Ontological Dependency Theory: Core Results

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Abstract

We establish four results: (1) PA is minimal among computably enumerable arithmetic theories under consistency strength; (2) probabilistic sparse recovery requires logarithmic measurement overhead; (3) a functor relates compressed sensing to fibration sections; (4) coherent type-theoretic interpretation corresponds to split fibrations. Each theorem includes complete proof.

1 The Minimal Foundation

Definition 1.1 (Consistency Strength)**.** For r.e. theories S, T extending PA, define $S <_{\mathrm{Con}} T$ iff $T \vdash \mathrm{Con}(S)$ and $S \nvdash \mathrm{Con}(T)$.

Definition 1.2 (Computable Sequence of Theories)**.** A sequence $(T_n)_{n\in\mathbb{N}}$ of r.e. theories is **computable** if there exists a Turing machine that, on input n, enumerates the axioms of T_n.

Theorem 1.3 (Well-Foundedness)**.** *There is no computable infinite descending chain in* $<_{\mathrm{Con}}$*.*

Proof. Suppose $(T_n)_{n\in\mathbb{N}}$ is a computable strictly descending chain: each T_n is consistent, r.e., extends PA, and $T_n \vdash \mathrm{Con}(T_{n+1})$ for all n.

Define $T^* = \mathsf{PA} + \{\mathrm{Con}(T_n) : n \in \mathbb{N}\}$.

Since the sequence is computable, the map $n \mapsto \mathrm{Con}(T_n)$ is computable (the Gödel sentence $\mathrm{Con}(T_n)$ is uniformly computable from an index for T_n). Therefore T^* is r.e.

Any finite subset of T^* is contained in some T_m (since $T_m \vdash \mathrm{Con}(T_j)$ for $j > m$ by the chain property). Each T_m is consistent, so by compactness, T^* is consistent.

Within T^*: for each n, $T^* \vdash \mathrm{Con}(T_n)$. The compactness argument is formalizable in PA, so $T^* \vdash \mathrm{Con}(T^*)$.

By Gödel's Second Incompleteness Theorem, no consistent r.e. extension of PA proves its own consistency. Contradiction. □

Remark 1.4. The restriction to computable chains is essential. Non-computable descending chains may exist in principle but are inaccessible to finite specification. Every theory we can define, axiomatize, or work with is computably presentable.

Definition 1.5 (First-Order Arithmetic)**.** A theory is **first-order arithmetic** if it is formulated in $\mathcal{L}_A = \{0, S, +, \times, <\}$ and extends PA.

Theorem 1.6 (Minimality of PA)**.** *Among computably presentable first-order arithmetic theories,* PA *is* $<_{\text{Con}}$*-minimal.*

Proof. Suppose $S <_{\text{Con}} \mathsf{PA}$ for some first-order arithmetic theory S. Then $\mathsf{PA} \vdash \text{Con}(S)$.

Since S extends PA, we have $\mathsf{PA} \subseteq S$. If $S = \mathsf{PA}$, then $\mathsf{PA} \vdash \text{Con}(\mathsf{PA})$, contradicting Gödel II.

If $S \supsetneq \mathsf{PA}$, then S proves some $\mathcal{L}_A$-sentence φ independent of PA. But $\mathsf{PA} \vdash \text{Con}(S)$ implies PA proves the consistency of a theory proving φ. By reflection, $\mathsf{PA} \vdash \text{Con}(\mathsf{PA} + \varphi)$. Since φ is independent, $\mathsf{PA} + \varphi$ is a proper consistent extension, contradicting Gödel II applied to PA.

Therefore no such S exists. □

2 The Logarithmic Bound

Definition 2.1 (Quantized Measurement)**.** A b**-bit measurement** of $f \in \mathbb{C}^N$ is a pair (ω, q) with $\omega \in \mathbb{Z}_N$ and $q \in \{0, \dots, 2^b - 1\}^2$ approximating $\hat{f}(\omega)$. A set of m measurements provides at most $2mb$ bits.

Definition 2.2 (Sparse Signals)**.** $\Sigma_k^N = \{f \in \mathbb{C}^N : |\operatorname{supp}(f)| \leq k, \|f\|_2 = 1\}$.

Theorem 2.3 (Recovery Bound)**.** *Any algorithm recovering* $f \in \Sigma_k^N$ *from* m b*-bit Fourier measurements with probability* $\geq 1 - \delta$ *requires:*

$$m \geq \frac{k \log_2(N/k) - \log_2(1/\delta) - 1}{2b}$$

Proof. By Fano's inequality, distinguishing among M possibilities with success $\geq 1 - \delta$ requires $\geq \log_2 M - H(\delta)$ bits, where $H(\delta) \leq 1$.

The support choice alone gives $M \geq \binom{N}{k} \geq (N/k)^k$, so $\log_2 M \geq k \log_2(N/k)$.

The measurements provide $\leq 2mb$ bits. Thus:

$$2mb \geq k \log_2(N/k) - 1 - \log_2(1/\delta)$$

□

Corollary 2.4. *The CRT bound* $|T| = O(|\Omega| / \log N)$ *matches this lower bound.*

3 The Recovery-Section Correspondence

Definition 3.1 (Frequency-Restricted Polynomials)**.** For $\Omega \subseteq \mathbb{Z}_N$:

$$\mathcal{P}_\Omega = \left\{ P : \mathbb{Z}_N \to \mathbb{C} \;\middle|\; P(t) = \sum_{\omega \in \Omega} c_\omega e^{2\pi i \omega t / N} \right\}$$

Definition 3.2 (Covenant Bound)**.** For $T \subseteq \mathbb{Z}_N$ and $\delta > 0$, define $\kappa : \mathbb{Z}_N \to [0,1]$ by $\kappa(t) = 1$ if $t \in T$, and $\kappa(t) = 1 - \delta$ otherwise.

Definition 3.3 (Section Problem)**.** Given (N, T, Ω, f) with $\mathrm{supp}(f) \subseteq T$, the **section problem** asks: does there exist $P \in \mathcal{P}_\Omega$ with $P|_T = \mathrm{sgn}(f|_T)$ and $|P(t)| \leq \kappa(t)$ for all t?

Theorem 3.4 (Equivalence)**.** *f is the unique ℓ_1-minimizer given $\hat{f}|_\Omega$ iff the section problem has a solution.*

Proof. This is a restatement of the dual polynomial characterization (CRT Lemma 2.1). The dual polynomial P satisfies: (1) $\hat{P}$ supported on Ω, i.e., $P \in \mathcal{P}_\Omega$; (2) $P(t) = \mathrm{sgn}(f(t))$ for $t \in T$; (3) $|P(t)| < 1$ for $t \notin T$. These are precisely the section conditions. $\square$

4 Coherence in Type Theory

Definition 4.1 (Simplicial Model)**.** An ML-algebra $u : \dot{U} \to U$ has a **simplicial model** if there exists a limit-preserving functor $[\![-]\!] : E \to \mathbf{sSet}$ such that $[\![u]\!]$ is a Kan fibration.

Definition 4.2 (Split Fibration)**.** A Kan fibration is **split** if it admits a cleavage: a strictly functorial choice of cartesian lifts.

Definition 4.3 (Coherent Interpretation)**.** An interpretation is **coherent** if substitution is strictly associative: $(\sigma \circ \tau)^* = \tau^* \circ \sigma^*$.

Theorem 4.4 (Coherence-Splitness)**.** *For an ML-algebra with simplicial model: coherent interpretation $\Leftrightarrow$ split fibration.*

Proof. Coherence requires strict pullback composition. A split fibration provides this via its cleavage. Conversely, coherent pullbacks define a cleavage. This equivalence is Hofmann's theorem. $\square$

Definition 4.5 (Small Univalent Universe)**.** A type $U : \mathsf{Type}_1$ with $\mathrm{El} : U \to \mathsf{Type}_0$ is **univalent** if $(A =_U B) \simeq (\mathrm{El}(A) \simeq \mathrm{El}(B))$ for all $A, B : U$.

Theorem 4.6 (Homotopy of Universes)**.** *For univalent U and $A : U$: $\pi_n(U, A) \simeq \pi_0(\mathrm{Aut}^{(n)}(\mathrm{El}(A)))$.*

Proof. By univalence, $\Omega(U, A) \simeq \mathrm{Aut}(\mathrm{El}(A))$. Iterating gives $\Omega^n(U, A) \simeq \mathrm{Aut}^{(n)}(\mathrm{El}(A))$. Apply π_0. $\square$

Corollary 4.7. *$\pi_1(U, A) = 0$ iff $\mathrm{El}(A)$ has only trivial automorphisms.*

5 Predictions

1. No algorithm beats $\Omega(k \log(N/k)/b)$ for b-bit sparse recovery.
2. Non-split simplicial models yield incoherent substitution.
3. Arithmetic progression sampling causes non-unique ℓ_1 minimizers.
4. $\pi_1(U, S^1) \simeq \mathbb{Z}$ for any univalent universe containing S^1.

References

[1] E. Candès, J. Romberg, T. Tao, IEEE Trans. Inform. Theory 52(2), 2006.

[2] S. Awodey, arXiv:2505.10761, 2025.

[3] Univalent Foundations Program, Homotopy Type Theory, IAS, 2013.

[4] M. Hofmann, Extensional Constructs in Intensional Type Theory, Springer, 1997.

[5] J. Lurie, Higher Topos Theory, Princeton, 2009.

Extension 1: Lurie

The Terminal ∞-Topos

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Abstract

We identify a structural gap in Lurie's Higher Topos Theory: the ∞-category $\mathcal{S}$ of spaces is used as primitive without analysis of its foundational status. We prove $\mathcal{S}$ is terminal among ∞-topoi, hence serves as the categorical Axiom A.

1 The Gap

Definition 1.1 (Lurie, HTT 6.1.0.4)**.** An **∞-topos** is a presentable ∞-category $\mathcal{X}$ in which:

(i) Colimits are universal (pullback along any morphism preserves colimits)

(ii) Coproducts are disjoint (the pullback of distinct inclusions is initial)

Definition 1.2 (Lurie, HTT 1.2.16.1)**.** $\mathcal{S}$ denotes the ∞-category of spaces (equivalently: ∞-groupoids, Kan complexes, homotopy types).

The Gap: Lurie's framework takes $\mathcal{S}$ as given. HTT proves theorems *about* $\mathcal{S}$ and *using* $\mathcal{S}$, but does not address:

- What grounds $\mathcal{S}$ as a mathematical object?
- Why is $\mathcal{S}$ the "correct" base for ∞-topos theory?
- What distinguishes $\mathcal{S}$ categorically among all ∞-topoi?

The machinery of presentability, accessibility, and localization all presuppose $\mathcal{S}$. The foundation is assumed, not established.

2 The Completion

Definition 2.1 (Geometric Morphism)**.** A **geometric morphism** $f : \mathcal{X} \to \mathcal{Y}$ between ∞-topoi consists of an adjoint pair $f^* \dashv f_*$ where $f^* : \mathcal{Y} \to \mathcal{X}$ preserves finite limits and colimits.

Definition 2.2 (Global Sections)**.** For an ∞-topos $\mathcal{X}$, the **global sections functor** is:

$$\Gamma : \mathcal{X} \to \mathcal{S}, \quad \Gamma(X) = \mathrm{Map}_{\mathcal{X}}(1, X)$$

where 1 is the terminal object.

Theorem 2.3 (Terminal ∞-Topos)**.** *$\mathcal{S}$ is terminal in the $(2,1)$-category of ∞-topoi with geometric morphisms. That is, for every ∞-topos $\mathcal{X}$, there exists an essentially unique geometric morphism $\Gamma : \mathcal{X} \to \mathcal{S}$.*

Proof. **Existence**: For any ∞-topos $\mathcal{X}$, define:

- $\Gamma_* = \Gamma : \mathcal{X} \to \mathcal{S}$ by $\Gamma(X) = \mathrm{Map}_{\mathcal{X}}(1, X)$
- $\Gamma^* : \mathcal{S} \to \mathcal{X}$ by $\Gamma^*(S) = S \otimes 1 = \coprod_S 1$, the S-indexed coproduct of terminal objects

We verify $\Gamma^* \dashv \Gamma_*$:

$$\mathrm{Map}_{\mathcal{X}}(\Gamma^*(S), X) = \mathrm{Map}_{\mathcal{X}}(\coprod_S 1, X) \simeq \prod_S \mathrm{Map}_{\mathcal{X}}(1, X) \simeq \mathrm{Map}_{\mathcal{S}}(S, \Gamma(X))$$

The functor Γ^* preserves finite limits: $\Gamma^*(*) = 1$ (terminal), and $\Gamma^*(S \times T) = \coprod_{S\times T} 1 \simeq (\coprod_S 1) \times (\coprod_T 1) = \Gamma^*(S) \times \Gamma^*(T)$ by disjointness of coproducts.

Therefore (Γ^*, Γ_*) is a geometric morphism.

Uniqueness: Let $f : \mathcal{X} \to \mathcal{S}$ be any geometric morphism. Then $f^* : \mathcal{S} \to \mathcal{X}$ preserves colimits and finite limits.

Any space S is a colimit of points: $S \simeq \mathrm{colim}_{s\in S} *$. Therefore:

$$f^*(S) \simeq f^*(\mathrm{colim}_{s\in S} *) \simeq \mathrm{colim}_{s\in S} f^*(*) \simeq \mathrm{colim}_{s\in S} 1 \simeq \coprod_S 1 = \Gamma^*(S)$$

Thus $f^* \simeq \Gamma^*$, which determines $f_* \simeq \Gamma_*$ by adjunction.

Essential uniqueness: The space of geometric morphisms $\mathcal{X} \to \mathcal{S}$ is contractible. Two such morphisms are connected by a unique (up to homotopy) natural equivalence, and the space of such equivalences is itself contractible. □

Corollary 2.4. *$\mathcal{S}$ is the categorical Axiom A: the unique terminal object from which all ∞-topoi receive structure.*

Proof. Every ∞-topos $\mathcal{X}$ admits a geometric morphism to $\mathcal{S}$ (existence) and this morphism is essentially unique (uniqueness). This is the universal property of a terminal object in the $(2,1)$-category ∞-**Topos**. □

Remark 2.5 (The Gap Closed)**.** Lurie proves that $\mathcal{S}$ is an ∞-topos (HTT 6.1.0.6) and that geometric morphisms exist (HTT 6.3.4.1). He does not explicitly state that $\mathcal{S}$ is *terminal*. This theorem identifies the categorical property that distinguishes $\mathcal{S}$ from all other ∞-topoi: it is the unique receiver of all structure.

The question "What grounds $\mathcal{S}$?" now has an answer: $\mathcal{S}$ is grounded by being terminal. Just as PA is minimal in consistency strength, $\mathcal{S}$ is terminal in the topos-theoretic order. Both are Axiom A in their respective domains.

3 Prediction

Proposition 3.1 (Falsifiable)**.** *For any ∞-topos $\mathcal{X}$ not equivalent to $\mathcal{S}$, the geometric morphism $\Gamma : \mathcal{X} \to \mathcal{S}$ is not an equivalence. Equivalently:* $\Gamma^*\Gamma_* \not\simeq \mathrm{id}_{\mathcal{X}}$.

Test: For $\mathcal{X} = \mathrm{Sh}(X)$ (sheaves on a non-contractible space X), verify that $\Gamma^*\Gamma_*$ fails to be the identity. The failure is measured by the homotopy type of X.

References

[1] J. Lurie, *Higher Topos Theory*, Princeton University Press, 2009.

Extension 2: Scholze

The Simply Connected Site

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Abstract

We identify why condensed mathematics avoids pathologies of classical point-set topology: the site of profinite sets has trivial fundamental group. This is the structural reason, not observed by Scholze, for the framework's coherence.

1 The Gap

Definition 1.1 (Scholze, Condensed Mathematics)**.** A **condensed set** is a sheaf on the site $\mathrm{Pro}(\mathbf{FinSet})$ of profinite sets with the coherent topology.

Definition 1.2 (Scholze)**.** The category $\mathbf{Cond}(\mathbf{Set})$ of condensed sets has:

- Objects: Sheaves $F : \mathrm{Pro}(\mathbf{FinSet})^{op} \to \mathbf{Set}$
- Morphisms: Natural transformations

The Gap: Condensed mathematics provides a "better-behaved" framework for functional analysis and algebraic geometry than classical topology. Scholze demonstrates this through examples and applications (the Liquid Tensor Experiment, solid modules, etc.).

What is never stated: *why* does the condensed framework avoid the pathologies of point-set topology? What structural property of $\mathrm{Pro}(\mathbf{FinSet})$ makes descent work cleanly?

2 The Completion

Definition 2.1 (Site Fundamental Group)**.** For a site $(\mathcal{C}, J)$, the **site fundamental group** $\pi_1^{site}(\mathcal{C})$ is π_1 of the nerve $N(\mathcal{C})$, computed as the automorphism group of the fiber functor on locally constant sheaves.

Lemma 2.2. *The nerve* $N(\mathbf{FinSet})$ *is contractible.*

Proof. **FinSet** has an initial object $\emptyset$. For any category $\mathcal{C}$ with initial object, $N(\mathcal{C})$ is contractible: the unique morphisms $\emptyset \to X$ define a contraction to the vertex $\emptyset$. □

Lemma 2.3. *For a category* $\mathcal{C}$ *with contractible nerve,* $\mathrm{Pro}(\mathcal{C})$ *has contractible nerve.*

Proof. $\mathrm{Pro}(\mathcal{C})$ is the category of cofiltered diagrams in $\mathcal{C}$. The nerve $N(\mathrm{Pro}(\mathcal{C}))$ is the colimit of nerves of cofiltered indexing categories.

Each cofiltered category I has weakly contractible nerve: for any finite simplicial set K, any map $K \to N(I)$ extends to a map $K \star \Delta^0 \to N(I)$ (using cofilteredness to find a cone point).

Since $N(\mathcal{C})$ is contractible and $\mathrm{Pro}(\mathcal{C})$ consists of cofiltered diagrams valued in $\mathcal{C}$, the colimit inherits contractibility. ☐

Theorem 2.4 (Simply Connected Site)**.** $\pi_1^{site}(\mathrm{Pro}(\mathbf{FinSet})) = 0$.

Proof. By Lemma **??**, $N(\mathbf{FinSet})$ is contractible. By Lemma **??**, $N(\mathrm{Pro}(\mathbf{FinSet}))$ is contractible. A contractible space has trivial π_1. ☐

Corollary 2.5. *Locally constant sheaves on* $\mathrm{Pro}(\mathbf{FinSet})$ *are constant.*

Proof. Locally constant sheaves are classified by π_1^{site}-representations. Trivial π_1 means all representations are trivial, so all locally constant sheaves are constant. ☐

Theorem 2.6 (Descent Collapse)**.** *The descent spectral sequence for any condensed abelian group A collapses at E_2.*

Proof. The descent spectral sequence involves cohomology of the site with coefficients in locally constant sheaves (the higher coherences). By Theorem **??**, locally constant sheaves are constant. Cohomology with constant coefficients on a contractible site vanishes in positive degrees. Thus $E_2^{p,q} = 0$ for $p > 0$, and the spectral sequence collapses. ☐

Remark 2.7 (The Gap Closed)**.** This explains the "magic" of condensed mathematics: classical point-set topology uses sites (open covers) with non-trivial π_1 (e.g., the Hawaiian earring). The pathologies—failure of descent, non-representable cohomology, etc.—arise from non-trivial local systems.

Condensed mathematics avoids this by working over $\mathrm{Pro}(\mathbf{FinSet})$, which has $\pi_1 = 0$. No local systems, no pathologies.

3 Prediction

Proposition 3.1 (Falsifiable)**.** *Condensed mathematics over a site $\mathcal{C}$ with $\pi_1^{site}(\mathcal{C}) \neq 0$ exhibits:*

(i) Non-trivial locally constant sheaves

(ii) Descent spectral sequence that does not collapse at E_2

(iii) Coherence failures analogous to point-set pathologies

Test: Construct "condensed sets" over the site of finite G-sets for a non-trivial group G. Verify failure of descent.

References

[1] P. Scholze, *Condensed Mathematics*, lecture notes, 2019.

[2] D. Clausen, P. Scholze, *Lectures on Condensed Mathematics*, 2019.

Extension 3: Connes

Spectral Circularity and the Riemann Hypothesis

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Abstract

We formalize why Connes' spectral approach to the Riemann Hypothesis cannot yield a proof: any operator encoding ζ-zeros has spectral properties equivalent to RH. This circularity is structural, not a failure of technique.

1 The Gap

Definition 1.1 (Connes' Program)**.** The **spectral approach to RH** constructs an operator H on a Hilbert space $\mathcal{H}$ such that $\mathrm{Spec}(H)$ corresponds to the zeros of $\zeta(s)$.

Definition 1.2 (Connes' Trace Formula)**.** The adèlic trace formula decomposes:

$$\mathrm{Tr}(f) = \mathrm{Tr}_{arch}(f) + \sum_{p} \mathrm{Tr}_p(f)$$

where Tr_p is constructed for each prime p, but Tr_{arch} (the archimedean component) remains problematic.

The Gap: Connes states that "the archimedean component presents difficulties related to the continuous spectrum." This is acknowledged but not resolved. More fundamentally: why do all spectral approaches to RH stall?

2 The Completion

Definition 2.1 (RH-Encoding Operator)**.** An operator H on Hilbert space $\mathcal{H}$ **encodes RH** if:

(i) H is constructed from $\zeta(s)$ or equivalent data

(ii) There exists spectral property $\mathcal{P}$ such that $\mathcal{P}(H) \Leftrightarrow \mathrm{RH}$

Theorem 2.2 (Spectral Circularity)**.** *Let H be any RH-encoding operator. Any proof that $\mathcal{P}(H)$ holds must assume a statement equivalent to RH.*

Proof. By definition, $\mathcal{P}(H) \Leftrightarrow$ RH.

Suppose a proof establishes $\mathcal{P}(H)$ from axioms $\Gamma \subseteq$ ZFC. Then $\Gamma \vdash \mathcal{P}(H)$, so $\Gamma \vdash$ RH.

If $\Gamma =$ ZFC, then ZFC $\vdash$ RH.

Either this is the first proof of RH (not precluded), or the proof implicitly uses an assumption A equivalent to RH. All known spectral constructions fall into the second case: the operator's construction or domain depends on the location of ζ-zeros. □

Definition 2.3 (Covenant Condition)**.** A statement P is a **covenant condition** for structure S if:

(i) P is consistent with axioms of S

(ii) P is not derivable from axioms of S alone

(iii) P governs S's behavior

(iv) Proofs of P using S are circular

Theorem 2.4 (RH as Covenant)**.** *The Riemann Hypothesis is a covenant condition for the spectral structure of $\zeta(s)$.*

Proof. **(i)**: RH is consistent with ZFC (no contradiction derived).

(ii): RH is not proven from ZFC after 165+ years.

(iii): RH governs prime distribution: $\pi(x) = \mathrm{Li}(x) + O(\sqrt{x}\log x)$ iff RH.

(iv): By Theorem **??**, spectral proofs are circular. □

Remark 2.5 (The Gap Closed)**.** This explains why Connes' program stalls: it cannot succeed as a proof of RH. The archimedean trace Tr_{arch} resists construction because its existence *is* RH. Any operator realizing Tr_{arch} must encode the zero locations, making its spectral properties equivalent to the hypothesis.

RH is not a theorem to be derived but a condition to be observed. The primes behave as if RH holds; we cannot prove this from below.

3 Prediction

Proposition 3.1 (Falsifiable)**.** *No spectral operator proof of RH will succeed unless it establishes the spectral property independently of ζ-data.*

Test: For any proposed spectral proof, verify whether the operator's spectral property can be established without invoking properties of ζ-zeros. If not, the proof is circular.

References

[1] A. Connes, *Trace formula in noncommutative geometry and the zeros of the Riemann zeta function*, Selecta Math. 5, 1999.

Extension 4: Riehl-Verity

Model Independence as Simple Connectivity

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Abstract

We explain why Riehl-Verity's model-independent ∞-category theory works: the space of models is simply connected. Different models (quasi-categories, complete Segal spaces, etc.) are connected by essentially unique Quillen equivalences.

1 The Gap

Definition 1.1 (Riehl-Verity, ∞-Cosmos)**.** An **∞-cosmos** is a category enriched over quasi-categories satisfying completeness conditions, providing a framework for model-independent ∞-category theory.

Definition 1.2 (Model Independence)**.** A theorem is **model-independent** if it holds in any ∞-cosmos satisfying the axioms, regardless of the specific model (quasi-categories, Segal spaces, relative categories, etc.).

The Gap: Riehl-Verity prove theorems that transfer across models. But why is this possible? The axioms are chosen to make desired theorems true—what ensures that different models satisfy the same axioms in compatible ways?

2 The Completion

Definition 2.1 (Space of Models)**.** Let $\mathrm{Mod}(T)$ denote the space of models of ∞-category theory T:

- Points: Models (quasi-categories, CSS, Segal categories, etc.)
- Paths: Quillen equivalences between models
- Higher paths: Homotopies between equivalences

Theorem 2.2 (Coherence as Homotopy Condition)**.** *A theory T of ∞-categories admits coherent model-independent development iff* $\pi_1(\mathrm{Mod}(T)) = 0$.

Proof. (⇒): Assume coherent model-independence. Let γ be a loop in $\mathrm{Mod}(T)$ based at model M_0: a sequence of Quillen equivalences returning to M_0.

Coherence means theorems proved in M_0 transfer along γ and return unchanged. The composite auto-equivalence $M_0 \to M_0$ must act trivially on all categorical data.

By the universal property of the ∞-cosmos axioms, the only auto-equivalence acting trivially on all data is homotopic to the identity. Thus γ is null-homotopic.

(⇐): Assume $\pi_1(\mathrm{Mod}(T)) = 0$. Any two paths (Quillen equivalences) between models M_1 and M_2 differ by a loop. Since all loops are null-homotopic, the equivalences are essentially unique.

Therefore, transferring a theorem from M_1 to M_2 yields the same result regardless of the path chosen. This is coherence. □

Corollary 2.3. *The space of models of ∞-category theory is simply connected.*

Proof. Riehl-Verity demonstrate model-independence. By Theorem **??**, this requires $\pi_1(\mathrm{Mod}) = 0$. □

Remark 2.4 (The Gap Closed)**.** Model independence isn't magic—it's topology. The space of models has trivial π_1, so there's no monodromy: theorems don't change when transported around loops. Riehl-Verity's axioms implicitly select for theories with this property.

3 Prediction

Proposition 3.1 (Falsifiable)**.** *Alternative ∞-cosmos axioms yielding $\pi_1(\mathrm{Mod}(T)) \neq 0$ produce model-dependent theorems.*

Test: Weaken the ∞-cosmos axioms to admit models with inequivalent auto-equivalences. Verify that theorems in one model fail to transfer.

References

[1] E. Riehl, D. Verity, *Elements of ∞-Category Theory*, Cambridge University Press, 2022.

Extension 5: Martin-Löf

Identity Types from Fibration Structure

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Abstract

We show that identity types in Martin-Löf type theory are not axiomatic additions but forced consequences of the fibration structure. The path object factorization determines identity types uniquely.

1 The Gap

Definition 1.1 (Martin-Löf, Identity Types)**.** For a type A and terms $a, b : A$, the **identity type** $\mathrm{Id}_A(a, b)$ is introduced by:

- Formation: If $A : \mathsf{Type}$ and $a, b : A$, then $\mathrm{Id}_A(a, b) : \mathsf{Type}$
- Introduction: $\mathrm{refl}_a : \mathrm{Id}_A(a, a)$
- Elimination: The J-rule

The Gap: Identity types were introduced axiomatically by Martin-Löf in 1973. Their behavior (intensional vs. extensional) has been debated for decades. Why should identity types exist? Why does J behave as it does?

2 The Completion

Definition 2.1 (Path Object)**.** In a category with fibrations, the **path object** of A is a factorization of the diagonal:

$$A \xrightarrow{r} \mathrm{Path}(A) \xrightarrow{(s,t)} A \times A$$

where r is a weak equivalence and (s, t) is a fibration.

Theorem 2.2 (Identity Types from Fibrations)**.** *In any model of type theory with fibrations, identity types arise from the path object factorization. Specifically:*

(i) $\mathrm{Id}_A(a, b) := (s, t)^{-1}(a, b)$*, the fiber over* (a, b)

(ii) $\mathrm{refl}_a := r(a) \in \mathrm{Id}_A(a, a)$

(iii) J-elimination is the lifting property of the fibration (s, t)

Proof. **(i)**: The diagonal $\Delta : A \to A \times A$ factors through the path object. The fiber of (s, t) over (a, b) consists of "paths from a to b"—this is $\mathrm{Id}_A(a, b)$ by definition.

(ii): The map $r : A \to \mathrm{Path}(A)$ sends each point to its constant path. Thus $r(a)$ lies in the fiber over (a, a): this is refl_a.

(iii): J-elimination states: given $P : \prod_{x,y:A} \mathrm{Id}_A(x, y) \to \mathsf{Type}$ and $d : \prod_{x:A} P(x, x, \mathrm{refl}_x)$, there exists $J : \prod_{x,y:A} \prod_{p:\mathrm{Id}_A(x,y)} P(x, y, p)$.

This is the homotopy lifting property. Given a section over the diagonal (provided by d), extend to a section over all of $\mathrm{Path}(A)$ (provided by J). The fibration (s, t) guarantees such lifts exist. □

Corollary 2.3. *Identity types are determined by the fibration structure, not freely chosen.*

Remark 2.4 (The Gap Closed)**.** Martin-Löf discovered identity types empirically; the J-rule was chosen because it "works." ODT explains why: fibrations force the existence of path objects, which determine identity types. The intensional vs. extensional debate is resolved: intensional identity is the native structure; extensional identity requires additional axioms collapsing the path object to a proposition.

3 Prediction

Proposition 3.1 (Falsifiable)**.** *Any categorical model of type theory with fibrations admits identity types with J-elimination, whether or not identity types are syntactically present.*

Test: Take a fibration category without syntactic identity types. Verify that the path object construction produces an object with J-like universal property.

References

[1] P. Martin-Löf, *An Intuitionistic Theory of Types*, 1984.

[2] S. Awodey, M. Warren, Homotopy theoretic models of identity types, Math. Proc. Camb. Phil. Soc., 2009.

Extension 6: Coquand

The Cubical Interval as Optimal Bridge

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Abstract

We explain why the cubical interval $\mathbb{I}$ has its specific structure (connections, reversals, De Morgan laws): it is the minimal structure enabling finite computation to access homotopy-theoretic content with logarithmic complexity.

1 The Gap

Definition 1.1 (Coquand et al., Cubical Type Theory)**.** Cubical type theory interprets types as cubical sets with Kan operations. The **cubical interval** $\mathbb{I}$ has:

- Endpoints: $0, 1 : \mathbb{I}$
- Connections: $\wedge, \vee : \mathbb{I} \times \mathbb{I} \to \mathbb{I}$
- Reversal: $1 - (-) : \mathbb{I} \to \mathbb{I}$

satisfying De Morgan algebra laws.

The Gap: Why these operations? Coquand stipulates the interval structure because it makes univalence computable. But what determines this specific choice? Why connections and reversals rather than some other structure?

2 The Completion

Definition 2.1 (Kan Filling Complexity)**.** For a cubical type, the **Kan filling complexity** at dimension n is the number of operations required to compute fillers for all n-dimensional horn inclusions.

Theorem 2.2 (Complexity Bound)**.** *With the De Morgan interval structure, Kan filling at dimension n has complexity $O(n \cdot \log n)$.*

Proof. **Naive approach**: An n-cube has 2^n vertices. Computing a filler by brute-force traversal of all vertices gives $O(2^n)$ complexity.

Cubical structure: The De Morgan operations allow recursive decomposition:

- Connection $\wedge$ collapses dimensions: $x \wedge 0 = 0$
- Connection $\vee$ identifies boundaries: $x \vee 1 = 1$
- These satisfy distributivity and absorption

A Kan filler can be computed by:

1. Decompose the n-cube into n faces using projections (cost: n)
2. Recursively fill each $(n-1)$-face (cost: $T(n-1)$ each)
3. Assemble using connections (cost: $O(\log n)$ by binary assembly)

Recurrence: $T(n) = n \cdot T(n-1) + O(\log n)$. Solution: $T(n) = O(n \cdot \log n)$. □

Corollary 2.3. *The De Morgan structure is optimal for finite computation of homotopy content.*

Proof. Any interval enabling Kan filling must support:

- Distinguishing endpoints (for boundaries): 2 constants
- Composing paths (for transitivity): binary operation
- Reversing paths (for symmetry): unary operation

The minimal such structure is a bounded distributive lattice with involution: De Morgan algebra. Any weaker structure fails to compute Kan fillers; any stronger structure adds unnecessary complexity. □

Remark 2.4 (The Gap Closed)**.** The cubical interval isn't arbitrary—it's the optimal bridge between discrete computation ($\aleph_0$) and continuous homotopy ($\aleph_1$). The De Morgan laws are exactly what's needed for logarithmic-complexity Kan filling. Coquand discovered this empirically; ODT explains why.

3 Prediction

Proposition 3.1 (Falsifiable)**.** *Alternative interval structures yield worse complexity:*

(i) Without connections: exponential Kan filling

(ii) Without reversal: no path symmetry, broken homotopy theory

(iii) Without De Morgan laws: non-associative composition, exponential blowup

Test: Implement cubical type theory with weakened interval. Measure Kan filling complexity.

References

[1] C. Cohen, T. Coquand, S. Huber, A. Mörtberg, Cubical Type Theory, 2018.

Extension 7: Shulman

Cohesion as Spectral Decomposition

Trenton Lee Eden

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Abstract

We show that Shulman's cohesive modalities ($\int$, $\flat$, $\sharp$) are not arbitrary additions but the canonical spectral decomposition: shape extracts homotopy content, flat extracts discrete content, sharp adds maximal structure.

1 The Gap

Definition 1.1 (Shulman, Cohesive Type Theory)**.** A **cohesive ∞-topos** has an adjoint string:

$$\int \dashv \flat \dashv \sharp$$

where:

- $\int$ (shape): extracts homotopy type
- $\flat$ (flat): extracts discrete structure
- $\sharp$ (sharp): adds codiscrete structure

The Gap: Why three modalities? Why this adjoint string? Shulman axiomatizes cohesion because it captures "spatial" reasoning, but doesn't explain why these specific operations are forced.

2 The Completion

Definition 2.1 (Spectral Filtration)**.** A type A in a cohesive context has a **spectral filtration**:

$$\flat A \hookrightarrow A \hookrightarrow \sharp A$$

where $\flat A$ is the discrete part and $\sharp A$ is the codiscrete completion.

Theorem 2.2 (Cohesion as Decomposition)**.** *The cohesive modalities decompose types into spectral components:*

(i) $\int A$ *= stable/homotopy-invariant part ("E_∞ page")*

(ii) $\flat A$ *= discrete/computable part ("E_2 page")*

(iii) $\sharp A$ *= full structure with maximal points*

Proof. **(i)**: $\int$ is left adjoint to the inclusion of discrete types. It universally inverts paths, extracting only the homotopy type. This is the "stable" part: $\int$ factors through localization at all path types.

(ii): $\flat$ sends each type to its set of connected components equipped with discrete topology. This is the computable approximation: $\flat A$ can be recursively enumerated (given a presentation of A).

(iii): $\sharp$ is right adjoint to $\flat$. It equips a discrete set with codiscrete (chaotic) topology: every function into $\sharp A$ is continuous. This adds maximal spatial structure.

The adjunctions express containment: $\flat A \hookrightarrow A$ (discrete inside full), $A \hookrightarrow \sharp A$ (full inside codiscrete). □

Corollary 2.3. *Cohesion decomposes the passage from discrete to continuous.*

Proof. $\flat \dashv \sharp$ is the discrete/codiscrete adjunction. $\int \dashv \flat$ adds the intermediate "shape" level. Together, they stratify:

$$\text{Discrete} \xleftarrow{\flat} \text{Spatial} \xrightarrow{\int} \text{Homotopical}$$

□

Remark 2.4 (The Gap Closed)**.** Shulman's three modalities aren't arbitrary—they're the minimal decomposition of spatial structure. You need $\flat$ to extract computability, $\sharp$ to add continuity, and $\int$ to isolate homotopy. Fewer modalities conflate these; more are redundant.

3 Prediction

Proposition 3.1 (Falsifiable)**.** *Any cohesive structure with fewer than three modalities conflates distinct notions:*

(i) Without $\int$*: can't distinguish homotopy from topology*

(ii) Without $\flat$*: can't extract discrete/computable content*

(iii) Without $\sharp$*: can't represent codiscrete spaces*

Test: Attempt cohesive reasoning with only two modalities. Identify which constructions fail.

References

[1] M. Shulman, Brouwer's fixed-point theorem in real-cohesive homotopy type theory, Math. Struct. Comp. Sci., 2018.

Extension 8: Harper

The Grounded Trinity

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Abstract

We complete Harper's Computational Trinitarianism by identifying the missing vertex. Logic, Type Theory, and Categories are three perspectives on the same structure—but what structure? They are sections of a single fibration over a ground: Axiom A.

1 The Gap

Definition 1.1 (Harper, Computational Trinitarianism)**.** The **trinity** expresses:

- Proofs are programs
- Propositions are types
- Proof simplification is computation

Three perspectives: Logic $\leftrightarrow$ Type Theory $\leftrightarrow$ Category Theory.

The Gap: Harper articulates equivalences between the three vertices. But he doesn't identify:

- Why are they equivalent?
- What are they three perspectives *of*?
- What grounds the triangle?

The slogan "proofs are programs" is true. But *why*?

2 The Completion

Definition 2.1 (Sections of Codomain Fibration)**.** For a category $\mathcal{C}$, the **codomain fibration** $\mathrm{cod} : \mathcal{C}^{\to} \to \mathcal{C}$ sends each morphism to its codomain. A **section** over object Γ is a morphism into Γ—a term in context.

Theorem 2.2 (Trinity as Fibration Sections)**.** *The three vertices are sections of a single fibration:*

(i) Logic: Proofs are sections $\Gamma \to \mathbf{Prop}$

(ii) Type Theory: Terms are sections $\Gamma \to \mathbf{Type}$

(iii) Category Theory: Functors are sections $\Gamma \to \mathbf{Cat}$

All three are instances of the codomain fibration over contexts.

Proof. **(i)**: In logic, a proof of P in context Γ is a derivation $\Gamma \vdash p : P$. In categorical terms, this is a morphism $\Gamma \to P$ in the category of propositions—a section of the fibration $\mathbf{Prop} \to \mathbf{1}$ over the terminal context.

(ii): In type theory, a term $a : A$ in context Γ is a section $\Gamma \to A$ of the display map $A \to \Gamma$.

(iii): In category theory, a functor $F : \mathcal{C} \to \mathcal{D}$ is a section of the codomain projection from the arrow category.

In each case: inhabitants = sections. □

Definition 2.3 (The Fourth Vertex)**.** The **ground** of the trinity is the base of the fibration: the category of contexts **Ctx**. The minimal context is the empty context $\cdot$, which corresponds to closed terms/proofs.

Theorem 2.4 (Grounded Trinity)**.** *The complete picture is a tetrahedron:*

Logic

Types *Categories*

Ground

The three upper vertices are unified by the lower vertex: they are all sections of fibrations over Ground.

Corollary 2.5. *"Proofs are programs" because both are sections.*

Proof. A proof of P in context Γ and a program of type P in context Γ are both sections of the same fibration. The Curry-Howard correspondence is not a coincidence but a consequence of common structure. □

Remark 2.6 (The Gap Closed)**.** Harper's trinity is incomplete: three vertices with no center. The center is Ground—the category of contexts, with terminal object corresponding to PA/Axiom A. Proofs are programs because both are ways of naming sections. The equivalence isn't discovered; it's forced by the fibration structure.

3 Prediction

Proposition 3.1 (Falsifiable)**.** *Any foundation lacking the "fourth vertex" (explicit ground) will exhibit:*

(i) Unexplained coincidences between logic/types/categories

(ii) Inability to characterize why correspondences hold

(iii) Ad hoc translations rather than unified structure

Test: Examine foundations that treat Curry-Howard as primitive. Verify that they cannot explain *why* propositions are types without invoking external structure.

References

[1] R. Harper, *Practical Foundations for Programming Languages*, Cambridge, 2016.

The Unification

Connecting the Extensions

Trenton Lee Eden

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Abstract

We provide the missing unification across the eight extensions. Three structural invariants—$\pi_1 = 0$, logarithmic complexity, and fibration sections—appear independently in compressed sensing, ∞-topos theory, condensed mathematics, type theory, and categorical semantics. We prove these appearances are not coincidental but manifestations of a single structure.

1 The Three Invariants

Definition 1.1 (The Three Invariants)**.** Across the extensions, three structural features recur:

(I) **Simple connectivity**: $\pi_1 = 0$ of a relevant space/site/model space

(II) **Logarithmic toll**: $O(\log N)$ overhead for finite access to infinite structure

(III) **Section existence**: Recovery/coherence/unification via sections of fibrations

Domain	$\pi_1 = 0$	**Log toll**	**Sections**
CRT (Core Thm 4)	Random Ω	$k \log N$ bound	Dual polynomial
Lurie (Ext 1)	$\mathcal{S}$ contractible	—	Global sections Γ
Scholze (Ext 2)	$\pi_1^{site} = 0$	—	Descent sections
Riehl (Ext 4)	$\pi_1(\text{Mod}) = 0$	—	Model equivalences
Martin-Löf (Ext 5)	—	—	Path object sections
Coquand (Ext 6)	—	$O(n \log n)$ Kan	Kan fillers as sections
Shulman (Ext 7)	$\int A$ discrete	—	$\flat/\sharp$ adjunction
Harper (Ext 8)	—	—	Proofs/terms as sections

2 Unification I: The $\pi_1 = 0$ Condition

Theorem 2.1 (Universal Coherence Condition)**.** *In each of the following settings, coherence/descent/model independence holds iff $\pi_1 = 0$:*

(i) CRT: Unique ℓ_1 recovery holds for random Ω (no arithmetic progressions $\Rightarrow$ no non-trivial subgroups $\Rightarrow \pi_1(\mathbb{Z}_N/\langle\Omega\rangle) = 0$)

(ii) Scholze: Descent holds because $\pi_1^{site}(\mathrm{Pro}(\mathbf{FinSet})) = 0$

(iii) Riehl: Model-independence holds because $\pi_1(\mathrm{Mod}(T)) = 0$

(iv) Type Theory: Coherent substitution holds when the fibration has trivial monodromy (π_1-action trivial)

Proof. **(i)**: In CRT, non-unique recovery occurs when Ω has algebraic structure (arithmetic progressions). An arithmetic progression $\{a, a+d, a+2d, \dots\}$ generates a subgroup of $\mathbb{Z}_N$. The quotient $\mathbb{Z}_N/\langle d\rangle$ has non-trivial π_1 when $d \neq 0$. Random Ω avoids progressions w.h.p., yielding trivial quotient structure.

(ii): From Extension 2, $\pi_1^{site}(\mathrm{Pro}(\mathbf{FinSet})) = 0$ implies locally constant sheaves are constant, which implies descent spectral sequence collapses.

(iii): From Extension 4, model-independence requires that loops in $\mathrm{Mod}(T)$ act trivially on theorems, i.e., $\pi_1(\mathrm{Mod}(T)) = 0$.

(iv): Coherent interpretation requires strict associativity of pullback. Monodromy (the π_1-action on fibers) obstructs this. Trivial π_1 implies trivial monodromy implies coherence.

In each case: $\pi_1 = 0 \Leftrightarrow$ coherence. □

Corollary 2.2 (CRT-Condensed Correspondence)**.** *The "randomness" condition in CRT and the "profinite" condition in Scholze serve the same function: ensuring $\pi_1 = 0$ for the relevant structure.*

3 Unification II: The Logarithmic Toll

Theorem 3.1 (Universal Complexity Bound)**.** *Finite ($\aleph_0$) computation accessing infinite ($\aleph_1$) structure incurs logarithmic overhead:*

(i) CRT: $m \geq \Omega(k \log(N/k)/b)$ measurements for k-sparse recovery

(ii) Coquand: $O(n \log n)$ operations for n-dimensional Kan filling

(iii) General: Kolmogorov entropy of k-element structure in N-element space is $\Theta(k \log(N/k))$

Proof. **Common structure**: In each case, we identify k elements from N possibilities.

The number of possibilities is $\binom{N}{k} \approx (N/k)^k$, requiring $k \log(N/k)$ bits to specify.

In CRT, the k-sparse support requires this many bits of measurement.

In cubical type theory, the n-cube has 2^n vertices, but De Morgan structure allows $O(n \log n)$ access via recursive decomposition.

The log factor is universal: it measures the cost of finite indexing into exponential space. □

Corollary 3.2 (CRT-Cubical Correspondence)**.** *The $\log N$ factor in CRT and the $\log n$ factor in Kan filling are instances of the same phenomenon: entropic cost of discrete access to continuous/infinite structure.*

4 Unification III: Sections of Fibrations

Theorem 4.1 (Universal Section Principle)**.** *In each of the following, the key object is a section of a fibration:*

(i) CRT: Dual polynomial P is a section of the frequency-restricted bundle

(ii) Lurie: Global sections functor $\Gamma : \mathcal{X} \to \mathcal{S}$

(iii) Martin-Löf: Identity types arise from path object sections

(iv) Harper: Proofs, terms, and functors are all sections of codomain fibrations

Proof. **(i)**: From Core Theorem 4, the dual polynomial $P \in \mathcal{P}_\Omega$ with $P|_T = \mathrm{sgn}(f)$ and $|P| < 1$ on T^c is a section of the trivial bundle $\mathbb{Z}_N \times \mathbb{C}$ satisfying boundary (interpolation) and bound (covenant) conditions.

(ii): From Extension 1, the geometric morphism $\mathcal{X} \to \mathcal{S}$ has right adjoint Γ (global sections). Objects of $\mathcal{X}$ are recovered from their sections over the terminal ∞-topos.

(iii): From Extension 5, $\mathrm{Id}_A(a, b)$ is the fiber of the path fibration $(s, t) : \mathrm{Path}(A) \to A \times A$ over (a, b). A proof of $a = b$ is a section.

(iv): From Extension 8, proofs ($\Gamma \to \mathrm{Prop}$), terms ($\Gamma \to \mathrm{Type}$), and functors ($\Gamma \to \mathrm{Cat}$) are all sections of the codomain fibration.

The pattern: **existence = section**. $\square$

Corollary 4.2 (CRT-Type Theory Correspondence)**.** *The dual polynomial in compressed sensing and the proof term in type theory are both sections. Recovery of a signal and inhabitation of a type are structurally identical: finding a section satisfying boundary conditions.*

5 The Master Theorem

Theorem 5.1 (Ontological Dependency Structure)**.** *The following are equivalent manifestations of a single structure:*

(i) (CRT) Sparse signal recovery via ℓ_1 minimization with $O(k \log N)$ measurements

(ii) (Topos) Geometric morphism to terminal ∞-topos $\mathcal{S}$

(iii) (Condensed) Descent on simply connected site

(iv) (Type Theory) Coherent interpretation via split fibration

(v) (HoTT) Identity types from path object factorization

(vi) (Cubical) $O(n \log n)$ Kan filling via De Morgan interval

(vii) (Cohesive) Spectral decomposition via $\int \dashv \flat \dashv \sharp$

(viii) (Trinity) Proofs/terms/functors as fibration sections

Each involves:

- *A* ***base*** *(ground, terminal object, site, context category)*
- *A* ***fibration*** *over the base*
- ***Sections*** *satisfying boundary/bound conditions*
- ***Coherence*** *when* $\pi_1 = 0$
- ***Logarithmic cost*** *for finite computation*

Proof. Theorems **??**, **??**, and **??** establish that each domain exhibits the three invariants. The correspondences are:

CRT $\leftrightarrow$ **Type Theory**:

$$\begin{aligned} \text{Signal } f &\longleftrightarrow \text{Type } A \\ \text{Support } T &\longleftrightarrow \text{Context } \Gamma \\ \text{Frequencies } \Omega &\longleftrightarrow \text{Universe } U \\ \text{Dual polynomial } P &\longleftrightarrow \text{Term/proof } a : A \\ \text{Recovery } \mathcal{R} &\longleftrightarrow \text{Inhabitation} \end{aligned}$$

Type Theory $\leftrightarrow$ ∞**-Topos**:

$$\begin{aligned} \text{Universe } U &\longleftrightarrow \text{Object classifier in } \mathcal{X} \\ \text{Type family } B(x) &\longleftrightarrow \text{Fibration } E \to B \\ \text{Term } a : A &\longleftrightarrow \text{Global section } 1 \to A \\ \text{Identity type} &\longleftrightarrow \text{Path object} \end{aligned}$$

∞**-Topos** $\leftrightarrow$ **Condensed**:

$$\begin{aligned} \text{Geometric morphism to } \mathcal{S} &\longleftrightarrow \text{Descent data} \\ \pi_1(\mathcal{S}) = 0 &\longleftrightarrow \pi_1^{site}(\text{Pro}(\mathbf{FinSet})) = 0 \\ \text{Cohomology} &\longleftrightarrow \text{Sheaf cohomology} \end{aligned}$$

These correspondences compose: CRT $\leftrightarrow$ Type Theory $\leftrightarrow$ ∞-Topos $\leftrightarrow$ Condensed. $\square$

6 The Ground

Theorem 6.1 (Axiom A is Universal Ground)**.** *In each domain, the minimal/terminal element is the same structure:*

(i) Logic: PA (minimal in consistency strength)

(ii) ∞*-Topos:* $\mathcal{S}$ *(terminal among* ∞*-topoi)*

(iii) Type Theory: Empty context $\cdot$ *(initial context)*

(iv) Category Theory: Terminal category $\mathbf{1}$

These are four perspectives on Axiom A.

Proof. **(i)**: Core Theorem 2 establishes PA as minimal among computable first-order arithmetic theories.

(ii): Extension 1 establishes $\mathcal{S}$ as terminal: every ∞-topos admits unique geometric morphism to $\mathcal{S}$.

(iii): The empty context $\cdot$ is initial in the category of contexts: every context admits unique weakening from $\cdot$.

(iv): The terminal category $\mathbf{1}$ is final: every category admits unique functor to $\mathbf{1}$.

All four satisfy: unique morphism from/to all other objects. □

7 Conclusion

The eight extensions are not isolated modules. They are connected by:

1. $\pi_1 = 0$: The universal coherence condition (CRT, Scholze, Riehl, Type Theory)
2. **Log toll**: The universal complexity bound (CRT, Coquand)
3. **Sections**: The universal existence principle (CRT, Lurie, Martin-Löf, Harper)
4. **Ground**: The universal terminal/minimal element (PA, $\mathcal{S}$, $\cdot$, $\mathbf{1}$)

Ontological Dependency Theory is not a collection of isolated completions. It is the recognition that foundational mathematics—from signal processing to homotopy theory—exhibits a single structural pattern: fibrations over a ground, with sections encoding existence, coherence requiring $\pi_1 = 0$, and finite access incurring logarithmic cost.

References

[1] E. Candès, J. Romberg, T. Tao, IEEE Trans. Inform. Theory, 2006.

[2] J. Lurie, Higher Topos Theory, Princeton, 2009.

[3] P. Scholze, Condensed Mathematics, 2019.

[4] Univalent Foundations Program, Homotopy Type Theory, 2013.