# Worksheet 3
# The Virial Theorem Strikes Back
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Last time, we derived the complete form of the Virial Theorem:

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2K + \sum_i F_i r_i \tag{1}$$

We also argued (but did not prove) that the third term represents the potential energy $U$ of the system, giving us:

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2K - (n+1)U \tag{2}$$

Finally, we showed that the time-average of the left-hand term goes to zero if the system is in equilibrium (and the time-average is over an appropriately long time period), giving us:

$$2\langle K\rangle = (n+1)\langle U\rangle \tag{3}$$

For the familiar case of a gravitational system ($n = -2$), we recover:

$$2\langle K\rangle = -\langle U\rangle \tag{4}$$

In these final exercises, we will play with these forms of the Virial Theorem in a variety of scenarios.

**Dark Matter in Galaxy Clusters**

The most well-known and frequently-cited use of the Virial Theorem was in the discovery of dark matter. By applying the theorem to the velocities of galaxies in a galaxy cluster, we can make an estimate of the total mass of the cluster.

1. We will model a galaxy cluster as a sphere with mass $M$ (comprised of many galaxies) and radius $R$.

   (a) We can use the redshifts of each galaxy to try to measure the velocity-dispersion $\langle v^2\rangle$ of the cluster. But we can only actually measure the component $\langle v_r^2\rangle$ *along the line of sight*; we can't hope to measure their velocity in the plane of the sky (we'd have to wait for millions of years). How does the true 3D *velocity dispersion* $\langle v^2\rangle_{3D}$ relate to the observed radial velocity dispersion $\langle v_r^2\rangle$, if we assume the clusters are on randomly-oriented orbits?

   (b) Using the Virial Theorem to connect the kinetic and potential energies of the cluster, derive an estimate of the total mass $M$ as a function of the velocity dispersion $\langle v^2\rangle_{3D}$ and the radius $R$.

   (c) You observe the galaxies in the cluster to have a radial velocity dispersion $\langle v_r^2\rangle = 10^6$ km$^2$s$^{-2}$, and estimate it to have a size $R = 1.5$ Mpc. What is your prediction for the total mass $M$ in solar masses ($M_\odot$)?

(d) Measuring the light coming from all the cluster galaxies, you measure the total luminosity of the cluster to be $L = 10^{13} L_\odot$. What is the "mass-to-light" ratio $M/L$ in solar units. Does anything seem strange?

(e) The Virial Theorem implies there is much more mass in clusters than you can explain from the observed galaxies. Congratulations, you've just discovered Dark Matter!

But, being the skeptical scientist you are, you want to make sure you've considered all other possibilities. What are some alternative ways you could explain your results?

- Think about the assumptions you've made in this analysis. What ways can you think of to argue that they are valid?
- Maybe there were errors in the data you used. How far off would different measurements have to be to explain your results? Think critically about how each value was *actually measured* by astronomers, and where the most likely places are to have gotten things wrong.

## Fritz Zwicky, Vera Rubin, and Dark Matter

The analysis you just completed was first published by Caltech Astronomer Fritz Zwicky in 1933, not long after it was first conclusively proven that the "nebulae" seen around the Milky Way were in fact distant galaxies. Zwicky compiled measurements of galactic redshifts in several clusters, most notably the Coma cluster, and used the Virial Theorem to argue that there must be huge amounts of "dark matter" there to account for the large masses.

2. Read the provided pages from Zwicky's 1933 paper (translated from German) which first argued for the existence of dark matter. It should be quite straight-forward, as you've just worked through the same exact problem!

   If you're interested, you can find the full article at

   http://spiff.rit.edu/classes/phys440/lectures/gal_clus/zwicky_1933_en.pdf.

Zwicky was an edgy, often confrontational personality, whose life now makes for a very entertaining study. But at the time, Zwicky's discovery of dark matter was largely ignored, as were many of his other groundbreaking ideas. In a short biography of Zwicky, Professor Stephen Maurer wrote: "When researchers talk about neutron stars, dark matter, and gravitational lenses, they all start the same way: 'Zwicky noticed this problem in the 1930s. Back then, nobody listened..."'.

Zwicky's dark matter hypothesis finally gained acceptance in the 1970s, with huge credit due to Vera Cooper Rubin and Kent Ford of the Carnegie Institute. Rubin and Ford measured the rotation curve of the Andromeda Galaxy and showed that dark matter must exist around galaxies (a problem you also have tackled, back in Worksheet 1). Rubin was a true vanguard for women in astronomy (only the second woman elected to the National Academy of Sciences), and overcame large obstacles in a field determined to deter her from success. Sadly, she passed away last December (2016). Many in the field believed her work strongly deserving of the Nobel Prize, but since they are not awarded posthumously, this recognition will never be granted to her.

3. Spend some time researching the story of Fritz Zwicky's entertaining personality, and Vera Rubin's pioneering journey.

## Hook's Law

4. Consider the classic "mass on a spring" problem (1D, no gravity or friction). A mass $m$ is attached to a horizontal spring that has spring constant $k$ and is in equilibrium at position $x = 0$.

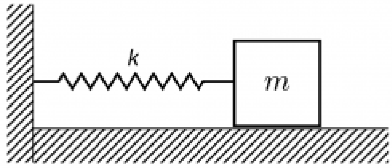


Figure 1: Figure for problem 4..

Hook's law tells us the force on the mass when it is at position $x$ is:

$$F = -kx \tag{5}$$

or alternatively,

$$m\frac{d^2x}{dt^2} = -kx \tag{6}$$

One solution to this equation of motion is sinusoidal motion:

$$x(t) = A\cos(\omega t)\,, \tag{7}$$

where the frequency is given by $\omega = \pm\sqrt{k/m}$.

$$\tag{8}$$

(a) Solve for the kinetic energy $K(t)$ of the mass as a function of time.

(b) Solve for $U(x)$ as a function of position, using $x = 0$ as the potential energy reference point.

(c) Solve for the potential energy $U(t)$ as a function of time.

(d) Solve for the total energy $E(t)$. Does this energy change in time?

(e) Confirm that equation 3 is correct for this system ($n = +1$) by taking the time average of $K$ and $U$ over a period $P = 2\pi\omega$. For your reference, the needed time-averages are:

$$\left\langle \cos^2(\omega t)\right\rangle_{2\pi\omega} = \left\langle \sin^2(\omega t)\right\rangle_{2\pi\omega} = \frac{1}{2} \tag{9}$$

(f) Compute the moment-of-inertia $I(t) = mx^2(t)$ as a function of time.

(g) Confirm that the non-averaged Virial Theorem (eq. 2) is correct at *all times* for this system.

5. **Dynamical Timescale: How Long is Long Enough?**

We've talked about applying eq. 3 whenever time-averages are taken over "long enough" timescales where a system can be said to be in equilibrium. Here, we'll derive a good rule-of-thumb for determining whether a system is in equilibrium.

Imagine that the kinetic energy of a gravitational system (star, cluster, etc.) suddenly disappeared ($K = 0$). How long would it take before the system collapses entirely?

Equation 2 tells us that, in this case:

$$\frac{1}{2}\frac{d^2I}{dt^2} = U\,. \tag{10}$$

Let's now make an order-of-magnitude estimate (dropping the factor of $\frac{1}{2}$) for this process takes. If the system experiences a change in moment-of-inertia $\Delta I$ in time $\Delta t$, we estimate:

$$\frac{d^2I}{dt^2} \approx \frac{\Delta I}{(\Delta t)^2} = U\,. \tag{11}$$

If the system collapses entirely, then the moment-of-inertia will go to zero, and $\Delta I = -I$. How much time will this take?

$$\frac{-I}{(\Delta t)^2} \approx U \tag{12}$$

$$(\Delta t)^2 \approx \frac{-I}{U} \tag{13}$$

Substituting in the moment of inertia $I$ and potential energy $U$ of a constant-density sphere:

$$(\Delta t)^2 \approx \frac{-\frac{2}{5}MR^2}{-\frac{3}{5}\frac{GM^2}{R}} \tag{14}$$

$$\boxed{\tau_d \equiv \sqrt{\frac{2R^3}{3GM}}} \tag{15}$$

This is the definition of the *dynamical time* $\tau_d$, the shortest timescale over which the moment of inertia could change significantly. As long as the system does not change drastically (any $\Delta I \ll I$) on the timescale of $\tau_d$, then it can be considered in dynamical equilibrium, and it is valid to use eq. 3.

(a) Compute $\tau_d$ for the following objects:

- The Sun (use the values on the final page of the worksheet).
- The Milky Way ($M \approx 10^{12} M_\odot$, $R \approx 100$ kpc).
- The Coma Cluster ($M \approx 10^{15} M_\odot$, $R \approx 1$ Mpc).
- A theoretical "Super-Duper-Cluster" with mass $M \approx 10^{16} M_\odot$ and radius $R = 30$ Mpc.

(b) Which of these objects do you think are likely to be in equilibrium? Think about whether you would expect these systems to change drastically within the dynamical timescales calculated above. How might you judge the "stability" of each system? Rank them in order of decreasing "stability" according to this judgment.

6. **Ideal Gas Law**

Consider a cubic box (side length $s$) with $N$ gas particles at temperature $T$ and average pressure $P$, with the center of the box at the origin.

Because the box is in equilibrium, we will use the Virial Theorem of the form:

$$2\langle K \rangle = -\left\langle \sum_i F_i r_i \right\rangle \tag{16}$$

(a) What is $2\langle K \rangle$? (*HINT: think thermal energy*)

(b) What is the average *TOTAL* force $\sum_i F_i$ applied by the gas particles on all surfaces of the box? By Newton's third law, the gas particles will feel a total force with the same magnitude, but in the opposite direction.

(c) At what position $r$ from the center of the box is this force *always* applied to the particles?

(d) Use your answers to the above, along with eq. 16 to derive the Ideal Gas Law:

$$PV = NkT\,.$$

If you're getting stuck with a minus sign, remember to consider what direction the force $\sum_i F_i$ on the gas particles is in, relative to their displacement $r$ from the center of the box.

7. **Radius of Electron Orbitals**

We will now use the Virial Theorem to derive the radius $r_n$ of the n$^{\mathrm{th}}$ Hydrogen orbital. We are playing fast-and-loose with Quantum Mechanics (the "time-averages" $\langle \cdots \rangle$ are really "quantum expectation values" $\langle \cdots \rangle$) but our answers are accurate to factors of a few.

Note: this problem is done in CGS units.

(a) The electron is bound to the Hydrogen nucleus via an $r^{-2}$ central force. The potential energy of the electron at radius $r$ is given by:

$$U(r) = -\frac{e^2}{r} \,. \tag{17}$$

Use the Virial Theorem to derive the expectation value of total energy $\langle E(r) \rangle$ of an electron as a function of position.

(b) From quantum mechanics, we can derive that an electron in the n$^{\mathrm{th}}$ eigenstate has energy:

$$E_n = -\frac{1}{2}\frac{m_e e^4}{n^2 \hbar^2} \,. \tag{18}$$

To an order of magnitude, we can say:

$$\left\langle \frac{1}{r_n} \right\rangle \approx \frac{1}{\langle r_n \rangle} \,. \tag{19}$$

Using the above, prove that:

$$\langle r_n \rangle \approx n^2 a_0 \,, \tag{20}$$

where $a_0$ is the "Bohr Radius":

$$a_0 \equiv \frac{\hbar^2}{m_e e^2} \,. \tag{21}$$

**Useful Constants and Conversions**

| Name | Symbol | Value (SI) | Value (CGS) |
|---|---|---|---|
| Mass of Sun | $M_\odot$ | $2.0 \times 10^{30}$ kg | $2.0 \times 10^{33}$ g |
| Radius of Sun | $R_\odot$ | $7.0 \times 10^{8}$ m | $7.0 \times 10^{10}$ cm |
| Luminosity of Sun | $L_\odot$ | $3.9 \times 10^{26}$ J s$^{-1}$ | $3.9 \times 10^{33}$ erg s$^{-1}$ |
| Gravitational Constant | $G$ | $6.7 \times 10^{-11}$ m$^3$kg$^{-1}$s$^{-2}$ | $6.7 \times 10^{-8}$ cm$^3$g$^{-1}$s$^{-2}$ |
| Kilometer | km | $10^3$ m | $10^5$ cm |
| Kiloparsec | kpc | $3.1 \times 10^{19}$ m | $3.1 \times 10^{21}$ cm |
| Megaparsec | Mpc | $3.1 \times 10^{22}$ m | $3.1 \times 10^{24}$ cm |
| Year | yr | $3.1 \times 10^{7}$ s | – |
| Giga-year | Gyr | $3.1 \times 10^{16}$ s | – |