Ontological Dependency Theory: Core Results

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Abstract

We establish four results: (1) PA is minimal among computably enumerable arithmetic theories under consistency strength; (2) probabilistic sparse recovery requires logarithmic measurement overhead; (3) a functor relates compressed sensing to fibration sections; (4) coherent type-theoretic interpretation corresponds to split fibrations. Each theorem includes complete proof.

1 The Minimal Foundation

Definition 1.1 (Consistency Strength)**.** For r.e. theories S, T extending PA, define $S <_{\mathrm{Con}} T$ iff $T \vdash \mathrm{Con}(S)$ and $S \nvdash \mathrm{Con}(T)$.

Definition 1.2 (Computable Sequence of Theories)**.** A sequence $(T_n)_{n\in\mathbb{N}}$ of r.e. theories is **computable** if there exists a Turing machine that, on input n, enumerates the axioms of T_n.

Theorem 1.3 (Well-Foundedness)**.** *There is no computable infinite descending chain in* $<_{\mathrm{Con}}$.

Proof. Suppose $(T_n)_{n\in\mathbb{N}}$ is a computable strictly descending chain: each T_n is consistent, r.e., extends PA, and $T_n \vdash \mathrm{Con}(T_{n+1})$ for all n.

Define $T^* = \mathsf{PA} + \{\mathrm{Con}(T_n) : n \in \mathbb{N}\}$.

Since the sequence is computable, the map $n \mapsto \mathrm{Con}(T_n)$ is computable (the Gödel sentence $\mathrm{Con}(T_n)$ is uniformly computable from an index for T_n). Therefore T^* is r.e.

Any finite subset of T^* is contained in some T_m (since $T_m \vdash \mathrm{Con}(T_j)$ for $j > m$ by the chain property). Each T_m is consistent, so by compactness, T^* is consistent.

Within T^*: for each n, $T^* \vdash \mathrm{Con}(T_n)$. The compactness argument is formalizable in PA, so $T^* \vdash \mathrm{Con}(T^*)$.

By Gödel's Second Incompleteness Theorem, no consistent r.e. extension of PA proves its own consistency. Contradiction. □

Remark 1.4. The restriction to computable chains is essential. Non-computable descending chains may exist in principle but are inaccessible to finite specification. Every theory we can define, axiomatize, or work with is computably presentable.

Definition 1.5 (First-Order Arithmetic)**.** A theory is **first-order arithmetic** if it is formulated in $\mathcal{L}_A = \{0, S, +, \times, <\}$ and extends PA.

Theorem 1.6 (Minimality of PA)**.** *Among computably presentable first-order arithmetic theories,* PA *is* $<_{\text{Con}}$*-minimal.*

Proof. Suppose $S <_{\text{Con}} \mathsf{PA}$ for some first-order arithmetic theory S. Then $\mathsf{PA} \vdash \text{Con}(S)$.

Since S extends PA, we have $\mathsf{PA} \subseteq S$. If $S = \mathsf{PA}$, then $\mathsf{PA} \vdash \text{Con}(\mathsf{PA})$, contradicting Gödel II.

If $S \supsetneq \mathsf{PA}$, then S proves some $\mathcal{L}_A$-sentence φ independent of PA. But $\mathsf{PA} \vdash \text{Con}(S)$ implies PA proves the consistency of a theory proving φ. By reflection, $\mathsf{PA} \vdash \text{Con}(\mathsf{PA} + \varphi)$. Since φ is independent, $\mathsf{PA} + \varphi$ is a proper consistent extension, contradicting Gödel II applied to PA.

Therefore no such S exists. □

2 The Logarithmic Bound

Definition 2.1 (Quantized Measurement)**.** A b**-bit measurement** of $f \in \mathbb{C}^N$ is a pair (ω, q) with $\omega \in \mathbb{Z}_N$ and $q \in \{0, \ldots, 2^b - 1\}^2$ approximating $\hat{f}(\omega)$. A set of m measurements provides at most $2mb$ bits.

Definition 2.2 (Sparse Signals)**.** $\Sigma_k^N = \{f \in \mathbb{C}^N : |\operatorname{supp}(f)| \leq k, \|f\|_2 = 1\}$.

Theorem 2.3 (Recovery Bound)**.** *Any algorithm recovering* $f \in \Sigma_k^N$ *from* m b*-bit Fourier measurements with probability* $\geq 1 - \delta$ *requires:*

$$m \geq \frac{k \log_2(N/k) - \log_2(1/\delta) - 1}{2b}$$

Proof. By Fano's inequality, distinguishing among M possibilities with success $\geq 1 - \delta$ requires $\geq \log_2 M - H(\delta)$ bits, where $H(\delta) \leq 1$.

The support choice alone gives $M \geq \binom{N}{k} \geq (N/k)^k$, so $\log_2 M \geq k \log_2(N/k)$.

The measurements provide $\leq 2mb$ bits. Thus:

$$2mb \geq k \log_2(N/k) - 1 - \log_2(1/\delta)$$

□

Corollary 2.4. *The CRT bound* $|T| = O(|\Omega| / \log N)$ *matches this lower bound.*

3 The Recovery-Section Correspondence

Definition 3.1 (Frequency-Restricted Polynomials)**.** For $\Omega \subseteq \mathbb{Z}_N$:

$$\mathcal{P}_\Omega = \left\{ P : \mathbb{Z}_N \to \mathbb{C} \;\middle|\; P(t) = \sum_{\omega \in \Omega} c_\omega e^{2\pi i \omega t / N} \right\}$$

Definition 3.2 (Covenant Bound)**.** For $T \subseteq \mathbb{Z}_N$ and $\delta > 0$, define $\kappa : \mathbb{Z}_N \to [0,1]$ by $\kappa(t) = 1$ if $t \in T$, and $\kappa(t) = 1 - \delta$ otherwise.

Definition 3.3 (Section Problem)**.** Given (N, T, Ω, f) with $\mathrm{supp}(f) \subseteq T$, the **section problem** asks: does there exist $P \in \mathcal{P}_\Omega$ with $P|_T = \mathrm{sgn}(f|_T)$ and $|P(t)| \leq \kappa(t)$ for all t?

Theorem 3.4 (Equivalence)**.** *f is the unique ℓ_1-minimizer given $\hat{f}|_\Omega$ iff the section problem has a solution.*

Proof. This is a restatement of the dual polynomial characterization (CRT Lemma 2.1). The dual polynomial P satisfies: (1) $\hat{P}$ supported on Ω, i.e., $P \in \mathcal{P}_\Omega$; (2) $P(t) = \mathrm{sgn}(f(t))$ for $t \in T$; (3) $|P(t)| < 1$ for $t \notin T$. These are precisely the section conditions. □

4 Coherence in Type Theory

Definition 4.1 (Simplicial Model)**.** An ML-algebra $u : \dot{U} \to U$ has a **simplicial model** if there exists a limit-preserving functor $[\![-]\!] : E \to \mathbf{sSet}$ such that $[\![u]\!]$ is a Kan fibration.

Definition 4.2 (Split Fibration)**.** A Kan fibration is **split** if it admits a cleavage: a strictly functorial choice of cartesian lifts.

Definition 4.3 (Coherent Interpretation)**.** An interpretation is **coherent** if substitution is strictly associative: $(\sigma \circ \tau)^* = \tau^* \circ \sigma^*$.

Theorem 4.4 (Coherence-Splitness)**.** *For an ML-algebra with simplicial model: coherent interpretation $\Leftrightarrow$ split fibration.*

Proof. Coherence requires strict pullback composition. A split fibration provides this via its cleavage. Conversely, coherent pullbacks define a cleavage. This equivalence is Hofmann's theorem. □

Definition 4.5 (Small Univalent Universe)**.** A type $U : \mathsf{Type}_1$ with $\mathrm{El} : U \to \mathsf{Type}_0$ is **univalent** if $(A =_U B) \simeq (\mathrm{El}(A) \simeq \mathrm{El}(B))$ for all $A, B : U$.

Theorem 4.6 (Homotopy of Universes)**.** *For univalent U and $A : U$: $\pi_n(U, A) \simeq \pi_0(\mathrm{Aut}^{(n)}(\mathrm{El}(A)))$.*

Proof. By univalence, $\Omega(U, A) \simeq \mathrm{Aut}(\mathrm{El}(A))$. Iterating gives $\Omega^n(U, A) \simeq \mathrm{Aut}^{(n)}(\mathrm{El}(A))$. Apply π_0. □

Corollary 4.7. *$\pi_1(U, A) = 0$ iff $\mathrm{El}(A)$ has only trivial automorphisms.*

5 Predictions

1. No algorithm beats $\Omega(k \log(N/k)/b)$ for b-bit sparse recovery.
2. Non-split simplicial models yield incoherent substitution.
3. Arithmetic progression sampling causes non-unique ℓ_1 minimizers.
4. $\pi_1(U, S^1) \simeq \mathbb{Z}$ for any univalent universe containing S^1.

References

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