\mu K_\nu = \nabla_{[\mu} K_{\nu]} \tag{4.1}$$

Conserved energy

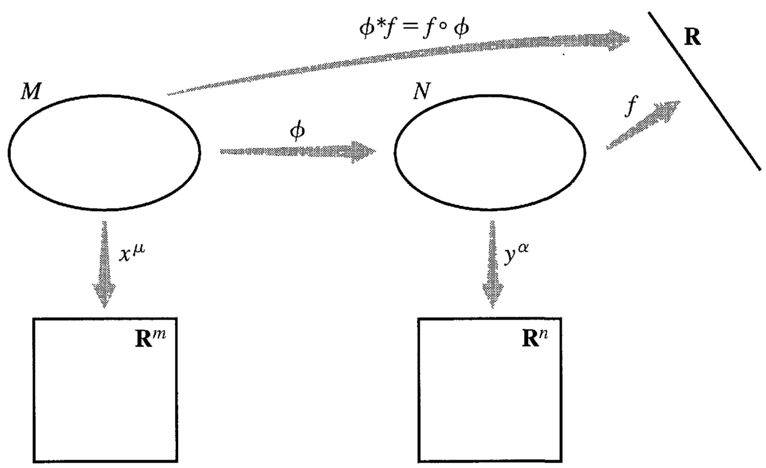

Figure 4.1: $\phi^* f = f \circ \phi$

Chapter 5.

Geodesic Congruences

5.1 Energy condition

5.2 Kinematics of a deformable medium

5.3 Congruence of timelike geodesics

5.4 Congruence of null geodesics

Einstein's Equation

Referring to the book [0] of Chapter 4.

6.1 Newtonian limit

satisfy

1. weakness of gravitational field $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $|h_{\mu\nu}| \ll 1$. and $g^{\mu\nu} g_{\nu\sigma} = \delta^{\mu}_{\sigma} \Rightarrow g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$
2. slow moving test particles $\dfrac{dx^i}{d\tau} \ll \dfrac{dt}{d\tau} \quad \Rightarrow \quad \dfrac{d^2 x^\mu}{d\tau^2} + \Gamma^{\mu}_{00} \left(\dfrac{dt}{d\tau}\right)^2 = 0$
3. static field $\partial_0 g_{\mu\nu} = 0 \quad \Rightarrow \quad \Gamma^{\mu}_{00} = -\dfrac{1}{2} g^{\mu\lambda} \partial_\lambda g_{00} = -\dfrac{1}{2}(\eta^{\mu\lambda} - h^{\mu\lambda})\partial_\lambda(\eta_{00} + h_{00}) = -\dfrac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}$

Therefore, $\dfrac{d^2 x^\mu}{d\tau^2} = \dfrac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}\left(\dfrac{dt}{d\tau}\right)^2$. if $\partial_0 h_{00} = 0$, then $\dfrac{d^2 t}{d\tau^2} = 0 \quad \Rightarrow \quad \dfrac{dt}{d\tau} = \text{const}$,
Rewrite as $\dfrac{d^2 x^i}{dt^2} = \dfrac{1}{2}\partial_i h_{00}$, compare it to $\vec{a} = -\nabla\Phi$, therefore

$$h_{00} = -2\Phi, \qquad g_{00} = 1 - 2\Phi$$

6.2 The Einstein field equation

By Bianchi identies $\nabla^\mu G_{\mu\nu} = 0$, assume $G_{\mu\nu} \equiv R_{\mu\nu} - R g_{\mu\nu}/2 = \kappa T_{\mu\nu}$, since $\nabla_\mu T = 0$. From $R = -\kappa T$,

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right)$$

consider $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}$ and Newnonian limit we neglect pressure, then $g_{00} = 1 + h_{00}, g^{00} = -1 - h_{00}$, and $T_{\mu\nu} = \rho U_\mu U_\nu$ where $U^\mu = (U^0, 0, 0, 0)$.
since $g_{\mu\nu} U^\mu U^\nu = -1$, then $U^0 = 1 + \dfrac{1}{2} h_{00}$. let $U^0 = 1, U_0 = -1 \quad \Rightarrow \quad T = \rho, T^{00} = g^{00} T_{00} = -T_{00} = -\rho$, then

$$R^{\lambda}{}_{0\lambda 0} = R_{00} = \frac{1}{2}\kappa\rho$$

since $R^{i}{}_{0j0} = \partial_j \Gamma^{i}_{00} - \partial_0 \Gamma^{i}_{j0} + \Gamma^{i}_{j\lambda}\Gamma^{\lambda}_{00} - \Gamma^{i}_{0\lambda}\Gamma^{\lambda}_{j0} = \partial_j \Gamma^{i}_{00}$, then

$$R^{i}{}_{0i0} = \partial_i \left[\frac{1}{2} g^{i\lambda}(\partial_0 g_{0\lambda} + \partial_0 g_{\lambda 0} - \partial_\lambda g_{00})\right] = -\frac{1}{2}\delta^{ij}\partial_i\partial_j h_{00} = -\frac{1}{2}\nabla^2 h_{00}$$

therefore $\nabla^2 h_{00} = -\kappa\rho$. plug $h_{00} = -2\Phi$ into and compare to $\nabla^2\Phi - 4\pi G\rho$, we got $\kappa = 8\pi G$, i.e.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \qquad \text{or} \qquad R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right) \tag{6.1}$$

when $T_{\mu\nu} = 0$ (vaccume), then $R_{\mu\nu} = 0$

Properties of Einstein's equation

6.3 Langrangian formulation

similarly, action $S = \int \mathscr{L}(\Phi^i, \nabla_\mu \Phi^i) d^n x$,

6.4

Hypersurfaces

Referring to the book [0] of Chapter 3. From intrinsic and extrinsic geometry of a hypersurfae, to discontinuities of the metric. according to the book here I use $\alpha\beta$ instead of $\mu\nu$

7.1 Description of hypersurfaces

- Hypersurface (Σ): a 3-dim submanifold in a 4-dim spacetime manifold (either timelike, spacelike or null).
- Σ like restriction $\Phi(x^\alpha) = 0$ or parametric equation $x^\alpha = x^\alpha(y^a)$, where y^a $(a = 1,2,3)$ are coordinates intrinsics to Σ.

Normal vector

7.2 Integration on hypersurfaces

Surface element (non-null case)

Surface element (null case)

7.3 Gauss-Stokes theorem

First version: Gauss' theorem

$$\int_{\mathcal{V}} A^\alpha{}_{;\alpha}\sqrt{-g}\,d^4x = \oint_{\partial\mathcal{V}} A^\alpha d\Sigma_\alpha \tag{7.1}$$

Second version: Stokes' theorem

$$\int_{\Sigma} B^{\alpha\beta}{}_{;\beta}\,d\Sigma_\alpha = \frac{1}{2}\oint_{\partial\Sigma} B^{\alpha\beta} dS_{\alpha\beta} \tag{7.2}$$

7.4 Differentiation of tangent vector fields

Linear Perturbation and Gravitational Radiation

Refering to Carroll_chap7 [0] ,

8.1 Linearized gravity and gauge transformation

express weakness of the gravitational field to decompose the metric into the flat Minkowski metric plus a small perturbation,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1$$

we also have $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ where $h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$

> **Note.** we only consider first order, effects of higher than first order in $h_{\mu\nu}$ are neglected

Equation of motion by $h_{\mu\nu}$

$$\begin{aligned}
\Gamma^{\rho}_{\mu\nu} &= \frac{1}{2}g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) = \frac{1}{2}\eta^{\rho\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}) \\
R_{\mu\nu\rho\sigma} &= \eta_{\mu\lambda}\partial_\rho\Gamma^{\lambda}_{\ \nu\sigma} - \eta_{\mu\lambda}\partial_\sigma\Gamma^{\lambda}_{\ \nu\rho} = \frac{1}{2}\eta_{\mu\lambda}\left\{\partial_\rho\left[\eta^{\lambda\alpha}\left(\partial_\nu h_{\sigma\alpha} + \partial_\sigma h_{\alpha\nu} - \partial_\alpha h_{\nu\sigma}\right)\right] - (\rho \leftrightarrow \sigma)\right\} \\
&= \frac{1}{2}\left(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\sigma\partial_\nu h_{\mu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma}\right) \\
R_{\mu\nu} &= \eta^{\rho\lambda}R_{\lambda\nu\rho\sigma} = \frac{1}{2}\left(\partial_\rho\partial_\nu h^{\rho}_{\ \sigma} + \partial_\sigma\partial_\lambda h^{\lambda}_{\ \nu} - \partial_\sigma\partial_\nu h^{\rho}_{\ \rho} - \eta^{\rho\lambda}\partial_\rho\partial_\lambda h_{\nu\sigma}\right) \\
&= \frac{1}{2}\left(\partial_\sigma\partial_\nu h^{\sigma}_{\ \mu} + \partial_\sigma\partial_\mu h^{\sigma}_{\ \nu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}\right) \\
R &= \eta^{\mu\nu}R_{\mu} = \frac{1}{2}\left(\partial_\sigma\partial_\nu h^{\sigma\nu} + \partial_\sigma\partial_\mu h^{\sigma\mu} - \Box h - \Box h\right) \\
&= \delta_\mu\partial_\nu h^{\mu\nu} - \Box h
\end{aligned} \tag{8.1}$$

the $\Gamma\Gamma$ components are vanished, $h = h^{\mu}_{\ \mu} = \eta^{\mu\nu}h_{\mu\nu}$, $\Box = \eta^{\rho\lambda}\partial_\rho\partial_\lambda$

then we derive the linear Einstein tensor,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R =$$

Gauge invariance issue

$$h^{(\epsilon)}_{\mu\nu} = h_{\mu\nu} + 2\epsilon\partial_{(\mu}\xi_{\nu)}$$

gauge transformation in linearized theory, the change of the metric perturbation under infinitesimal diffeomorphism along the $\epsilon\xi^\mu$.

it can be verified by

$$\delta R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_\rho\partial_\nu\partial_\mu\xi_\sigma + \partial_\rho\partial_\nu\partial_\sigma\xi_\mu + \partial_\sigma\partial_\mu\partial_\nu\xi_\rho + \partial_\sigma\partial_\mu\partial_\rho\xi_\nu - (\rho \leftrightarrow \sigma)\right) = 0$$

8.2 Degrees of freedom

8.2.1 Einstein equation of

consider spatial rotational symmetric, decompose $h_{\mu\nu}$ to

$$h_{00} = -2\Phi$$

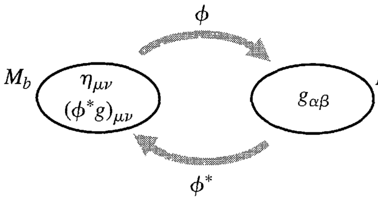

Figure 8.1: a diffeomorphism

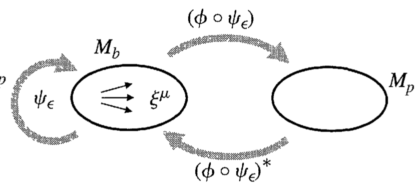

Figure 8.2: ψ_ϵ generated by ξ^μ on M_b

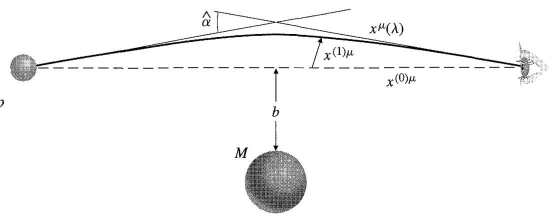

Figure 8.3: deflected geodesics

$$
\begin{aligned}
h_{0i} &= w_i \\
h_{ij} &= 2s_{ij} - 2\Psi\delta_{ij}
\end{aligned}
$$

- w_i: 3-vector
- $\Psi = -\dfrac{1}{6}\delta^{ij}h_{ij}$: the trace of spacial vector h_{ij}
- $s_{ij} = \dfrac{1}{2}\left(h_{ij} - \dfrac{1}{3}\delta^{kl}h_{kl}\delta_{ij}\right)$: the traceless of h_{ij} (strain)

then the metric is

$$ds^2 = -(1+2\Phi)dt^2 + \omega_i(dt + dx^i + dx^i dt) + [(1-2\Psi)\delta_{ij} + 2\delta_{ij}]dx^i dx^j$$

use Eqn(8.1), we have

$$
\begin{aligned}
&\Gamma^0_{00} = \partial_0\Phi, \qquad \Gamma^0_{j0} = \partial_j\Phi, \qquad \Gamma^0_{ij} = -\partial_{(j}w_{k)} + \frac{1}{2}\partial_0 h_{jk} \\
&\Gamma^i_{00} = \partial_i\Phi + \partial_0 w_i, \qquad \Gamma^i_{j0} = \partial_{[j}w_{i0} + \frac{1}{2}\partial_0 h_{ij}, \qquad \Gamma^i_{jk} = \partial_{(j}h_{k)i} - \frac{1}{2}\partial_i h_{jk}
\end{aligned}
\tag{8.2}
$$

fixed an inertial frame, $p^\mu = \dfrac{dx^\mu}{d\lambda}$, $p^0 = \dfrac{dt}{d\lambda} = E$, $p^i = Ev^i$, $v^i = \dfrac{dx^i}{dt}$. for massive particel, $\lambda = \tau/m$. the geodesic equation is

$$\frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\rho\sigma}p^\rho o^\sigma = 0 \quad \Rightarrow \quad \frac{dp^\mu}{dt} = -\Gamma^\mu_{\rho\sigma}\frac{p^\rho p^\sigma}{E}$$

8.2.2 Gauge transformation

Transverse gauge

$$
\begin{aligned}
G_{00} &= 2\nabla^2\Psi = 8\pi G T_{00} \\
G_{0j} &= -\frac{1}{2}\nabla^2 w_j + 2\partial_0\partial_j\Psi = 8\pi G T_{0j} \\
G_{ij} &= \left(\delta_{ij}\nabla^2 - \partial_i\partial_j\right)(\Phi - \Psi) - \partial_0\partial_{(i}w_{j)} + 2\delta_{ij}\partial_0^2\Psi - \Box s_{ij} = 8\pi G T_{ij}
\end{aligned}
$$

Synchronous gauge

8.2.3 Degree of freedom of the metric

8.3 Newtonian field and photon trajectries

consider static gravitating sources, dust, perfect fluid, pressure vanished, then $T_{\mu\nu} = \rho U_\mu U_\nu = \mathrm{diag}(\rho, 0, 0, 0)$

$$ds^2 = -(1+2\Phi)dt^2 + (1+2\Phi)(dx^2 + dy^2 + dz^2)$$

8.4 Gravitational wave solutions

freely-propagating degrees of freedom of the gravitatinal field, no local source
here, keep time derivative, and $T_{\mu\nu} = 0$
for the trace-free part,

$$\Box h^{TT}_{\mu\nu} = 0$$

plane wave solution

$$h^{TT}_{\mu\nu} = C_{\mu\nu} e^{k_\sigma x^\sigma}$$

s.t.

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

1. "plus" polarization, $h_\times = 0$
2. "cross" plolarization, $h_+ = 0$

8.5 Production of gravitational waves

Einstein equation coupled to matter,
define a trace-reverse perturbation as

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$$

notice that $\overline{h} = \eta^{\mu\nu}\overline{h}_{\mu\nu} = \eta^{\mu\nu} h - \frac{1}{2} h \eta^{\mu\nu}\eta_{\mu\nu} = -h$
Therefore, we got Einstein equation

$$\Box \overline{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$$

next, we use Green function[1] $G(x^\sigma - y^\sigma)$ to find the solution,

$$\overline{h}_{ij} = \frac{8GM}{r} \Omega^2 R^2 \begin{pmatrix} -\cos 2\Omega t_r & -\sin 2\Omega t_r & 0 \\ -\sin 2\Omega t_r & \cos 2\Omega t_r & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

8.6 Energy loss due to gravitational radiation

8.7 Detection of gravitational waves

receive

[1] $\widehat{L}(x) G(x, x') = \delta(x - x')$, $u(x) = \int f(x') G(x, x')$, $\widehat{L}(x) u(x) = \int f(x') \delta(x') \widehat{L}(x) G(x, x') = f(x)$, where $u(x)$ is the solution and $G(x, x')$ is Green function.

PART II

BLACK HOLE

it is a individual part, can be skipped

Lagrangian and Hamiltonian Formulation of GR

Referring to the book [0] of Chapter 4.

9.1 Lagrangian formulation

9.2 Hamiltonian formulation

9.3 Mass and angular momentum

The Schwarzschild Solution

10.1 Schwarazshild metric

unique spherically symmetric vaccume solution

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{10.1}$$

10.2 Birkhoff's theorem

Proof Schwarzschild metric is the unique vaccume solution with spherical symmetry

10.3 Geodesics of Schwarzschild solution

10.4 Classical experimental tests

10.5 Spherical Stars and their evolution

10.6 The Kruskal extension and Schwarazchild black holes

More General Black Holes

11.1 Schwarazshild metric

unique spherically symmetric vaccume solution

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{11.1}$$

11.2 Birkhoff's theorem

Proof Schwarzschild metric is the unique vaccume solution with spherical symmetry

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PART III

HOMOGENEOUS ISOTROPIC UNIVERSE

have the fundamental in part, here we introduc from the dynamics of cosmos to the early age.

The Expanding Universe

12.1 Spacetime geometry

12.1.1 Maxi

12.1.2 Robertson-Walker metric

12.2 The metric for an expanding universe

- proper-time interval: $ds^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu$
- for a Euclidean universe: $g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t) & 0 \\ 0 & 0 & 0 & a^2(t) \end{pmatrix}$
- four-dimensional energy-momentum vector: $p^\alpha = (E, \vec{p})$, where $p^\alpha = \frac{dx^\alpha}{d\lambda}$
- geodesic equation: $\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0$
  where $\Gamma^0_{00} = 0$, $\quad \Gamma^0_{0i} = \Gamma^0_{i0} = 0$, $\quad \Gamma^0_{ij} = \delta_{ij} \dot{a} a$, $\quad \Gamma^i_{0j} = \Gamma^i_{j0} = \delta_{ij} \frac{\dot{a}}{a}$

12.3 Kinematics

12.3.1 Geodesics

12.3.2 The cosmological redshift

scale factor: a for expansion, physical distance = comoving distance $\times a$
stretching factor: z, observed wavelength = emit wavelength $\times 1/a_{\text{emit}}$
Universe geometry: Euclidean, open, closed.
history of universe: evolution of a with t, determined by energy density.

- redshift:
$$1 + z \equiv \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} = \frac{a_{\text{obs}}}{a_{\text{emit}}} = \frac{1}{a_{\text{emit}}}$$
- Hubble rate: (measure how rapidly the scale factor changes)
$$H(t) \equiv \frac{1}{a} \frac{da}{dt}$$
  Hubble constant: $H_0 \equiv H(t=0) = \dot{a} = 100\, h\, \text{km}\, \text{s}^{-1}\, \text{Mpc}^{-1} = \frac{h}{0.98 \times 10^{10}\, \text{years}} = 2.13 \times 10^{-33} \frac{\text{eV}}{\hbar} h$ $1\,\text{Mpc} = 3.856 \times 10^{-24}\,\text{cm}$
- Critical density:
$$\rho_{\text{cr}} \equiv \frac{3H_0^2}{8\pi G}$$
  $\rho_{\text{cr}} = 1.88\, h^2 \times 10^{-29}\, \text{g}\, \text{cm}^{-3}$

Friedmann equation:

$$H^2(t) = \frac{8\pi G}{3}\left[\rho(t) + \frac{\rho_{\rm cr} - \rho(t_0)}{a^2(t)}\right]$$

e.g. Euclidean and matter dominated, $a \approx t^{2/3}$

Hubble found Distance galaxies are receding from us, i.e. redshifted. comoving motion $\dot{x} = 0$

Hubble-Lemaitre law: distance-redshift relation, the relative velocity is

$$v = \frac{dt}{d(ax)} = \dot{a}x = H_0 d \qquad (v \ll c)$$

12.3.3 Distance

- total comoving distance:

$$\chi(t) = \int_t^{t_0} \frac{dt'}{a(t')} = \int_{a(t)}^1 \frac{da'}{a'^2 H(a')} = \int_0^z \frac{dz'}{H(z')}$$

- comoving horizon:

$$\eta(t) \equiv \int_0^t \frac{dt'}{a(t')}$$

  where t' is conformal time

- angular diameter distance:

$$d_A = \frac{l}{\theta}$$

  where l: physical size, θ: angle subtended.
  i.e. in Euclidean universe:

$$d_A^{\rm Euc} = a\chi = \frac{\chi}{1+z}$$

- luminosity distance in a Euclidean expanding universe:

$$d_L^{\rm Euc} \equiv \frac{\chi}{a}$$

12.4 Dynamics of expansion

12.4.1 Friedmann equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{12.1}$$

where Einstein tensor:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

From $R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\nu\mu} - \partial_\nu \Gamma^\alpha_{\alpha\mu} + \Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{\nu\mu} - \Gamma^\alpha_{\nu\beta}\Gamma^\beta_{\alpha\mu} \quad \Rightarrow \quad R_{00} = -3 dota/a, \quad R_{ij} = \delta_{ij}[2\dot{a}^2 + adota]$

Consider time-time component of the Einstein equations:

$$R_{00} - \frac{1}{2} g_{00} R = 8\pi G T_{00}$$

which lead to first Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho$$

FLRW metric and assuming a Euclidean universe,

$$\frac{H^2(t)}{H_0^2} = \frac{\rho(t)}{\rho_{\rm cr}} = \sum_{s={\rm r,m},\nu,{\rm DE}} \Omega_s [a(t)]^{-3(1+w_s)}$$

To calculate the evolution of the homogeneous universe.

$$\frac{H^2(t)}{H_0^2} = \sum_{s={\rm r,m},\nu,{\rm DE}} \Omega_s [a(t)]^{-3(1+w_s)} + \Omega_{\rm K}[a(t)]^{-2}$$

12.4.2 Evolution of energy

Energy-momentum tensor in isotropic smooth universe:

$$T^{\mu}{}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

Local energy and momentum conservation: $\nabla_{\mu} T^{\mu}{}_{\nu} \equiv 0$

conservation law in an expanding universe,

$$\frac{\partial \rho}{\partial t} + \frac{\dot{a}}{a}[3\rho + 3\mathscr{P}] = 0$$

- equation of state parameter

$$w_s \equiv \frac{P_s}{\rho_s}$$

- evolution of any constituent of s

$$\rho_s \propto \exp -3 \int^a \frac{da'}{a} [q + w_s(a')] \overset{w_s = \text{const}}{\propto} a^{-3(1+w_s)}$$

12.5 Our universe

ΛCDM

Early universe hottter and denser, temperature 1MeV/k_B, no neutral atoms or bound nuclei. Then light forms (Big Bang Nuceosynthesis, BBN), -> CMB smooth -> not completely smooth -> inhomogeneities (universe structure)

Chapter 13.

The Hot Universe

the thermal universe and the very early universe.

Inflation and the Very Early Universe

14.1 Evolution of fluctuations: matter & radiation

Derive how matter, photons and neutrinos behave in a given expanding spacetime with perturbation.

14.1.1 The collision Boltzmann equation for photons

Distribution function

$E = p$, distribution function of Bose-Einstein:

$$f(\boldsymbol{x}, p, \hat{\boldsymbol{p}}, t) = [\exp \frac{p}{T(t)[1+\Theta(\boldsymbol{x}, \hat{\boldsymbol{p}}, t)]}]^{-1}$$

- Θ: perturbation to the distribution function, temperature perturbation

Zeroth-order distribution function is set preciseby by the requirement that the collision term vanished.

$$\left.\frac{df}{dt}\right|_{\text{first order}} = -p\frac{\partial f^{(0)}}{\partial p}[\dot{\Theta} + \frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i} + \dot{\Phi} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}]$$

Collision term: Compton Scattering

Definition 4. Monopole:

$$\Theta_0(\boldsymbol{x}, t) \equiv \frac{1}{4\pi}\int d\Omega' \Theta(\hat{\boldsymbol{p}}', \boldsymbol{x}, t)$$

monopole+dipole -> photon behave like a fluid.

The Boltzmann equation for phtoton

$\eta(t) \equiv \int_0^t \frac{dt'}{a(t')}$
In terms of the conformal time,

Definition 5. The angle between the wavenumber $\boldsymbol{k}$ and the photon direction $\hat{\boldsymbol{p}}$ is denoted as μ:

$$\mu \equiv \frac{\boldsymbol{k}\cdot\hat{\boldsymbol{p}}}{k}$$

Definition 6. Optical depth

$$\tau(\eta) \equiv \int_\eta^{\eta_0} d\eta' n_e \sigma_{\mathrm{T}} a$$

$$\tau' \equiv \frac{d\tau}{d\eta} = -n_e \sigma_{\mathrm{T}} a$$

In Fourier mode,

$$\Theta' + ik\mu\Theta + \Phi' + ik\mu\Psi = -\tau'[\Theta_0 - \Theta + \mu u_b]$$

14.1.2 The Boltzmann equation for cold dark matter

Collisionless,

Definition 7. Fluid velocity

$$u_{\mathrm{c}}^i \equiv \frac{1}{n_{\mathrm{c}}} \int \frac{d^3 p}{(2\pi)^3} f_{\mathrm{c}} \frac{p \hat{p}^i}{E(p)}$$

continuity equation:

$$\frac{\partial \delta_{\mathrm{c}}}{\partial t} + \frac{1}{a} \frac{\partial u_{\mathrm{c}}^i}{\partial x^i} + 3\dot{\Phi} = 0$$

Euler equation:

$$\frac{\partial u_{\mathrm{c}}^j}{\partial t} + H u_{\mathrm{c}}^j + \frac{1}{a} \frac{\partial \Psi}{\partial x^j} = 0$$

In terms of conformal time η and in Fourier space,

$$\delta_{\mathrm{c}}' + iku_{\mathrm{c}} = -3\Phi',$$
$$u_{\mathrm{c}}' + \frac{a'}{a} u_{\mathrm{c}} = -ik\Psi,$$

14.1.3 The Boltzmann equation for baryon

Compton scattering.

Definition 8. Dipole

$$\Theta_1(k,\eta) \equiv i \int_{-1}^{1} \frac{d\mu}{2} \Theta(\mu, k, \eta)$$

14.1.4 The Boltzmann equation for newtrinos

Colissionless.

Summary

Equation:

$$\Theta' + ik\mu\Theta + \Phi' + ik\mu\Psi = -\tau'[\Theta_0 - \Theta + \mu u_b - \frac{1}{2}\mathcal{P}_2(\mu)\Pi]$$

And

$$\delta_{\mathrm{c}}' + iku_{\mathrm{c}} = -3\Phi', \tag{14.1}$$
$$u_{\mathrm{c}}' + \frac{a'}{a} u_{\mathrm{c}} = -ik\Psi, \tag{14.2}$$
$$\delta_{\mathrm{b}}' + iku_{\mathrm{b}} = -3\Phi', \tag{14.3}$$
$$u_{\mathrm{b}}' + \frac{a'}{a} u_{\mathrm{b}} = -ik\Psi + \frac{\tau'}{R}[u_{\mathrm{b}} + 3i\Theta_1], \tag{14.4}$$
$$\mathcal{N}' = \tag{14.5}$$

14.2 Evolution of fluctuations: gravity

Non-gravitational to gravitational field

14.2.1 Scalar vector-tensor decomposition

Generally, FLRW spacetime perturbed by a small ammount:

$$
\begin{aligned}
g_{00}(t,\boldsymbol{x}) &= -1 + h_{00}(t,\boldsymbol{x}),\\
g_{0i}(t,\boldsymbol{x}) &= a(t)h_{0i}(t,\boldsymbol{x}) = a(t)h_{i0}(t,\boldsymbol{x}),\\
g_{ij}(t,\boldsymbol{x}) &= a^2(t)[\delta_{ij} + h_{ij}(t,\boldsymbol{x})]
\end{aligned}
$$

Theorem 1. **Decomposition theorem:** perturbation of each type – scalar, vector, and tensor – evolve independently at linear order.

14.2.2 From gauge to gauge

Consider a scalar field $\phi(x)$, separate into background and perturbation:

$$\phi(x) = \bar{\phi}(t) + \delta\phi(t,\boldsymbol{x})$$

14.2.3 The Einstein equation for scalar perturbations

Conformal-Newtonian gauge,

Ricci tensor

$$
\begin{aligned}
\delta R = -12\Psi(H^2 + \frac{dota}{a}) + \frac{2k^2}{a^2}\Psi + 6\Phi_{,00}\\
-6H(\Psi,0 - 4\Phi_{,0}) + 4\frac{k^2\Phi}{a^2}
\end{aligned}
$$

Two components of the Einstein equations

$$\delta G = -6H\Phi_{,0} + 6\Phi H^2 - 2\frac{k^2\Phi}{a^2}$$

Evolution equation for Ψ and Φ

14.2.4 Tensor perturbations

14.3 Initial conditions

Equations governing perturbation around a smooth background,

14.3.1 The horizon problems and a solution

- $k \gg aH$, leave the horizon
- $k \ll aH$, at the end of inflation for all modes (can observe)

14.3.2 Inflation

Scalar field energy-momentum tensor:

$$T^{\alpha}{}_{\beta} = g^{\alpha\nu}\frac{\partial\phi}{\partial x^{\nu}}\frac{\partial\phi}{\partial x^{\beta}} - \delta^{\alpha}{}_{\beta}[\frac{1}{2}g^{\mu\nu}\frac{\partial\phi}{\partial x^{\mu}}\frac{\partial\phi}{\partial x^{\nu}} + V(\phi)]$$

Assume zero-order: homogeneous; first-order: perturbation;

$$\phi'' + 2aH\phi' + a^2 V_{,\phi} = 0 \tag{14.6}$$

14.3.3 Gravitational wave production

harmonic oscillator

$$\frac{d[}{d2}]xt + \omega^2 x^2 = 0$$

Tensor perturbation

From 14.6,

$$v'' + (k^2 - \frac{a''}{a})v = 0$$

Power spectrum of the primordial tensor perturbation

$$P_h = 16\pi G\frac{|v(k,\eta)|^2}{a^2}$$

14.3.4 Salar perturbations

Find the spectrum of scalar perturbations emerging from inflation.

Salar field perturbations around an unexpected background

$$\delta\phi'' + 2aH\delta\phi' + k^2\delta\phi = 0$$

Super-horizon perturbation

Definition 9. Curvature perturbation,

Spatially flat slicing

14.3.5 The Einstein-Boltzmann equations at early times

Reduce the Boltzmann equations to when $k\eta \ll 1$,

$$\begin{aligned}\Theta_0' + \Phi' &= 0\\ \mathcal{N}_0' + \Phi' &= 0\\ \delta_{\mathrm{c}}' &= -3\Phi'\\ \delta_{\mathrm{b}} &= -3\Phi'\end{aligned}$$

Einstein equation to

$$3\frac{a'}{a}(\Phi' - \frac{a'}{a}\Psi) = 16\pi G a^2 \rho_{\mathrm{r}}\Theta_{\mathrm{r},0}$$

PART IV

INHOMOGENEOUS UNIVERSE

This part is the late universe

Cosmological Fluctuations

15.1 Boltzmann equation

$$\frac{df}{dt} = C[f]$$

Boltzmann equation in an expanding universe

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \cdot \frac{dx^i}{dt} + \frac{\partial f}{\partial p} \cdot \frac{dp}{dt} + \frac{\partial f}{\partial \hat{p}^i} \cdot \frac{d\hat{p}^i}{dt}$$

In homogeneous universe,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{p}{E}\frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - Hp\frac{\partial f}{\partial p} \tag{15.1}$$

Eq. **??** is valid for all particles.

Collision term

15.2 General theory of small fluctuations

General theory of small fluctuations

Perturbed spacetime

Perturbed metric:

$$\begin{aligned}
g_{00}(\vec{x}, t) &= -1 - \Psi(\vec{x}, t), \\
g_{0i}(\vec{x}, t) &= 0, \\
g_{ij}(\vec{x}, t) &= a^2(t)\delta_{ij}[1 + 2\Phi(\vec{x}, t)].
\end{aligned}$$

- Ψ: corresponds to Newtonian potential and governs the motion of slow-moving (non-relativistic) bodies;
- Φ: the perturbation to the spatial curvature
  $a(t) \to$
  $a(\boldsymbol{x}, t) = q(t)\sqrt{1 + \Phi(\vec{x}, t)}$

The geodesic equation

Via perturbed metric,

$$P^\mu = [E(1-\Psi), p^i\frac{1-\Phi}{a}]$$

The change in the magnitude of the momentum of a particle as it moves through a perturbed FLRW universe,

$$\frac{dp}{dt} = -[H + \dot{\Phi}]p - \frac{E}{a}\hat{p}^i\Psi_{,i}$$

Energy-momentum tensor in the perturbed universe for a single species with degeneracy factor g:

$$T^0{}_0(\vec{x}, t) = -g$$

The collisionless Boltzmann equation for photons

$m = 0, E = p$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - [H + \dot{\Phi} + \frac{1}{a}\hat{p}^i \Psi_{,i}]\, p \frac{\partial f}{\partial p} \tag{15.2}$$

The collisionless Boltzmann equation for massive particles

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{p}{E}\frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - [H + \dot{\Phi} + \frac{E}{ap}\hat{p}^i \Psi_{,i}]\, p \frac{\partial f}{\partial p} \tag{15.3}$$

Chapter 16.

The Cosmic Microwave Background

Chapter 17.

The Polarized CMB

Chapter 18.

The Growth of Structure

Chapter 19.

Gravitational Lenses

Chapter 20.

Gravitational Wave

Chapter 21.

Analysis and Inference

PART V

QFT

Chapter 22.

Field theory

Chapter 23.

QFT in Curved Spacetime

Chapter 24.

The Standard Inflationary Universe

Chapter 25.

Inflation and the Cosmological Perturbation

.Recommended Resources

Books

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