Fuck Logic ICPC Team Notebook

9 Math Extra

9.1 Seldom used combinatorics

- Fibonacci in $O(\log(N))$ with memoization is:

$$f(0) = f(1) = 1$$

$$f(2k) = f(k)^2 + f(k-1)^2$$

$$f(2k+1) = f(k) \times (f(k) + 2 * f(k-1))$$

- Wilson's Theorem Extension:

  $B = b_1 b_2 \ldots b_m (\mod n) = \pm 1$, all $b_i \leq n$ such that $\gcd(b_i, n) = 1$.

  If $n \leq 4$ or $n = (\text{odd prime})^k$ or $n = 2(\text{odd prime})^k)$ then $B = -1$ for any k.

  Else $B = 1$.

- Stirling numbers of the second kind, denoted by $S(n, k) =$ number of ways to split n numbers into k non-empty sets.

$$S(n, 1) = S(n, n) = 1$$

$$S(n, k) = kS(n-1, k) + S(n-1, k-1)$$

$S(n - d + 1, k - d + 1) = S(n, k)$ where if indexes i, j belong to the same set, then $|i - j| \geq d$.

- Burnside's Lemma: $|\text{classes}| = \frac{1}{|G|} \sum_{\forall\; g \in G}(K^{C(g)})$, where:

  G = different permutations possible,

  $C(g)$ = number of cycles on the permutation g,

  and K = Number of states for each element

- Different ways to paint a necklace with N beads and K colors:

$$G = (1, 2, \ldots N), (2, 3, \ldots N, 1), \ldots (N, 1, \ldots N - 1)$$

$g_i = (i, i + 1, \ldots i + N)$, (taking mod N to get it right) $i = 1 \ldots N$ with i per step, that is, $i \to 2i \to 3i \ldots$

Cycles in g_i all have size $n/\gcd(i, n)$, so

$$C(g_i) = \gcd(i, n)$$

$$\text{ans} = \frac{1}{N} \sum_{i=1\ldots N} (K^{\gcd(i,n)})$$

$$\text{ans} = \frac{1}{N} \sum_{\forall\; d|N} (\phi(\frac{N}{d}) K^d)$$

9.2   Combinatorial formulas

$$\sum_{k=0}^{n} k^2 = n(n+1)(2n+1)/6$$

$$\sum_{k=0}^{n} k^3 = n^2(n+1)^2/4$$

$$\sum_{k=0}^{n} k^4 = (6n^5 + 15n^4 + 10n^3 - n)/30$$

$$\sum_{k=0}^{n} k^5 = (2n^6 + 6n^5 + 5n^4 - n^2)/12$$

$$\sum_{k=0}^{n} x^k = (x^{n+1} - 1)/(x - 1)$$

$$\sum_{k=0}^{n} kx^k = (x - (n+1)x^{n+1} + nx^{n+2})/(x - 1)^2$$

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k}$$

$$\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1}$$

$$\binom{n+1}{k} = \frac{n+1}{n-k+1}\binom{n}{k}$$

$$\binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k}$$

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$$

$$\sum_{k=1}^{n} k^2\binom{n}{k} = (n + n^2)2^{n-2}$$

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

$$\binom{n}{k} = \prod_{i=1}^{k} \frac{n-k+i}{i}$$

9.3   Number theory identities

Lucas' Theorem: For non-negative integers m and n and a prime p,

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} \pmod{p},$$

where

$$m = m_k p^k + m_{k-1} p^{k-1} + \cdots + m_1 p + m_0$$

is the base p representation of m, and similarly for n.

9.4   Stirling Numbers of the second kind

Number of ways to partition a set of n numbers into k non-empty subsets.

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{(k-j)} \binom{k}{j} j^n$$

Recurrence relation:

$$\left\{ {0 \atop 0} \right\} = 1$$

$$\left\{ {n \atop 0} \right\} = \left\{ {0 \atop n} \right\} = 1$$

$$\left\{ {n+1 \atop k} \right\} = k \left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\}$$

9.5   Burnside's Lemma

Let G be a finite group that acts on a set X. For each g in G let X^g denote the set of elements in X that are fixed by g, which means $X^g = \{x \in X | g(x) = x\}$. Burnside's lemma assers the following formula for the number of orbits, denoted $|X/G|$:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

9.6   Numerical integration

RK4: to integrate $\dot{y} = f(t, y)$ with $y_0 = y(t_0)$, compute

$$k_1 = f(t_n, y_n)$$
$$k_2 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2} k_1)$$
$$k_3 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2} k_2)$$
$$k_4 = f(t_n + h, y_n + h k_3)$$
$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$