Worksheet 4
Return of the Virial Theorem
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Last time, we derived the complete form of the Virial Theorem:

$$\frac{1}{2}\frac{d^2I}{dt^2} = 2K + \sum_i F_i r_i \tag{1}$$

We also argued (but did not prove) that the third term represents the potential energy U of the system, giving us:

$$\frac{1}{2}\frac{d^2I}{dt^2} = 2K - (n+1)U \tag{2}$$

Finally, we showed that the time-average of the left-hand term goes to zero if the system is in equilibrium (and the time-average is over an appropriately long time period), giving us:

$$2\langle K\rangle = (n+1)\langle U\rangle \tag{3}$$

For the familiar case of a gravitational system ($n = -2$), we recover:

$$2\langle K\rangle = -\langle U\rangle \tag{4}$$

In these final exercises, we will play with these forms of the Virial Theorem in a variety of scenarios.

1. **Hook's Law**

   Consider the classic "mass on a spring" problem (1D, no gravity or friction). A mass m is attached to a horizontal spring that has spring constant k and is in equilibrium at position $x = 0$.

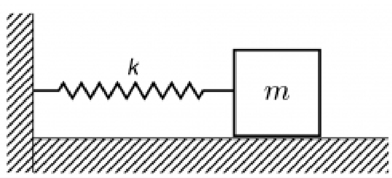

Figure 1: Figure for problem 1.

   Hook's law tells us the force on the mass when it is at position x is:

$$F = -kx$$

   or alternatively,

$$m\frac{d^2x}{dt^2} = -kx$$

   One solution to this equation of motion is sinusoidal motion:

$$x(t) = A\cos(\omega t)\,,$$

   where the frequency is given by $\omega = \pm\sqrt{k/m}$.

(a) Solve for the kinetic energy $K(t)$ of the mass as a function of time.

(b) Solve for $U(x)$ as a function of position, using $x = 0$ as the potential energy reference point.

(c) Solve for the potential energy $U(t)$ as a function of time.

(d) Solve for the total energy $E(t)$. Does this energy change in time?

(e) Confirm that equation 3 is correct for this system ($n = +1$) by taking the time average of K and U over a period $P = 2\pi\omega$. For your reference, the needed time-averages are:

$$\left\langle \cos^2(\omega t) \right\rangle_{2\pi\omega} = \left\langle \sin^2(\omega t) \right\rangle_{2\pi\omega} = \frac{1}{2}$$

(f) Compute the moment-of-inertia $I(t) = mx^2(t)$ as a function of time.

(g) Confirm that the non-averaged Virial Theorem (eq. 2) is correct at *all times* for this system.

2. **Dynamical Timescale: How Long is Long Enough?**

We've talked about applying eq. 3 whenever time-averages are taken over "long enough" timescales where a system can be said to be in equilibrium. Here, we'll derive a good rule-of-thumb for determining whether a system is in equilibrium.

Imagine that the kinetic energy of a gravitational system (star, cluster, etc.) suddenly disappeared ($K = 0$). How long would it take before the system collapses entirely?

Equation 2 tells us that, in this case:

$$\frac{1}{2}\frac{d^2 I}{dt^2} = U\,.$$

Let's now make an order-of-magnitude estimate (dropping the factor of $\frac{1}{2}$) for this process takes. If the system experiences a change in moment-of-inertia ΔI in time Δt, we estimate:

$$\frac{d^2 I}{dt^2} \approx \frac{\Delta I}{(\Delta t)^2} = U\,.$$

If the system collapses entirely, then the moment-of-inertia will go to zero, and $\Delta I = -I$. How much time will this take?

$$\frac{-I}{(\Delta t)^2} \approx U$$
$$(\Delta t)^2 \approx \frac{-I}{U}$$

Substituting in the moment of inertia I and potential energy U of a constant-density sphere:

$$(\Delta t)^2 \approx \frac{-\frac{2}{5}MR^2}{-\frac{3}{5}\frac{GM^2}{R}}$$
$$\boxed{\tau_d \equiv \sqrt{\frac{2R^3}{3GM}}}$$

This is the definition of the *dynamical time* τ_d, the shortest timescale over which the moment of inertia could change significantly. <u>As long as the system does not change drastically (any $\Delta I \ll I$) on the timescale of τ_d, then it can be considered in dynamical equilibrium, and it is valid to use eq. 3.</u>

(a) Compute τ_d for the following objects:

- The Sun (use the values on the final page of the worksheet).
- The Milky Way ($M \approx 10^{12} M_\odot$, $R \approx 100$ kpc).
- The Coma Cluster ($M \approx 10^{15} M_\odot$, $R \approx 1$ Mpc).
- A theoretical "Super-Duper-Cluster" with mass $M \approx 10^{16} M_\odot$ and radius $R = 30$ Mpc.

(b) Which of these objects do you think are likely to be in equilibrium? Think about whether you would expect these systems to change drastically within the dynamical timescales calculated above. How might you judge the "stability" of each system? Rank them in order of decreasing "stability" according to this judgment.

3. **<u>Ideal Gas Law</u>**

Consider a cubic box (side length s) with N gas particles at temperature T and average pressure P, with the center of the box at the origin.

Because the box is in equilibrium, we will use the Virial Theorem of the form:

$$2\langle K\rangle = -\left\langle \sum_i F_i r_i \right\rangle \tag{5}$$

(a) What is $2\langle K\rangle$? (*HINT: think thermal energy*)

(b) What is the average *TOTAL* force $\sum_i F_i$ applied by the gas particles on all surfaces of the box? By Newton's third law, the gas particles will feel a total force with the same magnitude, but in the opposite direction.

(c) At what position r from the center of the box is this force *always* applied to the particles?

(d) Use your answers to the above, along with eq. 5 to derive the Ideal Gas Law:

$$PV = NkT\,.$$

If you're getting stuck with a minus sign, remember to consider what direction the force $\sum_i F_i$ on the gas particles is in, relative to their displacement r from the center of the box.

4. **<u>Radius of Electron Orbitals</u>**

We will now use the Virial Theorem to derive the radius r_n of the n$^{\text{th}}$ Hydrogen orbital. We are playing fast-and-loose with Quantum Mechanics (the "time-averages" $\langle\cdots\rangle$ are really "quantum expectation values" $\langle\cdots\rangle$) but our answers are accurate to factors of a few.

Note: this problem is done in CGS units.

(a) The electron is bound to the Hydrogen nucleus via an r^{-2} central force. The potential energy of the electron at radius r is given by:

$$U(r) = -\frac{e^2}{r}\,. \tag{6}$$

Use the Virial Theorem to derive the expectation value of total energy $\langle E(r)\rangle$ of an electron as a function of position.

(b) From quantum mechanics, we can derive that an electron in the n$^{\text{th}}$ eigenstate has energy:

$$E_n = -\frac{1}{2}\frac{m_e e^4}{n^2\hbar^2}\,. \tag{7}$$

To an order of magnitude, we can say:

$$\left\langle \frac{1}{r_n}\right\rangle \approx \frac{1}{\langle r_n\rangle}\,.$$

Using the above, prove that:

$$\langle r_n \rangle \approx n^2 a_0 \,,$$

where a_0 is the "Bohr Radius":

$$a_0 \equiv \frac{\hbar^2}{m_e e^2} \,.$$

Useful Constants and Conversions

Name	Symbol	Value (SI)	Value (CGS)
Mass of Sun	$M_\odot$	2.0×10^{30} kg	2.0×10^{33} g
Radius of Sun	$R_\odot$	7.0×10^{8} m	7.0×10^{10} cm
Luminosity of Sun	$L_\odot$	3.9×10^{26} J s^{-1}	3.9×10^{33} erg s^{-1}
Gravitational Constant	G	6.7×10^{-11} m^{3}kg^{-1}s^{-2}	6.7×10^{-8} cm^{3}g^{-1}s^{-2}
Kilometer	km	10^3 m	10^5 cm
Kiloparsec	kpc	3.1×10^{19} m	3.1×10^{21} cm
Megaparsec	Mpc	3.1×10^{22} m	3.1×10^{24} cm
Year	yr	3.1×10^{7} s	–
Giga-year	Gyr	3.1×10^{16} s	–