日本综合研究院大学

KEK 素粒子专攻

一般入试过去问试题尝试解答

by *heyingqiu*

主页：heyingqiu.github.io

邮箱： he.yingqiu@hotmail.com

2016～2023

前：物理 ；后：数学

免责申明：只解答了80%左右，不会的用红色已标出，不保证正确；若有不解或是错误之处，非常欢迎交流。另外，解答免费，请获取后勿用作营利性用途。

SOKENDAI KEK 入试 2023 解答

物理

第1問

[問1]

$$[\hat{x},\hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$$

$$\hat{a}^+\hat{a} = \frac{m\omega}{2\hbar}\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} - \frac{i}{m\omega}\hat{x}\hat{p} + \frac{i}{m\omega}\hat{p}\hat{x}\right) = \frac{m\omega}{2\hbar}\left(\hat{x}^2+\frac{\hat{p}^2}{m^2\omega^2} - \frac{\hbar}{m\omega}\right)$$

$$\Rightarrow \hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} = \frac{2\hbar}{m\omega}\hat{a}^+\hat{a} + \frac{\hbar}{m\omega}$$

$$\hat{H} = \frac12 m\omega^2\left(\hat x^2 + \frac{\hat p^2}{m^2\omega^2}\right) = \frac12 m\omega^2\left(\frac{2\hbar}{m\omega}\hat a^+\hat a + \frac{\hbar}{m\omega}\right) = \hbar\omega\left(\hat a^+\hat a+\frac12\right)$$

[問2]

$$\hat H\phi_n = E_n\phi_n \rightarrow \hbar\omega\left(\hat a^+\hat a + \frac12\right)\phi_n = E_n\phi_n$$

$$\text{l.h.s} = \hbar\omega \hat a^+\hat a\phi_n + \frac12\phi_n = \hbar\omega\left(\sqrt n\, \hat a^+\phi_{n-1} + \frac12\phi_n\right) = \hbar\omega\left(\sqrt n\sqrt n\,\phi_n + \frac12\phi_{n-1}\right)$$

$$= \hbar\omega\left(n+\frac12\right)\phi_n$$

Therfore, $E_n = \hbar\omega(n+\frac12)$

[問3]

$$\hat a\psi(x) = \sqrt{\frac{m\omega}{2\hbar}}\left[x + \frac{i}{m\omega}(-i\hbar)\frac{\partial}{\partial x}\right]A\exp\left(-\frac{m\omega}{2\hbar}x^2 + ikx\right)$$

$$= \sqrt{\frac{m\omega}{2\hbar}}\left[x + \frac{\hbar}{m\omega}\left(-\frac{m\omega x}{\hbar} + ik\right)\right]\psi(x)$$

$$= \sqrt{\frac{m\omega}{2\hbar}}\cdot ik\,\psi(x)$$

$$\Rightarrow \alpha = ik\sqrt{\frac{m\omega}{2\hbar}}$$

[問4]

Let $C^0 = 1$, and we have $\hat a\phi_0(x) = 0$

$$\psi(x) = B\left[\sum_{n=0}^\infty \frac{1}{\sqrt{(n+1)!}}C^{n+1}\phi_{n+1}(x) + C^0\phi_0(x)\right]$$

$$\hat a\psi(x) = B\left[\sum_{n=0}^\infty \frac{1}{\sqrt{(n+1)!}}C^{n+1}\sqrt{n+1}\,\phi_n(x)\right]$$

Since $\alpha\psi(x) = B\left[\sum_{n=0}^\infty \frac{1}{\sqrt{n!}}C^n\alpha\phi_n(x)\right]$, therefore $\frac{1}{\sqrt{(n+1)!}}C^{n+1} = \frac{1}{\sqrt{n!}}C^n\alpha$,

$$C^{n+1} = \sqrt{n+1}\,\alpha C^n \Rightarrow C^n = \sqrt{n!}\,\alpha^{n-1} \quad (n>1)$$

$$\psi(x) = B\left[\sum_{n=1}^\infty \alpha^{n-1}\phi_n(x) + \phi_0(x)\right]$$

第2問

[問1]

$$\hat H\varphi(x,y,z) = E\varphi(x,y,z)$$

$$\text{l.h.s} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial}{\partial z^2}\right)\left(\frac2L\right)^{\frac32}\sin\left(\frac{\pi n_x x}{L}\right)\sin\left(\frac{\pi n_y y}{L}\right)\sin\left(\frac{\pi n_z z}{L}\right)$$

$$= \frac{\hbar^2}{2m}\left(\frac2L\right)^{\frac32}\left[\left(\frac{\pi n_x}{L}\right)^2 + \left(\frac{\pi n_y}{L}\right)^2 + \left(\frac{\pi n_z}{L}\right)^2\right]\sin\left(\frac{\pi n_x x}{L}\right)\sin\left(\frac{\pi n_y y}{L}\right)\sin\left(\frac{\pi n_z z}{L}\right)$$

$$= \frac{\hbar^2\pi^2}{2mL^2}\left(n_x^2+n_y^2+n_z^2\right)\varphi(x,y,z)$$

Therefore, $E(n_x,n_y,n_z) = \frac{\hbar^2\pi^2}{2mL^2}(n_x^2+n_y^2+n_z^2)$

[問2]

$$E = \sum_{j=1}^N E^{(j)} = \frac{\hbar^2\pi^2}{2mL^2}\sum_{j=1}^N\left(n_x^{(j)2}+n_y^{(j)2}+n_z^{(j)2}\right) = \frac{\hbar^2\pi^2}{2mL^2}\sum_{j=1}^{3N} n^{(j)2}$$, assume $n_x=n_y=n_z=n$

Then,

$$Z_{V,N}(T) = \frac{1}{N!}\sum_{n=1}^\infty \exp\left(-\frac{\frac{\hbar^2\pi^2}{2mL^2}\sum_{j=1}^{3N}n^{(j)2}}{k_BT}\right)$$

$$= \frac{1}{N!}\prod_{j=1}^{3N}\sum_{n=1}^\infty\exp\left(-\frac{\frac{\hbar^2\pi^2}{2mL^2}n^{(j)2}}{k_BT}\right)$$

$$= \frac{1}{N!}\left[\sum_{n=1}^\infty \exp\left(-\frac{\varepsilon_0 n^2}{k_BT}\right)\right]^{3N}$$

" $\frac{1}{N!}$ " is caused by particles are identical with each other

Besides, $\varepsilon_0 = \frac{\hbar^2\pi^2}{2mL^2}$

[問3]

$$F(T,V,N) = -k_BT\ln Z_{V,N}(T)$$

$$= -k_BT\cdot 3N\left[\ln\left(\frac{1}{N!}\right) + \ln\sum_{n=1}^\infty e^{-\frac{\varepsilon_0 n^2}{k_BT}}\right]$$

$$\sim 3k_BTN\left[N(\ln N - 1) - \ln\sum_{n=1}^\infty e^{-\frac{\varepsilon_0 n^2}{k_BT}}\right]$$

$$F(T,\lambda V,\lambda N) = \ ?$$

第3問

[問1]

(1) $\begin{cases} \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{cases}$ $\quad T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)$

(2) $\mathcal{L} = \frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2) - U(r)$

$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}$, $\frac{\partial \mathcal{L}}{\partial r} = mr\dot{\theta}^2 - \frac{dU}{dr} \Rightarrow m\ddot{r} = mr\dot{\theta}^2 - \frac{dU}{dr} = \frac{l^2}{mr^3} - \frac{dU}{dr}$

$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta}$, $\frac{\partial \mathcal{L}}{\partial \theta} = 0 \Rightarrow mr\ddot{\theta} \equiv 0 \Rightarrow \frac{dl}{dt} \equiv 0 \Rightarrow l$ is conserved

And $\frac{dE}{dt} = m\dot{r}\ddot{r} + \frac{l^2}{2m}\dot{r}(-2)\frac{1}{r^3} + \frac{dU}{dr}\cdot\dot{r}$

$= \dot{r}\left[m\ddot{r} - \left(\frac{l^2}{mr^3} - \frac{dU}{dr}\right)\right]$

$\equiv 0 \qquad \Rightarrow E$ is conserved.

[問2]

$l = mr^2\frac{d\theta}{dt} \Rightarrow d\theta = \frac{l}{m}u^2 dt \qquad du = -u^2 dr \rightarrow \frac{dr}{dt} = -\frac{du}{u^2 dt}$

$E = \frac{1}{2}m\left(-\frac{du}{u^2 dt}\right)^2 + \frac{l^2u^2}{2m} - ku \qquad \frac{du}{dt} = \frac{du}{d\theta}\frac{d\theta}{dt} = \frac{l}{m}u^2\frac{du}{d\theta}$

$= \frac{1}{2}m\left(\frac{1}{u^2}\cdot\frac{l}{m}u^2\frac{du}{d\theta}\right)^2 + \frac{l^2}{2m}u^2 - ku$

$\Rightarrow \frac{2mE}{l^2} + \frac{2mk}{l^2}u - u^2 = \frac{2m}{l^2}\cdot\frac{m}{2}\cdot\frac{l^2}{m^2}\left(\frac{du}{d\theta}\right)^2$

$\Rightarrow \left(\frac{du}{d\theta}\right)^2 = -u^2 + \frac{2mk}{l^2}u + \frac{2mE}{l^2} \qquad$ Since $\dot{r} > 0$, $\dot{u} < 0$

$\Rightarrow d\theta = -\frac{du}{\sqrt{-u^2 + \frac{2mk}{l^2}u + \frac{2mE}{l^2}}}$

(2)

$$\int \frac{du}{\sqrt{-u^2+\frac{2mk}{l^2}u+\frac{2mE}{l^2}}} = -\arccos\left(\frac{2u-\frac{2mk}{l^2}}{\sqrt{\left(\frac{2mk}{l^2}\right)^2+\frac{8mE}{l^2}}}\right) + C$$

we can have $\quad \cos(\theta-\theta_0) = \frac{u - \frac{mk}{l^2}}{\sqrt{\left(\frac{mk}{l^2}\right)^2 + \frac{2mE}{l^2}}}$

$\Rightarrow \frac{mk}{l^2} + \frac{mk}{l^2}\sqrt{1+\frac{2El^2}{mk^2}}\cos(\theta-\theta_0) = u$

$\Rightarrow r = \frac{l^2}{mk}\frac{1}{1+\sqrt{1+\frac{2El^2}{mk^2}}\cos(\theta-\theta_0)}$

(3) $\varepsilon = \sqrt{1+\frac{2l^2}{mk^2}E}$

(i) $\varepsilon = 0$, $E = \frac{-mk^2}{2l^2}$, orbit: circle

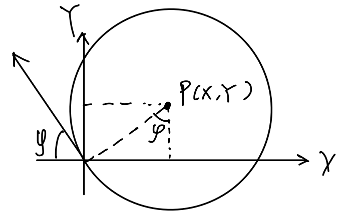
(ii) $0 < \varepsilon < 1$, $-\frac{mk^2}{2l^2} < E < 0$, orbit: ellipse.

(iii) $\varepsilon = 1$, $E = 0$, orbit: parabola

(iv) $\varepsilon > 1$, $E > 0$, orbit: hyperbola

第4問

[問1]



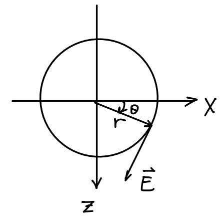
(1) $\oint \vec{B}\cdot d\vec{l} = 2\pi\sqrt{x^2+y^2}|\vec{B}^0| = \mu_0 I$

$\Rightarrow |\vec{B}^0| = \frac{\mu_0 I}{2\pi\sqrt{x^2+y^2}}$

(2) $B_x^0 = -|\vec{B}^0|\cdot\cos\varphi = -\frac{y}{\sqrt{x^2+y^2}}|\vec{B}^0| = -\frac{\mu_0 y I}{2\pi\sqrt{x^2+y^2}}$

$B_y^0 = |\vec{B}^0|\sin\varphi = \frac{x}{\sqrt{x^2+y^2}}|\vec{B}^0| = \frac{\mu_0 x I}{2\pi\sqrt{x^2+y^2}}$

[問2]



(1) $-\frac{d\vec{B}}{dt} = k$

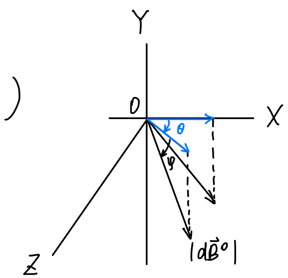
$\int \mathrm{rot}\vec{E}\cdot d\vec{S} = k\cdot\pi r^2$

$\oint \vec{E}\cdot d\vec{l} = |\vec{E}|\cdot 2\pi r$

$\Rightarrow |\vec{E}| = \frac{kr}{2}$, $\vec{E} = \left(-\frac{kr}{2}\sin\theta, -\frac{kr}{2}\cos\theta\right)$

(2) $\vec{I} = a\vec{j} = a\sigma\vec{E} = \left(-\frac{kra\sigma}{2}\sin\theta, -\frac{kra\sigma}{2}\cos\theta\right)$

(3) $|\vec{I}| = \frac{a\sigma kr}{2}$



$dB_x^0 = |d\vec{B}^0|\cdot\cos\varphi\cdot\sin\theta$

$= \frac{1}{4\pi}\frac{\mu_0|\vec{I}|r\,d\theta}{r^2+y^2}\frac{y}{\sqrt{r^2+y^2}}\sin\theta$

$= \frac{\mu_0|\vec{I}|ry}{4\pi(r^2+y^2)^{\frac{3}{2}}}\sin\theta\,d\theta$

$dB_y^0 = -|d\vec{B}^0|\sin\varphi = -\frac{\mu_0|\vec{I}|r^2}{4\pi(r^2+y^2)^{\frac{3}{2}}}d\theta$

$$dB_z^\circ = |d\vec{B}^\circ| \cos\varphi \cos\theta = \frac{\mu_0 |\vec{I}| r y}{4\pi (r^2+y^2)^{\frac{3}{2}}} \cos\theta \, d\theta$$

(4) $B_x^\circ = \frac{\mu_0 |\vec{I}| r y}{4\pi (r^2+y^2)^{\frac{3}{2}}} \int_0^{2\pi} \sin\theta \, d\theta = 0$, $B_z^\circ = 0$

$$B_y^\circ = -\frac{\mu_0 |\vec{I}| r^2}{4\pi (r^2+y^2)^{\frac{3}{2}}} \cdot \int_0^{2\pi} d\theta = -\frac{\mu_0 |\vec{I}| r^2}{2 (r^2+y^2)^{\frac{3}{2}}} = -\frac{a\sigma \mu_0 r^3}{4 (r^2+y^2)^{\frac{3}{2}}}$$

$$\vec{B} = \left(0, -\frac{a\sigma \mu_0 r^3}{4 (r^2+y^2)^{\frac{3}{2}}}, 0\right)$$

数学

[問1]

$$A^2 = \begin{pmatrix} -2 & -3 & 2 \\ 2 & 4 & -2 \\ 2 & 5 & -2 \end{pmatrix} \begin{pmatrix} -2 & -3 & 2 \\ 2 & 4 & -2 \\ 2 & 5 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 4 & -2 \\ 0 & 0 & 0 \\ 2 & 4 & -2 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} -2 & -3 & 2 \\ 2 & 4 & -2 \\ 2 & 5 & -2 \end{pmatrix} \begin{pmatrix} 2 & 4 & 2 \\ 0 & 0 & 0 \\ 2 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

[問2]

$$\exp(xA) = \sum_{n=0}^{\infty} \frac{1}{n!} (Ax)^n = I + Ax + \frac{A^2}{2} x^2$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -2 & -3 & 2 \\ 2 & 4 & -2 \\ 2 & 5 & -2 \end{pmatrix} x + \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & -1 \end{pmatrix} x^2$$

$$= \begin{pmatrix} 1-2x+x^2 & -3x+2x^2 & 2x-x^2 \\ 2x & 1+4x & -2x \\ 2x+x^2 & 5x+2x^2 & 1-2x-x^2 \end{pmatrix}$$

[問3]

Let $F(x) = \begin{pmatrix} f(x) \\ g(x) \\ h(x) \end{pmatrix}$, we know $\frac{dF(x)}{dt} = AF$, then $F(x) = e^{Ax} C$

Since $\begin{pmatrix} f(0) \\ g(0) \\ h(0) \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, therefore $C = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, and

$$\begin{pmatrix} f(x) \\ g(x) \\ h(x) \end{pmatrix} = \begin{pmatrix} 1-2x+x^2 & -3x+2x^2 & 2x-x^2 \\ 2x & 1+4x & -2x \\ 2x+x^2 & 5x+2x^2 & 1-2x-x^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow \begin{cases} f(x) = a + (-2a-3b+2c)x + (a+2b-c)x^2 \\ g(x) = b + (2a+4b-2c)x \\ h(x) = c + (2a+5b-2c)x + (a+2b-c)x^2 \end{cases}$$

第2問

[問1]

$$\mathcal{L}[e^{at}](s) = \int_0^\infty dt \, e^{at} e^{-st} dt = \frac{1}{a-s} e^{(a-s)t} \Big|_0^\infty = \frac{1}{s-a} \quad (s>a)$$

$$\mathcal{L}[\sin t](s) = \int_0^\infty dt \, \sin t \, e^{-st} dt = -\frac{1}{s} e^{-st} \sin t \Big|_0^\infty + \frac{1}{s} \int_0^\infty \cos t \, e^{-st} dt$$

$$= \frac{1}{s}\left[-\frac{1}{s} e^{-st} \cos t \Big|_0^\infty + \frac{1}{s} \int_0^\infty -\sin t \, e^{-st} dt \right]$$

$$\Rightarrow \int_0^\infty dt \, \sin t \, e^{-st} dt \left(1 + \frac{1}{s^2}\right) = -\frac{1}{s^2}(0-1)$$

$$\Rightarrow \int_0^\infty dt \, \sin t \, e^{-st} dt = \frac{1}{1+s^2}$$

$$\mathcal{L}[\cos t](s) = \int_0^\infty dt \, \cos t \, e^{-st} dt = \sin t \, e^{-st} \Big|_0^\infty - \int_0^\infty (-s) \sin t \, e^{-st} dt$$

$$= s\left[-\cos t \, e^{-st} \Big|_0^\infty + (-s) \int_0^\infty \cos t \, e^{-st} dt \right]$$

$$\Rightarrow \int_0^\infty dt \, \cos t \, e^{-st} dt \, (1+s^2) = (-s)(0-1)$$

$$\Rightarrow \int_0^\infty dt \, \cos t \, e^{-st} dt = \frac{s}{1+s^2}$$

[問2]

$$\mathcal{L}[f'(x)] = \int_0^\infty dt \, f'(t) e^{-st} = f(t) e^{-st} \Big|_0^\infty - (-s) \int_0^\infty dt \, f(t) e^{-st} = -f(0) + s\mathcal{L}[f(t)]$$

$$\mathcal{L}[f''(x)] = -f'(0) + s\mathcal{L}[f'(t)] = -f'(0) + s[-f(0) + s\mathcal{L}[f(t)]]$$

$$= -f'(0) - s f(0) + s^2 \mathcal{L}[f(t)]$$

Note

Since $f(0) = 0$, $f'(0) = 1$, then $\mathcal{L}[f'(x)] = s\mathcal{L}[f(t)]$, $\mathcal{L}[f''(t)] = -1 + s^2 \mathcal{L}[f(t)]$

$$\mathcal{L}[f''(x) - 3f'(t) + 2f(t)] = -1 + s^2 \mathcal{L}[f(t)] - 3s\mathcal{L}[f(t)] + 2\mathcal{L}[f(t)]$$

$$= (s^2 - 3s + 2)\mathcal{L}[f(t)] + 1$$

And $\mathcal{L}[\frac{1}{\sin t}] = \frac{1}{s^2+1}$, then $\mathcal{L}[f(t)] = \frac{-s^2}{(s^2+1)(s-1)(s-2)} = \frac{\frac{3}{10}s + \frac{1}{10}}{s^2+1} + \frac{\frac{1}{2}}{s-1} + \frac{-\frac{4}{5}}{s-2}$

$A_1 = \frac{1}{2}$, $A_2 = -\frac{4}{5}$, $B = \frac{3}{10}$, $C = \frac{1}{10}$, $S_1 = 1$, $S_2 = 2$

[問3]

$$f(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t} + B\cos t + C \sin t$$

$$= \frac{1}{2} e^t - \frac{4}{5} e^{2t} + \frac{3}{10} \cos t + \frac{1}{10} \sin t$$

第3問

[問1] $\delta f = \delta g = g(z+\delta z, \bar{z}+\delta\bar{z}) - g(z,\bar{z})$ $\quad d(re^{i\theta}) = e^{i\theta}dr + ire^{i\theta}d\theta$

$$S = \lim_{\delta r \to 0} \frac{\delta f}{\delta z} = \lim_{\substack{r\to 0 \\ d\theta = 0}} \frac{\delta u + i\delta v}{e^{i\theta}\delta r} = e^{-i\theta}\frac{df}{dr}$$

[問2]

$$\lim_{\substack{\delta x\to 0\\ \delta y = 0}} \frac{\delta f}{\delta z} = \lim_{\delta x\to 0} \frac{\delta u + i\delta v}{\delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\lim_{\substack{\delta x = 0\\ \delta y\to 0}} \frac{\delta f}{\delta z} = \lim_{\delta y\to 0} \frac{\delta u + i\delta v}{i\delta y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\Rightarrow \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

[問3]

Equate $\frac{df}{dz}$ in x and y, $\frac{du+idv}{dx} = \frac{du+idv}{-idy}$,

$\frac{df}{dz}$ does not exist unless the equation are satified.

[問4]

$$\frac{\partial u}{\partial x} = 2x+3 \Rightarrow v(x,y) = (2x+3)y + q, \quad q = q(x)$$

$$\frac{\partial u}{\partial y} = -2y+1 \Rightarrow v(x,y) = (-2y+1)x + p, \quad p = p(y)$$

Then, $v(x,y) = 2xy + 3y + x$

And $h(z) = x^2 - y^2 + 3x + y + i(2xy + x + 3y)$

[問5]

$$\frac{\partial u}{\partial x} = \cos x \Rightarrow v(x,y) = (\cos x)y + q$$

$$\frac{\partial u}{\partial y} = 0 \Rightarrow v(x,y) = x + p$$

$>$ obviously, impossible.

第4問

[問1]

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta = \cos\theta\, dr - r\sin\theta\, d\theta$$

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta = \sin\theta\, dr + r\cos\theta\, d\theta$$

$$A_x(x,y)dx + A_y(x,y)dy = (r^2\cos^2\theta - r^2\sin^2\theta - 3r\cos\theta)(\cos\theta\, dr - r\sin\theta\, d\theta) + (2r^2\cos\theta\sin\theta - 3r\sin\theta)(\sin\theta\, dr + r\cos\theta\, d\theta)$$

$$\Rightarrow A_r(r,\theta) = r^2\cos\theta - 3r, \quad A_\theta(r,\theta) = r^3\sin\theta$$

[問2]

$$I_C = \int_{C_1} d\vec{s}\cdot\vec{A} + \int_{C_2} d\vec{s}\cdot\vec{A}$$

$$= \int_0^\pi R^3\sin\theta\, d\theta + \int_{-R}^{R}(r^2-3r)\,dr = 2R^3 + \frac{2}{3}R^3 = \frac{8}{3}R^3$$

[問3]

$$\frac{\partial A_\theta}{\partial r} = 3r^2\sin\theta, \quad \frac{\partial A_r}{\partial \theta} = -r^2\sin\theta$$

$$I_D = \iint_D dr\,d\theta\,(3r^2\sin\theta + r^2\sin\theta) = 4\int_0^\pi \sin\theta\, d\theta \int_0^R r^2\,dr = \frac{8}{3}R^3$$

SOKENDAI KEK 入试 2022 解答

物理

第1問

[問1]

(1) $\frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q}$, $\frac{\partial \mathcal{L}}{\partial q} = 0 \Rightarrow m\ddot{q} = 0$

(2) $p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q}$, $H = p\dot{q} - \mathcal{L} = m\dot{q}^2 - \frac{m}{2}\dot{q}^2 = \frac{p^2}{2m}$

(3) $\hat{p} = -i\hbar\frac{\partial}{\partial q}$

$\hat{H}\Psi(q) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2}\Psi(q) \Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2}\Psi(q) = E\Psi(q)$

(4) $\hat{H}\Psi(q) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} C_n \exp\left(in\frac{q}{R}\right) = -\frac{\hbar^2}{2m}\left(i\frac{n}{R}\right)^2\Psi(q)$

$\Rightarrow E = n^2\frac{\hbar^2}{2mR^2}$

$\int_0^{2\pi R} dq\, \Psi_n^* \Psi = \int_0^{2\pi R} dq\, |C_n|^2 = 1 \Rightarrow C_n = \sqrt{2\pi R}$

[問2]

(1) $\frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q} + \frac{\hbar\theta}{2\pi R}$, $\frac{\partial \mathcal{L}}{\partial q} = 0 \Rightarrow m\ddot{q} = 0$, not depend on θ.

(2) $p = m\dot{q} + \frac{\hbar\theta}{2\pi R}$, $H = m\dot{q}^2 + \frac{\hbar\theta}{2\pi R}\dot{q} - \frac{m}{2}\dot{q}^2 - \frac{\hbar}{2\pi R}\theta\dot{q} = \frac{m}{2}\dot{q}^2 = \frac{1}{2m}\left(p - \frac{\hbar\theta}{2\pi R}\right)^2$

$\hat{H}\Psi = \frac{1}{2m}\left(\hat{p}^2 - \frac{\hbar\theta}{\pi R}\hat{p} + \left(\frac{\hbar\theta}{2\pi R}\right)^2\right)\Psi = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + \frac{i\hbar^2\theta}{2m\pi R}\frac{\partial}{\partial q} + \frac{(\hbar\theta)^2}{2m(2\pi R)^2}\right]\Psi = E\Psi$

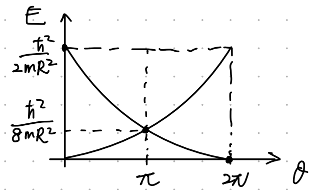
(3)

① $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2}\Psi = E'\Psi \Rightarrow E' = \frac{n^2\hbar^2}{2mR^2}$

② $\frac{i\hbar^2\theta}{2m\pi R}\frac{\partial}{\partial q}\Psi = \frac{i\hbar^2\theta}{2m\pi R}\left(\frac{in}{R}\right)\Psi = -\frac{n\hbar^2\theta}{2m\pi R^2}\Psi$, $\Rightarrow E'' = -\frac{n\hbar^2\theta}{2m\pi R^2}$

$\Rightarrow E_n = \frac{\hbar^2}{2mR^2}\left(n^2 - \frac{\theta}{\pi}n + \frac{\theta^2}{(2\pi)^2}\right)$

(4) $E_0 = \frac{\hbar^2\theta^2}{2mR^2(2\pi)^2}$, $E_1 = \frac{\hbar^2}{2mR^2}\left(1 - \frac{\theta}{\pi} + \frac{\theta^2}{(2\pi)^2}\right)$



(5-1) no idea

(5-2) $\hat{H}_{\theta+2\pi} = \frac{1}{2m}\left[\hat{p} - \frac{\hbar}{2\pi R}(\theta+2\pi)\right]^2 = \frac{1}{2m}\left[\left(p - \frac{\hbar}{R}\right) - \frac{\hbar^2}{2\pi R}\theta\right]^2$

$e^{-iq/R}\hat{H}_\theta e^{-i\hat{q}/R} = e^{-iq/R}\frac{1}{2m}\left(\hat{p} - \frac{\hbar}{2\pi R}\theta\right)^2 e^{-i\hat{q}/R}$

$= \frac{1}{2m}e^{-iq/R}\left(\hat{p}^2 - \hat{p}\frac{\hbar}{\pi R}\theta + \left(\frac{\hbar}{2\pi R}\theta\right)^2\right)e^{iq/R}$

$= \frac{1}{2m}\left[\left(\hat{p} - \frac{\hbar}{R}\right)^2 - \frac{\hbar^2}{\pi R}\theta\left(\hat{p} - \frac{\hbar}{R}\right) + \left(\frac{\hbar\theta}{2\pi R}\right)^2\right]$

$= \frac{1}{2m}\left[\left(p - \frac{\hbar}{R}\right) - \frac{\hbar^2}{2\pi R}\theta\right]^2$ $\square$

(5-3) $\hat{H}_{\theta+2\pi}\Psi = e^{i\hat{q}/R}\hat{H}_\theta e^{-i\hat{q}/R}\Psi$

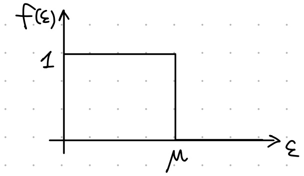
$= \hat{H}_\theta e^{-i\hat{q}/R}\Psi e^{i\hat{q}/R}$?

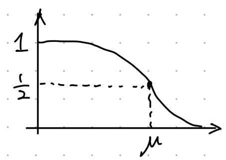
$= \hat{H}_\theta\Psi$

第2問

[問1]

① when $T=0$ ② when $T>0$





[問2]

$D(\varepsilon)\Delta\varepsilon = \int_{\Delta\varepsilon}\frac{d\omega}{h^3} = \frac{1}{h^3}\iiint_V dx\,dy\,dz\iiint dp_x\,dp_y\,dp_z = \frac{4\pi V p^2}{h^3}\Delta p$

$\varepsilon = \frac{p^2}{2m}$, $\Delta\varepsilon = \frac{p}{m}\Delta p \Rightarrow p = \sqrt{2m\varepsilon}$, $\Delta p = \frac{m}{\sqrt{2m\varepsilon}}\Delta\varepsilon$

$\therefore D(\varepsilon)\Delta\varepsilon = \frac{4\pi V}{(2\pi\hbar)^3}2m\varepsilon\frac{m}{\sqrt{2m\varepsilon}}\Delta\varepsilon \Rightarrow D(\varepsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}\varepsilon^{\frac{1}{2}}$

[問3] when $T=0$, $N = 2\int_0^{\varepsilon_F} d\varepsilon\,\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}\varepsilon^{\frac{1}{2}}$

let $\frac{2}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} = A$, then $N = AV\int_0^{\varepsilon_F}\varepsilon^{\frac{1}{2}}d\varepsilon = AV\frac{2}{3}\varepsilon_F^{\frac{3}{2}}$

$\Rightarrow \varepsilon_F = \left(\frac{3n}{2A}\right)^{\frac{2}{3}} = \frac{\hbar^2}{2m}(3n\pi^2)^{\frac{2}{3}}$

[問4] $T_F = \frac{\varepsilon_F}{k_B} = \frac{\hbar^2}{2mk_B}(3n\pi^2)^{\frac{2}{3}} \simeq \frac{(1.0546\times10^{-34})^2}{2\times9.11\times10^{-31}\times1.38\times10^{-23}}\times3.1\times10^{20} \simeq 1.37\times10^5\,\mathrm{K}$

[問5] $D(\varepsilon) = \frac{A}{2}V\varepsilon^{\frac{1}{2}}$ $\frac{dD(\varepsilon)}{d\varepsilon} = \frac{A}{4}V\varepsilon^{-\frac{1}{2}}$ $\int_0^\mu d\varepsilon\,D(\varepsilon) = \frac{A}{3}V\mu^{\frac{3}{2}}$

$N = 2\int_0^\infty d\varepsilon\,D(\varepsilon)f(\varepsilon) = 2\left(\frac{A}{3}V\mu^{\frac{3}{2}} + \frac{\pi^2}{6}(k_BT)^2\frac{A}{4}V\mu^{-\frac{1}{2}}\right) = \frac{2A}{3}V\mu^{\frac{3}{2}}\left(1 + \frac{\pi^2}{8}\left(\frac{k_BT}{\mu}\right)^2\right)$

$$\Rightarrow \mu = \left(\frac{3n}{2A}\right)^{\frac{2}{3}} \left(1+\frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right)^{-\frac{2}{3}} \qquad \mu_0 = \varepsilon_F$$

$$= \mu_0 \left(1+\frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right)^{-\frac{2}{3}}$$) since μ is changing around μ_0

$$\approx \mu_0 \left(1+\frac{\pi^2}{8}\left(\frac{kT}{\mu_0}\right)^2\right)^{-\frac{2}{3}}$$

using Taylor expansion around $T=0$, we got $\mu \simeq \mu_0 \left[1-\frac{\pi^2}{6}\left(\frac{kT}{\mu_0}\right)^2\right]$

[問6]

$$C = \frac{\pi^2}{4\varepsilon_F} k_B^2 N \cdot 2T = \frac{\pi^2}{2} N k_B \left(\frac{k_B T}{\varepsilon_F}\right)$$

[問7]

Quatum statistics and classical statistics differences.

第3問

[問1] $\mathcal{L} = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} k x^2 \quad \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \quad \frac{\partial \mathcal{L}}{\partial x} = -kx \Rightarrow m\ddot{x} = -kx$

(1)

(2) $p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \quad H(x,p) = m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{p^2}{2m} + \frac{1}{2}kx^2$

$\dot{x} = \frac{p}{m}, \quad \dot{p} = -kx$

[問2]

(1) when $k=0$, $\dot{p}=0 \Rightarrow p(t) = p(t=0) = p(0)$

$\dot{x} = \frac{p(0)}{m} \Rightarrow x(t) = \frac{p(0)}{m} t + x(0)$

$$\therefore M = \begin{pmatrix} 1 & \frac{t}{m} \\ 0 & 1 \end{pmatrix}$$

(2) when $k = m\omega^2 > 0$

$\ddot{x} = \frac{\dot{p}}{m} = -\omega^2 x \Rightarrow x = A_1 \cos\omega t + A_2 \sin(\omega t)$

$\Rightarrow p = m\dot{x} = -m\omega A_1 \sin\omega t + m\omega A_2 \cos(\omega t)$

take into initial condition, we have

$\begin{cases} A_1 = x(0) \\ m\omega A_2 = p(0) \end{cases}$, therefore $\begin{cases} x = x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin(\omega t) \\ p = -m\omega x(0)\sin\omega t + p(0)\cos(\omega t) \end{cases}$

$$\therefore M = \begin{pmatrix} \cos(\omega t) & \frac{\sin(\omega t)}{m\omega} \\ -m\omega\sin(\omega t) & \cos(\omega t) \end{pmatrix}$$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\sinh x = \frac{e^x - e^{-x}}{2}$

(3) when $k = -m\omega^2 < 0$

$\ddot{x} = \omega^2 x \Rightarrow x = B_1 \cosh\omega t + B_2 \sinh\omega t$

$\Rightarrow p = -m\omega B_1 \sinh\omega t + m\omega B_2 \cosh\omega t$

take into initial condition, we have

$\begin{cases} B_1 = x(0) \\ m\omega B_2 = p(0) \end{cases}$, therefore $\begin{cases} x = x(0)\cosh\omega t + \frac{p(0)}{m\omega}\sinh\omega t \\ p = -m\omega x(0)\sinh\omega t + p(0)\cosh\omega t \end{cases}$

$$\therefore M = \begin{pmatrix} \cosh\omega t & \frac{\sinh\omega t}{m\omega} \\ -m\omega\sinh\omega t & \cosh\omega t \end{pmatrix}$$

[問3]

(1) $\dot{W}(t) = \dot{x}_2 p_1 + x_2 \dot{p}_1 - \dot{p}_2 x_1 - p_2 \dot{x}_1$

$= \frac{p_2}{m} p_1 + x_2(-kx_1) - (-kx_2)x_1 - p_2 \frac{p_1}{m}$

$= 0$

(2) $\begin{cases} x_n(t) = a x_n(0) + b p_n(0) \\ p_n(t) = c x_n(0) + d p_n(0) \end{cases}, \quad n = 1, 2.$

$W(t) = (a x_2(0) + b p_2(0))(c x_1(0) + d p_1(0)) - (c x_2(0) + d p_2(0))(a x_1(0) + b p_1(0))$

$= \cancel{ac\, x_1(0) x_2(0)} + ad\, x_2(0) p_1(0) + bc\, p_2(0) x_1(0) + \cancel{bd\, p_1(0) p_2(0)}$

$\quad - \cancel{ac\, x_1(0) x_2(0)} - bc\, p_1(0) x_2(0) - da\, x_1(0) p_2(0) - \cancel{db\, p_1(0) p_2(0)}$

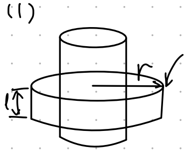
$= (ad - bc) x_2(0) p_1(0) + (bc - da) x_1(0) p_2(0)$

$= \det T \cdot W(t=0)$

$\Rightarrow \det T = 1.$

第4問

(1)



Gassian Surface, $\iint_S \vec{E}\cdot d\vec{S} = 2\pi r l E$

$\frac{Q}{\varepsilon_0} = \frac{\lambda l \cdot 2\pi a}{\varepsilon_0}$

$\Rightarrow E = \frac{a\lambda}{\varepsilon_0 r}$

(2)

$E_A(r) = \frac{a\lambda}{\varepsilon_0 r} \quad E_B(r) = \frac{-a\lambda}{\varepsilon_0 (d-r)}$

$$V_{AB} = \int_{d-a}^{d+a} (E_A + E_B)\, dr = \frac{a\lambda}{\varepsilon_0} \ln\frac{d+a}{d-a}$$

(3)

$$C = \frac{Q'}{V_{AB}} = \frac{2\cdot\lambda 2\pi a}{\frac{a\lambda}{\varepsilon_0}\ln\frac{d+a}{d-a}} = \frac{4\pi\varepsilon_0}{\ln\frac{d+a}{d-a}}$$

(4) For A: $\oint_C \vec{H}\cdot d\vec{s} = 2\pi r H_A$

$\iint_S \vec{J}\cdot d\vec{S} = I$

$\Rightarrow H_A = \frac{I}{2\pi r}$, same as $H_B = \frac{-I}{2\pi(d-r)}$

Therefore $H = \frac{I}{2\pi}\left(\frac{1}{r} - \frac{1}{d-r}\right)$

(5) $B=\mu_0 H$

$$\Phi = \oint \vec{B}\cdot d\vec{S} = l\int_{d-a}^{d+a} \mu_0 \frac{I}{2\pi}\left(\frac{1}{r} - \frac{1}{d-r}\right)dr = \frac{l\mu_0 I}{2\pi}\ln\frac{d+a}{d-a}$$

(6)

$$2LI = \frac{\Phi}{l} \Rightarrow L = \frac{\Phi}{2lI} = \frac{\mu_0}{4\pi}\ln\frac{d+a}{d-a}$$

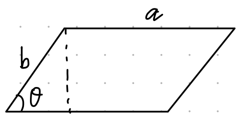

unit length

数学

第1問

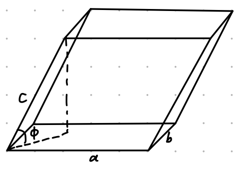
[問1]

$$S = b\sin\theta\cdot a = |\vec{a}\times\vec{b}|$$



[問2]

$$V = S\cdot c\cdot\cos\phi = |(\vec{a}\times\vec{b})\cdot\vec{c}|$$



[問3]

$$M = \vec{a}\times\vec{b}\cdot\vec{c} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}\cdot(c_x\vec{e}_x + c_y\vec{e}_y + c_z\vec{e}_z)$$

$$= (a_y b_z - a_z b_y)c_x + (a_z b_x - a_x b_z)c_y + (a_x b_y - a_y b_x)c_z$$

第2問

[問1]

Take (2) into (1), we have $\nabla^2 G(\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{|\vec{k}|^2}\nabla^2 e^{i\vec{k}\cdot\vec{x}} = \int\frac{d^3\vec{k}}{(2\pi)^3}\frac{(i\vec{k})^2}{|\vec{k}|^2}e^{i\vec{k}\cdot\vec{x}}$

$$= -\int\frac{d^3\vec{k}}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}$$

For $\delta^{(1)}(x) = \int\frac{dk}{2\pi}e^{-ikx}$, therefore $\nabla^2 G(\vec{x}) = -\delta^{(3)}(\vec{x})$

[問2]

Cauchy theorem: $\oint_{C_I+C_\varepsilon+C_{II}+C_{III}} f(z)=0$

$\int_0^\infty dz\frac{\sin z}{z} = \int_0^\infty dz\frac{e^{iz}-e^{-iz}}{2iz} = \int_{-\infty}^{\infty} dz\frac{e^{iz}}{2iz}$, let $f(z)=\frac{e^{iz}}{z}$, $z=Re^{i\theta}$

$dz = e^{i\theta}dR$

$dz = iRe^{i\theta}d\theta$

$$\lim_{R\to\infty}\int_{C_{III}} dz\frac{e^{iz}}{z} = \lim_{R\to\infty}\int_0^\pi \frac{e^{iR(\cos\theta+i\sin\theta)}}{Re^{i\theta}}\, iRe^{i\theta}d\theta \to 0$$

$$\lim_{\varepsilon\to0}\int_{C_\varepsilon}dz\frac{e^{iz}}{z} = \lim_{\varepsilon\to0}\int_\pi^0 \frac{e^{i(\varepsilon e^{i\theta})}}{\varepsilon e^{i\theta}}\, i\varepsilon e^{i\theta}d\theta = i\int_\pi^0 d\theta = -i\pi$$

$$\lim_{\substack{R\to\infty\\ \varepsilon\to0}}\int_{C_I+C_{II}} dz\frac{e^{iz}}{z} = \lim_{\substack{R\to\infty\\ \varepsilon\to0}}\int_{-R}^{-\varepsilon}\frac{e^{iz}}{z}dz + \int_\varepsilon^R \frac{e^{iz}}{z}dz = \int_{-\infty}^{+\infty}\frac{e^{iz}}{z}dz$$

$$\therefore -i\pi + \int_{-\infty}^{+\infty}\frac{e^{iz}}{z}dz = 0 \Rightarrow \int_0^\infty dz\frac{\sin z}{z} = \frac{1}{2i}\int_{-\infty}^{+\infty}dz\frac{e^{iz}}{z} = \frac{\pi}{2}$$

[問3、問4]. No idea

第3問

[問1]

$$\frac{d^2f(x)}{d^2x} = \frac{2x\frac{df}{dx}-2f}{1-x^2}$$

(a)

$$\frac{dW(x)}{dx} = \frac{df_1}{dx}\frac{df_2}{dx} + f_1\frac{d^2f_2}{d^2x} - \frac{df_2}{dx}\frac{df_1}{dx} - f_2\frac{d^2f_2}{d^2x}$$

$$= f_1\frac{2}{1-x^2}\left(x\frac{df_2}{dx}-f_2\right) - f_2\frac{2}{1-x^2}\left(x\frac{df_1}{dx}-f_1\right)$$

$$=\frac{2x}{1-x^2}\left(f_1\frac{df_2}{dx}-f_2\frac{df_1}{dx}\right)$$

$$=\frac{2x}{1-x^2}W(x)$$

(b)

$$\int\frac{1}{W(x)}dW(x) = \int\frac{2x}{1-x^2}dx \Rightarrow \ln|W(x)| = \ln|1-x^2|^{-1}+C$$

$$\Rightarrow W(x) = C\frac{1}{1-x^2}$$

[問2] $2x\frac{df}{dx}-2f=0 \Rightarrow \int\frac{1}{f}df = \int\frac{1}{x}dx \Rightarrow f_1 = C_1 x$

take condition of $f_1(1)=1$ into, we got $f_1 = x$

[問3]

$$W(x) = x\frac{df_2}{dx} - f_2 = \frac{C}{1-x^2} \Rightarrow \frac{df_2}{dx} - \frac{1}{x}f_2 - \frac{C}{x(1-x^2)} = 0$$

[問4] $p(x) = -\frac{1}{x}$, $q(x) = \frac{-C}{x(1-x^2)}$

$$\alpha(x) = e^{\int^x p(x)dx} = \frac{1}{x}$$

$$f_2(x) = \frac{1}{\alpha(x)}\left[\int^x\alpha(x)q(x)dx + C_1\right] = x\left[-C\int^x\left(\frac{1}{x^2}-\frac{1}{1-x^2}\right)dx + C_1\right]$$

$$= x\left\{-C\left[\left(-\frac{1}{x}\right) - \frac{1}{2}\ln\frac{1+x}{1-x}\right]+C_1\right\}$$

$$=\frac{Cx}{2}\ln\frac{1+x}{1-x} + C_1x + C$$

$$\therefore f(x) = C_2 f_1(x) + C_3 f_2(x) = C_2x + \frac{CC_3x}{3}\ln\frac{1+x}{1-x} + C_3C_1x + CC_3$$

$$= A_1 x\ln\frac{1+x}{1-x} + A_2x + A_3$$

第4問

[問1]

$$U^*U = I \Rightarrow \det(U^*U) = \det I \Rightarrow (\det U)^2 = 1 \Rightarrow \det U = 1$$

[問2]

$$UX = \lambda X \Rightarrow (U-\lambda I)X = 0 \quad \det(U-\lambda I) = 0 \Rightarrow \det U - \lambda \det I = 0 \Rightarrow \lambda = 1$$

[問3]

$$\begin{cases} \det U = ad-cb = e^{i\varphi} \\ a+d = 1 \end{cases} \Rightarrow \begin{cases} c = \dfrac{a-a^2-e^{i\varphi}}{b} \\ d = 1-a \end{cases}$$

[問4]

$$\begin{vmatrix} \frac{1}{\sqrt2}-\lambda & \frac{1}{\sqrt2} \\ -\frac{i}{\sqrt2} & \frac{i}{\sqrt2}-\lambda \end{vmatrix} = \left(\frac{1}{\sqrt2}-\lambda\right)\left(\frac{i}{\sqrt2}-\lambda\right)+\frac{i}{2} = 0 \Rightarrow \lambda^2+\left(-\frac{1}{\sqrt2}-\frac{i}{\sqrt2}\right)\lambda+i = 0$$

$$\Rightarrow \lambda = \frac{(1+i)\pm\sqrt{-6i}}{2\sqrt2}, \quad (1-i)^2 = -2i \Rightarrow \sqrt{-2i} = 1-i \Rightarrow \sqrt{-6i} = (1-i)\sqrt3$$

$$= \frac{(1+i)\pm(1-i)\sqrt3}{2\sqrt2}$$

$$\therefore \lambda_1 = \frac{1+\sqrt3}{2\sqrt2}+i\frac{1-\sqrt3}{2\sqrt2}, \quad \lambda_2 = \frac{1-\sqrt3}{2}+i\frac{1+\sqrt3}{2\sqrt2}$$

SOKENDAI KEK 入试 2021 解答

物理

第1問

[問1]

$$\hbar = \sqrt{[q][\mathcal{E}][L]^3[m]}$$

[問2]

$$i\hbar\frac{d}{dt}\Psi(x,t) = \hat{H}\Psi(x,t) \Rightarrow \frac{d}{dt}\Psi(x,t) = \frac{\hat{H}}{i\hbar}\Psi(x,t)$$

$$-i\hbar\frac{d}{dt}\Psi^*(x,t) = \hat{H}\Psi^*(x,t) \Rightarrow \frac{d}{dt}\Psi^*(x,t) = \frac{\hat{H}}{-i\hbar}\Psi(x,t)$$

$$\frac{d\langle x\rangle}{dt} = \frac{d}{dt}\langle\Psi|x|\Psi\rangle = \left\langle\frac{d\Psi}{dt}\middle|x\middle|\frac{\Psi}{dt}\right\rangle + \left\langle\Psi\middle|\frac{\partial x}{\partial t}\middle|\Psi\right\rangle + \left\langle\Psi\middle|\frac{dx}{dt}\middle|\frac{d\Psi}{dt}\right\rangle$$

$$= \frac{1}{-i\hbar}\langle\Psi|\hat{H}\hat{x}|\Psi\rangle + \frac{1}{i\hbar}\langle\Psi|\hat{x}\hat{H}|\Psi\rangle$$

$$= \frac{1}{i\hbar}\langle[\hat{x},\hat{H}]\rangle$$

$$= \frac{1}{i\hbar}\left\langle\left[\hat{x},\frac{\hat{p}^2}{2m}+V(x)\right]\right\rangle \qquad \Rightarrow \qquad \langle a\rangle = \frac{d}{dt}\left(\frac{d\langle x\rangle}{dt}\right) = \frac{1}{m}\frac{d\langle\hat{p}\rangle}{dt}$$

$$= \frac{1}{i\hbar 2m}\langle[\hat{x},\hat{p}^2]\rangle$$

$$= \frac{1}{i\hbar 2m}\cdot 2i\hbar\langle\hat{p}\rangle$$

$$= \frac{\langle\hat{p}\rangle}{m}$$

[問3]

$$\frac{d}{dt}\left(e^{iS(x)/\hbar}e^{-iEt/\hbar}\right) = \frac{1}{i\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}-q\mathcal{E}x\right)\left(e^{iS(x)/\hbar}e^{-iEt/\hbar}\right)$$

$$\text{l.h.s} = e^{iS(x)/\hbar}\left(\frac{-iE}{\hbar}\right)e^{-iEt/\hbar} = \frac{-iE}{\hbar}\Psi_E(x,t)$$

$$\text{r.h.s} = \frac{1}{i\hbar}\left[-\frac{\hbar^2}{2m}\left(\frac{i}{\hbar}\frac{\partial^2 S}{\partial x^2}+\left(\frac{i}{\hbar}\frac{\partial S}{\partial x}\right)^2\right)-q\mathcal{E}x\right]\Psi_E(x,t)$$

$$\Rightarrow E = -\frac{\hbar^2}{2m}\left[\frac{i}{\hbar}\frac{\partial^2 S}{\partial x^2}+\left(\frac{i}{\hbar}\frac{\partial S}{\partial x}\right)^2\right]-q\mathcal{E}x$$

$$\Rightarrow \frac{d^2S(x)}{dx^2}+\frac{i}{\hbar}\left(\frac{dS(x)}{dx}\right)^2-\frac{2mi}{\hbar}(E+q\mathcal{E}x) = 0$$

[問4]

$$\frac{dS(x)}{dx} = \frac{dS_0(x)}{dx}+\hbar\frac{dS_1(x)}{dx}, \quad \frac{d^2S(x)}{dx^2} = \frac{d^2S_0(x)}{dx^2}+\hbar\frac{d^2S_1(x)}{dx^2}$$

$$\frac{d^2S_0}{dx^2}+\hbar\frac{d^2S_1}{dx^2}+\frac{i}{\hbar}\left[\left(\frac{dS_0}{dx}\right)^2+2\hbar\frac{dS_0}{dx}\frac{dS_1}{dx}+\hbar^2\left(\frac{S_1}{dx}\right)^2\right]-\frac{2mi}{\hbar}(E+q\mathcal{E}x) = 0$$

multiply $\hbar$ in both sides, and ignore the h^2 terms, we got

$$\hbar \frac{d^2S_0}{dx^2} + i\left(\frac{dS_0}{dx}\right)^2 + 2i\hbar \frac{dS_0}{dx}\frac{dS_1}{dx} - 2mi(E+q\mathcal{E}x) = 0$$

$$\Rightarrow \hbar\left(\frac{d^2S_0}{dx^2} + 2i\frac{dS_0}{dx}\frac{dS_1}{dx}\right) + i\left[\left(\frac{dS_0}{dx}\right)^2 - 2m(E+q\mathcal{E}x)\right] = 0$$

$$\Rightarrow \begin{cases} E = \frac{1}{2m}\left(\frac{dS_0(x)}{dx}\right)^2 - q\mathcal{E}x \\ 0 = \frac{dS_0(x)}{dx}\frac{dS_1(x)}{dx} - \frac{i}{2}\frac{d^2S_0(x)}{dx^2}\end{cases}$$

[問5]

$$\frac{dS_0}{dx} = \pm\sqrt{2m(E+q\mathcal{E}x)} \Rightarrow S_0(x) = \frac{2\sqrt{2m}}{3q\mathcal{E}}\left[(E+q\mathcal{E}L)^{\frac{3}{2}} - E^{\frac{3}{2}}\right]$$

$$\frac{d^2S_0}{dx^2} = \pm\sqrt{2m}\,\frac{q\mathcal{E}}{2}(E+q\mathcal{E}x)^{-\frac{1}{2}}$$

$$0 = \sqrt{2m(E+q\mathcal{E}x)}\,\frac{dS_1(x)}{dx} - \frac{i}{2}\sqrt{2m}\,\frac{q\mathcal{E}}{2}(E+q\mathcal{E}x)^{-\frac{1}{2}}$$

$$\Rightarrow \frac{dS_1(x)}{dx} = \frac{iq\mathcal{E}}{4}(E+q\mathcal{E}x)^{-1} \Rightarrow S_1(x) = \frac{i}{4}\ln\left(1+\frac{q\mathcal{E}}{E}x\right)\Big|_0^L = \frac{i}{4}\ln\left(1+\frac{q\mathcal{E}L}{E}\right)$$

$$\therefore \psi(x) = e^{i(S_0(x)+\hbar S_1(x))/\hbar} = e^{i\frac{2\sqrt{2m}}{3q\mathcal{E}\hbar}\left[(E+q\mathcal{E}L)^{\frac{3}{2}} - E^{\frac{3}{2}}\right]}\left(1+\frac{q\mathcal{E}L}{E}\right)^{-\frac{1}{4}}$$

[問6] no idea

第2問

[問1]

(1)

$$Z = \sum_{\{\sigma_i\}} e^{-\mathcal{H}/(k_BT)} = \sum_{\sigma_i=\pm1} e^{\sum_{i=1}^{N} \mu\sigma_i H/(k_BT)}$$

$$= \prod_{i=1}^{N}\left(e^{\mu H/(k_BT)} + e^{-\mu H/(k_BT)}\right) = \left[2\cosh\left(\frac{\mu H}{k_BT}\right)\right]^N$$

$$F = -k_BT\ln Z = -k_BTN\left[\ln 2 + \ln\left(\cosh\left(\frac{\mu H}{k_BT}\right)\right)\right]$$

$$S = -\frac{\partial F}{\partial T} = k_BN\left[\ln 2 + \ln\left(\cosh\left(\frac{\mu H}{k_BT}\right)\right)\right] + k_BTN\frac{1}{\cosh\frac{\mu H}{k_BT}}\sinh\left(\frac{\mu H}{k_BT}\right)\left(\frac{-\mu H}{k_B}\right)T^{-2}$$

$$= k_BN\left\{\left[\ln 2 + \ln\left(\cosh\left(\frac{\mu H}{k_BT}\right)\right)\right] - \tanh\left(\frac{\mu H}{k_BT}\right)\frac{\mu H}{k_BT}\right\}$$

$$\bar{E} = k_BT^2\frac{\partial}{\partial T}\left(\frac{F}{k_BT}\right) = -k_BT^2\frac{\partial}{\partial T}\left(N\cdot\ln\cosh\left(\frac{\mu H}{k_BT}\right)\right)$$

$$= k_BT^2\cdot N\cdot\tanh\left(\frac{\mu H}{k_BT}\right)\left(\frac{\mu H}{k_BT^2}\right) = \mu HN\tan\frac{\mu H}{k_BT}$$

(2)

$$M = \mu\sum_{\{\sigma_i\}}\sigma_i = -\frac{1}{V}\left(\frac{\partial F}{\partial H}\right)_T = k_BTn\cdot\tanh\left(\frac{\mu H}{k_BT}\right)\frac{\mu}{k_BT}$$

$$= n\mu\tan(\beta\mu H)$$

(3) ?

$$M_0 = n\mu\tan(\beta\mu aM_0) \approx n\mu\left[\beta\mu aM_0 - (\beta\mu aM_0)^3/3\right]$$

$$= M_0\left[n\mu^2a\beta - n\mu^4a^3\beta^3M_0^2/3\right]$$

$$\Rightarrow M_0 \sim \sqrt{\frac{n\mu^2a\beta-1}{n\mu^4a^3\beta^3}} = \sqrt{\frac{(T_c-T)\frac{1}{T}}{\frac{aT_c^2}{nkT^3}}}$$

$$d\varepsilon = \frac{p}{m}dp \qquad \varepsilon = \frac{p^2}{2m} \qquad p = \sqrt{2m\varepsilon}$$

[問2]

(1)

$$D(\varepsilon)d\varepsilon = \int_{d\varepsilon}\frac{d\omega}{h^3} = \frac{V}{h^3}\int dp_x dp_y dp_z = \frac{4\pi V}{h^3}p^2dp$$

$$= \frac{4\pi V}{(2\pi\hbar)^3}(2m\varepsilon)\cdot\frac{m}{\sqrt{2m\varepsilon}}d\varepsilon$$

$$\Rightarrow D(\varepsilon) = \frac{m^{\frac{3}{2}}V}{\sqrt{2}\pi^2\hbar^3}\varepsilon^{\frac{1}{2}}$$

(2)

$$\sqrt{(\varepsilon+\mu\sigma H)2m} = p$$

$$D_+(\varepsilon)d\varepsilon = \frac{4\pi V}{(2\pi\hbar)^3}p^2dp \qquad dp = (\varepsilon+\mu\sigma H)^{-\frac{1}{2}}(2m)^{\frac{1}{2}}\cdot\frac{1}{2}\,d\varepsilon$$

$$= \frac{V}{2\pi^2\hbar^3}(\varepsilon+\mu H)\cdot 2m\cdot 2^{-\frac{1}{2}}m^{\frac{1}{2}}(\varepsilon+\mu H)^{-\frac{1}{2}}d\varepsilon$$

$$= \frac{m^{\frac{3}{2}}V}{\sqrt{2}\pi^2\hbar^3}(\varepsilon+\mu H)^{\frac{1}{2}}d\varepsilon$$

$$\therefore D_+(\varepsilon) = \frac{m^{\frac{3}{2}}V}{\sqrt{2}\pi^2\hbar^3}(\varepsilon+\mu H)^{\frac{1}{2}},\quad D_-(\varepsilon) = \frac{m^{\frac{3}{2}}V}{\sqrt{2}\pi^2\hbar^3}(\varepsilon-\mu H)^{\frac{1}{2}}$$

(3)

$$N_+ = \int_0^{\varepsilon_F}D_+(\varepsilon)d\varepsilon = \frac{m^{\frac{3}{2}}V}{\sqrt{2}\pi^2\hbar^3}\int_0^{\varepsilon_F}(\varepsilon+\mu H)^{\frac{1}{2}}d\varepsilon$$

$$= \frac{2m^{\frac{3}{2}}V}{3\sqrt{2}\pi^2\hbar^3}\left[(\varepsilon_F+\mu H)^{\frac{3}{2}} - (\mu H)^{\frac{3}{2}}\right]$$

$$N_- = \frac{m^{\frac{3}{2}}N}{\sqrt{2}\pi^2\hbar^3}\int_{\mu H}^{\varepsilon_F}(\varepsilon-\mu H)^{\frac{1}{2}}d\varepsilon = \frac{2m^{\frac{3}{2}}V}{3\sqrt{2}\pi^2\hbar^3}(\varepsilon_F-\mu H)^{\frac{3}{2}}$$

$$M = N_+\sigma_+ + N_-\sigma_- = N_+ - N_-$$
$$= \frac{2m^{\frac{3}{2}}V}{3\sqrt{2}\pi^2\hbar^3}\left[(\varepsilon_F+\mu H)^{\frac{3}{2}} - (\varepsilon_F-\mu H)^{\frac{3}{2}} - (\mu H)^{\frac{3}{2}}\right]$$

Since $(\varepsilon_F+\mu H)^{\frac{3}{2}} \simeq \varepsilon_F^{\frac{3}{2}} + \frac{3}{2}\mu(\varepsilon_F+\mu H)^{\frac{1}{2}}\Big|_{H=0} H + O(H^2)$

$(\varepsilon_F-\mu H)^{\frac{3}{2}} \simeq \varepsilon_F^{\frac{3}{2}} - \frac{3}{2}\mu(\varepsilon_F-\mu H)^{\frac{1}{2}}\Big|_{H=0} H + O(H^2)$.

$$\Rightarrow M \simeq \frac{2m^{\frac{3}{2}}V}{3\sqrt{2}\pi^2\hbar^3}\mu\varepsilon_F H + o(H) \sim H \qquad \square$$

第3問

[問1]

(1) $\dot{x} = a(2\dot\theta + 2\dot\theta\cos 2\theta)$ $\dot{z} = 2a\dot\theta\sin 2\theta$

$$K = \frac{1}{2}m(\dot{x}^2+\dot{z}^2) = \frac{m}{2}\left[4a^2\dot\theta^2(1+\cos 2\theta)^2 + 4a^2\dot\theta^2(\sin 2\theta)^2\right]$$
$$= 2a^2\dot\theta^2 m\left[4\cos^4\theta + 4\cos^2\theta(1-\cos^2\theta)\right]$$
$$= 8ma^2\dot\theta^2\cos^2\theta$$

(2) $U = mgz = 2mga\sin^2\theta$

(3) $\frac{dx}{d\theta} = a(2+2\cos 2\theta)$ $\frac{dz}{d\theta} = 2a\sin 2\theta$

$$s = \int_0^\theta 2a\sqrt{(1+\cos 2\theta)^2+\sin^2 2\theta}\,d\theta = 2a\int_0^\theta\sqrt{2+2\cos 2\theta}\,d\theta \qquad \begin{pmatrix}\theta<0\\ s<0\end{pmatrix}$$
$$= -4a\sin\theta$$

(4) $\dot{s} = \frac{ds}{d\theta} = -4a\cos\theta$

$$\mathcal{L} = K - U = 8ma^2\dot\theta^2\cos^2\theta - 2mga\sin^2\theta$$
$$= \frac{1}{2}m\dot{s}^2 - \frac{mg}{8a}s^2$$

(5) $\frac{\partial\mathcal{L}}{\partial\dot{s}} = m\dot{s}$ $\frac{\partial\mathcal{L}}{\partial s} = -\frac{mg}{4a}s$ $\Rightarrow$ $\ddot{s} = -\frac{gs}{4a}$

(6) $s(t) = A_1\cos\sqrt{\frac{g}{4a}}\,t + A_2\sin\sqrt{\frac{g}{4a}}\,t$ $T = \frac{2\pi}{\sqrt{\frac{g}{4a}}} = 4\pi\sqrt{\frac{a}{g}}$

(7) when $\theta = \frac{\pi}{2}$, $x = 2a\theta = a\pi = 1.54/2$

$$t = \frac{T}{2} = 2\pi\sqrt{\frac{a}{g}} = 4\pi/\sqrt{10}$$

問2

(1)
$$x = \sin\phi\, a(2\theta+\sin 2\theta)$$
$$y = \cos\phi\, a(2\theta+\sin 2\theta)$$
$$\dot{x} = a\dot\phi\cos\phi(2\theta+\sin 2\theta) + 2a\sin\phi\,\dot\theta(1+\cos 2\theta)$$
$$\dot{y} = -a\dot\phi\sin\phi(2\theta+\sin 2\theta) + 2a\cos\phi\,\dot\theta(1+\cos 2\theta)$$
$$\dot{z} = 2a\dot\theta\sin 2\theta$$
$$K = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$
$$= \frac{1}{2}m\Big\{a^2\dot\phi^2(2\theta+\sin 2\theta)^2 + 4a^2\dot\theta^2\sin^2\phi(1+\cos 2\theta)^2 + 4a^2\dot\theta^2\cos^2\phi(1+\cos 2\theta)^2 + 4a^2\dot\theta^2\sin^2 2\theta\Big\}$$
$$= \frac{1}{2}ma^2\Big\{\dot\phi^2(4\theta^2+4\theta\sin 2\theta+\sin^2 2\theta) + 8\dot\theta^2 + 8\dot\theta^2\cos 2\theta\Big\}$$

we have $s = 4a\sin\theta$ $\dot{s} = 4a\dot\theta\cos\theta$ $\frac{s}{a} = 4\sin\theta \sim 4\theta$

$$K = \frac{1}{2}ma^2\cdot 16\dot\theta^2\cos^2\theta + \frac{1}{2}ma^2\dot\phi^2(4\theta^2+8\theta^2+4\theta^2)$$
$$= \frac{1}{2}m\dot{s}^2 + \frac{1}{2}m\dot\phi^2 s^2$$
$$\mathcal{L} = K - U = \frac{1}{2}m\dot{s}^2 - \frac{mg}{8a}s^2 + \frac{1}{2}m\dot\phi^2 s^2$$

(2) $\frac{\partial\mathcal{L}}{\partial\phi} = 0$ $\frac{\partial\mathcal{L}}{\partial\dot\phi} = l$ $\Rightarrow$ $\frac{dl}{dt} = 0$, $\therefore$ l is conserved

(3) $l = \frac{\partial \mathcal{L}}{\partial \dot\phi} = ms^2\dot\phi \Rightarrow \dot\phi = \frac{l}{ms^2}$

$E = \frac{1}{2}m\dot s^2 + \frac{mg}{8a}s^2 + \frac{l^2}{2ms^2}$

$\frac{\partial \mathcal{L}}{\partial s} = -\frac{mg}{4a}s + m\dot\phi^2 s, \quad \frac{\partial \mathcal{L}}{\partial \dot s} = m\dot s \Rightarrow \ddot s = \frac{-gs}{4a} + \frac{l^2}{m^2 s^3}$

$\frac{dE}{dt} = m\dot s\ddot s + \frac{mgs}{4a}\dot s + \frac{l^2}{2m}(-2)s^{-3}\dot s$

$= \dot s\left\{\frac{-gms}{4a} + \frac{l^2}{ms^3} + \frac{mgs}{4a} - \frac{l^2}{ms^3}\right\} \equiv 0$

$\Rightarrow E$ is conserved.

(4)(5) no idea

第4問

[問1]

(1) ① $Q = -en_0 Sx$

② $E \cdot 2S = -2en_0 Sx/\varepsilon_0 \Rightarrow E = -\frac{en_0 x}{\varepsilon_0}$

(2) $m_e \frac{d\vec v}{dt} = e\vec E$

(3) $\frac{d^2x}{dt^2} + \frac{e^2 n_0}{\varepsilon_0 m_e}x = 0$, compare to $\frac{d^2x}{dt^2} + \omega_p^2 x = 0$

$\Rightarrow \omega_p = e\sqrt{\frac{n_0}{\varepsilon_0 m_e}}$

[問2]

(1) $\nabla\times(\nabla\times\vec E) = \nabla(\nabla\cdot\vec E) - \nabla^2\vec E = \nabla\left(\frac{\rho}{\varepsilon_0}\right) - \nabla^2\vec E$

$\nabla\times\left(-\frac{\partial\vec B}{\partial t}\right) = -\frac{\partial}{\partial t}\nabla\times\vec B = -\frac{\partial}{\partial t}\left[\mu_0(\vec J + \varepsilon_0\frac{\partial}{\partial t}\vec E\right]$

$\Rightarrow \nabla^2\vec E - \varepsilon_0\mu_0\frac{\partial^2\vec E}{\partial t^2} - \nabla\left(\frac{\rho}{\varepsilon_0}\right) - \frac{\partial}{\partial t}\mu_0\vec J = 0$

Since $\rho = 0$, $\frac{d}{dt}\vec J = -en_0\frac{e\vec E}{m_e} = -\varepsilon_0\omega_p^2\vec E$

$\therefore \nabla^2\vec E - \varepsilon_0\mu_0\left(\omega_p^2\vec E + \frac{\partial^2\vec E}{\partial t^2}\right) = 0$

(2) $\vec E = \vec E_0\cos(\vec k\cdot\vec r - \omega t + \varphi)$ $\quad \vec r = (x, y, z)$

$= E_0\cos(kz - \omega t + \varphi)$ $\quad \vec k = (0, 0, k)$

Take it into the equation, then

$-E_0 k^2\cos(kz - \omega t + \varphi) - \varepsilon_0\mu_0(\omega_p^2 E_0\cos(kz - \omega t + \varphi))$

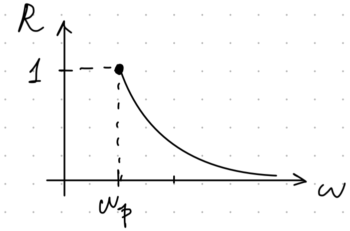
$+ \varepsilon_0\mu_0\omega^2 E_0\cos(kz - \omega t + \varphi) = 0$

$\Rightarrow -k^2 - \varepsilon_0\mu_0(\omega_p^2 - \omega^2) = 0 \Rightarrow \omega = \sqrt{\omega_p^2 + \frac{k^2}{\varepsilon_0\mu_0}}$

(3) $v_p = \frac{\omega}{k} = \left[\varepsilon_0\mu_0\left(1 - \left(\frac{\omega_p}{\omega}\right)^2\right)\right]^{-\frac{1}{2}}$ $\quad k = \sqrt{\varepsilon_0\mu_0(\omega^2 - \omega_p^2)}$

$n = \frac{c}{v_p} = \sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2}$

(4) $R = \left|\frac{1 - \sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2}}{1 + \sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2}}\right|$



数学

第一問

[問1] $\int_{-\infty}^{+\infty}dx\, e^{-ax^2}\int_{-\infty}^{+\infty}dy\, e^{-ay^2} = \int_0^{2\pi}d\theta\int_0^\infty dr\, re^{-ar^2} = 2\pi\int_0^\infty \frac{1}{2}d(r^2)\, e^{-ar^2} = \frac{\pi}{a}$

$\Rightarrow \int_{-\infty}^{+\infty}dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$

[問2] let $x - b = t$, $I_1 = \int_{-\infty}^{+\infty}dt\, e^{-at^2}(t+b)^2 = \int_{-\infty}^{+\infty}dt\, e^{-at^2}t^2 + 2b\int_{-\infty}^{+\infty}e^{-at^2}t\,dt + b^2\int_{-\infty}^{+\infty}e^{-at^2}dt$

$\frac{d}{da}\int_{-\infty}^{+\infty}dx\, e^{-ax^2} = \int_{-\infty}^{+\infty}-x^2 e^{-ax^2}dx = -\frac{\sqrt\pi}{2}a^{-\frac{3}{2}}$

$\therefore I_1 = \frac{\sqrt\pi}{2}a^{-\frac{3}{2}} + b^2\sqrt\pi\, a^{-\frac{1}{2}}$

[問3] $I_2=\int d^3\vec{x}\, e^{-a(x^2+y^2+z^2)}\left[(x+b_1)^2+(y+b_2)^2+(z+b_3)^2\right]$

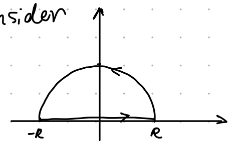
$\int dx\, e^{-ax^2}(x+b_1)^2\int dy\, e^{-ay^2}\int dz\, e^{-az^2} = I_1^2\cdot\frac{\pi}{a}$ $\Rightarrow I_2 = 3\pi^2\left(\frac{a^{-1}}{2}+b^2\right)^2$

問4 ~

第2問 let $f(z)=\frac{z}{z^2-(p+iq)^2}$

[問1] The pole of function are at $z=p+iq$ and $z=-p-iq$, consider

$\oint dz\, \frac{z}{z^2-(p+iq)^2}e^{iaz}$ $\quad z=Re^{i\theta}$, $dz=e^{i\theta}dR$, $dz=iRe^{i\theta}d\theta$



$I_R=\int_0^\pi iRe^{i\theta}f(z)e^{iaR\cos\theta-aR\sin\theta}d\theta$

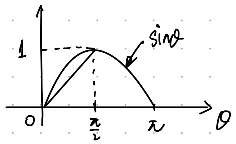
when $R\to\infty$ $|f(z)|=|f(Re^{i\theta})|<\varepsilon$ for all θ within the integration range.
$\varepsilon\to 0$. Then,

$|I_R|\le \varepsilon R\int_0^\pi e^{-aR\sin\theta}d\theta = 2\varepsilon R\int_0^{\frac{\pi}{2}} e^{-aR\sin\theta}d\theta$, $(a>0)$

since $\frac{2}{\pi}\theta\le\sin\theta$ $\quad\le 2\varepsilon R\int_0^{\frac{\pi}{2}} e^{-2aR\theta/\pi}d\theta$

$=\varepsilon\pi/a\,(1-e^{-aR})$

$<\frac{\pi}{a}\varepsilon$ $\quad\Rightarrow \lim_{R\to\infty} I_R=0$

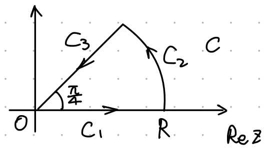


Therefore $\int_{-\infty}^{+\infty} f(x)e^{iax} = 2\pi i\,\mathrm{Res}[f(x)e^{iax}, p+iq] = 2\pi i\lim_{x\to p+iq}\frac{xe^{-iax}}{x^2-(p+iq)^2}(x-(p+iq))$

$=\pi i e^{-iap+aq}$

[問2] $f(z)=e^{iz^2}=\cos(z^2)+i\sin(z^2)$,

consider $\oint_{C_1+C_2+C_3} f(z)dz=0$



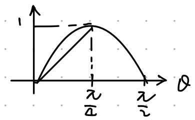
$C_1=\int_{C_1}f(z)\,dz=\int_0^R\cos(z^2)dz+i\int_0^R\sin(z^2)dz$ $\quad z=re^{i\theta}=r$, $dz=dr$

$=\int_0^R\cos(r^2)dr+i\int_0^R\sin(r^2)dr$

$\lim_{R\to\infty}C_1=\frac12(I_2+iI_3)$

$C_2=\int_{C_2}f(z)dz=\int_0^{\frac{\pi}{4}}e^{iR^2e^{2i\theta}}iRe^{i\theta}d\theta = iR\int_0^{\frac{\pi}{4}}e^{i\theta}\cdot e^{iR^2\cos(2\theta)-R^2\sin2\theta}d\theta$, $dz=iRe^{i\theta}d\theta$

$\le iR\int_0^{\frac{\pi}{4}}e^{-R^2\sin2\theta}d\theta \le iR\int_0^{\frac{\pi}{4}}e^{-R^2\frac{4}{\pi}\theta}d\theta = iR\cdot\frac{\pi}{4R^2}(1-e^{-R^2})$



$\lim_{R\to\infty}|C_2|\to 0$

$C_3=\int_{C_3}f(z)dz=\int_R^0 e^{i(re^{i\frac{\pi}{4}})^2}e^{i\frac{\pi}{4}}dr$ $\quad z=re^{i\frac{\pi}{4}}$, $dz=e^{i\frac{\pi}{4}}dr$

$=\int_R^0\left(\frac{\sqrt2}{2}+i\frac{\sqrt2}{2}\right)e^{-r^2}dr$ $\quad \lim_{R\to\infty}C_3=\frac{\sqrt2}{2}(1+i)\left(-\frac{\sqrt\pi}{2}\right)$

Therefore: $\frac12(I_2+iI_3)+0-\frac{\sqrt{2\pi}}{4}(1+i)=0$

$\Rightarrow I_2=\sqrt{\frac{\pi}{2}}$, $I_3=\sqrt{\frac{\pi}{2}}$

第3問

[問1] $|A-\lambda I| = \begin{vmatrix} -\lambda & \frac{1}{\sqrt2} & 0 \\ \frac{1}{\sqrt2} & -\lambda & \frac{1}{\sqrt2} \\ 0 & \frac{1}{\sqrt2} & -\lambda \end{vmatrix} = -\lambda\left(\lambda^2-\frac12\right)-\frac{1}{\sqrt2}\left(\frac{-\lambda}{\sqrt2}\right)=0$ $\Rightarrow \begin{cases}\lambda_1=0\\ \lambda_2=1\\ \lambda_3=-1\end{cases}$

① $\lambda_1=0$, $(A-\lambda I)\vec{x} = \begin{pmatrix} 0 & \frac{1}{\sqrt2} & 0 \\ \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \\ 0 & \frac{1}{\sqrt2} & 0 \end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0$ $\Rightarrow \vec{x}_1=\begin{pmatrix}\frac{1}{\sqrt2}\\0\\-\frac{1}{\sqrt2}\end{pmatrix}$

② $\lambda=1$, $(A-\lambda I)\vec{x} = \begin{pmatrix} -1 & \frac{1}{\sqrt2} & 0 \\ \frac{1}{\sqrt2} & -1 & \frac{1}{\sqrt2} \\ 0 & \frac{1}{\sqrt2} & -1 \end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0$ $\Rightarrow \vec{x}_2=\begin{pmatrix}\frac12\\\frac{1}{\sqrt2}\\\frac12\end{pmatrix}$

③ $\lambda=-1$, $(A-\lambda I)\vec{x} = \begin{pmatrix} 1 & \frac{1}{\sqrt2} & 0 \\ \frac{1}{\sqrt2} & 1 & \frac{1}{\sqrt2} \\ 0 & \frac{1}{\sqrt2} & 1 \end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0$ $\Rightarrow \vec{x}_3=\begin{pmatrix}\frac12\\-\frac{1}{\sqrt2}\\\frac12\end{pmatrix}$

[問2] $\Lambda=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$ $\quad O^{-1}AO=\Lambda$, $O=\begin{pmatrix}\frac12&\frac{1}{\sqrt2}&\frac12\\\frac{1}{\sqrt2}&0&-\frac{1}{\sqrt2}\\\frac12&-\frac{1}{\sqrt2}&\frac12\end{pmatrix}$

$O^{-1}=$ ∘∘∘

[問3] $O^{-1}AOO^{-1}A\cdots AO = O^{-1}A^nO=\Lambda^n$

$\Rightarrow A^n=O\Lambda^nO^{-1}$ ∘∘∘

第4問

[問1] $e^{ax}\frac{d}{dx}\left(e^{-ax}f(x)\right) = e^{ax}\left(-ae^{-ax}f(x)+\frac{df}{dx}e^{-ax}\right) = -af(x)+\frac{df}{dx}=(\hat{D}-a)f(x)$

[問2] $z^2+2pz+q=0$ $\xrightarrow{\text{solution}}$ $\alpha,\beta=\frac{-2p\pm\sqrt{4p^2-4q}}{2}$, $\quad\alpha=-p+\sqrt{p^2-q}$, $q=-p-\sqrt{p^2-q}$

$y_1=C_1e^{\alpha x}+C_2e^{\beta x}=C_1e^{(-p+\sqrt{p^2-q})x}+C_2e^{(-p-\sqrt{p^2-q})x}$

[問3] $y_2 = C_1 e^{i(-p+\sqrt{p^2 q})x} + C_1 e^{i(-p-\sqrt{p^2 q})x}$

[問4] $y_3 = C_1 e^{-px}$

[問5] 。。。

SOKENDAI KEK 入試 2019 解答

物理

第1問

[問1]

(1) $\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}x^2$

(2) $\hat{H}\psi_0 = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}x^2\right) a e^{-bx^2} = \left\{-\frac{\hbar^2}{2m}\left[-2b + (-2bx)^2\right] + \frac{m\omega^2}{2}x^2\right\}\psi_0 = E_0\psi_0$

$-\frac{\hbar^2}{2m}4b^2 + \frac{m\omega^2}{2} = 0 \Rightarrow b = \frac{m\omega}{2\hbar}, \quad E_0 = \frac{\hbar^2 b}{m} = \frac{\hbar\omega}{2}$

$\hat{H}\psi_1 = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}x^2\right) c x e^{-bx^2} = \left\{-\frac{\hbar^2}{2m}\left[-6b + (-2bx)^2\right] + \frac{m\omega^2}{2}x^2\right\}\psi_1 = E_1\psi_1$

$E_1 = \frac{3\hbar^2 b}{m} = \frac{3}{2}\hbar\omega$

(3) $\int \psi_0^*\psi_0\, dx = a^2\int e^{-2bx^2}dx = a^2\sqrt{\frac{\pi}{2b}} = 1 \Rightarrow a = \left(\frac{2b}{\pi}\right)^{\frac{1}{4}}$

$\int \psi_1^*\psi_1\, dx = c^2\int x^2 e^{-2bx^2}dx = \frac{1}{4}\sqrt{\frac{\pi}{2}}\, b^{-\frac{3}{2}}\cdot c^2 = 1 \Rightarrow c = 2\left(\frac{2b^3}{\pi}\right)^{\frac{1}{4}}$

[問2]

(1) $H' = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega'^2}{2}x^2 \quad \psi_0' = \left(\frac{m\omega'}{\pi\hbar}\right)e^{-\frac{m\omega'}{2\hbar}x^2}$

(2) $\int_0^\infty |\psi|^2 dx = 1 \Rightarrow a = \left(\frac{m\omega}{4\hbar\pi}\right)^{\frac{1}{4}} \quad \tilde{\psi}_0 = \left(\frac{m\omega}{4\hbar\pi}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}, \quad E_0 = \frac{\hbar\omega}{2}$

[問3]

(1) $\begin{cases} \psi(x,t) = \psi(x)\phi(t) \\ i\hbar\frac{d}{dt}\psi(x,t) = \hat{H}\psi(x,t) \end{cases} \Rightarrow i\hbar\psi(x)\frac{d\phi(t)}{dt} = \hat{H}\psi(x)\phi(t)$

$i\hbar\frac{1}{\phi(t)}\frac{d\phi(t)}{dt} = \frac{1}{\psi(x)}\hat{H}\psi(x) = E$

$\Rightarrow \begin{cases} \frac{d\phi(t)}{dt} = \frac{1}{i\hbar}E\phi(t) \\ \hat{H}\psi(x) = E\psi(x) \end{cases} \quad \therefore \phi(t) = e^{-iEt/\hbar}$

$\therefore \psi(x,t) = \sum_{n=0}^{\infty} a_n\psi_n(x)e^{-iE_nt/\hbar}$

(2) $e^{-iE_nt/\hbar} = e^{-i(E_nt/\hbar + 2\pi)} \Rightarrow T = 2\pi\hbar/E_n$

(3) ?

第2問

[問1]

$$H=\sum_{i=1}^{N}\frac{p_i^2}{2m}+V(r)+f\cdot r$$

[問2]

$$Z=(2\pi\hbar)^{-N}\int d^N x\, d^N p\, e^{-H/(k_BT)}=(2\pi\hbar)^{-N}\int d^N p\, e^{\sum\frac{p_i^2}{2mk_BT}}\int d^3 x\, e^{-\frac{V(r)+f\cdot r}{k_BT}}$$

when $r<a$, $e^{-\frac{V(r)+f\cdot r}{k_BT}}\to 0$

$$=(2\pi\hbar)^{-N}(2mk_BT\pi)^{\frac{N}{2}}\left[\int_a^{+\infty}e^{-\frac{(b+f)r-ab}{k_BT}}dr\right]^N$$

$$=\left(\frac{2mk_BT\pi}{4\pi^2\hbar^2}\right)^{\frac{N}{2}}\left(\frac{k_BT}{b+f}\right)^N e^{\frac{ab}{k_BT}N}e^{\frac{-a(b+f)N}{k_BT}}$$

$$=\left(\frac{m}{2\pi\hbar^2}\right)^{\frac{N}{2}}\frac{(k_BT)^{\frac{3}{2}N}}{(b+f)^N}e^{\frac{-Nfa}{k_BT}}$$

[問3]

$$F=-k_BT\ln Z=-k_BTN\left\{\frac{1}{2}\ln\frac{m}{2\pi\hbar^2}+\ln\frac{(k_BT)^{\frac{3}{2}}}{b+f}-\frac{fa}{k_BT}\right\}$$

$$U=-k_BT^2\frac{\partial}{\partial T}\left(\frac{F}{k_BT}\right)=k_BT^2N\frac{\partial}{\partial T}\left(\frac{3}{2}\ln k_BT-\frac{fa}{k_BT}\right)=-\frac{3}{2}Nk_BT+faN$$

$$S=-\frac{\partial F}{\partial T}=k_BN\left\{\left[\frac{1}{2}\ln\frac{m}{2\pi\hbar^2}+\ln\frac{(k_BT)^{\frac{3}{2}}}{b+f}-\frac{fa}{k_BT}\right]+T\left[\frac{3}{2}\frac{1}{T}+\frac{fa}{k_BT^2}\right]\right\}$$

$$=k_BN\left\{\frac{1}{2}\ln\frac{m}{2\pi\hbar^2}+\ln\frac{(k_BT)^{\frac{3}{2}}}{b+f}+\frac{3}{2}\right\}$$

[問4]

[問5]

第3問

[問1]

$\dot{x}=\dot{r}\cos\theta-\dot{\theta}r\sin\theta$, $\dot{y}=\dot{r}\sin\theta+\dot{\theta}r\cos\theta$

$$\mathcal{L}=\frac{1}{2}m\{(\dot{r}\cos\theta-\dot{\theta}r\sin\theta)^2+(\dot{r}\sin\theta+\dot{\theta}r\cos\theta)^2+\dot{z}^2\}-mgz$$

$$=\frac{1}{2}m(\dot{r}^2+\dot{\theta}^2r^2+\dot{z}^2)-mgz$$

[問2]

(1) $\dot{r}=-\dot{\phi}a\sin\phi$ $\quad\dot{z}=a\dot{\phi}\cos\phi$

$$\mathcal{L}=\frac{1}{2}m\left(\dot{\phi}^2a^2\sin^2\phi+\dot{\theta}^2(R+a\cos\phi)^2+a^2\dot{\phi}^2\cos^2\phi\right)-mgz$$

$$=\frac{1}{2}m[a^2\dot{\phi}^2+\dot{\theta}^2(R+a\cos\phi)^2]-mga\sin\phi$$

(2) $L_z=xp_y-yp_x=m(x\dot{y}-y\dot{x})=m(r\cos\theta\cdot(\dot{r}\sin\theta+\dot{\theta}r\cos\theta)-r\sin\theta(\dot{r}\cos\theta-\dot{\theta}r\sin\theta))$

$$=mr^2\dot{\theta}=m(R+a\cos\phi)^2\cdot\dot{\theta}$$

$\frac{\partial\mathcal{L}}{\partial\theta}=0$ $\quad\frac{\partial\mathcal{L}}{\partial\dot{\theta}}=m\dot{\theta}(R+a\cos\phi)^2\Rightarrow\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right)=\frac{d}{dt}L_z=0$

Therefore, L_z is conserved.

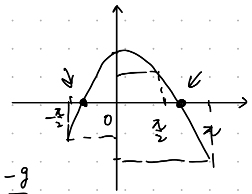
(3) $\frac{\partial\mathcal{L}}{\partial\phi}=-ma\dot{\theta}^2\sin\phi(R+a\cos\phi)-mga\cos\phi$, $\quad\frac{\partial\mathcal{L}}{\partial\dot{\phi}}=ma^2\dot{\phi}$

$$\Rightarrow ma^2\ddot{\phi}=-ma\dot{\theta}^2\sin\phi(R+a\cos\phi)-mga\cos\phi$$

Since $m\frac{d}{dt}(R+a\cos\phi)^2\dot{\theta}=0\Rightarrow m\dot{\theta}(R+a\cos\theta)^2=C_1$

$$\therefore f(\phi)=\frac{1}{a}\cdot\frac{C_1\sin\phi}{m(R+a\cos\phi)}+\frac{g}{a}\cos\phi$$

[問3]



(1) $\frac{d\phi}{dt}=0\Rightarrow f(\phi)=0$

$f(0)=\frac{g}{a}$ $\quad f(-\frac{\pi}{2})=\frac{-C_1}{mRa}$ $\quad f(\frac{\pi}{2})=\frac{C_1}{mRa}$ $\quad f(\pi)=\frac{-g}{a}$

(2) $f(\phi_0)=0$

$$\frac{d^2(\phi_0+\delta)}{dt^2}=\ddot{\delta}=-f(\phi+\delta)=-f(\phi_0)-f'(\phi_0)\delta$$

$$\Rightarrow\ddot{\delta}=-f'(\phi_0)\delta$$

(3) $\delta=A_1\cos[\sqrt{f'(\phi_0)}t]+A_2\sin[\sqrt{f'(\phi_0)}t]$

$\dot{\delta}=-A_1\sqrt{f'(\phi_0)}\sin\sqrt{f'(\phi_0)}t+A_2\sqrt{f'(\phi_0)}\cos\sqrt{f'(\phi_0)}t$

$\dot{\delta}(0)=A_2\sqrt{f'(\phi_0)}=0\Rightarrow A_2=0$

$\delta(t)=A_1\cos\sqrt{f'(\phi_0)}t$

$|\delta(0)|=|A_1|\ll 1$, Therefore δ is osicllating around zero.

第4問

[問1]

$\mathrm{rot\,rot}\,\vec{E}=\mathrm{grad\,div}\,\vec{E}-\Delta\vec{E}=-\Delta\vec{E}$

$\mathrm{rot}\left(-\frac{\partial}{\partial t}\vec{B}\right)=-\varepsilon_0\mu_0\frac{\partial^2}{\partial t^2}\vec{E}$ $\quad\Rightarrow\Delta\vec{E}-\varepsilon_0\mu_0\frac{\partial^2}{\partial t^2}\vec{E}=0$,

take $E_x(x,y,z,t)=E_x(y,z)e^{i(\omega t-kx)}$ into,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_x(y,z)e^{i(\omega t-kx)}-\varepsilon_0\mu_0\frac{\partial^2}{\partial t^2}E_x(y,z)e^{i(\omega t-kx)}=0$$

$$\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_x+(-k^2)E_x+\varepsilon_0\mu_0\omega^2E_x=0 \Rightarrow \frac{\partial^2E_x}{\partial y^2}+\frac{\partial^2E_x}{\partial z^2}+\gamma^2E_x=0$$

[問2]

$E_x|_{y=0,a}=0$ ①

$E_x|_{z=0,b}=0$ ②

[問3]

$-\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]E_x+r^2E_x=0 \Rightarrow \frac{\omega^2}{c^2}-k^2=\left(\frac{m\pi}{b}\right)^2+\left(\frac{n\pi}{L}\right)^2$

when $m=n=1$, $\omega=c\sqrt{k^2+\pi(\frac{1}{a^2}+\frac{1}{b^2})}$

[問4]

$$\nabla\times\vec{E}=\begin{vmatrix}\vec{e}_x & \vec{e}_y & \vec{e}_z\\ \partial_x & \partial_y & \partial_z\\ E_x & E_y & E_z\end{vmatrix}=(\partial_yE_z-\partial_zE_y)\vec{e}_x+(\partial_zE_x-\partial_xE_z)\vec{e}_y+(\partial_xE_y-\partial_yE_x)\vec{e}_z$$

$-\frac{\partial\vec{B}}{\partial t}=-i\omega\vec{B}$, $B_x=0 \Rightarrow \partial_yE_z-\partial_zE_y=0$

$$\begin{cases}-i\omega B_y=\partial_zE_x-\partial_xE_z=\partial_zE_x+ikE_z & \text{——— ①}\\ -i\omega B_z=\partial_xE_y-\partial_yE_x=-ikE_y-\partial_yE_x & \text{——— ②}\end{cases}$$

$$\nabla\times\vec{B}=(\partial_yB_z-\partial_zB_y)\vec{e}_x+(\partial_z\cancel{B_x}-\partial_xB_z)\vec{e}_y+(\partial_xB_y-\cancel{\partial_yB_x})\vec{e}_z$$

$\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}=i\omega\mu_0\varepsilon_0\vec{E}$

$$\begin{cases}i\omega\mu_0\varepsilon_0E_y=-\partial_xB_z=ikB_z & \text{——— ③}\\ i\omega\mu_0\varepsilon_0E_z=\partial_xB_y=-ikB_y & \text{——— ④}\end{cases}$$

From ①②③④, we can have

$-i\omega B_y=\partial_zE_x+ik\frac{-kB_y}{\omega\mu_0\varepsilon_0} \Rightarrow B_y=\frac{\partial_zE_x}{-i\omega+\frac{ik^2}{\omega\mu_0\varepsilon_0}}$

$-i\omega B_z=-ik\frac{kB_z}{\omega\mu_0\varepsilon_0}-\partial_yE_x \Rightarrow B_z=\frac{-\partial_yE_x}{-i\omega+\frac{ik^2}{\omega\mu_0\varepsilon_0}}$

$E_y=\frac{k}{\omega\mu_0\varepsilon_0}\cdot\frac{-\partial_yE_x}{-i\omega+\frac{ik^2}{\omega\mu_0\varepsilon_0}}=\frac{-k\partial_yE_x}{-i\omega^2\mu_0\varepsilon_0+ik^2}$

$E_z=\frac{-k}{\omega\mu_0\varepsilon_0}\cdot\frac{\partial_zE_x}{-i\omega+\frac{ik^2}{\omega\mu_0\varepsilon_0}}=\frac{-k\partial_zE_x}{-i\omega^2\mu_0\varepsilon_0+ik^2}$

数学

第1問

[問1]

$|A-\lambda I|=\begin{vmatrix}-\lambda & m\\ m & M-\lambda\end{vmatrix}=-\lambda(M-\lambda)-m^2=0 \Rightarrow \lambda=\frac{M\pm\sqrt{M^2+4m^2}}{2}$

[問2]

$(A-\lambda I)\vec{x}=0$

$\begin{pmatrix}-\lambda & m\\ m & M-\lambda\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=0$, $\Rightarrow x_+\propto\begin{pmatrix}m\\\lambda_+\end{pmatrix}$, $x_-\propto\begin{pmatrix}m\\\lambda_-\end{pmatrix} \Rightarrow P=\begin{pmatrix}m & m\\\lambda_+ & \lambda_-\end{pmatrix}$

[問3]

$m=M\epsilon$, $\lambda_-=\frac{M-\sqrt{M^2+4M^2\epsilon^2}}{2}=\frac{M}{2}(1-\sqrt{1+4\epsilon^2})$

$\frac{d\lambda_-}{d\epsilon}=-\frac{M}{2}\cdot\frac{1}{2}\cdot8\epsilon(1+4\epsilon^2)^{-\frac{1}{2}}=-2M\epsilon(1+4\epsilon^2)^{-\frac{1}{2}}$

$\frac{d^2\lambda_-}{d\epsilon^2}=-2M(1+4\epsilon^2)^{-\frac{1}{2}}+8M\epsilon^2(1+4\epsilon^2)^{-\frac{3}{2}}$

$\lambda_-=\frac{M}{2}-M\epsilon$

第2問

[問1]

$\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\,e^{-(x^2+y^2)}=\int_0^{\infty}dr\int_0^{2\pi}d\theta\,re^{-r^2}=2\pi\frac{1}{2}\int_0^{\infty}d(r^2)e^{-r^2}=\pi$

[問2] $\int_{-\infty}^{+\infty}dx\,e^{-x^2}=\sqrt{\pi} \Rightarrow \int_{-\infty}^{+\infty}dx\,e^{-\frac{1}{2}x^2}=\frac{2}{\sqrt{2}}\int_{-\infty}^{\infty}d(\frac{\sqrt{2}}{2}x)e^{-(\frac{\sqrt{2}}{2}x)^2}=\sqrt{2\pi}$

[問3]

$\frac{\partial^n}{\partial\alpha^n}\int_{-\infty}^{+\infty}e^{-\alpha x^2}dx=(-1)^n\int_{-\infty}^{+\infty}x^{2n}e^{-\alpha x^2}dx \Rightarrow \int_{-\infty}^{\infty}x^{2n}e^{-\alpha x^2}dx=\sqrt{\pi}\frac{(2n-1)!!}{2^n}\alpha^{-(\frac{1}{2}+n)}$

$\frac{\partial^n}{\partial\alpha^n}\sqrt{\frac{\pi}{\alpha}}=\sqrt{\pi}(-1)^n\frac{(2n-1)!!}{2^n}\alpha^{-(\frac{1}{2}+n)}$

$\therefore \int_{-\infty}^{+\infty}dx\,x^{2n}e^{-\frac{1}{2}x^2}=\sqrt{\pi}\frac{(2n-1)!!}{2^n}\left(\frac{1}{2}\right)^{-(\frac{1}{2}+n)}$

第3問

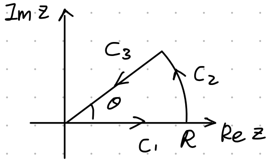
[問1] $z=e^{i\theta}$, $z^6+1=e^{6i\theta}+1=0$, $\theta=\frac{\pi+2k\pi}{6}$, $z=e^{i\frac{\pi+2k\pi}{6}}$

$k=0,1,2$

[問2] $\theta=\frac{2\pi}{n}$, $|z|=R$, $z=xe^{i\theta}$, consider $f(z)=\frac{1}{z^n+1}$

$\oint_{C_1+C_2+C_3}f(z)dz=2\pi i\,\mathrm{Res}[f(z),z=e^{i\pi/n}]$

Along C_2, as $R\to\infty$, $\int_{C_2} f(z)dz \to 0$



Along C_1, $\int_0^R \frac{1}{x^n+1}dx$

Along C_3, $\int_R^0 \frac{1}{(xe^{i\frac{2\pi}{n}})^n+1} e^{\frac{2\pi i}{n}} dx = -e^{\frac{2\pi i}{n}}\int_0^R \frac{1}{x^n+1}dx$

when $R\to\infty$, therefore $\left(1-e^{\frac{2\pi i}{n}}\right)\int_0^\infty \frac{1}{x^n+1}dx = 2\pi i\, \mathrm{Res}[f(z), z=e^{\frac{i\pi}{n}}]$

let $I(n)=\int_0^\infty \frac{1}{x^n+1}dx \Rightarrow I(n) = 2\pi i \frac{1}{1-e^{2\pi i/n}} \lim_{z\to e^{\frac{i\pi}{n}}} \frac{z-e^{\frac{i\pi}{n}}}{z^n+1} = \frac{2\pi i}{-n}\frac{1}{e^{-i\pi/n}-e^{i\pi/n}} = \frac{\pi}{n\sin(\frac{\pi}{n})}$

when $n=6$, $\int_{-\infty}^{+\infty}\frac{1}{x^6+1}dx = \frac{2\pi}{6\sin(\frac{\pi}{6})} = \frac{2\pi}{3}$

第4問

[問1] $y'(x) = e^{-F(x)}g(x)e^{F(x)} + \left(-\frac{dF(x)}{dx}\right)e^{-F(x)}\int dx\, g(x)e^{F(x)}$

$= g(x) - f(x)y$

[問2]

$f(x)=-x \Rightarrow F(x) = -\frac12x^2+C_1$

$g(x)=x$

$\therefore y = e^{\frac12x^2-C_1}\int dx\, xe^{-\frac12x^2+C_1} = e^{\frac12x^2}\int dx\, xe^{-\frac12x^2} = e^{\frac12 x^2}\cdot -\int d\left(-\frac12x^2\right)e^{-\frac12x^2}$

$= e^{\frac12x^2}\left(-e^{-\frac12x^2}+C_1\right)$

$y(0) = (-1+C_1) = 0 \Rightarrow C_1=1 \quad \therefore y = e^{\frac12x^2}-1$

SOKENDAI KEK 入试 2018 解答

物理

第1問

[問1]

(1) $q\vec v\times\vec B = q\begin{pmatrix}\vec e_x & \vec e_y & \vec e_z\\ v_x & v_y & v_z\\ 0&0&B\end{pmatrix} = qv_yB\vec e_x - qv_xB\vec e_y$

$\begin{cases} m\frac{dv_x}{dt} = qv_yB \\ m\frac{dv_y}{dt} = -qv_xB\end{cases} \Rightarrow \frac{d^2v_x}{dt^2} = \omega\frac{dv_y}{dt} = -\omega^2 v_x \Rightarrow \frac{dx}{dt} = v_x = A\cos(\omega t+\alpha)$

$\frac{dy}{dt} = v_y = \frac1\omega\frac{dv_x}{dt} = -A\sin(\omega t+\alpha)$

$\therefore \begin{cases} x = R\sin(\omega t+\alpha)+x_c \\ y = R\cos(\omega t+\alpha)+y_c\end{cases}$

(2) $\begin{cases} v_x = \omega R\cos(\omega t+\alpha)\\ v_y = -\omega R\sin(\omega t+\alpha)\end{cases}$

$\begin{cases} x_c = x - R\sin(\omega t+\alpha) = x+\frac1\omega v_y \\ y_c = y-\frac1\omega v_x\end{cases}$

$\frac{dx_c}{dt} = \frac{dx}{dt}+\frac1\omega\frac{dv_y}{dt}$

$= v_x+\frac1\omega\cdot(-\omega v_x)$

$=0$

$\frac{dx_y}{dt}=0$

☆ $[\hat A, f(\hat B)] = [\hat A,\hat B]\frac{\partial f}{\partial B}$

[問2]

(1) $[\hat\pi_x,\hat\pi_y] = [p_x-qA_x, p_y-qA_y] = [p_x-qA_x, p_y] - [p_x-qA_x, qA_y]$

$= -q[A_x,p_y] - q[p_x,A_y]$

$= q[p_y,y]\frac{\partial A_x}{\partial y} - q[p_x,x]\frac{\partial A_y}{\partial x}$

$= -i\hbar q\frac{\partial A_x}{\partial y} + i\hbar q\frac{\partial A_y}{\partial x}$

$= i\hbar qB$

(2) $\hat H_\perp = \frac{1}{2m}\hat\pi_\perp^2 = \frac1{2m}\left(\hat\pi_x^2+\hat\pi_y^2\right) = \frac{qB}{2m}\left(\hat X^2+\hat P^2\right)$

(3) $\hat a^\dagger\hat a = \frac{1}{2\hbar}\left(\hat X^2 + i\hat X\hat P - i\hat P\hat X + \hat P^2\right) = \frac1{2\hbar}\left(\hat X^2+\hat P^2\right) - \frac12$

$\hat H_\perp = \hbar\omega\left(\hat a^\dagger\hat a+\frac12\right)$, Since $\hat a^\dagger\hat a\psi_n = n\psi_n$, $\hat H\psi_n = E_n\psi_n$,

Therefore, $E_n = \hbar\omega\left(n+\frac12\right)$

(4) $0 = \langle \hat{x} | \hat{a} | 0 \rangle = \langle \hat{x} | \frac{1}{\sqrt{2\hbar}} (\hat{x} + i\hat{p}) | 0 \rangle = \frac{1}{\sqrt{2\hbar}} [\langle \hat{x} | \hat{x} | 0 \rangle + i \langle \hat{x} | \hat{p} | 0 \rangle]$

$= \frac{1}{\sqrt{2\hbar}} [X\varphi_0(X) + i(-i\hbar \frac{d}{dx}) \varphi_0(x)]$

$\Rightarrow \quad X\varphi_0(x) = -\hbar \frac{d}{dx} \varphi_0(x) \quad \Rightarrow \quad \frac{1}{\varphi_0} d\varphi_0 = -\frac{1}{\hbar} X dx \quad \Rightarrow \quad \varphi_0(x) = N \exp\{-\frac{X^2}{2\hbar}\}$

(5) $[\hat{x}_c, \hat{H}_\perp] = [\hat{x} + \frac{1}{qB}\hat{\pi}_y, \frac{1}{2m}(\hat{\pi}_x^2 + \hat{\pi}_y^2)] = \frac{1}{2m}\{[\hat{x}, \hat{\pi}_x^2 + \hat{\pi}_y^2] + \frac{1}{qB}[\hat{\pi}_y, \hat{\pi}_x^2 + \hat{\pi}_y^2]\}$

$[\hat{x}, \hat{\pi}_x^2] = 2[\hat{x}, \hat{\pi}_x]\hat{\pi}_x = 2[\hat{x}, \hat{p}_x - qAx]\hat{\pi}_x = 2i\hbar\hat{\pi}_x$

$[\hat{x}, \hat{\pi}_y^2] = 2[\hat{x}, \hat{p}_y - q\hat{A}_y]\hat{\pi}_y = 0$

$[\hat{\pi}_y, \hat{\pi}_x^2] = 2[\hat{\pi}_y, \hat{\pi}_x]\hat{\pi}_x = -2i\hbar qB\hat{\pi}_x$

Then, $[\hat{x}_c, \hat{H}_\perp] = \frac{1}{2m}\{2i\hbar\hat{\pi}_x + \frac{1}{qB}(-2i\hbar qB)\hat{\pi}_x\} = 0$

Same as $[\hat{y}_c, \hat{H}_\perp] = 0$

(6) $\hat{R}^2\psi = (\hat{x} - \hat{x}_c)^2\psi + (\hat{y} - \hat{y}_c)^2\psi = (\frac{1}{qB})^2(\hat{\pi}_y^2 + \hat{\pi}_x^2)\psi = 2m(\frac{1}{qB})^2\hat{H}_\perp\psi$

$\Rightarrow \quad \lambda_n = 2m(\frac{1}{qB})^2 \hbar \cdot \frac{qB}{m}(n + \frac{1}{2}) = \frac{2\hbar}{qB}(n + \frac{1}{2})$

第2問

[問1]

(1) Introduce $\Xi = \prod_l (1 - e^{-E/kT})^{-\omega_l}$

$\varepsilon = pc, \; D(\varepsilon)d\varepsilon = \int_{d\omega} \frac{d\omega}{h^3} = \frac{4\pi V}{h^3c^3}\varepsilon^2 d\varepsilon$, let $x = \frac{\varepsilon}{kT}$

$\ln\Xi_1 = -2\int_{-\infty}^{+\infty} \frac{d\omega}{h^3} \ln(1 - e^{-E/kT}) = -\frac{8\pi V}{h^3c^3}(kT)^3 \int_0^\infty x^2 \ln(1 - e^{-x}) dx$

$= \frac{-8\pi V}{h^3c^3}(kT)^3 \{\frac{1}{3}x^3 \ln(1 - e^{-x})\Big|_0^\infty - \int_0^\infty \frac{1}{3}x^3 \frac{e^{-x}}{1 - e^{-x}} dx\}$

$= \frac{8\pi V}{h^3c^3}(kT)^3 \cdot \frac{1}{3} \cdot \frac{\pi^4}{15} = \frac{8\pi^5 V}{45}(\frac{kT}{hc})^3$

$F_1 = -kT\ln\Xi_1 = -\frac{1}{3}aVT^4$. let $a \equiv \frac{8\pi^5 k^4}{45(hc)^3}$

$u_1 = -\frac{1}{V}T^2\frac{\partial}{\partial T}(\frac{F_1}{T}) = aT^4$, $P_1 = -(\frac{\partial F}{\partial V})_T = \frac{1}{3}aT^4$

(2) $\ln\Xi_2 = \frac{8\pi V}{h^3c^3}(kT)^3 \cdot \frac{7}{8}\frac{\pi^4}{15} = \frac{7\pi^5 V}{45}(\frac{kT}{hc})^3$

$F_2 = \frac{7}{8}F_1 \quad \therefore u_2 = \frac{7}{8}u_1 = \frac{7}{8}aT^4$, $P_2 = \frac{7}{8}P_1 = \frac{7}{24}aT^4$

[問2] $S_r = -(\frac{\partial F}{\partial T})_V \cdot \frac{1}{V} = CT^3 \quad S_e = \frac{7}{8}S_r = \frac{7}{8}CT^3$

[問3-6] ×

第3問

[問1]

(1) $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

(2) $\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z) \qquad p_x = \frac{\partial\mathcal{L}}{\partial\dot{x}} = m\dot{x}$

$H(x, y, z, p_x, p_y, p_z) = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + U(x, y, z)$

(3) $\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \quad \begin{cases} p_r = p_x\sin\theta\cos\phi + p_y\sin\theta\sin\phi + p_z\cos\theta \\ p_\theta = p_x r\cos\theta\cos\phi + p_y r\cos\theta\sin\phi - p_z r\sin\theta \\ p_\phi = -p_x r\sin\theta\sin\phi + p_y r\sin\theta\cos\phi \end{cases}$

$\mathcal{H} = \frac{1}{2m}(p_r^2 + \frac{1}{r^2}p_\theta^2 + \frac{1}{r^2\sin^2\theta}p_\phi^2) + U(r)$ From Cartesian coordinate, To spherical coordinate.

(4) $\dot{r} = \frac{\partial\mathcal{H}}{\partial p_r} = \frac{p_r}{m} \quad \Rightarrow p_r = m\dot{r}$

$\dot{\theta} = \frac{p_\theta}{mr^2} \quad \Rightarrow p_\theta = mr^2\dot{\theta}$

$\dot{\phi} = \frac{p_\phi}{mr^2\sin^2\theta} \quad \Rightarrow p_\phi = mr^2\sin^2\theta\,\dot{\phi}$

$\frac{dp_\phi}{dt} = -\frac{\partial\mathcal{H}}{\partial\phi} = 0 \quad \Rightarrow p_\phi$ is conserved. $\quad \frac{dp_\theta}{dt} = -\frac{1}{2m}p_\phi^2\frac{1}{r^2}(-2)\cos\theta(\sin\theta)^{-3}$

$= \frac{p_\phi^2}{mr^2}\cos\theta(\sin\theta)^{-3}$

$\frac{dl^2}{dt} = 2p_\theta\frac{dp_\theta}{dt} + \frac{d}{dt}(\frac{p_\phi^2}{\sin^2\theta})$

$= 2p_\phi \cdot \frac{dp_\phi}{dt}\frac{1}{\sin^2\theta} + p_\phi^2(-2)(\sin\theta)^{-3}(\cos\theta)\dot{\theta} \quad = 0$

[問2]

(1) $\frac{dp_r}{dt} = m\ddot{r} \quad , p_\theta = mr^2\dot{\theta} = 0 \quad , p_\phi = h \quad , \sin\theta = 1$

$-\frac{\partial\mathcal{H}}{\partial r} = \frac{-1}{2m}(p_\theta^2(-2)r^{-3} + \frac{p_\phi^2}{\sin^2\theta}(-2)r^{-3}) - \frac{dU}{dr} = \frac{h^2}{mr^3} - \frac{\mu}{r^2}$

$\Rightarrow \quad m\ddot{r} = \frac{h^2}{mr^3} - \frac{\mu}{r^2}$

(2) $|r| = a \quad , m\ddot{r} = \frac{h^2}{ma^3} - \frac{\mu}{a^2} \quad \Rightarrow r = \frac{1}{2}(\frac{h^2}{m^2a^3} - \frac{\mu}{ma^2})t^2 + A_1t + A_2$

when $x = 2\pi a = \frac{1}{2}(\frac{h^2}{m^2a^3} - \frac{\mu}{ma^2})T^2$

$\Rightarrow T = \sqrt{\frac{1}{4\pi}(\frac{h^2}{m^2a^4} - \frac{\mu}{ma^3})}$

?

(3) $-\frac{\partial u}{\partial r} = \frac{h^2}{mr^3} - kr \Rightarrow m\ddot{r} = \frac{h^2}{mr^3} - kr$

$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}$

第4問

[問1]

(1) $2\pi r\cdot l\cdot E(r) = \lambda l\cdot 2\pi a/\varepsilon_0 \Rightarrow E(r) = \frac{a\lambda}{\varepsilon_0 r}$

$\int_a^b E(r)dr = \frac{a\lambda}{\varepsilon_0}\int_a^b \frac{1}{r}dr = \frac{a\lambda}{\varepsilon_0}\ln\frac{b}{a} = V \Rightarrow \lambda = \frac{\varepsilon_0 V}{a\ln\frac{b}{a}}$

$\therefore E(r) = \frac{V}{r\ln\frac{b}{a}}$

(2) $C_0 = \frac{Q_0}{V} = \frac{2\pi a\lambda}{V}$ $\therefore U = \frac{1}{2}\frac{Q_0}{V}V^2 = \frac{\pi\varepsilon_0 V^2}{\ln\frac{b}{a}}$ (unit length energy)

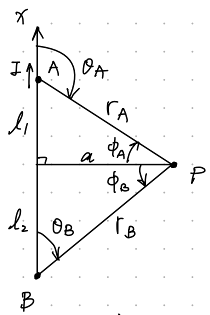
(3) $\Delta U = F\Delta z$, when $\Delta z = 1$.

$F = \frac{\pi\varepsilon V^2}{\ln\frac{b}{a}} - \frac{\pi\varepsilon_0 V^2}{\ln\frac{b}{a}} = \frac{\pi V^2}{\ln\frac{b}{a}}(\varepsilon - \varepsilon_0)$

[問2]

(1) $\oint_C \vec{H}\cdot d\vec{s} = H\cdot 2\pi r$

$\iint_S \vec{J}\cdot d\vec{S} = I$ $\Rightarrow H = \frac{I}{2\pi r}$



$\sin\theta = \cos\phi = \frac{a}{r}$

(2) $H = \int dH = \int \frac{I}{4\pi}\frac{dx\sin\theta}{r^2} = \frac{I}{4\pi}\int_{\phi_B}^{\phi_A}\frac{\cos\phi\cdot a\,d\phi}{\cos^2\phi\cdot\left(\frac{a}{\cos\phi}\right)^2}$

$= \frac{I}{4\pi a}\int_{\phi_B}^{\phi_A}\cos\phi\,d\phi$

$= \frac{I}{4\pi a}(\sin\phi_A - \sin\phi_B) = \frac{I}{4\pi a}\left(\frac{a}{\sqrt{a^2+l_1^2}} - \frac{a}{\sqrt{a^2+l_2^2}}\right) = \frac{I}{4a}\left(\frac{1}{\sqrt{a^2+l_1^2}} - \frac{1}{\sqrt{a^2+l_2^2}}\right)$

数学

第1問

[問1]

$\frac{dE}{dt} = \frac{dx(t)}{dt}\cdot\frac{d^2x(t)}{dt^2} + \frac{1}{x(t)^2}\frac{dx(t)}{dt} = \frac{dx}{dt}\left(\frac{d^2x(t)}{dt^2} + \frac{1}{x(t)^2}\right) \equiv 0$

[問2] $\frac{1}{2}\left(\frac{dx}{dt}\right)^2 - \frac{1}{x} = C$, when $t=0$, $\frac{1}{2}2 - 1 = 0 = C$

$\therefore \frac{dx}{dt} = \pm\sqrt{\frac{2}{x}} \Rightarrow \int\sqrt{x}\,dx = \pm\sqrt{2}\int dt \Rightarrow x = \pm\left(\frac{3\sqrt{2}}{2}t + C_1\right)^{\frac{2}{3}}$

$x \overset{x(0)=1}{=} \pm\left(\frac{3\sqrt{2}}{2}t+1\right)^{\frac{2}{3}} \overset{\frac{dx}{dt}|_0=\sqrt{2}}{=} \left(\frac{3\sqrt{2}}{2}t+1\right)^{\frac{2}{3}}$

[問3]

$\frac{dx}{dt} = \frac{3\sqrt{2}}{2}\cdot\frac{2}{3}\left(\frac{3\sqrt{2}}{2}t+1\right)^{-\frac{1}{3}} = \sqrt{2}\left(\frac{3\sqrt{2}}{2}t+1\right)^{-\frac{1}{3}}$

$\frac{d^2x}{dt^2} = \sqrt{2}\cdot\frac{3\sqrt{2}}{2}\left(-\frac{1}{3}\right)\left(\frac{3\sqrt{2}}{2}t+1\right)^{-\frac{4}{3}} = -\left(\frac{3\sqrt{2}}{2}t+1\right)^{-\frac{4}{3}}$

$\frac{d^3x}{dt^3} = (-1)\frac{3\sqrt{2}}{2}\left(-\frac{4}{3}\right)\left(\frac{3\sqrt{2}}{2}+1\right)^{-\frac{7}{3}} = 2\sqrt{2}\left(\frac{3\sqrt{2}}{2}t+1\right)^{-\frac{7}{3}}$

$\therefore x = 1 + \sqrt{2}t + \frac{-1}{2}t^2 + \frac{2\sqrt{2}}{6}t^3 + O(t^3)$

$= 1 + \sqrt{2}t - \frac{1}{2}t^2 + \frac{\sqrt{2}}{3}t^3 + O(t^3)$

第2問

[問1] (a) $\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x = -\frac{1}{y}$ (b) $\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x = 0$

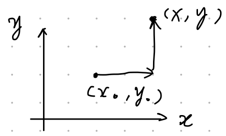
[問2]

$\oint_C \vec{A}(x',y')d\vec{r}' = \iint\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)dxdy$

(a) $\oint_C \vec{A}(x',y')d\vec{r}' \neq 0$, depend on trace

(b) $\oint_C \vec{A}(x',y')d\vec{r}' = 0$, independ on trace

[問3]



$S(x,y) = \int_{(x_0,y_0)}^{(x,y_0)}\frac{1}{x}dx + \int_{(x,y_0)}^{(x,y)} -\frac{1}{y}dy$

$= \ln\frac{x}{x_0} + \left(-\ln\frac{y}{y_0}\right) = \ln\frac{y_0}{x_0}\frac{x}{y}$

第3問

[問1]

$\int_{-\infty}^{\infty}dx\,e^{ikx}\frac{\partial}{\partial x}F(x,t) = F(x,t)e^{ikx}\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}ike^{ikx}F(x,t) = -ik\hat{F}(k)$

$\therefore \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}\right)G(x,t) \to (-k^2 - i\omega)\hat{G}(k,\omega)$ $\therefore \hat{G}(k,\omega) = \frac{1}{k^2+i\omega}$

$-\delta(x)\delta(t) \to -1$

[問2] $G(k,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}d\omega\,e^{it\omega}\frac{1}{k^2+i\omega} = \frac{1}{2\pi}\cdot 2\pi i\cdot\mathrm{Res}\left[\frac{e^{it\omega}}{k^2+i\omega}, \omega = ik^2\right]$

$= i\cdot\lim_{\omega\to ik^2}(\omega - ik^2)\frac{e^{it\omega}}{k^2+i\omega} = e^{-k^2t}$

[問3] $G(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{-ixk} \cdot e^{-k^2 t} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{-t(k+\frac{ix}{2t})^2 + t(\frac{ix}{2t})^2}$

$$= \frac{1}{2\pi} e^{-x^2/2t}\sqrt{\frac{\pi}{t}}$$

第4問

[問1]

$$\alpha_i\alpha_j + \alpha_j\alpha_i = \begin{pmatrix}0&\sigma_i\\ \sigma_i&0\end{pmatrix}\begin{pmatrix}0&\sigma_j\\ \sigma_j&0\end{pmatrix} + \begin{pmatrix}0&\sigma_j\\ \sigma_j&0\end{pmatrix}\begin{pmatrix}0&\sigma_i\\ \sigma_i&0\end{pmatrix} = \begin{pmatrix}\sigma_i\sigma_j&0\\0&\sigma_i\sigma_j\end{pmatrix} + \begin{pmatrix}\sigma_j\sigma_i&0\\0&\sigma_j\sigma_i\end{pmatrix} = \begin{pmatrix}\sigma_i\sigma_j+\sigma_j\sigma_i&0\\0&\sigma_i\sigma_j+\sigma_j\sigma_i\end{pmatrix}$$

$$= \begin{pmatrix}2\delta_{ij}I&0\\0&2\delta_{ij}I\end{pmatrix} = 2\delta_{ij}\mathbb{1}_{4\times4}$$

$$\alpha_i\beta + \beta\alpha_i = \begin{pmatrix}0&\sigma_i\\ \sigma_i&0\end{pmatrix}\begin{pmatrix}I&0\\0&-I\end{pmatrix} + \begin{pmatrix}I&0\\0&-I\end{pmatrix}\begin{pmatrix}0&\sigma_i\\ \sigma_i&0\end{pmatrix} = \begin{pmatrix}0&-\sigma_i I\\ \sigma_i I&0\end{pmatrix} + \begin{pmatrix}0&\sigma_i I\\-\sigma_i I&0\end{pmatrix} = 0_{4\times4}$$

$$\beta^2 = \begin{pmatrix}I&0\\0&-I\end{pmatrix}\begin{pmatrix}I&0\\0&-I\end{pmatrix} = \begin{pmatrix}I&0\\0&I\end{pmatrix} = \mathbb{1}_{4\times4}$$

[問2]

$$A^2 = \sum_i A_iA_i = \sum_i(\alpha_i n_i\sin\theta + \beta\cos\theta)(\alpha_i n_i \sin\theta + \beta\cos\theta)$$

$i = x, y, z$

$\vec n = (n_x, n_y, n_z)$

$$= \sum_i\left\{[\alpha_i^2\sin^2\theta + \sin\theta\cos\theta(\alpha_i\beta+\beta\alpha_i)] + \beta^2\cos^2\theta\right\}n_i^2$$

$$= \sum_i \mathbb{1}\, n_i^2$$

$$= \mathbb{1}$$

[問3]

$$|A - \lambda\mathbb{1}_{4\times4}| = \left|\sin\theta\begin{pmatrix}0&\sigma\\ \sigma&0\end{pmatrix}n + \cos\theta\begin{pmatrix}I&0\\0&-I\end{pmatrix} - \begin{pmatrix}\lambda I&0\\0&\lambda I\end{pmatrix}\right|$$

$$= \begin{vmatrix}(\cos\theta-\lambda)I & \sin\theta\,\sigma n\\ \sin\theta\,\sigma n & (-\cos\theta-\lambda)I\end{vmatrix} = \ ?$$

SOKENDAI KEK 入试 2017 解答

物理

第1問

第2問

[問1]

[問2] $F = \sum_\nu P_\nu E_\nu + T\sum_\nu P_\nu \ln P_\nu + \lambda\left(\sum_\nu P_\nu - 1\right) = \sum_\nu(E_\nu + T\ln P_\nu + \lambda)P_\nu - \lambda$

$$\delta F = \sum_\nu (E_\nu + T\ln P_\nu + \lambda)\delta P_\nu = 0$$

$$\therefore E_\nu + T\ln P_\nu + \lambda = 0 \Rightarrow P_\nu = e^{-\frac{\lambda}{T}} e^{-E_\nu/T}$$

Since $\sum_\mu P_\mu = e^{-\frac{\lambda}{T}}\sum_\mu e^{-\frac{E_\mu}{T}} = 1$ $\quad\therefore e^{-\frac{\lambda}{T}} = \frac{1}{\sum_\mu e^{-E_\mu/T}}$

$$\therefore P_\nu = \frac{e^{-E_\nu/T}}{\sum_\mu e^{-E_\mu/T}}$$

[問3]

$$Z = \sum_{\{\sigma_i\}} e^{-\frac{H}{kT}} = \sum_{\{\sigma_i\}} e^{J\sum_{\langle i,j\rangle}\sigma_i\sigma_j/kT}, \quad Z_0^d = e^{dJN/kT}$$

$$F_0^d = -kT\ln Z_0^d = -kT(dJN/kT) = -dJN$$

[問4][問5]

第3問

[問1] (1) $\frac{\partial U}{\partial x} = U_0\{-k\sin(kx+\theta_0)\} = 0$, when $x=0$, $\Rightarrow \theta_0 = 0, \pi$

(2) $\mathcal{L} = \frac{m\dot x^2}{2} - U_0\{\cos(kx+\theta_0) - \cos\theta_0\}$

$\frac{\partial\mathcal{L}}{\partial x} = kU_0\sin(kx+\theta_0)$ $\quad \frac{\partial\mathcal{L}}{\partial \dot x} = m\dot x$ $\quad\Rightarrow m\ddot x = kU_0\sin(kx+\theta_0)$

(3) $\theta_0 = 0$

$m\ddot x \simeq kU_0(kx) = k^2U_0 x$?

[問2]

第4問

[問1] $I - C'\Delta x\frac{d(V+\Delta V)}{dt} - (I+\Delta I) = 0$, $\quad V - \left(-L'\Delta x\frac{dI}{dt}\right) - (V+\Delta V) = 0$

[問2]

$$\frac{dI}{dx} = \lim_{\Delta x \to 0} \frac{(I+\Delta I)-I}{\Delta x} = -C' \frac{d(V+\Delta V)}{dt} = -C' \frac{dV}{dt}$$

$$\frac{dV}{dx} = \lim_{\Delta x \to 0} \frac{(V+\Delta V)-V}{\Delta x} = -L' \frac{dI}{dt}$$

[問3]

$$\frac{d^2 I}{dx^2} = -C' \frac{d}{dt}\frac{dV}{dx} = L'C' \frac{d^2 I}{dt^2} \quad \Rightarrow \quad c = \frac{1}{\sqrt{L'C'}}$$

[問4] $\int_S \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_0} \quad \Rightarrow 2\pi r|\vec{E}| = \frac{Q}{\varepsilon_0} \quad \Rightarrow |\vec{E}| = \frac{Q}{2\pi r \varepsilon_0}$

[問5] $V_{ab} = \int_a^b |\vec{E}|\, dr = \frac{Q}{2\pi\varepsilon_0}\int_a^b \frac{1}{r} dr = \frac{Q}{2\pi\varepsilon_0} \ln\frac{b}{a}$

$$C' = \frac{Q}{V_{ab}} = \frac{2\pi\varepsilon_0}{\ln\frac{b}{a}}$$

[問6] $\frac{1}{\mu_0}\oint_C \vec{B}\cdot d\vec{s} = I \quad \Rightarrow \quad \frac{1}{\mu_0} 2\pi r |\vec{B}| = I \Rightarrow \quad |\vec{B}| = \frac{\mu_0 I}{2\pi r}$

[問7] $U_B = \frac{1}{2\mu_0}\int_V |\vec{B}|^2 dV = \frac{1}{2\mu_0}\left(\frac{\mu_0 I}{2\pi}\right)^2 \iiint \frac{1}{r^2} r \sin\theta\, dr\, d\theta\, dz$

$$= \frac{\mu_0 I^2}{4\pi} \ln\frac{b}{a} \qquad \Rightarrow L' = \frac{2U_B}{I^2} = \frac{\mu_0}{2\pi} \ln\frac{b}{a}$$

[問8] $\frac{1}{\sqrt{L'C'}} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

数学

第1問

[問1] $T_1^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ T_1^3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad T_1^n = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & n = 2, 4, 6 \cdots \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & n = 1, 3, 5 \cdots \end{cases}$

[問2]

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} a_{12} & a_{11} & 0 \\ a_{22} & a_{21} & 0 \\ a_{32} & a_{31} & 0 \end{pmatrix} = \begin{pmatrix} a_{21}-a_{12} & a_{22}-a_{11} & a_{23} \\ a_{11}-a_{22} & a_{12}-a_{21} & a_{13} \\ -a_{32} & -a_{31} & 0 \end{pmatrix} = O$$

$\{T_8\}$ is changable.

[問3]

$$e^{iAx} = I + A(ix) + \frac{A^2}{2}(ix)^2$$

第2問

[問1] $\lambda^2 - 4 = 0 \quad \lambda = \pm 2 \quad \Rightarrow \quad f_1(x) = e^{2x}, \ f_2 = e^{-2x}$

[問2]

$$\frac{dC_1}{dx}e^{2x} + \frac{dC_2}{dx}e^{-2x} = 0$$

$$f(x) = C_1(x)e^{2x} + C_2(x)e^{-2x}$$

$$\frac{df(x)}{dx} = e^{2x}\frac{dC_1}{dx} + 2C_1 e^{2x} + e^{-2x}\frac{dC_2}{dx} - 2C_2 e^{-2x} = 2C_1 e^{2x} - 2C_2 e^{-2x}$$

$$\frac{d^2 f(x)}{dx^2} = 2\left[e^{2x}\frac{dC_1}{dx} + 2C_1 e^{2x} - e^{-2x}\frac{dC_2}{dx} + 2C_2 e^{-2x}\right]$$

$$\Rightarrow 2\left[e^{2x}\left(\frac{dC_1}{dx} + \cancel{2C_1}\right) + e^{-2x}\left(-\frac{dC_2}{dx} + \cancel{2C_2}\right)\right] - \cancel{4C_1 e^{2x}} - \cancel{4C_2 e^{-2x}}$$

$$= 2\left(e^{2x}\frac{dC_1}{dx} - e^{-2x}\frac{dC_2}{dx}\right) = \frac{8}{e^{2x}+e^{-2x}} \quad \Rightarrow \quad \left(e^{2x}\frac{dC_1}{dx} - e^{-2x}\frac{dC_2}{dx}\right)(e^{2x}+e^{-2x}) = 4$$

① $2e^{2x}\frac{dC_1}{dx}(e^{2x}+e^{-2x}) = 4 \quad \Rightarrow \quad \frac{dC_1}{dx}(e^{4x}+1) = 2$

② $-2e^{-2x}\frac{dC_2}{dx}(e^{2x}+e^{-2x}) = 4 \quad \Rightarrow \quad \frac{dC_2}{dx}(1+e^{-4x}) = -2$

[問3]

$$C_1 = \int \frac{2}{e^{4x}+1} dx$$

第3問

[問1]

$$F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{-ikx} e^{-|x|} = \frac{1}{2\pi}\left[\int_0^{+\infty} dx\, e^{-ikx} e^{-x} + \int_{-\infty}^0 dx\, e^{-ikx} e^{x}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{-1-ik} e^{(-1-ik)x}\Big|_0^{\infty} + \frac{1}{1-ik} e^{(1-ik)x}\Big|_{-\infty}^0\right]$$

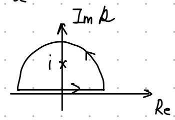
$$= \frac{1}{2\pi}\left[\frac{-1}{-1-ik} + \frac{1}{1-ik}\right]$$

$$= \frac{1}{\pi(1+k^2)}$$

[問2]

$$f(x) = \int_{-\infty}^{\infty} \frac{1}{\pi(1+k^2)} e^{ikx} dk = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{1}{1+ik} + \frac{1}{1-ik}\right) e^{ikx} dk$$

① $x > 0$



$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{1+ik} dk = 2\pi i \,\mathrm{Res}\left[\frac{e^{ikx}}{1+ik}, \ k = i\right]$$

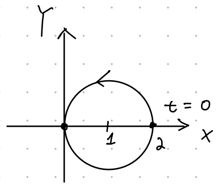
$$= 2\pi i \lim_{k \to i} \frac{e^{ikx}}{1+ik}(k-i)$$

$$= 2\pi e^{-x}$$

② $x<0$: $\displaystyle\int_{-\infty}^{+\infty}\frac{e^{ikx}}{1-ik}dk = 2\pi i\,\mathrm{Res}\left[\frac{e^{ikx}}{1-ik}, k=-i\right] = 2\pi e^{x}$

$\displaystyle \therefore f(x) = \frac{1}{2\pi}\cdot 2\pi\, e^{-|x|} = e^{-|x|}$

第4問.



[問1] $(x-1)^2+\sin^2 t = 1 \Rightarrow x = \pm\cos t + 1$

$x = \cos t + 1$

$z(t) = 0$

[問2]

$$\oint_c d\vec{r}\cdot\vec{A} = \oint_c y\,dx = \int_2^0\sqrt{1-(x-1)^2}\,dx - \int_0^2\sqrt{1-(x-1)^2}\,dx = 2\int_2^0\sqrt{1-(x-1)^2}\,dx$$

$$= 2\int_1^{-1}\sqrt{1-x^2}\,dx \qquad \text{let } x=\sin\theta \quad dx=\cos\theta\,d\theta$$

$$= 2\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}}(\cos\theta)^2\,d\theta = 2\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}}\frac{\cos 2\theta+1}{2}\,d\theta = -2\pi$$

[問3]

$$\nabla\times\vec{A} = \begin{pmatrix}\vec{e}_x & \vec{e}_y & \vec{e}_z\\ \partial x & \partial y & \partial z\\ y & z & x\end{pmatrix} = -\vec{e}_x - \vec{e}_y - \vec{e}_z$$

$$\frac{\partial y}{\partial\theta}\frac{\partial z}{\partial\varphi} - \frac{\partial z}{\partial\theta}\frac{\partial y}{\partial\varphi} = \sin\theta\sin\theta\cos\varphi = \sin^2\theta\cos\varphi$$

$$\frac{\partial z}{\partial\theta}\frac{\partial x}{\partial\varphi} - \frac{\partial x}{\partial\theta}\frac{\partial z}{\partial\varphi} = -\sin\theta\sin\theta(-\sin\varphi) = \sin^2\theta\sin\varphi$$

$$\frac{\partial x}{\partial\theta}\frac{\partial y}{\partial\varphi} - \frac{\partial y}{\partial\theta}\frac{\partial x}{\partial\varphi} = \cos\theta\cos\varphi\sin\theta\cos\varphi - \cos\theta\sin\varphi\sin\theta(-\sin\varphi) = \cos\theta\sin\theta$$

[問4]

$$\int_S dS\cdot\mathrm{rot}\,\vec{A} = -\int[\sin^2\theta(\cos\varphi+\sin\varphi) + \cos\theta\sin\theta]\,d\theta\,d\varphi$$

$$\int_0^{\frac{\pi}{2}}\sin^2\theta\,d\theta\int_0^{2\pi}(\cos\varphi+\sin\varphi)\,d\varphi = 0$$

$$\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta\,d\theta\int_0^{2\pi}d\varphi = 2\pi$$

$$\Rightarrow \int_S dS\cdot\mathrm{rot}\,\vec{A} = -2\pi$$

SOKENDAI KEK 入试 2016 解答

物理

第1問

[問1]

(1)

$$H|\psi_n\rangle = (H_0+\lambda H')(|n\rangle + \lambda|\psi_n^{(1)}\rangle + \lambda^2|\psi_n^{(2)}\rangle + \cdots)$$

$$= H_0|n\rangle + H_0\lambda|\psi_1^{(1)}\rangle + H'\lambda|n\rangle + \cdots$$

$$= H_0|n\rangle + \lambda(H_0|\psi_1^{(1)}\rangle + H'|n\rangle) + \cdots$$

$$E_n|\psi_n\rangle = (\varepsilon_n + \lambda E_n^{(1)} + \cdots)(|n\rangle + \lambda|\psi_n^{(1)}\rangle + \cdots)$$

$$= \varepsilon_n|n\rangle + \lambda(E_n^{(1)}|n\rangle + \varepsilon_n|\psi_n^{(1)}\rangle + \cdots)$$

multiply each with $\langle k|$

$$\cancel{\langle k|H_0|n\rangle} + \lambda(\langle k|H'|n\rangle + \langle k|H_0|\psi_1^{(1)}\rangle) + \cdots = \cancel{\langle k|\varepsilon_n|n\rangle} + \lambda(\langle k|E_n^{(1)}|n\rangle + \langle k|\varepsilon_n|\psi_n^{(1)}\rangle$$

for $k=n$, $\langle n|H_0| = \langle n|\varepsilon_0$, $\therefore E_n^{(1)} = \langle n|H'|n\rangle$ ——— ①

(2) $H_0|\psi_n^{(1)}\rangle = H_0 C_i^{(1)}|n\rangle + \sum_{m\neq n}H_0 C_m^{(1)}|m\rangle$ $\qquad \langle k|\varepsilon_n|\psi_n^{(1)}\rangle = \varepsilon_n\delta_{kn} + \sum_{m\neq n}C_m^{(1)}\varepsilon_n\delta_{mk}$

$\langle k|H_0|\psi_n^{(1)}\rangle = C_i^{(1)}\varepsilon_n\delta_{kn} + \sum_{m\neq n}C_m^{(1)}\varepsilon_m\delta_{km}$

when $m=k$, $m\neq n$, $k\neq n$, ① $\Rightarrow \langle k|H'|n\rangle + C_k^{(1)}\varepsilon_k = C_k^{(1)}\varepsilon_n + E_n^{(1)}\delta_{kn}$

$$\therefore C_m^{(1)} = C_k^{(1)} = \frac{\langle k|H'|n\rangle}{\varepsilon_n - \varepsilon_m}$$

(3) $\langle\psi_n|\psi_n\rangle = 1$

$$\Rightarrow (\langle n| + \lambda\langle\psi_n^{(1)}| + \lambda^2\langle\psi_n^{(1)}| + \cdots)(|n\rangle + \lambda|\psi_n^{(1)}\rangle + \lambda^2|\psi_n^{(1)}\rangle + \cdots) = 1$$

$$\Rightarrow \langle n|n\rangle + \lambda(\langle n|\psi_n^{(1)}\rangle + \langle\psi_n^{(1)}|n\rangle) + \cdots = 1$$

$$\therefore \langle n|\psi_n^{(1)}\rangle + \langle\psi_n^{(1)}|n\rangle = 0$$

$$\Rightarrow \langle n|(C_n^{(1)}|n\rangle + \sum_{m\neq n}C_m^{(1)}|m\rangle) + (\langle n|C_n^{(1)} + \sum_{m\neq n}\langle m|C_m^{(1)})|n\rangle = 0 \Rightarrow C_n^{(1)} = 0 \,?$$

(4)

$$O|\psi_n\rangle = O|n\rangle + \lambda O|\psi_n^{(1)}\rangle = O|n\rangle + \lambda C_n^{(1)}O|n\rangle + \lambda\sum_{m\neq n}C_m^{(1)}O|m\rangle$$

$$\langle\psi_n|O|\psi_n\rangle = (\langle n| + \lambda\langle n|C_n^{(1)} + \lambda\sum_{m\neq n}\langle m|C_m^{(1)})(O|n\rangle + \lambda C_n^{(1)}O|n\rangle + \lambda\sum_{m\neq n}C_m^{(1)}O|n\rangle)$$

$$= \langle O\rangle\,? + \lambda\langle n|C_n^{(1)}O|n\rangle + \lambda\sum_{m\neq n}\langle n|C_m^{(1)}O|n\rangle + \lambda\langle n|C_n^{(1)}O|n\rangle + \lambda\sum_{m\neq n}\langle m|C_m^{(1)}O|n\rangle$$

問2

(1) $\Delta\psi(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}A\exp\left(-\frac{r}{a_0}\right)\right) = \frac{1}{a_0 r}\left(\frac{r}{a_0}-2\right)\psi(r,\theta,\phi)$

$\left[\frac{-\hbar^2}{2m}\left(\frac{1}{a_0 r}\left(\frac{r}{a_0}-2\right)\right)-\frac{C}{r}\right]\psi(r,\theta,\phi) = E\psi(r,\theta,\phi) \Rightarrow \begin{cases} a_0 = \frac{\hbar^2}{mC} \\ E = -\frac{mC^2}{2\hbar^2}\end{cases}$

(2) $\int \psi^*\psi\, dx\,dy\,dz = \int_0^\infty dr\int_0^\pi d\theta\int_0^{2\pi} r^2\sin\theta A^2 e^{-\frac{2r}{a_0}} = 4\pi A^2\int_0^\infty r^2 e^{-\frac{2r}{a_0}}$

(3) ⊗ $\quad = 8\pi A^2\left(\frac{a_0}{2}\right)^3 = 1 \Rightarrow A = \sqrt{\frac{\pi}{a_0^3}}$

第2問

[問1]

$\int_{-\infty}^{+\infty} dx\, x^2 e^{-\alpha x^2} = \frac{1}{2}\pi^{\frac{1}{2}}\alpha^{-\frac{3}{2}}$

(1) $Z = \frac{1}{h}\iint e^{-\beta\left(\frac{p^2}{2m}+\frac{ax^2}{2}\right)}dp\,dx = \frac{1}{h}\sqrt{\frac{2m\pi}{\beta}}\sqrt{\frac{2\pi}{a\beta}} = \frac{2\pi}{h\beta}\sqrt{\frac{m}{a}}$

$\left\langle\frac{p^2}{2m}\right\rangle = \frac{\frac{1}{h}\iint\frac{p^2}{2m}e^{-\beta\left(\frac{p^2}{2m}+\frac{ax^2}{2}\right)}dp\,dx}{Z} = \frac{\frac{1}{h}\cdot\frac{1}{2m}\cdot\frac{1}{2}\pi^{\frac{1}{2}}\cdot\left(\frac{2m}{\beta}\right)^{\frac{3}{2}}\sqrt{\frac{\pi}{a\beta}}}{Z} \Rightarrow \left\langle\frac{p^2}{2m}\right\rangle = \left\langle\frac{ax^2}{2}\right\rangle$

$\left\langle\frac{ax^2}{2}\right\rangle = \frac{\frac{1}{h}\iint\frac{ax^2}{2}e^{-\beta\left(\frac{p^2}{2m}+\frac{ax^2}{2}\right)}dp\,dx}{Z} = \frac{\frac{1}{h}\frac{a}{2}\pi^{\frac{1}{2}}\cdot\left(\frac{2}{a\beta}\right)^{-\frac{3}{2}}\sqrt{\frac{2m\pi}{\beta}}}{Z}$

(2) $Z = \frac{1}{h}\iint e^{-\beta\sum_{i=1}^{N}\{(p_{i,x}^2+p_{i,y}^2+p_{i,z}^2)/(2m)+a(x_i^2+y_i^2+z_i^2)/2\}}dp\,dx = \left(\frac{2\pi}{h\beta}\sqrt{\frac{m}{a}}\right)^{3N}$

$\langle E\rangle = \frac{\frac{1}{h}\int He^{-\beta H}dp\,dx}{\frac{1}{h}\int e^{-\beta H}dp\,dx} = -\frac{\partial\ln Z}{\partial\beta} = -3N\frac{1}{Z}\cdot\frac{2\pi}{h\frac{m}{a}}(-1)\frac{1}{\beta^2}\quad 3Nk_BT \Rightarrow C = \frac{\partial\langle E\rangle}{\partial T} = 3Nk$

[問2]

(1) $Z = \sum_{n_x}\sum_{n_y}\sum_{n_z}e^{-\beta(n_x+n_y+n_z+3/2)/\hbar\omega} = e^{-\beta\frac{3}{2}\hbar\omega}\left(\frac{1}{1-e^{-\beta/\hbar\omega}}\right)^3$

$\langle E\rangle = -N\frac{\partial\ln Z}{\partial\beta} = -N\left\{\frac{\partial}{\partial\beta}\left(-\frac{3}{2}\hbar\omega\beta - 3\ln\left(1-e^{-\beta\hbar\omega}\right)\right)\right\}$

$= \frac{3}{2}\hbar\omega N + 3N\frac{1}{1-e^{-\beta\hbar\omega}}\left(\hbar\omega e^{-\beta/\hbar\omega}\right)$

$= \frac{3}{2}\hbar\omega N + 3N\hbar\omega\frac{1}{e^{\beta\hbar\omega}-1}$

(2) $\lim_{T\to\infty}\frac{\partial\langle E\rangle}{\partial T} = \lim_{T\to\infty}3N\hbar\omega\frac{-1}{(e^{\hbar\omega/(kT)}-1)^2}\cdot e^{\frac{\hbar\omega}{kT}}\cdot\left(\frac{-\hbar\omega}{k}\right)T^{-2}$, let $\frac{\hbar\omega}{kT} = x \to 0$

$= \lim_{x\to 0}3Nk\, x^2e^x\frac{1}{(e^x-1)^2} = 3Nk\lim_{x\to 0}\frac{x^2e^x+2xe^x}{2(e^x-1)e^x} = 3Nk$

[問3]

(1) $Z = \frac{1}{h^3}\int e^{-\beta H}dp\,dx = \frac{L^3}{h^3}\int e^{-\beta(p_x^2+p_y^2+p_z^2)/2m_e}dp = \left(\frac{L}{h}\right)^3\left(2m_e\pi\frac{1}{\beta}\right)^{\frac{3}{2}}$

$\langle E\rangle = -\frac{\partial\ln Z}{\partial\beta} = -\frac{1}{Z}\left(\frac{L}{h}\right)^3\frac{3}{2}\left(2m_e\pi\frac{1}{\beta}\right)^{\frac{1}{2}}\cdot 2m_e\pi\cdot(-1)\beta^{-2} = \frac{3}{2}kT$

$C = \frac{\partial\langle E\rangle}{\partial T} = \frac{3}{2}k$

(2) $D(\varepsilon)d\varepsilon = \frac{L^3}{h^3}4\pi p^2dp = \frac{4\pi L^3}{(2\pi)^3\hbar^3}(2m\varepsilon)^{\frac{1}{2}}m\,d\varepsilon$

$N = 2\int_0^{\mu_0}D(\varepsilon)d\varepsilon = \frac{L^3}{\pi^2\hbar^3}2^{\frac{1}{2}}m^{\frac{3}{2}}\frac{2}{3}\mu_0^{\frac{3}{2}} \Rightarrow \mu_0 = \frac{\hbar^2}{2m}(3\pi^2n_e)^{\frac{2}{3}}$

(3) ⊗

第3問

[問1]

(1) $\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

(2) $\frac{\partial\mathcal{L}}{\partial x} = -kx \quad \frac{\partial\mathcal{L}}{\partial\dot{x}} = m\dot{x} \Rightarrow \ddot{x}+\omega_0^2x = 0,\ \omega_0 = \sqrt{k/m}$

Solution: $x = A_1\cos\omega_0t + A_2\sin\omega_0t = A\cos(\omega_0t+a)$

(3) $\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = 0$

[問2]

(1) Assume $x = a_1e^{ir_1t}+a_2e^{ir_2t}$, take into $\ddot{x}+2\lambda\dot{x}+\omega_0^2x = 0$,

$(-a_1r_1^2e^{ir_1t}-a_2r_2^2e^{ir_2t})+2\lambda(ir_1a_1e^{ir_1t}+ir_2a_2e^{ir_2t})+\omega_0^2(a_1e^{ir_1t}+a_2e^{ir_2t}) = 0$

$\Rightarrow a_1(-r_1^2+2\lambda ir_1+\omega_0^2)e^{ir_1t}+a_2(-r_2^2+2\lambda ir_2+\omega_0^2)e^{ir_2t} = 0$

consider $r^2-2i\lambda r-\omega_0^2 = 0 \Rightarrow r = \frac{2i\lambda\pm\sqrt{-4\lambda^2+4\omega_0^2}}{2} = i\lambda\pm\omega_a$

$\therefore x = a_1e^{i(i\lambda+\omega_a)t}+a_2e^{i(i\lambda-\omega_a)t}$

$= e^{-\lambda t}(a_1e^{i\omega_at}+a_2e^{-i\omega_at})$

$= Ae^{-\lambda t}\cos(\omega_at+a)$

(2) $\bar{U}(t) = \frac{1}{t_a}\int_t^{t+t_a}\frac{m\omega_0^2A^2e^{-2\lambda t'}\cos^2(\omega_at'+a)}{2}dt' = \frac{1}{4}m\omega_0^2A^2e^{-2\lambda t}$

$\int_t^{t+t_a}e^{-2\lambda t'}\cos^2(\omega_at'+a)dt' \approx e^{-2\lambda t}\int_t^{t+t_a}\frac{1+\cos(2\omega_at'+2a)}{2}dt' = \frac{1}{2}e^{-2\lambda t}t_a$

(3) $\dot{x} = -Ae^{-\lambda t}\{\lambda\cos(\omega_at+a)+\omega_a\sin(\omega_at+a)\}$

$\bar{T} = \frac{1}{t_a}\int_a^{t+t_a}\frac{mA^2e^{-2\lambda t'}\{\lambda\cos(\omega_at'+a)+\omega_a\sin(\omega_at'+a)\}^2}{2}dt' = \frac{1}{4}m\omega_0^2A^2e^{-2\lambda t}$

(4)

$$\frac{d}{dt}[\bar U+\bar T] = -\lambda m\omega_0^2A^2e^{-2\lambda t}$$

$$\bar D(t) = \frac{1}{t_a}\int_t^{t+t_a}\frac{\alpha\dot x^2}{2} = \lambda m\omega_0^2A^2e^{-2\lambda t}$$

$$\begin{cases} 2\lambda = \alpha/m \\ t_a = 2\pi/\omega_\alpha \end{cases}$$

(5)

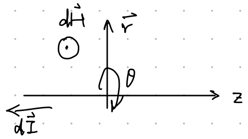
$$f_T = -\alpha\dot x = -\frac{\partial D}{\partial \dot x} \Rightarrow D = \frac12\alpha\dot x^2 = \frac12\alpha\frac{p^2}{m^2}$$

$$\frac{dE}{dt} = \frac{p}{m}\dot p + \frac{p}{m}kx = \left\{\left(\frac{\partial L}{\partial x}+f_T\right)+kx\right\}\frac{p}{m} = f_x\frac{p}{m} = -\frac{\partial D}{\partial \dot x}\frac{p}{m} = -\alpha\dot x\frac{p}{m}$$

$$= -\alpha\left(\frac{p}{m}\right)^2 = -2D$$

第4問

[問1](a)



(b)

(c) $\vec e_n\times(\vec E_2-\vec E_1)=0$

[問2]

(a) $\oint_C \vec E\cdot d\vec s = \int_0^R(\cancel{E_r-E_r})dr + \int_{z_1}^{z_2}(E_{z1}-E_{z2})dz = 0$

(b) $E_r = \frac{\lambda}{2\pi r\varepsilon_0}$ $H_\theta(r) = \frac{\lambda v}{2\pi r}$, $\frac{E_r}{H_\theta} = \frac{1}{\varepsilon_0 v}$

(c) $\lim_{v\to c}\frac{E_r}{H_\theta} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = c\mu_0 = 120\pi$ $m\cdot N/(s\cdot A^2)$

数学

[問1]

$A\vec x = \lambda\vec x$

$$\frac13\begin{pmatrix}2\alpha+\beta & \alpha-\beta\\ 2\alpha-2\beta & \alpha+2\beta\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \alpha\begin{pmatrix}1\\1\end{pmatrix} \Rightarrow \text{eigenvalue } \alpha$$

$$\frac13\begin{pmatrix}2\alpha+\beta & \alpha-\beta\\ 2\alpha-2\beta & \alpha+2\beta\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix} = \beta\begin{pmatrix}-1\\2\end{pmatrix} \Rightarrow \text{eigenvalue } \beta$$

[問3]

$P=\begin{pmatrix}1&-1\\1&2\end{pmatrix}$ $P^{-1}=\frac13\begin{pmatrix}2&1\\-1&1\end{pmatrix}$ $A^n = P\begin{pmatrix}\alpha^n&0\\0&\beta^n\end{pmatrix}P^{-1}$

$$A^n\begin{pmatrix}-3\\18\end{pmatrix} = \frac13\begin{pmatrix}1&-1\\1&2\end{pmatrix}\begin{pmatrix}\alpha^n&0\\0&\beta^n\end{pmatrix}\begin{pmatrix}2&1\\-1&1\end{pmatrix}\begin{pmatrix}-3\\18\end{pmatrix} = \begin{pmatrix}4\alpha^n-7\beta^n\\4\alpha^n+14\beta^n\end{pmatrix} = \begin{pmatrix}a_n-b_n\\a_n+2b_n\end{pmatrix}$$

$$\Rightarrow \begin{cases}a_n = 4\alpha^n\\ b_n = 7\beta^n\end{cases}$$

第2問

[問1]

$$I_{n+1} = \int_0^\infty dx\, e^{-x}x^{n+1} = -e^{-x}x^{n+1}\Big|_0^\infty - \int_0^\infty -e^{-x}(n+1)x^n = (n+1)I_n$$

[問2]

$$\int_{-\infty}^{\infty}x^2e^{-ax^2} = \frac12\sqrt\pi a^{-\frac32}$$

let $x^{\frac12}=t$ $dx = 2t\,dt$

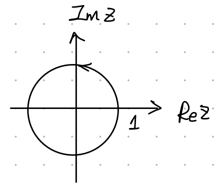
$$\int_0^\infty e^{-t^2}2t^2dt = 2\cdot\frac12\cdot\frac12\sqrt\pi = \frac{\sqrt\pi}{2}$$

第3問

[問1] $\frac{df(z)}{dz} = -2\pi\sin(2\pi z)$ $\frac{d^2f(z)}{dz^2} = -4\pi^2\cos(2\pi z)$ $\frac{d^3f(z)}{dz^3} = 8\pi^3\sin(2\pi z)$

$$f(z) = -1+\frac12 4\pi^2\left(z-\frac12\right)^2 = -1+2\pi^2\left(z-\frac12\right)^2$$

[問2]



$$\oint_{|z|=1}dz\frac{\cos2\pi z}{(z-\frac12)^4} = 2\pi i\frac{1}{3!}\left[\frac{d^3}{dz^3}\cos2\pi z\right]_{z=\frac12} = 0$$

第4問

[問1]

$z(y(x),g(x)) = y(x)\cdot g(x)$, $g(x)=\frac1x$, $y=xz$

$$\frac{dz}{dx} = \frac{\partial z}{\partial y}\frac{dy}{dx}+\frac{\partial z}{\partial g}\frac{dg}{dx} = \frac1x\frac{dy}{dx}+y(x)\left(-\frac1{x^2}\right) = \frac1x\cdot\frac{y^2-x^2}{2xy}+y\left(-\frac1{x^2}\right)$$

$$= \frac1{x^2}\left[\frac{x^2z^2-x^2}{2xz}-xz\right] = \frac{z^2-1}{2xz}-\frac zx = \frac1x\left[\frac{z^2-1}{2z}-z\right]$$

[問2]

$$\frac{dz}{\frac{z^2-1}{2z}-z} = \frac1x dx \Rightarrow -\int\frac{2z\,dz}{z^2+1} = \int\frac1x dx \Rightarrow -\ln(z^2+1) = \ln x + C$$

$$(z^2+1)^{-1} = Cx \Rightarrow y = \pm\sqrt{Cx-x^2}$$

heyingqiu

2024.5.20 ♡