Worksheet 2
Introducing: The Virial Theorem!
Instructor: Ben Cook

The Virial Theorem

The *Virial Theorem* is an incredibly useful relation, that connects the kinetic energy K of a system to its potential energy U. There are several different ways of writing down the theorem, and the most useful depends on the context you're dealing with. The most common (and simplified) version is the following, which is true for a system of objects which are *in gravitational equilibrium*:

$$\boxed{2\langle K\rangle = -\langle U\rangle} \quad \text{(V.T. for grav. equilibrium)} \tag{1}$$

The brackets $\langle\cdots\rangle$ indicate a *time-average* of the values inside the bracket. This equation tells us how the average kinetic energy $\langle K\rangle$ of a gravitational system connects to the average gravitational potential energy $\langle U\rangle$.

This equation applies to essentially all systems which are in gravitational equilibrium, which includes a huge range of astrophysical objects such as:

- Stars
- Galaxies
- Globular Clusters and Open Clusters
- Clusters of Galaxies
- Giant Molecular Clouds
- Planets
- Solar Systems
- $\cdots$

A corollary of the theorem above is how the time-average of the total energy ($\langle E\rangle = \langle K\rangle + \langle U\rangle$) is connected to the other energies:

$$\langle E\rangle = -\langle K\rangle \tag{2}$$

$$= \frac{1}{2}\langle U\rangle \tag{3}$$

1. Return to Problem 1 from yesterday's Worksheet 1 (satellite in circular orbit around Earth). Using your solutions from yesterday, check that all three of the Virial Theorem equations above are true. (*HINT: Since the satellite is in a circular orbit, the energies are not changing, so you can ignore the time-averages.*)

2. The Virial Theorem doesn't just apply to the "simple" system of a central object and a satellite. It is just as valid to use the Virial Theorem for a complicated system with many particles, such as a star or planet! Here, we'll derive an estimate of the average temperature of a star (or planet) from a so-called "Virial" argument.

   Consider a large, gravitationally-bound sphere (star or planet) with mass M and radius R. We will assume there are N particles, which are all in thermal equilibrium at an average temperature $\bar{T}$. For this system, the thermal kinetic energy of all the particles is given by:

   $$K = \frac{3}{2} N k \bar{T}\,.$$

   We can re-express this in terms of the total mass M by using the mean mass-per-particle:

   $$N = \frac{M}{\mu m_p}\,,$$

   where m_p is the mass of the proton, and μ is the "mean molecular weight", or the average mass of a particle in units of the proton mass ($\mu = 1$ for neutral Hydrogen, etc).

   (a) Use the Virial Theorem to equate the system's kinetic and gravitational potential energies, then solve for $\bar{T}$.

   (b) Using this approach, what is your estimate for the average temperature of the Sun? Use $\mu = 0.6$ and the other values at the back of the worksheet.

   (c) What is $\bar{T}$ for the Earth? Use $\mu = 100$ and other values at the back of the worksheet.

   (d) Hydrogen fusion initiates at temperatures around $T \approx 10^7$K (the exact values depend on the ambient pressure). Would the Virial Theorem lead you to expect the Sun to be fusing Hydrogen? What about the Earth?

Proving the Virial Theorem

It is now time to *prove* the Virial Theorem! Don't worry: while this may seem like a daunting task, it's actually quite straight-forward if you take it a step at a time.

3. We will begin with the definition of the *moment of inertia* I for a large system of particles, which should look somewhat familiar. The moment of inertia simply depends on the mass m and position r of each particle in the system. In both the vector and scalar versions:

   $$\begin{aligned} I &\equiv \sum_i m_i \mathbf{r}_i \cdot \mathbf{r}_i \quad \text{(vector)} \\ &\equiv \sum_i m_i r_i^2 \quad \text{(1-D)}\,. \end{aligned}$$

   Feel free to continue with the proof either in 1-D, or to work with the vector forms, if you're comfortable with vector dot-products.

   (a) Take one derivative w.r.t. time to show that:

   $$\begin{aligned} \frac{dI}{dt} &= \sum_i \mathbf{p}_i \cdot \mathbf{r}_i \quad \text{(vector)} \\ &= \sum_i p_i r_i \quad \text{(1-D)}, \end{aligned}$$

   where p_i is the momentum of particle i.

(b) Take a second derivative w.r.t time and show that:

$$\boxed{\frac{1}{2}\frac{d^2 I}{dt^2} = 2K + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i} \quad \text{(Virial Theorem, full vector form)} \tag{4}$$

$$\boxed{\frac{1}{2}\frac{d^2 I}{dt^2} = 2K + \sum_i F_i r_i} \quad \text{(Virial Theorem, full 1-D form)} \tag{5}$$

Final Steps

If the system of particles are interacting via a *power-law central force* which obeys:

$$F \propto -r^n \,,$$

for some power n[1], then the potential energy of the entire system can be proven to be:

$$U = -\frac{1}{n+1}\sum_i \mathbf{F}_i \cdot \mathbf{r}_i \quad \text{(vector)} \tag{6}$$

$$= -\frac{1}{n+1}\sum_i F_i r_i \quad \text{(1-D)} \tag{7}$$

The *time-average* of the left-hand side over some time interval τ is:

$$\begin{aligned}\left\langle \frac{1}{2}\frac{d^2 I}{dt^2}\right\rangle_\tau &= \frac{1}{2}\left(\frac{1}{\tau}\int_0^\tau \frac{d^2 I}{dt^2}dt\right)\\ &= \frac{1}{2\tau}\left(\left.\frac{dI}{dt}\right|_\tau - \left.\frac{dI}{dt}\right|_0\right)\end{aligned}$$

If the system is in equilibrium, then the term in parentheses is bounded (will never reach infinity). Then, by averaging over a large enough τ:

$$\boxed{\left\langle \frac{1}{2}\frac{d^2 I}{dt^2}\right\rangle_\tau \approx 0} \quad \text{(If } \tau \text{ large enough)} \tag{8}$$

If the system is periodic, then we can choose τ so that the time-averages $\langle \cdots \rangle$ are over a period, so that the difference in parentheses goes to zero. If it is not strictly periodic, then you can just choose an appropriately large value of τ. If the system is in "equilibrium", then the terms in the parentheses will remain small, and $\left\langle \frac{1}{2}\frac{d^2 I}{dt^2}\right\rangle$ will approach zero.

4. Substitute equations 6 or 7 into the complete Virial Theorem (4 or 5), and then take a time-average of both sides. Use the limit given in 8 to conclude:

$$\boxed{2\langle K\rangle = (n+1)\langle U\rangle} \tag{9}$$

(Virial Theorem, system in equilibrium, central force $F \propto -r^n$)

[1] $n = -2$ for gravity or electrostatics, $n = 1$ for Hook's Law for springs, etc.

Useful Constants

Name	Symbol	Value (SI)	Value (CGS)
Mass of Sun	$M_\odot$	2.0×10^{30} kg	2.0×10^{33} g
Mass of Earth	$M_\oplus$	6.0×10^{24} kg	6.0×10^{27} g
Mass of Proton	m_p	1.7×10^{-27} kg	1.7×10^{-24} g
Gravitational Constant	G	6.7×10^{-11} m^{3}kg^{-1}s^{-2}	6.7×10^{-8} cm^{3}g^{-1}s^{-2}
Boltzmann Constant	k	1.4×10^{-23} J k^{-1}	1.4×10^{-16} erg k^{-1}
Radius of Sun	$R_\odot$	7.0×10^{8} m	7.0×10^{10} cm
Radius of Earth	$R_\oplus$	6.4×10^{6} m	6.4×10^{8} cm