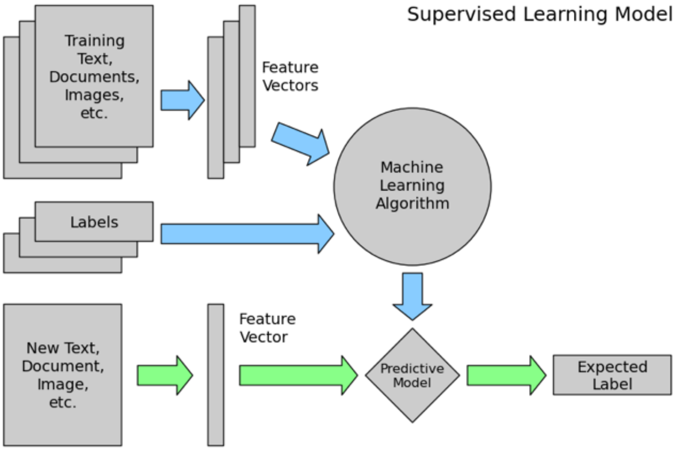
Lecture 2
Perceptron : Single Layer Neural Network

Naresh Manwani

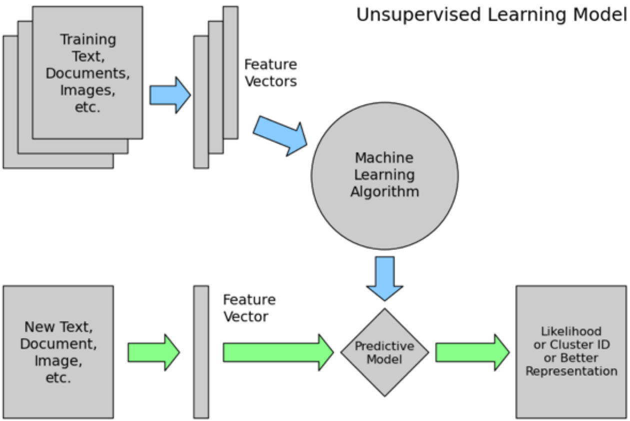
Deep Learning: Theory and Practices
CS7.601 (Monsson 2022)
IIIT-Hyderabad

August 2, 2022

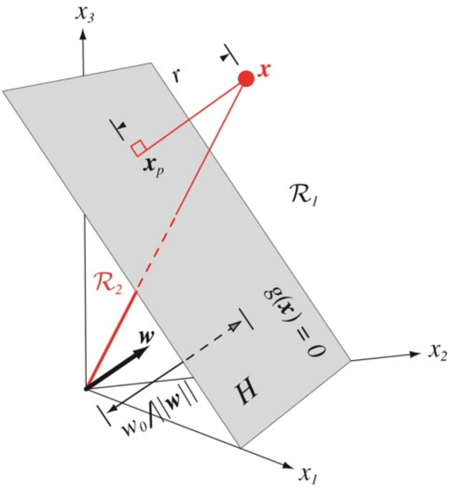
Supervised Learning



Unsupervised Learning

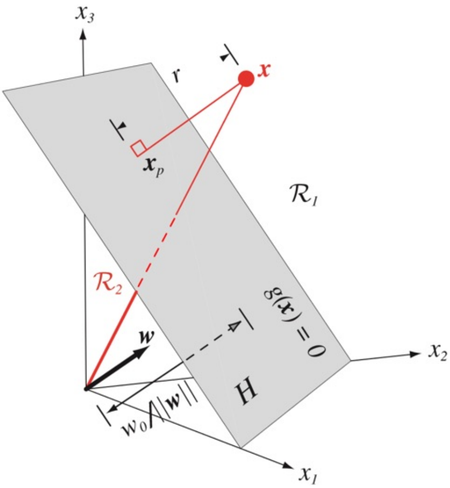


Hyperplane



- $\mathbf{w}$ is normal to the hyperplane.
- $\frac{\mathbf{w}^T\mathbf{x}+b}{\|\mathbf{w}\|}$ is the distance of $\mathbf{x}$ from the hyperplane $\mathbf{w}^T\mathbf{x} + b = 0$.

Hyperplane

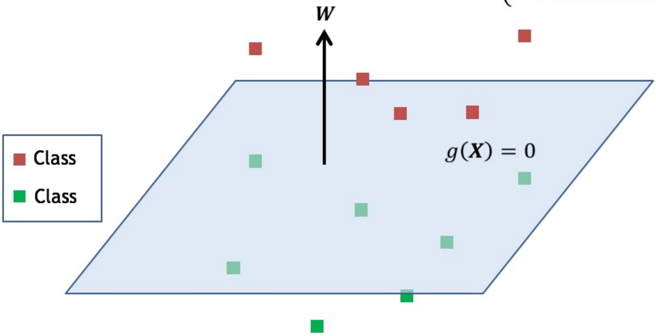


- $\mathbf{w}$ is normal to the hyperplane.
- $\frac{\mathbf{w}^T\mathbf{x}+b}{\|\mathbf{w}\|}$ is the distance of $\mathbf{x}$ from the hyperplane $\mathbf{w}^T\mathbf{x} + b = 0$. (**HW for you**)

Linear Discriminant Function

$$g(\boldsymbol{X}) = \boldsymbol{W}^T\boldsymbol{X} + w_0$$

$$g(\boldsymbol{X}) = \begin{cases} > 0 \quad (+ve) & class\ A \\ < 0 \quad (-ve) & class\ B \\ = 0\ Decision\ Boundary & \end{cases}$$



Supervised Learning Problem

- Let $\mathcal{X} \subset \mathbb{R}^d$ be the feature space.

Supervised Learning Problem

- Let $\mathcal{X} \subset \mathbb{R}^d$ be the feature space.
- We can categorise the task in hand based on the label space, denoted by $\mathcal{Y}$ as:

$$\mathcal{Y} = \begin{cases} \{+1, -1\} & \text{Binary Classification} \\ \{1, 2, 3, .., C\} & \text{Multi-class Classification} \\ \mathbb{R} & \text{Regression} \end{cases}$$

Supervised Learning Problem

- Let $\mathcal{X} \subset \mathbb{R}^d$ be the feature space.
- We can categorise the task in hand based on the label space, denoted by $\mathcal{Y}$ as:

$$\mathcal{Y} = \begin{cases} \{+1, -1\} & \text{Binary Classification} \\ \{1, 2, 3, .., C\} & \text{Multi-class Classification} \\ \mathbb{R} & \text{Regression} \end{cases}$$

- We assume the training data is $S = \{(\mathbf{x}_1, y_1), \cdots, (\mathbf{x}_N, y_N)\}$.

Supervised Learning Problem

- Let $\mathcal{X} \subset \mathbb{R}^d$ be the feature space.
- We can categorise the task in hand based on the label space, denoted by $\mathcal{Y}$ as:

$$\mathcal{Y} = \begin{cases} \{+1, -1\} & \text{Binary Classification} \\ \{1, 2, 3, .., C\} & \text{Multi-class Classification} \\ \mathbb{R} & \text{Regression} \end{cases}$$

- We assume the training data is $S = \{(\mathbf{x}_1, y_1), \cdots, (\mathbf{x}_N, y_N)\}$.
- We want to learn a function, $f : \mathcal{X} \to \mathcal{Y}$, using training set S .

Loss Function

- The hypothesis $f(x)$ predicts the label $(\hat{y})$.

Loss Function

- The hypothesis $f(x)$ predicts the label $(\hat{y})$.
- To measure the discrepancy between the prediction $(\hat{y})$ and the corresponding actual label y, we use a loss function.

$$L : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R}_+ \qquad \text{If considering } sign(f(\mathbf{x}))$$
$$L : \mathbb{R} \times \mathcal{Y} \rightarrow \mathbb{R}_+ \qquad \text{Considering simply } f(\mathbf{x})$$

Risk Function

- **Risk**: Risk is defined as the expectation of the loss function.

$$R_L(f) = \mathbb{E}[L(f(\mathbf{x}), y)] \qquad \text{Expected Risk}$$

Risk Function

- **Risk**: Risk is defined as the expectation of the loss function.

$$R_L(f) = \mathbb{E}[L(f(\mathbf{x}), y)] \qquad \text{Expected Risk}$$

- We are usually provided with a finite set of training examples, $S = \{(\mathbf{x}_1, y_1), \cdots, (\mathbf{x}_N, y_N)\}$.

Risk Function

- **Risk**: Risk is defined as the expectation of the loss function.

$$R_L(f) = \mathbb{E}[L(f(\mathbf{x}), y)] \qquad \text{Expected Risk}$$

- We are usually provided with a finite set of training examples, $S = \{(\mathbf{x}_1, y_1), \cdots, (\mathbf{x}_N, y_N)\}$.
- In such situation, we find the *empirical risk* as follows:

$$\hat{R}_L(f) = \frac{1}{N}\sum_{i=1}^{N} L(f(\mathbf{x}_i), y_i) \qquad \text{Empirical Risk}$$

Empirical Risk Minimization

- Minimizing the empirical risk of a learning algorithm can be achieved in two ways.

Empirical Risk Minimization

- Minimizing the empirical risk of a learning algorithm can be achieved in two ways.
- One is through **batch learning** where the risk over all the training examples is minimized at once. SVM, logistic regression algorithms minimize the risk through batch learning.

Empirical Risk Minimization

- Minimizing the empirical risk of a learning algorithm can be achieved in two ways.
- One is through **batch learning** where the risk over all the training examples is minimized at once. SVM, logistic regression algorithms minimize the risk through batch learning.
- The second way, is through **online learning** where, the risk is minimized in a sequential order and is fast and scalable. For example: **Perceptron**.

Generic Online Learning Framework

In an online learning algorithm, the learning takes place in a sequence of trials.

- Observes the example instance $\mathbf{x}_t$.

Generic Online Learning Framework

In an online learning algorithm, the learning takes place in a sequence of trials.

- Observes the example instance $\mathbf{x}_t$.
- Uses current hypothesis f_t to predict the output $\hat{y}_t = f_t(\mathbf{x}_t)$.

Generic Online Learning Framework

In an online learning algorithm, the learning takes place in a sequence of trials.

- Observes the example instance $\mathbf{x}_t$.
- Uses current hypothesis f_t to predict the output $\hat{y}_t = f_t(\mathbf{x}_t)$.
- Observe actual output y_t.

Generic Online Learning Framework

In an online learning algorithm, the learning takes place in a sequence of trials.

- Observes the example instance $\mathbf{x}_t$.
- Uses current hypothesis f_t to predict the output $\hat{y}_t = f_t(\mathbf{x}_t)$.
- Observe actual output y_t.
- Incurs a loss of $L(\hat{y}_t, y_t)$.

Generic Online Learning Framework

In an online learning algorithm, the learning takes place in a sequence of trials.

- Observes the example instance $\mathbf{x}_t$.
- Uses current hypothesis f_t to predict the output $\hat{y}_t = f_t(\mathbf{x}_t)$.
- Observe actual output y_t.
- Incurs a loss of $L(\hat{y}_t, y_t)$.
- Updates the current hypothesis.

In an online learning algorithm, the learning takes place in a sequence of trials.

- Observes the example instance $\mathbf{x}_t$.
- Uses current hypothesis f_t to predict the output $\hat{y}_t = f_t(\mathbf{x}_t)$.
- Observe actual output y_t.
- Incurs a loss of $L(\hat{y}_t, y_t)$.
- Updates the current hypothesis.
- There is an objective associated with the training, which is to minimize the total loss, in a sequence of T trials.

$$\text{Total Loss} = \sum_{t=1}^{T} L(\hat{y}_t, y_t)$$

Stochastic Gradient Descent

- In the above framework, there are some specifics left out.

Stochastic Gradient Descent

- In the above framework, there are some specifics left out.
  - What choice of loss function do we make?

Stochastic Gradient Descent

- In the above framework, there are some specifics left out.
  - What choice of loss function do we make?
  - How do we update the hypothesis?

Stochastic Gradient Descent

- In the above framework, there are some specifics left out.
  - What choice of loss function do we make?
  - How do we update the hypothesis?
- One widely used method to update the hypothesis is Stochastic Gradient Descent (SGD).

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \eta \nabla_{\mathbf{w}} L(f(\mathbf{x}_t), y_t)$$

Stochastic Gradient Descent

- In the above framework, there are some specifics left out.
  - What choice of loss function do we make?
  - How do we update the hypothesis?
- One widely used method to update the hypothesis is Stochastic Gradient Descent (SGD).

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \eta \nabla_{\mathbf{w}} L(f(\mathbf{x}_t), y_t)$$

- One model of the above framework is the linear Perceptron. Input to Perceptron is $\mathbf{x}$, hypothesis is given by:

$$f(\mathbf{x}) = \mathbf{w}^T \mathbf{x} + b \qquad \text{(linear classifier)}$$

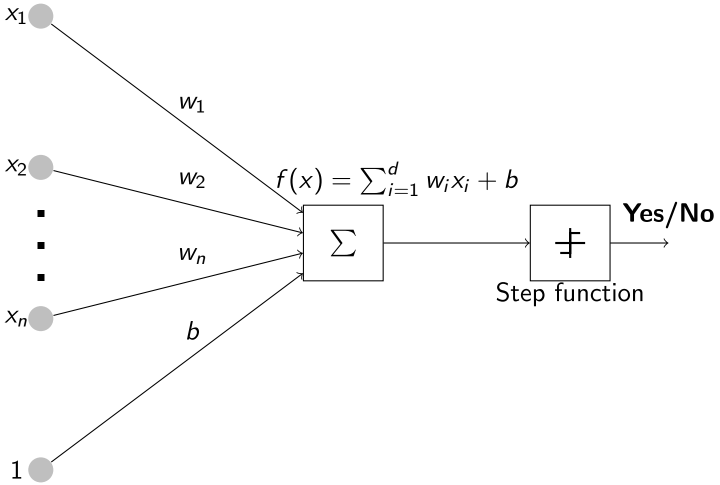


Figure: The Perceptron model

f predicts a real value, and $\mathrm{sign}(f(\mathbf{x}))$ gives the label.

Perceptron

- It used for classification purpose.

Perceptron

- It used for classification purpose.
- Given an example $\mathbf{x} = [x_1 \;\; x_2 \;\; \dots \;\; x_d]^T$, it predicts the output label

$$\hat{y} = \text{sign}(f(\mathbf{x})) = \text{sign}(\sum_{i=1}^{d} w_i x_i + b)$$

Perceptron

- It used for classification purpose.
- Given an example $\mathbf{x} = [x_1 \;\; x_2 \;\; \ldots \;\; x_d]^T$, it predicts the output label

$$\hat{y} = \text{sign}(f(\mathbf{x})) = \text{sign}(\sum_{i=1}^{d} w_i x_i + b)$$

- It then compares the predicted label $\hat{y}$ with the actual label y.

- It used for classification purpose.
- Given an example $\mathbf{x} = [x_1 \;\; x_2 \;\; \ldots \;\; x_d]^T$, it predicts the output label

$$\hat{y} = \text{sign}(f(\mathbf{x})) = \text{sign}(\sum_{i=1}^{d} w_i x_i + b)$$

- It then compares the predicted label $\hat{y}$ with the actual label y.
- Note that $\hat{y}, y \in \{+1, -1\}$.

Perceptron

- It used for classification purpose.
- Given an example $\mathbf{x} = [x_1 \;\; x_2 \;\; \ldots \;\; x_d]^T$, it predicts the output label

$$\hat{y} = \text{sign}(f(\mathbf{x})) = \text{sign}(\sum_{i=1}^{d} w_i x_i + b)$$

- It then compares the predicted label $\hat{y}$ with the actual label y.
- Note that $\hat{y}, y \in \{+1, -1\}$.
- If the predicted label is same as actual label, then $y\hat{y} = 1$, else $y\hat{y} = -1$.

Perceptron

- It used for classification purpose.
- Given an example $\mathbf{x} = [x_1 \;\; x_2 \;\; \ldots \;\; x_d]^T$, it predicts the output label

$$\hat{y} = \text{sign}(f(\mathbf{x})) = \text{sign}(\sum_{i=1}^{d} w_i x_i + b)$$

- It then compares the predicted label $\hat{y}$ with the actual label y.
- Note that $\hat{y}, y \in \{+1, -1\}$.
- If the predicted label is same as actual label, then $y\hat{y} = 1$, else $y\hat{y} = -1$.
- In other words, ff the predicted label is same as actual label, then $yf(\mathbf{x}) > 0$, else $yf(\mathbf{x}) < 0$.

Perceptron

- It used for classification purpose.
- Given an example $\mathbf{x} = [x_1 \ \ x_2 \ \ \ldots \ \ x_d]^T$, it predicts the output label

$$\hat{y} = \text{sign}(f(\mathbf{x})) = \text{sign}(\sum_{i=1}^{d} w_i x_i + b)$$

- It then compares the predicted label $\hat{y}$ with the actual label y.
- Note that $\hat{y}, y \in \{+1, -1\}$.
- If the predicted label is same as actual label, then $y\hat{y} = 1$, else $y\hat{y} = -1$.
- In other words, ff the predicted label is same as actual label, then $yf(\mathbf{x}) > 0$, else $yf(\mathbf{x}) < 0$.
- For correct prediction, we want to maximize $yf(\mathbf{x})$.

Perceptron

- It used for classification purpose.
- Given an example $\mathbf{x} = [x_1 \ \ x_2 \ \ \ldots \ \ x_d]^T$, it predicts the output label

$$\hat{y} = \text{sign}(f(\mathbf{x})) = \text{sign}(\sum_{i=1}^{d} w_i x_i + b)$$

- It then compares the predicted label $\hat{y}$ with the actual label y.
- Note that $\hat{y}, y \in \{+1, -1\}$.
- If the predicted label is same as actual label, then $y\hat{y} = 1$, else $y\hat{y} = -1$.
- In other words, ff the predicted label is same as actual label, then $yf(\mathbf{x}) > 0$, else $yf(\mathbf{x}) < 0$.
- For correct prediction, we want to maximize $yf(\mathbf{x})$.
- Or minimize $-yf(\mathbf{x})$.

Perceptron Objective Function

Loss Function

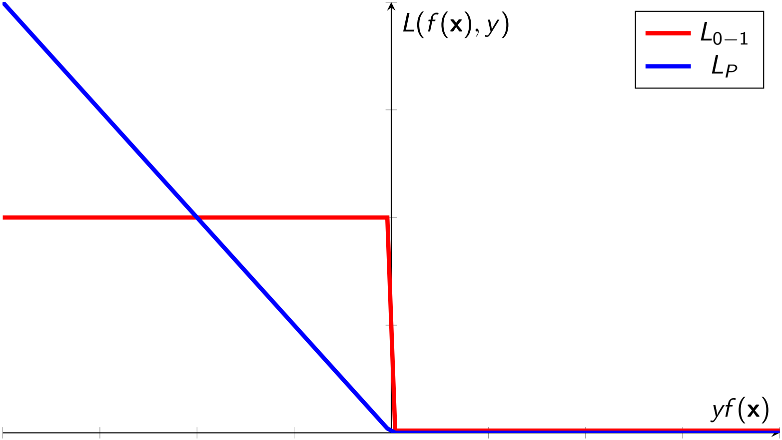
$$L_{\text{Perceptron}}(f(\mathbf{x}), y) = max(0, -yf(\mathbf{x}))$$

here $f(\mathbf{x}) = \mathbf{w}.\mathbf{x} + b$

- if $yf(\mathbf{x}) \geq 0$ (correct classification), then the loss is 0.
- if $yf(\mathbf{x}) < 0$ (misclassification), then the loss is $-yf(\mathbf{x})$
- Thus, the loss increases lineraly with the margin of misclassification[1]

[1] $yf(\mathbf{x})$ is denoted as margin

Perceptron Loss



How to update the hypothesis?

Stochastic Gradient Descent

$$\mathbf{w}^{t+1} = \mathbf{w}^t - \eta \nabla_f L(f(\mathbf{x}^t), y^t)$$

where ∇ denotes the gradient.

How to update the hypothesis?

Stochastic Gradient Descent

$$\mathbf{w}^{t+1} = \mathbf{w}^t - \eta \nabla_f L(f(\mathbf{x}^t), y^t)$$

where ∇ denotes the gradient.

Gradient of L

- Let $f(\mathbf{x}^t) = \mathbf{w}.\mathbf{x}^t + b$
- then

$$\begin{aligned}
\nabla_{\mathbf{w}} L(f(\mathbf{x}^t), y^t) &= \nabla_{\mathbf{w}} L(\mathbf{w}.\mathbf{x}^t + b, y^t) \\
&= \begin{pmatrix} \frac{\partial L}{\partial w_1} \\ \frac{\partial L}{\partial w_2} \\ \vdots \\ \frac{\partial L}{\partial w_d} \end{pmatrix}
\end{aligned}$$

Stochastic gradient descent on $L_{\text{Perceptron}}$

$$\nabla_{\mathbf{w}} L(\mathbf{w}.\mathbf{x}^t + b, y^t) = \begin{cases} \nabla_{\mathbf{w}} \left(-y^t(\mathbf{w}.\mathbf{x}^t + b)\right) & y_t(\mathbf{w}^t.\mathbf{x}^t + b^t) \leq 0 \\ 0 & y_t(\mathbf{w}^t.\mathbf{x}^t + b^t) > 0 \end{cases}$$

$$= \begin{cases} -y^t\mathbf{x}^t & y^t(\mathbf{w}^t.\mathbf{x}^t + b^t) \leq 0 \\ 0 & y^t(\mathbf{w}^t.\mathbf{x}^t + b^t) > 0 \end{cases}$$

$$\nabla_b L(\mathbf{w}.\mathbf{x}^t + b, y^t) = \begin{cases} \nabla_b \left(-y^t(\mathbf{w}.\mathbf{x}^t + b)\right) & y_t(\mathbf{w}^t.\mathbf{x}^t + b^t) \leq 0 \\ 0 & y_t(\mathbf{w}^t.\mathbf{x}^t + b^t) > 0 \end{cases}$$

$$= \begin{cases} -y^t & y^t(\mathbf{w}^t.\mathbf{x}^t + b^t) \leq 0 \\ 0 & y^t(\mathbf{w}^t.\mathbf{x}^t + b^t) > 0 \end{cases}$$

Update rule for Perceptron

$$\mathbf{w}^{t+1} = \begin{cases} \mathbf{w}^t + \eta y^t \mathbf{x}^t & y^t(\mathbf{w}^t.\mathbf{x}^t + b^t) \leq 0 \\ \mathbf{w}^t & y^t(\mathbf{w}^t.\mathbf{x}^t + b^t) > 0 \end{cases}$$

$$b^{t+1} = \begin{cases} b^t + \eta y^t & y^t(\mathbf{w}^t.\mathbf{x}^t + b^t) \leq 0 \\ b^t & y^t(\mathbf{w}^t.\mathbf{x}^t + b^t) > 0 \end{cases}$$

Update Rule for Perceptron

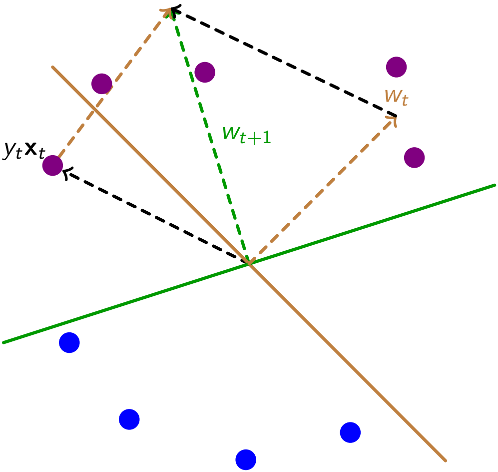


Figure: The weight update in Perceptron during misclassification

Perceptron Algorithm

Input: $S = \{(\mathbf{x}^1, y^1), \ldots, (\mathbf{x}^T, y^T)\}$
Output: $(\mathbf{w}^*, b^*)$

Initialize: $\mathbf{w}^1 = \mathbf{0},\ b^1 = 0$
For($t = 1$ to T)

Randomly draw an example $(\mathbf{x}^t, y^t)$ from S
If($y^t(\mathbf{w}^t.\mathbf{x}^t + b^t) \leq 0$)

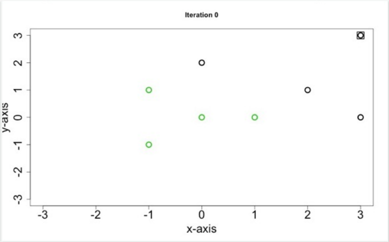
$$\mathbf{w}^{t+1} = \mathbf{w}^t + y^t\mathbf{x}^t$$
$$b^{t+1} = b^t + y^t$$

Else

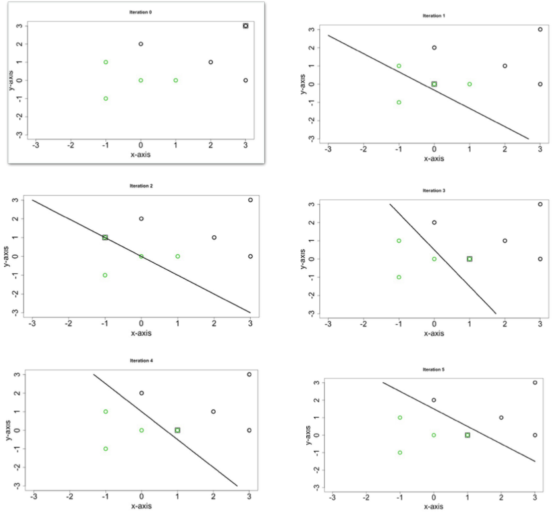
$$\mathbf{w}^{t+1} = \mathbf{w}^t$$
$$b^{t+1} = b^t$$

$$(\mathbf{w}^*, b^*) = (\mathbf{w}^{T+1}, b^{T+1})$$

$$X = \begin{pmatrix} 3 & 3 & 1 \\ 3 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad Y = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$



Example: Perceptron on 2-D Data



Implementation

Perceptron Implementation in Scikitlearn

Theorem

Finite mistake bound: *Let* $S = \{(\mathbf{x}^1, y^1), \ldots, (\mathbf{x}_T, y_T)\}$, *where* T *is number of samples drawn. Let* $\exists\ \mathbf{u} \in \mathbb{R}^d$, *s.t* $\|\mathbf{u}\|_2 = 1$ *and* $y_t \mathbf{u}^T \mathbf{x}_t \geq \gamma$, *where* $\gamma \geq 0,\ t = 1 \ldots T$. *Let* $R_2 = \max_t \|\mathbf{x}_t\|_2$. *The Perceptron algorithm converges in at most* $\left(\frac{R_2}{\gamma}\right)^2$ *iterations.*

Universal Approximation Theorem

Any **continuous function** defined in the **n-dimensional unit hypercube** may be approximated by a **finite sum** of the type:

$$\sum_{j=1}^{N} v_j \, \varphi \left(\vec{\omega}^{(j)} \cdot \vec{x} + b_j \right),$$

wherein $v_j, b_j \in \mathbb{R}$, $\vec{\omega}^{(j)} \in \mathbb{R}^n$, and φ is a **continuous discriminatory function**.