Worksheet 1
Dynamics Overview
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During this week's Astro-Skills module, we will be discussing one of the most useful (and simple) equations in astronomy, the *Virial Theorem*. Before we get there, however, it is important to warm-up by reviewing a few principles from dynamics.

Kinetic and Potential Energy

For an object in motion with mass m and velocity v, its kinetic energy K (the energy due to its motion) is given by:

$$K = \frac{1}{2}mv^2 \,. \tag{1}$$

If multiple objects in a system (such as multiple stars in a galaxy) are all in motion, then the total kinetic energy of the system is simply the sum of the individual kinetic energies of each object.

Potential energy U is a measure of how much stored energy an object has (technically, the amount of work it is capable of performing) at its current position. The difference in potential energy between two points ($\mathbf{r}_1$ and $\mathbf{r}_2$) is determined by the force ($\mathbf{F}$, as a vector) applied along the trajectory between the points:

$$\Delta U = -\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} \,. \tag{2}$$

If working in one-dimension, then this simplifies to:

$$\Delta U = -\int_{x_1}^{x_2} F dx \,. \tag{3}$$

To define an object's potential energy, we must always define a *reference location*, where we define potential energy to be zero. The choice of this location depends on the context of the problem being approached.

Unlike with kinetic energy, the presence of multiple objects in a system may change each object's potential energy, such as when a force must have been applied to move two charged particles close to one another. To calculate the total potential energy of a system, imagine building the system up in pieces: calculate the work required to move the first object into place, then the second, then the third, and so on. See the final section (**Self-bound Sphere**) for an example.

The total energy E of a system is simply the sum of its kinetic and potential energies (if we neglect other complicated energies, such as magnetic pressure, rotation, thermal energy, etc.):

$$E = K + U \,. \tag{4}$$

Gravitation and Circular Motion

Two massive objects (M and m) separated by a distance r (or displacement vector $\mathbf{r}$) will feel a gravitational force of attraction. If they are point-masses (or spheres) the force is simply:

$$\mathbf{F} = -\frac{GMm}{r^3}\mathbf{r} \quad \text{(vector)} \tag{5}$$

$$F = -\frac{GMm}{r^2} \quad \text{(scalar)}. \tag{6}$$

An object on a circular trajectory must be subject to a *centripetal acceleration* directed towards the center of the circle in order to remain on the trajectory. The magnitude of the acceleration is given by the radius r and velocity v of the trajectory:

$$a_{cen} = \frac{v^2}{r}\,. \tag{7}$$

1. A satellite (mass m) is on a circular orbit with radius r from the center of the Earth (mass M).

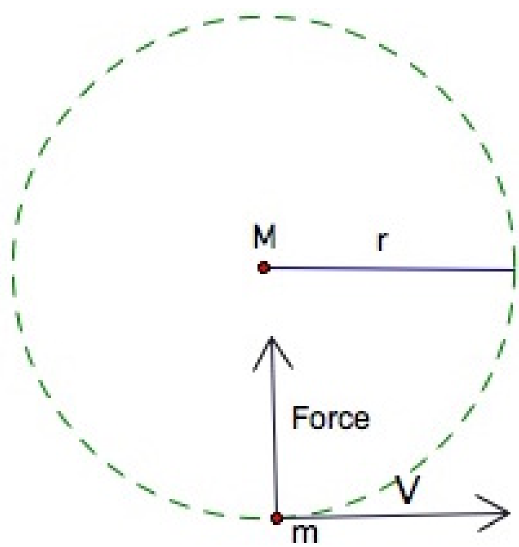

Figure 1: Figure for Problem 1.

(a) Solve for the satellite's velocity $v(r)$ as a function of radius, such that the gravitational force balances the required centripetal acceleration.

(b) Solve for the period P of the orbit.

(c) Solve for the satellite's kinetic energy $K(r)$ as a function of radius.

(d) When considering gravity, it is customary to set the potential energy reference point at $r \to \infty$, since the force is infinite at the origin. Using the fact that $U(r \to \infty) = 0$, solve for the potential energy $U(r)$ of the satellite.

This is important enough of a result that I'll give you the answer here, so you can check. The gravitational potential energy of an object with mass m near a mass M is:

$$U(r) = -\frac{GMm}{r} \tag{8}$$

(e) Compute the total energy of the satellite, $E(r)$.

<u>Gauss' Law</u>

Newton proved that the gravitational force on an object outside a uniform sphere of mass M is the same as that from a point-mass with the same mass. Even more generally, the gravitational force at a distance r from any object which is *spherically symmetric* simply depends on the **mass enclosed within that radius**, $M(<r)$:

$$F(r) = \frac{GmM(<r)}{r^2}\,. \tag{9}$$

This is a consequence of "Gauss' Law". If you're interested in a rigorous proof of this, talk to Ben or check Wikipedia[1].

2. Consider a hollow spherical shell, with all of its mass M contained at a radius R.

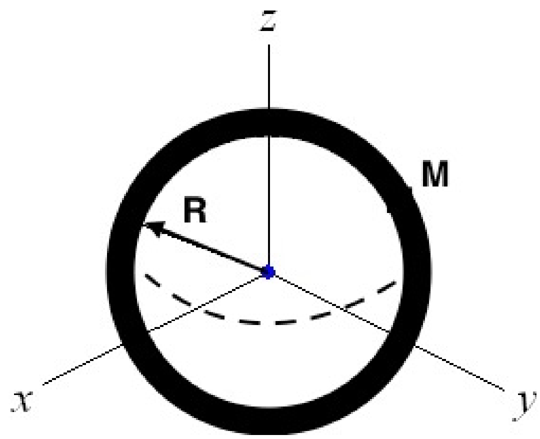

Figure 2: Figure for Problem 2.

   (a) What is the gravitational force $F(r)$ on a particle with mass m when *outside* the sphere ($r > R$)?
   (b) What is the force $F(r)$ on it when *inside* the sphere ($r < R$)?

Self-bound Sphere

3. In this next problem, we will compute the potential energy U of a gravitationally-bound sphere, a quantity which will arise extremely often in Astronomy. We will make this computation by imagining building up the sphere inside-out, and summing the potential energies required to move each slice of the sphere into position from infinity[2].

   Consider a sphere with mass M and radius R which has constant density ρ throughout the interior.

   (a) Use dimensional analysis to make an order-of-magnitude guess for the potential energy of the sphere. (*HINT: think about eq. 8, and what parameters are available in this problem.*)
   (b) Compute ρ, and use it to solve for the enclosed mass $M(< r)$ as a function of radius.
   (c) To compute the gravitational potential energy, let's imagine that we have already created a small portion of the sphere with radius r, and are now adding a thin shell with thickness Δr (see Figure 3).
   Use eq. 8 to compute the additional potential energy ΔU (in terms of Δr) from adding this shell.
   (d) Convert to the infinitesimal limit (dU in terms of dr) and integrate through the whole sphere to find U in terms of M and R. How does this answer compare to your initial guess (3a)?

4. In 3a, you used dimensional analysis to estimate the gravitational potential for a sphere of mass M and radius R should be something like $U = -\frac{GM^2}{R}$. The full solution is:

$$U = -\alpha \frac{GM^2}{R}\,, \tag{10}$$

[1] https://en.wikipedia.org/wiki/Gauss's_law_for_gravity

[2] This is not the only way to reach the correct answer. The same result can be derived outside-in, or in any haphazard order you want. Gauss' law requires the results to be the same.

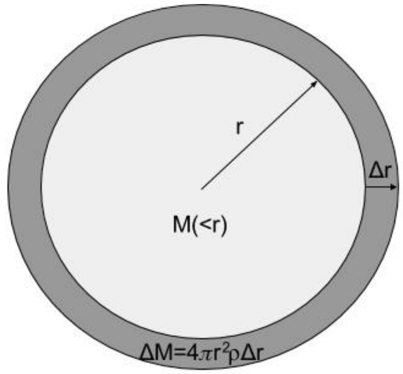

Figure 3: Figure for Problem 3c.

where α is a dimensionless factor of order 1 that is determined by how the mass is concentrated within the sphere.

(a) What is α for a sphere with constant density (see 3d)?

(b) If the sphere had higher density in the center than in the outer layers, would α be larger or smaller than the value above?

Milky Way Rotation Curve

5. Let's approximate the Milky Way galaxy as a sphere rather than a disk (this is not a terrible assumption, it turns out). We approximate it as having a constant density ρ which extends from $r = 0$ to some terminal radius $r = R$, at which point the density goes to zero.

   (a) What is the enclosed mass at each radius, $M(< r)$?

   (b) If a star or a blob of gas of mass m is in a circular orbit around the Milky Way (with $r < R$), what is its velocity $v(r)$? How does this scale with radius?[3]

   (c) What about at $r > R$? How does $v(r)$ scale with radius outside the galaxy?

   (d) Sketch the entire profile $v(r)$ as a function of radius, from $r = 0$ to $r \gg R$. (*HINT: this is most easily shown as a "log-log" plot, which Ben will discuss*).

   (e) Even though the vast majority of the visible mass of the Milky Way (stars, gas, etc.) is contained within about 15 kpc, there are a few objects orbiting far outside the central galaxy, allowing us to measure $v(r)$. Instead of decreasing with radius, $v(r)$ appears to remain constant out to several hundred kpc. What do you think this means?

[3]If a function "scales with radius" as r^α, this is commonly denoted as $f(r) \propto r^\alpha$.