AB

I. THE FOUR PATH PARADOX

This is a version of the three box paradox where instead of measuring one hermitian observable on a three level system we measure a sequential (non-hermitian) observable on a two level system. Apart from notation it is essentially the same setup as the sequential weak measurement example [1]

The system is prepared in the state

$$|\psi\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$$

and post-selected in the state

$$|\Phi\rangle = \cos\phi\,|+\rangle + \sin\phi\,|-\rangle$$

. In the intermediate time two measurements are made first a measurement of the state of the particle in the Z , $(0,1)$ basis and later its state in the X $(+,-)$ basis. There are four possible results to this measurement

$$(0,+),(1,+),(0,-),(1,-)$$

The measurement device is set in such a way (to be described later) that it clicks only for one of the four possible outcomes. In a similar way to the three box paradox the probability for a click on a measurement of two of the four pairs is always 1.

We begin by defining the four sequential operators to be measured

$$A = |+\rangle\langle+|\,|0\rangle\langle 0| = \frac{1}{\sqrt{2}}\,|+\rangle\langle 0| \tag{1}$$

$$B = |+\rangle\langle+|\,|1\rangle\langle 1| = \frac{1}{\sqrt{2}}\,|+\rangle\langle 1| \tag{2}$$

$$C = |-\rangle\langle-|\,|0\rangle\langle 0| = \frac{1}{\sqrt{2}}\,|-\rangle\langle 0| \tag{3}$$

$$D = |-\rangle\langle-|\,|1\rangle\langle 1| = \frac{-1}{\sqrt{2}}\,|-\rangle\langle 1| \tag{4}$$

these add up to the identity $A+B+C+D=\mathbb{1}$.

To simplify calculations later we will define four transition amplitudes

$$a = \langle\Phi|\,A\,|\psi\rangle = \frac{\cos\theta\cos\phi}{\sqrt{2}} \tag{5}$$

$$b = \langle\Phi|\,B\,|\psi\rangle = \frac{\sin\theta\cos\phi}{\sqrt{2}} \tag{6}$$

$$c = \langle\Phi|\,C\,|\psi\rangle = \frac{\cos\theta\sin\phi}{\sqrt{2}} \tag{7}$$

$$d = \langle\Phi|\,D\,|\psi\rangle = \frac{-\sin\theta\sin\phi}{\sqrt{2}} \tag{8}$$

We would now like to define θ and ϕ in such a way that B and C click with certainty.

Using the ABL formula we get

$$p(B) = \frac{|\langle\Phi|\,B\,|\psi\rangle|^2}{|\langle\Phi|\,B\,|\psi\rangle|^2 + |\langle\Phi|\,A+C+D\,|\psi\rangle|^2} = \frac{b^2}{b^2+(a+c+d)^2} \tag{9}$$

$$p(C) = \frac{|\langle\Phi|\,C\,|\psi\rangle|^2}{|\langle\Phi|\,C\,|\psi\rangle|^2 + |\langle\Phi|\,A+B+D\,|\psi\rangle|^2} = \frac{c^2}{c^2+(a+b+d)^2} \tag{10}$$

Our conditions for $p(B)=p(C)=1$ are

$$
\begin{aligned}
a+c+d&=0 \\
a+b+d&=0 \\
b\neq 0\; c&\neq 0
\end{aligned} \tag{11}
$$

Subtracting these we get $c = b$ which translates to $\cot(\theta) = \cot(\phi)$. Using this and dividing the first equation by $\sin\theta\sin\phi$ we get

$$\cot^2\phi + \cot\phi - 1 = 0 \tag{12}$$

and the solution is

$$\cot\theta = \cot\phi = \frac{-1\pm\sqrt{5}}{2} \tag{13}$$

Let us now look at weak values.

$$A_w = \frac{a}{a+b+c+d} \tag{14}$$

$$B_w = \frac{b}{a+b+c+d} \tag{15}$$

$$C_w = \frac{c}{a+b+c+d} \tag{16}$$

$$D_w = \frac{d}{a+b+c+d} \tag{17}$$

$$\tag{18}$$

The conditions $a + c + d = 0$, $b = c$ can now be derived using the theorem regarding weak values of dichotomic measurements so $B_w = C_w = 1$. Since the probabilities should add up to one we have $D_w + A_w = -1$ which leads again to eq (12).

using the solution (13) we have

$$A_w = \frac{a}{b} = \cot\theta = \frac{-1\pm\sqrt{5}}{2} \tag{19}$$

$$B_w = \frac{b}{b} = 1 \tag{20}$$

$$C_w = \frac{c}{b} = 1 \tag{21}$$

$$D_w = \frac{d}{b} = -\tan\theta = \frac{2}{1\mp\sqrt{5}} \tag{22}$$

A. experimental setup

We define the unitary operators $CNOT$ and $C_{ij}R(g)$ as follows:

$$CNOT\,|0\rangle\,|\xi\rangle = |0\rangle\,|\xi\rangle \tag{23}$$

$$CNOT\,|1\rangle\,|\xi\rangle = |1\rangle\,\sigma_x\,|\xi\rangle \tag{24}$$

$$\tag{25}$$

and

$$C_{ij}R(g)\,|i\rangle\,|j\rangle\,|\psi\rangle = |i\rangle\,|j\rangle\,e^{ig\sigma_x}\,|\psi\rangle \tag{26}$$

$$C_{ij}R(g)\,|l\rangle\,|m\rangle\,|\psi\rangle = |l\rangle\,|m\rangle\,|\psi\rangle \qquad \langle l|\,\langle m|\,|i\rangle\,|j\rangle = 0 \tag{27}$$

We will also define the operator

$$R_{ij,kl}(g) = R(g) \qquad if\ i = k \& j = l \tag{28}$$
$$\mathbb{1} \qquad otherwize \tag{29}$$

The measurement procedure requires an ancilla and a meter. It works in the following way: First we perform $CNOT$ on the system and ancilla the measure the system in the Z basis. Next for a we perform $C_{ij}R(g)$ with (i,j) set to be the inverted sequential operator we want for example $(i,j) = (+,0)$ for a measurement of A. This rotates themeter by g around σ_x. Finally we erase the first measurement by post-selecting the ancilla in the $|+\rangle$ state. Failing this final step the procedure returns a faithful readout but incorrect post-measurement state.

We will follow the circuit for an arbitrary pre-and post selection. $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, $|\phi\rangle = \gamma|+\rangle + \delta|-\rangle$

1. Initial state: 1-system; 2-Ancilla 3- Meter
$$[\alpha|0\rangle + \beta|1\rangle]|0\rangle|0\rangle \tag{30}$$

2. CNOT on 1,2
$$[\alpha|0\rangle|0\rangle + \beta|1\rangle|1\rangle]|0\rangle \tag{31}$$

3. $C_{ij}R(g)$
$$C_{ij}R(g)[\alpha|+\rangle|0\rangle|0\rangle + \alpha|-\rangle|0\rangle|0\rangle + \beta|+\rangle|1\rangle|0\rangle - \beta|-\rangle|1\rangle|0\rangle] \tag{32}$$
$$=\alpha|+\rangle|0\rangle R_{ij,+0}(g)|0\rangle + \alpha|-\rangle|0\rangle R_{ij,-0}(g)|0\rangle + \beta|+\rangle|1\rangle R_{ij,+1}(g)|0\rangle - \beta|-\rangle|1\rangle R_{ij,-1}(g)|0\rangle \tag{33}$$

4. Erasure: post selecting 2 on $|+\rangle$ and discarding
$$\alpha|+\rangle R_{ij,+0}(g)|0\rangle + \alpha|-\rangle R_{ij,-0}(g)|0\rangle + \beta|+\rangle R_{ij,+1}(g)|0\rangle - \beta|-\rangle R_{ij,-1}(g)|0\rangle \tag{34}$$
This state is not normalized.

5. Post selection in $|\phi\rangle = \gamma|+\rangle + \delta|-\rangle$
$$\alpha\gamma R_{ij,+0}(g)|0\rangle + \alpha\delta R_{ij,-0}(g)|0\rangle + \beta\gamma R_{ij,+1}(g)|0\rangle - \beta\delta R_{ij,-1}(g)|0\rangle \tag{35}$$
We now have four cases coresponding to the choices of i,j

   (A) $(i,j) = (+,0)$
$$\alpha\gamma R(g)|0\rangle + \alpha\delta|0\rangle + \beta\gamma|0\rangle - \beta\delta|0\rangle \tag{36}$$
   (B) $(i,j) = (+,1)$
$$\alpha\gamma|0\rangle + \alpha\delta|0\rangle + \beta\gamma R(g)|0\rangle - \beta\delta|0\rangle \tag{37}$$
   (C) $(i,j) = (-,0)$
$$\alpha\gamma|0\rangle + \alpha\delta R(g)|0\rangle + \beta\gamma|0\rangle - \beta\delta|0\rangle \tag{38}$$
   (D) $(i,j) = (-,1)$
$$\alpha\gamma|0\rangle + \alpha\delta|0\rangle + \beta\gamma|0\rangle - \beta\delta R(g)|0\rangle \tag{39}$$

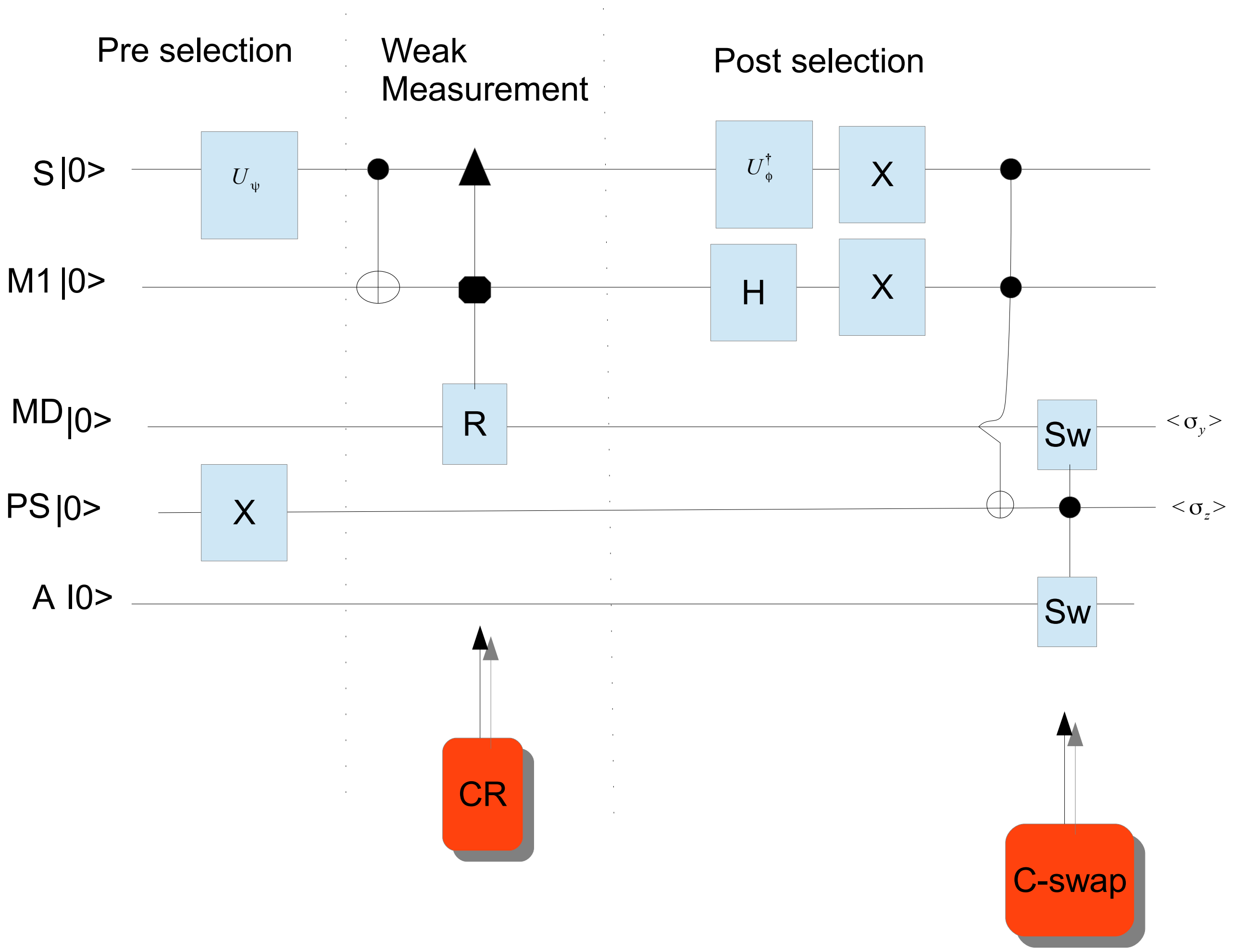
B. A different description

In principle one can take the state after step 2 i.e eq (2) to be the pre-selection and the state $[\gamma|+\rangle + \delta|-\rangle]|+\rangle$ as the post selection..............

II. THE NMR EXPERIMENT

The NMR version requires 5 -qubits labled S - the system,$M1$- the first meter (which will be erased later),MD- the second meter (which contains the result),PS- a pointer for post-selection and A an ancilla which will be used for re-setting MD when post selection fails.

The circuit is can be seen in the figure.



Before going into the procedure in details I will go over the non-trivial gates. For now we will leave θ and ϕ as parameters as they will each have a number of different values for the experiments. The two single gate unitaries are.

$$U_\psi |0\rangle = |\psi\rangle = \cos\theta |0\rangle + \sin\theta |1\rangle \tag{40}$$

likewize

$$U_\Phi |0\rangle = |\Phi\rangle = \cos\phi |+\rangle + \sin\phi |-\rangle \tag{41}$$

There is some freedom on how they are implemented.

Next we have the gate CR (triangle-octagon-R) first we define

$$R = e^{ig\sigma_x}$$

where g is a parameter to be chosen later next we have the control on S (triangle) to be in the $\left|+\right\rangle \left\langle +\right|$ for experiments A,B and $\left|-\right\rangle \left\langle -\right|$ for experiments C,D. The control on $M1$ (octagon) to be in the $\left|0\right\rangle \left\langle 0\right|$ for experiments A,C and $\left|1\right\rangle \left\langle 1\right|$ for experiments B,D.

CR is now a controlled controlled rotation about X with phase g. For example in A type experiments the corresponding interaction Hamiltonian would be $H_R = \left|+\right\rangle \left\langle +\right| \left|0\right\rangle \left\langle 0\right| \sigma_X$ and $CR = e^{igH_R}$. More generally

$$H_{ij} = \left|i\right\rangle \left\langle i\right|^S \otimes \left|j\right\rangle \left\langle j\right|^{M1} \sigma_x^{MD} \qquad i \in \{+,-\}\ j \in \{0,1\} \tag{42}$$

$$CR_{ij}(g) = e^{igH_{ij}} \tag{43}$$

with four possible settings: $A-(+0)$, $B-(+1)$, $C-(-0)$, $D-(-1)$

We are mainly interested in two possible pre-post selection pairs

$$\cot\theta = \cot\phi = \frac{-1-\sqrt{5}}{2} \tag{44}$$

and

$$\theta = \phi = \Pi/4$$

We can also try inverting the pre and post selection.

III. REFERENCES

[1] G. Mitchison, R. Jozsa, and S. Popescu, Physical Review A **76** (2007), URL `http://dx.doi.org/10.1103/PhysRevA.76.062105`.