



International Institute Of Information Technology - Hyderabad

# FLogic

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Contest (1)

.bashrc	3 lines
alias c='g++ -Wall -Wconversion -Wfatal-errors -g -std=c++14 \\\n-fsanitize=undefined,address'\n\nxmodmap -e 'clear lock' -e 'keycode 66=less greater' #caps = <>	
template.cpp	17 lines
#include <bits/stdc++.h>\n\nusing namespace std;\n\n#define rep(i, a, b) for(int i = a; i < (b); ++i)\n#define all(x) begin(x), end(x)\n#define sz(x) (int)(x).size()\n#define pb push_back\ntypedef long long ll;\ntypedef pair<int, int> pii;\ntypedef vector<int> vi;\n\nint main() {\n    // freopen("sample.in", "r", stdin);\n    // freopen("sample.out", "w", stdout);\n    cin.tie(0)->sync_with_stdio(0);\n    cin.exceptions(cin.failbit);\n}	

1	troubleshoot.txt	52 lines
1	Pre-submit:\nWrite a few simple test cases if sample is not enough.\nAre time limits close? If so, generate max cases.\nIs the memory usage fine?\nCould anything overflow?\nMake sure to submit the right file.	
3	Wrong answer:\nPrint your solution! Print debug output, as well.\nAre you clearing all data structures between test cases?\nCan your algorithm handle the whole range of input?\nRead the full problem statement again.\nDo you handle all corner cases correctly?\nHave you understood the problem correctly?\nAny uninitialized variables?\nAny overflows?\nConfusing N and M, i and j, etc.?\nAre you sure your algorithm works?\nWhat special cases have you not thought of?\nAre you sure the STL functions you use work as you think?\nAdd some assertions, maybe resubmit.\nCreate some testcases to run your algorithm on.\nGo through the algorithm for a simple case.\nGo through this list again.\nExplain your algorithm to a teammate.\nAsk the teammate to look at your code.\nGo for a small walk, e.g. to the toilet.\nIs your output format correct? (including whitespace)\nRewrite your solution from the start or let a teammate do it.	
5	Runtime error:\nHave you tested all corner cases locally?\nAny uninitialized variables?\nAre you reading or writing outside the range of any vector?\nAny assertions that might fail?\nAny possible division by 0? (mod 0 for example)\nAny possible infinite recursion?\nInvalidated pointers or iterators?\nAre you using too much memory?\nDebug with resubmits (e.g. remapped signals, see Various).	
	Time limit exceeded:\nDo you have any possible infinite loops?\nWhat is the complexity of your algorithm?\nAre you copying a lot of unnecessary data? (References)\nHow big is the input and output? (consider scanf)\nAvoid vector, map. (use arrays/unordered_map)\nWhat do your teammates think about your algorithm?	
	Memory limit exceeded:\nWhat is the max amount of memory your algorithm should need?\nAre you clearing all data structures between test cases?	
	Data structures (2)	
	OrderedSet.h	782797, 16 lines
	#include <bits/extc++.h>\nusing namespace __gnu_pbds;\n\ntemplate<class T>\nusing Tree = tree<T, null_type, less<T>, rb_tree_tag,\n    tree_order_statistics_node_update>;	

<pre>void example() {     Tree&lt;int&gt; t, t2; t.insert(8);     auto it = t.insert(10).first;     assert(it == t.lower_bound(9));     assert(t.order_of_key(10) == 1);     assert(t.order_of_key(11) == 2);     assert(*t.find_by_order(0) == 8);     t.join(t2); // assuming T &lt; T2 or T &gt; T2, merge t2 into t }</pre>	
<h2>Dsu.h</h2> <p><b>Description:</b> DSU with rollback</p> <p><b>Time:</b> <math>\mathcal{O}(\alpha(N))</math></p>	29a3e8, 18 lines
<pre>struct DSU {     int sets; vi p, s;     stack&lt;pii&gt; ss, sp;     DSU(int n) : p(n, -1), s(n, 1), sets(n) {}     bool IsSameSet(int a, int b) { return find(a) == find(b); }     int find(int x) {return p[x] == -1 ? x : p[x] = find(p[x]);}     void join(int a, int b) {         a = find(a), b = find(b);         if (a == b) return;         if (s[a] &lt; s[b]) swap(a, b);         ss.push({a, s[a]}); sp.push({b, p[b]});         sets--; s[a] += s[b]; p[b] = a;     }     void rollback() {         p[sp.top().first] = sp.top().second; sp.pop();         s[ss.top().first] = ss.top().second; ss.pop();     } };</pre>	
<h2>DsuBp.h</h2> <p><b>Description:</b> Graph, adding edges, checking bp color</p> <p><b>Time:</b> <math>\mathcal{O}(\alpha(N))</math></p>	8e325a, 18 lines
<pre>struct DSU {     int sets; vi p, s, l;     DSU(int n) : p(n, -1), s(n, 1), l(n, 0), sets(n) {}     bool IsSameColor(int a,int b) {         find(a); find(b); return l[a] == l[b];     }     bool IsSameSet(int a, int b) { return find(a) == find(b); }     int find(int x) {         if(p[x] == -1) return x;         int y = find(p[x]); l[x] ^= l[p[x]]; return p[x] = y;     }     void join(int a, int b) {         int ca = a, cb = b; a = find(a), b = find(b);         if (a == b) return;         if (s[a] &lt; s[b]) swap(a, b);         sets--; s[a] += s[b]; l[b] = 1 ^ l[ca] ^ l[cb]; p[b] = a;     } };</pre>	
<h2>MinQueue.h</h2> <p><b>Description:</b> DSU with rollback</p> <p><b>Time:</b> <math>\mathcal{O}(\alpha(N))</math></p>	68dd56, 24 lines
<pre>template&lt;class T&gt; struct MinQueue {     deque&lt;pair&lt;T, T&gt;&gt; q;     int ca = 0, cr = 0, plus = 0, size = 0;     void push(T x) {         x -= plus;         // change '&gt;' to '&lt;' and you get max-queue         while (!q.empty() &amp;&amp; q.back().first &gt; x)             q.pop_back();     } };</pre>	

```
    q.push_back({x, ca}); ca++; sze++;
}
T pop() {
    T re = 0;
    if (!q.empty() && q.front().second == cr) {
        re = q.front().first; q.pop_front();
    }
    cr++; sze--; return re + plus;
}
// Returns minimum in the queue
T min() { return q.front().first + plus; }
int size() { return sze; }
// Adds x to every element in the queue
void add(int x) { plus += x; }
};
```

Matrix.h

**Description:** Basic operations on square matrices.

**Usage:** Matrix<int, 3> A;

A.d = {{{{1,2,3}}, {{4,5,6}}, {{7,8,9}}}};

vector<int> vec = {1,2,3};

vec = (A^N) \* vec;

c43c7d, 26 lines

```
template<class T, int N> struct Matrix {
    typedef Matrix M;
    array<array<T, N>, N> d{};
    M operator*(const M& m) const {
        M a;
        rep(i,0,N) rep(j,0,N)
            rep(k,0,N) a.d[i][j] += d[i][k]*m.d[k][j];
        return a;
    }
    vector<T> operator*(const vector<T>& vec) const {
        vector<T> ret(N);
        rep(i,0,N) rep(j,0,N) ret[i] += d[i][j] * vec[j];
        return ret;
    }
    M operator^(ll p) const {
        assert(p >= 0);
        M a, b(*this);
        rep(i,0,N) a.d[i][i] = 1;
        while (p) {
            if (p&1) a = a*b;
            b = b*b;
            p >>= 1;
        }
        return a;
    }
};
```

SparseTable.h

**Description:** Range Minimum Queries on an array. Returns min(V[a], V[a + 1], ... V[b - 1]) in constant time.

**Usage:** RMQ rmq(values);

rmq.query(inclusive, exclusive);

**Time:**  $\mathcal{O}(|V|\log|V| + Q)$

4a61f2, 21 lines

```
template<class T>
struct SparseTable {
    T (*op)(T, T);
    vi log2s; vector<vector<T>> st;
    SparseTable (const vector<T>& arr, T (*op)(T, T))
        : op(op), log2s(sz(arr)+1), st(sz(arr)) {
        rep(i,2,sz(log2s)) { log2s[i] = log2s[i/2] + 1; }
        rep(i,0,sz(arr)) {
            st[i].assign(log2s[sz(arr) - i] + 1);
            st[i][0] = arr[i];
        }
        rep(p, 1, log2s[sz(arr)] + 1) rep(i,0,sz(arr))
```

```
        if(i+(1<<p) <= sz(arr)) {
            st[i][p] = op(st[i][p-1], st[i+(1<<(p-1))][p-1]);
        }
    }
    T query (int l, int r) {
        int p = log2s[r-l+1];
        return op(st[l][p], st[r-(1<<p)+1][p]);
    }
};
```

FenwickTree.h

**Description:** Computes partial sums a[0] + a[1] + ... + a[pos - 1], and updates single elements a[i], taking the difference between the old and new value.

**Time:** Both operations are  $\mathcal{O}(\log N)$ .

e62fac, 22 lines

```
struct FT {
    vector<ll> s;
    FT(int n) : s(n) {}
    void update(int pos, ll dif) { // a[pos] += dif
        for (; pos < sz(s); pos |= pos + 1) s[pos] += dif;
    }
    ll query(int pos) { // sum of values in [0, pos]
        ll res = 0;
        for (; pos > 0; pos &= pos - 1) res += s[pos-1];
        return res;
    }
    int lower_bound(ll sum) { // min pos st sum of [0, pos] >= sum
        // Returns n if no sum is >= sum, or -1 if empty sum is.
        if (sum <= 0) return -1;
        int pos = 0;
        for (int pw = 1 << 25; pw; pw >>= 1) {
            if (pos + pw <= sz(s) && s[pos + pw-1] < sum)
                pos += pw, sum -= s[pos-1];
        }
        return pos;
    }
};
```

FenwickTree2D.h

**Description:** Computes sums a[i,j] for all i<I, j<J, and increases single elements a[i,j]. Requires that the elements to be updated are known in advance (call fakeUpdate() before init()).

**Time:**  $\mathcal{O}(\log^2 N)$ . (Use persistent segment trees for  $\mathcal{O}(\log N)$ .)

"FenwickTree.h" 157f07, 22 lines

```
struct FT2 {
    vector<vi> ys; vector<FT> ft;
    FT2(int limx) : ys(limx) {}
    void fakeUpdate(int x, int y) {
        for (; x < sz(ys); x |= x + 1) ys[x].push_back(y);
    }
    void init() {
        for (vi& v : ys) sort(all(v)), ft.emplace_back(sz(v));
    }
    int ind(int x, int y) {
        return (int)(lower_bound(all(ys[x]), y) - ys[x].begin()); }
    void update(int x, int y, ll dif) {
        for (; x < sz(ys); x |= x + 1)
            ft[x].update(ind(x, y), dif);
    }
    ll query(int x, int y) {
        ll sum = 0;
        for (; x; x &= x - 1)
            sum += ft[x-1].query(ind(x-1, y));
        return sum;
    }
};
```

SegmentTree.h

**Description:** RMQ SegTree

**Time:**  $\mathcal{O}(\alpha(N))$

f19f05, 42 lines

```
const ll INF = 1e18;
struct node {
    ll x;
};

template<class T>
struct SegmentTrees {
    vector<node> st, lazy;
    node def;
    SegmentTrees(int n) : st(4*n, {INF}), lazy(4*n, {INF}), def({
        INF}) {}
    inline node combine(node a, node b) {
        node ret; ret.x = min(a.x, b.x); return ret;
    }
    void push(int pos) {
        if(lazy[pos].x != INF) {
            st[pos*2] = lazy[pos]; st[pos*2 + 1] = lazy[pos];
            lazy[pos*2] = lazy[pos]; lazy[pos*2+1] = lazy[pos];
            lazy[pos] = def;
        }
    }
    void update(int l,int r,T val,int left,int right,int pos=1) {
        if(l > r) return;
        if(l==left && r==right) {
            st[pos].x = val; lazy[pos] = {val};
        } else {
            push(pos);
            int mid = (left + right)/2;
            update(l, min(r,mid), val, left, mid, pos*2);
            update(max(l,mid+1), r, val, mid+1, right, pos*2+1);
            st[pos] = combine(st[pos*2], st[pos*2+1]);
        }
    }
    node query(int l,int r,int left,int right,int pos=1) {
        if(l>r) return def;
        if(l==left && r==right) return st[pos];
        else {
            push(pos); int mid = (left + right)/2;
            return combine(query(l, min(r,mid), left, mid, pos*2),
                query(max(l,mid+1), r, mid+1, right, pos*2+1));
        }
    }
};
```

Strings (3)

Hashing.h

**Description:** Various self-explanatory methods for string hashing. Use on Codeforces, which lacks 64-bit support and where solutions can be hacked.

<sys/time.h> 4b7e28, 36 lines

```
typedef uint64_t ull;
static int C; // initialized below

// Arithmetic mod two primes and 2^32 simultaneously.
// "typedef uint64_t H;" instead if Thue-Morse does not apply.
template<int M, class B>
struct A {
    int x; B b; A(int x=0) : x(x), b(x) {}
    A(int x, B b) : x(x), b(b) {}
    A operator+(A o){int y = x+o.x; return{y - (y>=M)*M, b+o.b};}
    A operator-(A o){int y = x-o.x; return{y + (y< 0)*M, b-o.b};}
    A operator*(A o) { return {(int)(1LL*x*o.x % M), b*o.b}; }
    explicit operator ull() { return x ^ (ull) b << 21; }
};
```

<pre><b>typedef</b> A&lt;10000000007, A&lt;10000000009, <b>unsigned</b>&gt;&gt; H;  <b>struct</b> HashInterval {     vector&lt;H&gt; ha, pw;     HashInterval(string&amp; str) : ha(SZ(str)+1), pw(ha) {         pw[0] = 1;         <b>REP</b>(i,0,SZ(str))             ha[i+1] = ha[i] * C + str[i],             pw[i+1] = pw[i] * C;     }     H hashInterval(<b>int</b> a, <b>int</b> b) { // hash [a, b)         <b>return</b> ha[b] - ha[a] * pw[b - a];     } };  <b>int</b> main() {     timeval tp;     gettimeofday(&amp;tp, 0);     C = (<b>int</b>)tp.tv_usec; // (less than modulo)     assert((ull)(H(1)*2+1-3) == 0);     // ... }</pre>
--

<p><b>Kmp.h</b> <b>Description:</b> pi[x] computes the length of the longest prefix of s that ends at x, other than s[0...x] itself (abacaba -&gt; 0010123). Can be used to find all occurrences of a string. <b>Time:</b> <math>\mathcal{O}(n)</math></p>	d4375e, 16 lines
--	------------------

```
vi pi(const string& s) {
    vi p(sz(s));
    rep(i,1,sz(s)) {
        int g = p[i-1];
        while (g && s[i] != s[g]) g = p[g-1];
        p[i] = g + (s[i] == s[g]);
    }
    return p;
}
```

```
vi match(const string& s, const string& pat) {
    vi p = pi(pat + '\0' + s), res;
    rep(i,sz(p)-sz(s),sz(p))
        if (p[i] == sz(pat)) res.push_back(i - 2 * sz(pat));
    return res;
}
```

<p><b>Manacher.h</b> <b>Description:</b> For each position in a string, computes p[0][i] = half length of longest even palindrome around pos i, p[1][i] = longest odd (half rounded down). <b>Time:</b> <math>\mathcal{O}(N)</math></p>	e7ad79, 13 lines
---	------------------

```
array<vi, 2> manacher(const string& s) {
    int n = sz(s);
    array<vi,2> p = {vi(n+1), vi(n)};
    rep(z,0,2) for (int i=0,l=0,r=0; i < n; i++) {
        int t = r-i!z;
        if (i<r) p[z][i] = min(t, p[z][l+t]);
        int L = i-p[z][i], R = i+p[z][i]-!z;
        while (L>=1 && R+1<n && s[L-1] == s[R+1])
            p[z][i]++, L--, R++;
        if (R>r) l=L, r=R;
    }
    return p;
}
```

<p><b>MinRotation.h</b> <b>Description:</b> Finds the lexicographically smallest rotation of a string. <b>Usage:</b> rotate(v.begin(), v.begin()+minRotation(v), v.end());</p>
--

## Kmp Manacher MinRotation SuffixArray Z

Time: $\mathcal{O}(N)$	d07a42, 8 lines
<pre>int minRotation(string s) {     int a=0, N=sz(s); s += s;     rep(b,0,N) rep(k,0,N) {         if (a+k == b    s[a+k] &lt; s[b+k]) { b += max(0, k-1); break; }         if (s[a+k] &gt; s[b+k]) { a = b; break; }     }     return a; }</pre>	

<p><b>SuffixArray.h</b> <b>Description:</b> Builds suffix array for a string. sa[i] is the starting index of the suffix which is <i>i</i>'th in the sorted suffix array. The returned vector is of size <math>n + 1</math>, and sa[0] = n. The lcp array contains longest common prefixes for neighbouring strings in the suffix array: lcp[i] = lcp(sa[i], sa[i-1]), lcp[0] = 0. The input string must not contain any zero bytes. <b>Time:</b> <math>\mathcal{O}(n \log n)</math></p>	38db9f, 23 lines
---	------------------

```
struct SuffixArray {
    vi sa, lcp;
    SuffixArray(string& s, int lim=256) { // or basic_string<int>
        int n = sz(s) + 1, k = 0, a, b;
        vi x(all(s)+1), y(n), ws(max(n, lim)), rank(n);
        sa = lcp = y, iota(all(sa), 0);
        for (int j = 0, p = 0; p < n; j = max(1, j * 2), lim = p) {
            p = j, iota(all(y), n - j);
            rep(i,0,n) if (sa[i] >= j) y[p++] = sa[i] - j;
            fill(all(ws), 0);
            rep(i,0,n) ws[x[i]]++;
            rep(i,1,lim) ws[i] += ws[i - 1];
            for (int i = n; i--;) sa[--ws[x[y[i]]]] = y[i];
            swap(x, y), p = 1, x[sa[0]] = 0;
            rep(i,1,n) a = sa[i - 1], b = sa[i], x[b] =
                (y[a] == y[b] && y[a + j] == y[b + j]) ? p - 1 : p++;
        }
        rep(i,1,n) rank[sa[i]] = i;
        for (int i = 0, j; i < n - 1; lcp[rank[i++]] = k)
            for (k && k--, j = sa[rank[i] - 1];
                s[i + k] == s[j + k]; k++);
    }
};
```

<p><b>Z.h</b> <b>Description:</b> z[x] computes the length of the longest common prefix of s[i:] and s, except z[0] = 0. (abacaba -&gt; 0010301) <b>Time:</b> <math>\mathcal{O}(n)</math></p>	ee09e2, 12 lines
---	------------------

```
vi Z(const string& S) {
    vi z(sz(S));
    int l = -1, r = -1;
    rep(i,1,sz(S)) {
        z[i] = i >= r ? 0 : min(r - i, z[i - 1]);
        while (i + z[i] < sz(S) && S[i + z[i]] == S[z[i]])
            z[i]++;
        if (i + z[i] > r)
            l = i, r = i + z[i];
    }
    return z;
}
```

## Mathematics (4)

### 4.1 Equations

$$ax^2+bx+c=0\Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The extremum is given by  $x=-b/2a$ .

$$\begin{matrix}ax+by=e\\cx+dy=f\end{matrix}\Rightarrow\begin{matrix}x=\frac{ed-bf}{ad-bc}\\y=\frac{af-ec}{ad-bc}\end{matrix}$$

In general, given an equation  $Ax=b$ , the solution to a variable  $x_i$  is given by

$$x_i=\frac{\det A_i'}{\det A}$$

where  $A_i'$  is  $A$  with the  $i$ 'th column replaced by  $b$ .

### 4.2 Recurrences

If  $a_n=c_1a_{n-1}+\cdots+c_ka_{n-k}$ , and  $r_1,\ldots,r_k$  are distinct roots of  $x^k+c_1x^{k-1}+\cdots+c_k$ , there are  $d_1,\ldots,d_k$  s.t.

$$a_n=d_1r_1^n+\cdots+d_kr_k^n.$$

Non-distinct roots  $r$  become polynomial factors, e.g.  
 $a_n=(d_1n+d_2)r^n$ .

### 4.3 Trigonometry

$$\sin(v+w)=\sin v\cos w+\cos v\sin w$$

$$\cos(v+w)=\cos v\cos w-\sin v\sin w$$

$$\tan(v+w)=\frac{\tan v+\tan w}{1-\tan v\tan w}$$

$$\sin v+\sin w=2\sin\frac{v+w}{2}\cos\frac{v-w}{2}$$

$$\cos v+\cos w=2\cos\frac{v+w}{2}\cos\frac{v-w}{2}$$

$$(V+W)\tan(v-w)/2=(V-W)\tan(v+w)/2$$

where  $V,W$  are lengths of sides opposite angles  $v,w$ .

$$a\cos x+b\sin x=r\cos(x-\phi)$$

$$a\sin x+b\cos x=r\sin(x+\phi)$$

where  $r=\sqrt{a^2+b^2}$ ,  $\phi=\operatorname{atan2}(b,a)$ .

### 4.4 Geometry

#### 4.4.1 Triangles

Side lengths:  $a,b,c$

Semiperimeter:  $p=\frac{a+b+c}{2}$

Area:  $A=\sqrt{p(p-a)(p-b)(p-c)}$

Circumradius:  $R=\frac{abc}{4A}$

Inradius:  $r=\frac{A}{p}$

Length of median (divides triangle into two equal-area triangles):  
 $m_a=\frac{1}{2}\sqrt{2b^2+2c^2-a^2}$

Length of bisector (divides angles in two):

$$s_a = \sqrt{bc \left[ 1 - \left( \frac{a}{b+c} \right)^2 \right]}$$
  
Law of sines:  $\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} = \frac{1}{2R}$   
Law of cosines:  $a^2 = b^2 + c^2 - 2bc \cos \alpha$   
Law of tangents:  $\frac{a+b}{a-b} = \frac{\tan \frac{\alpha+\beta}{2}}{\tan \frac{\alpha-\beta}{2}}$

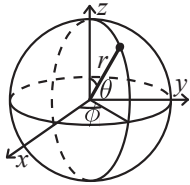
4.4.2 Quadrilaterals

With side lengths  $a, b, c, d$ , diagonals  $e, f$ , diagonals angle  $\theta$ , area  $A$  and magic flux  $F = b^2 + d^2 - a^2 - c^2$ :

$$4A = 2ef \cdot \sin \theta = F \tan \theta = \sqrt{4e^2 f^2 - F^2}$$

For cyclic quadrilaterals the sum of opposite angles is  $180^\circ$ ,  $ef = ac + bd$ , and  $A = \sqrt{(p-a)(p-b)(p-c)(p-d)}$ .

4.4.3 Spherical coordinates



$$\begin{aligned} x &= r \sin \theta \cos \phi & r &= \sqrt{x^2 + y^2 + z^2} \\ y &= r \sin \theta \sin \phi & \theta &= \arccos(z / \sqrt{x^2 + y^2 + z^2}) \\ z &= r \cos \theta & \phi &= \operatorname{atan2}(y, x) \end{aligned}$$

4.5 Derivatives/Integrals

$$\begin{aligned} \frac{d}{dx} \arcsin x &= \frac{1}{\sqrt{1-x^2}} & \frac{d}{dx} \arccos x &= -\frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx} \tan x &= 1 + \tan^2 x & \frac{d}{dx} \arctan x &= \frac{1}{1+x^2} \\ \int \tan ax &= -\frac{\ln |\cos ax|}{a} & \int x \sin ax &= \frac{\sin ax - ax \cos ax}{a^2} \\ \int e^{-x^2} &= \frac{\sqrt{\pi}}{2} \operatorname{erf}(x) & \int x e^{ax} dx &= \frac{e^{ax}}{a^2} (ax - 1) \end{aligned}$$

Integration by parts:

$$\int_a^b f(x)g(x)dx = [F(x)g(x)]_a^b - \int_a^b F(x)g'(x)dx$$

4.6 Sums

$$c^a + c^{a+1} + \dots + c^b = \frac{c^{b+1} - c^a}{c - 1}, c \neq 1$$
  
$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$
  
$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(2n+1)(n+1)}{6}$$
  
$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$
  
$$1^4 + 2^4 + 3^4 + \dots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

4.7 Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, (-\infty < x < \infty)$$
  
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, (-1 < x \leq 1)$$
  
$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{2x^3}{32} - \frac{5x^4}{128} + \dots, (-1 \leq x \leq 1)$$
  
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, (-\infty < x < \infty)$$
  
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, (-\infty < x < \infty)$$

4.8 Probability theory

Let  $X$  be a discrete random variable with probability  $p_X(x)$  of assuming the value  $x$ . It will then have an expected value (mean)  $\mu = \mathbb{E}(X) = \sum_x x p_X(x)$  and variance  $\sigma^2 = V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \sum_x (x - \mathbb{E}(X))^2 p_X(x)$  where  $\sigma$  is the standard deviation. If  $X$  is instead continuous it will have a probability density function  $f_X(x)$  and the sums above will instead be integrals with  $p_X(x)$  replaced by  $f_X(x)$ .

Expectation is linear:

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$$

For independent  $X$  and  $Y$ ,

$$V(aX + bY) = a^2 V(X) + b^2 V(Y).$$

4.8.1 Discrete distributions

Binomial distribution

The number of successes in  $n$  independent yes/no experiments, each which yields success with probability  $p$  is  $\operatorname{Bin}(n, p)$ ,  $n = 1, 2, \dots$ ,  $0 \leq p \leq 1$ .

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mu = np, \sigma^2 = np(1-p)$$

$\operatorname{Bin}(n, p)$  is approximately  $\operatorname{Po}(np)$  for small  $p$ .

First success distribution

The number of trials needed to get the first success in independent yes/no experiments, each wich yields success with probability  $p$  is  $\operatorname{Fs}(p)$ ,  $0 \leq p \leq 1$ .

$$p(k) = p(1-p)^{k-1}, k = 1, 2, \dots$$

$$\mu = \frac{1}{p}, \sigma^2 = \frac{1-p}{p^2}$$

Poisson distribution

The number of events occurring in a fixed period of time  $t$  if these events occur with a known average rate  $\kappa$  and independently of the time since the last event is  $\operatorname{Po}(\lambda)$ ,  $\lambda = t\kappa$ .

$$p(k) = e^{-\lambda} \frac{\lambda^k}{k!}, k = 0, 1, 2, \dots$$

$$\mu = \lambda, \sigma^2 = \lambda$$

4.8.2 Continuous distributions

Uniform distribution

If the probability density function is constant between  $a$  and  $b$  and 0 elsewhere it is  $\operatorname{U}(a, b)$ ,  $a < b$ .

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$\mu = \frac{a+b}{2}, \sigma^2 = \frac{(b-a)^2}{12}$$

Exponential distribution

The time between events in a Poisson process is  $\operatorname{Exp}(\lambda)$ ,  $\lambda > 0$ .

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\mu = \frac{1}{\lambda}, \sigma^2 = \frac{1}{\lambda^2}$$

Normal distribution

Most real random values with mean  $\mu$  and variance  $\sigma^2$  are well described by  $\mathcal{N}(\mu, \sigma^2)$ ,  $\sigma > 0$ .

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

If  $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$  and  $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$  then

$$aX_1 + bX_2 + c \sim \mathcal{N}(\mu_1 + \mu_2 + c, a^2\sigma_1^2 + b^2\sigma_2^2)$$

4.9 Markov chains

A *Markov chain* is a discrete random process with the property that the next state depends only on the current state. Let  $X_1, X_2, \dots$  be a sequence of random variables generated by the Markov process. Then there is a transition matrix  $\mathbf{P} = (p_{ij})$ , with  $p_{ij} = \Pr(X_n = i | X_{n-1} = j)$ , and  $\mathbf{p}^{(n)} = \mathbf{P}^n \mathbf{p}^{(0)}$  is the probability distribution for  $X_n$  (i.e.,  $p_i^{(n)} = \Pr(X_n = i)$ ), where  $\mathbf{p}^{(0)}$  is the initial distribution.

$\pi$  is a stationary distribution if  $\pi = \pi \mathbf{P}$ . If the Markov chain is *irreducible* (it is possible to get to any state from any state), then  $\pi_i = \frac{1}{\mathbb{E}(T_i)}$  where  $\mathbb{E}(T_i)$  is the expected time between two visits in state  $i$ .  $\pi_j / \pi_i$  is the expected number of visits in state  $j$  between two visits in state  $i$ .

For a connected, undirected and non-bipartite graph, where the transition probability is uniform among all neighbors,  $\pi_i$  is proportional to node  $i$ 's degree.

A Markov chain is *ergodic* if the asymptotic distribution is independent of the initial distribution. A finite Markov chain is ergodic iff it is irreducible and *aperiodic* (i.e., the gcd of cycle lengths is 1).  $\lim_{k \rightarrow \infty} \mathbf{P}^k = \mathbf{1}\pi$ .

A Markov chain is an A-chain if the states can be partitioned into two sets  $\mathbf{A}$  and  $\mathbf{G}$ , such that all states in  $\mathbf{A}$  are absorbing ( $p_{ii} = 1$ ), and all states in  $\mathbf{G}$  leads to an absorbing state in  $\mathbf{A}$ . The probability for absorption in state  $i \in \mathbf{A}$ , when the initial state is  $j$ , is  $a_{ij} = p_{ij} + \sum_{k \in \mathbf{G}} a_{ik} p_{kj}$ . The expected time until absorption, when the initial state is  $i$ , is  $t_i = 1 + \sum_{k \in \mathbf{G}} p_{ki} t_k$ .

Geometry (5)

5.1 Geometric primitives

Point.h	
<b>Description:</b> Class to handle points in the plane. T can be e.g. double or long long. (Avoid int.)	
<pre>template &lt;class T&gt; int sgn(T x) { return (x &gt; 0) - (x &lt; 0); } template&lt;class T&gt; struct Point {     typedef Point P;     T x, y;     explicit Point(T x=0, T y=0) : x(x), y(y) {}     bool operator&lt;(P p) const { return tie(x,y) &lt; tie(p.x,p.y); }     bool operator==(P p) const { return tie(x,y)==tie(p.x,p.y); }     P operator+(P p) const { return P(x+p.x, y+p.y); }     P operator-(P p) const { return P(x-p.x, y-p.y); }     P operator*(T d) const { return P(x*d, y*d); }     P operator/(T d) const { return P(x/d, y/d); }     T dot(P p) const { return x*p.x + y*p.y; }     T cross(P p) const { return x*p.y - y*p.x; }     T cross(P a, P b) const { return (a-*this).cross(b-*this); }     T dist2() const { return x*x + y*y; }     double dist() const { return sqrt((double)dist2()); }     // angle to x-axis in interval [-pi, pi]     double angle() const { return atan2(y, x); }     P unit() const { return *this/dist(); } // makes dist()==1</pre>	47ec0a, 28 lines

<pre>P perp() const { return P(-y, x); } // rotates +90 degrees P normal() const { return perp().unit(); } // returns point rotated 'a' radians ccw around the origin P rotate(double a) const {     return P(x*cos(a)-y*sin(a),x*sin(a)+y*cos(a)); } friend ostream&amp; operator&lt;&lt;(ostream&amp; os, P p) {     return os &lt;&lt; "(" &lt;&lt; p.x &lt;&lt; ", " &lt;&lt; p.y &lt;&lt; ")"; }</pre>	

lineDistance.h	
<b>Description:</b> Returns the signed distance between point p and the line containing points a and b. Positive value on left side and negative on right as seen from a towards b. a==b gives nan. P is supposed to be Point<T> or Point3D<T> where T is e.g. double or long long. It uses products in intermediate steps so watch out for overflow if using int or long long. Using Point3D will always give a non-negative distance. For Point3D, call .dist on the result of the cross product.	
<pre>template&lt;class P&gt; double lineDist(const P&amp; a, const P&amp; b, const P&amp; p) {     return (double) (b-a).cross(p-a) / (b-a).dist(); }</pre>	f6bf6b, 4 lines

SegmentDistance.h	
<b>Description:</b> Returns the shortest distance between point p and the line segment from point s to e.	
<b>Usage:</b> Point<double> a, b(2,2), p(1,1); bool onSegment = segDist(a,b,p) < 1e-10;	
<pre>"Point.h"</pre>	5c88fd, 6 lines

<pre>typedef Point&lt;double&gt; P; double segDist(P&amp; s, P&amp; e, P&amp; p) {     if (s==e) return (p-s).dist();     auto d = (e-s).dist2(), t = min(d,max(.0, (p-s).dot(e-s)));     return ((p-s)*d-(e-s)*t).dist() / d; }</pre>	
--	--

SegmentIntersection.h	
<b>Description:</b> If a unique intersection point between the line segments going from s1 to e1 and from s2 to e2 exists then it is returned. If no intersection point exists an empty vector is returned. If infinitely many exist a vector with 2 elements is returned, containing the endpoints of the common line segment. The wrong position will be returned if P is Point<ll> and the intersection point does not have integer coordinates. Products of three coordinates are used in intermediate steps so watch out for overflow if using int or long long.	
<b>Usage:</b> vector<P> inter = segInter(s1,e1,s2,e2); if (sz(inter)==1) cout << "segments intersect at " << inter[0] << endl;	
<pre>"Point.h", "OnSegment.h"</pre>	9d57f2, 13 lines

<pre>template&lt;class P&gt; vector&lt;P&gt; segInter(P a, P b, P c, P d) {     auto oa = c.cross(d, a), ob = c.cross(d, b),         oc = a.cross(b, c), od = a.cross(b, d);     // Checks if intersection is single non-endpoint point.     if (sgn(oa) * sgn(ob) &lt; 0 &amp;&amp; sgn(oc) * sgn(od) &lt; 0)         return {(a * ob - b * oa) / (ob - oa)};     set&lt;P&gt; s;     if (onSegment(c, d, a)) s.insert(a);     if (onSegment(c, d, b)) s.insert(b);     if (onSegment(a, b, c)) s.insert(c);     if (onSegment(a, b, d)) s.insert(d);     return {all(s)}; }</pre>	
---	--

lineIntersection.h	
<b>Description:</b> If a unique intersection point of the lines going through s1,e1 and s2,e2 exists {1, point} is returned. If no intersection point exists {0, (0,0)} is returned and if infinitely many exists {-1, (0,0)} is returned. The wrong position will be returned if P is Point<ll> and the intersection point does not have integer coordinates. Products of three coordinates are used in intermediate steps so watch out for overflow if using int or ll.	
<b>Usage:</b> auto res = lineInter(s1,e1,s2,e2); if (res.first == 1) cout << "intersection point at " << res.second << endl;	
<pre>"Point.h"</pre>	a01f81, 8 lines

<pre>template&lt;class P&gt; pair&lt;int, P&gt; lineInter(P s1, P e1, P s2, P e2) {     auto d = (e1 - s1).cross(e2 - s2);     if (d == 0) // if parallel         return {-(s1.cross(e1, s2) == 0), P(0, 0)};     auto p = s2.cross(e1, e2), q = s2.cross(e2, s1);     return {1, (s1 * p + e1 * q) / d}; }</pre>	
---	--

sideOf.h	
<b>Description:</b> Returns where $p$ is as seen from $s$ towards $e$ . $1/0/-1 \Leftrightarrow$ left/on line/right. If the optional argument $eps$ is given 0 is returned if $p$ is within distance $eps$ from the line. P is supposed to be Point<T> where T is e.g. double or long long. It uses products in intermediate steps so watch out for overflow if using int or long long.	
<b>Usage:</b> bool left = sideOf(p1,p2,q)==1;	
<pre>"Point.h"</pre>	3af81c, 9 lines

<pre>template&lt;class P&gt; int sideOf(P s, P e, P p) { return sgn(s.cross(e, p)); }</pre>	
<pre>template&lt;class P&gt; int sideOf(const P&amp; s, const P&amp; e, const P&amp; p, double eps) {     auto a = (e-s).cross(p-s);     double l = (e-s).dist()*eps;     return (a &gt; 1) - (a &lt; -1); }</pre>	

OnSegment.h	
<b>Description:</b> Returns true iff p lies on the line segment from s to e. Use (segDist(s,e,p)<=epsilon) instead when using Point<double>.	
<pre>"Point.h"</pre>	c597e8, 3 lines

<pre>template&lt;class P&gt; bool onSegment(P s, P e, P p) {     return p.cross(s, e) == 0 &amp;&amp; (s - p).dot(e - p) &lt;= 0; }</pre>	
---	--

linearTransformation.h	
<b>Description:</b> Apply the linear transformation (translation, rotation and scaling) which takes line p0-p1 to line q0-q1 to point r.	
<pre>"Point.h"</pre>	03a306, 6 lines

<pre>typedef Point&lt;double&gt; P; P linearTransformation(const P&amp; p0, const P&amp; p1,     const P&amp; q0, const P&amp; q1, const P&amp; r) {     P dp = p1-p0, dq = q1-q0, num(dp.cross(dq), dp.dot(dq));     return q0 + P((r-p0).cross(num), (r-p0).dot(num))/dp.dist2(); }</pre>	
---	--

Angle.h	
<b>Description:</b> A class for ordering angles (as represented by int points and a number of rotations around the origin). Useful for rotational sweeping. Sometimes also represents points or vectors.	

**Usage:** vector<Angle> v = {w[0], w[0].t360() ...}; // sorted

```
int j = 0; rep(i,0,n) { while (v[j] < v[i].t180()) ++j; }
// sweeps j such that (j-i) represents the number of positively oriented triangles with vertices at 0 and i
```

0f0602, 35 lines

```
struct Angle {
    int x, y;
    int t;
    Angle(int x, int y, int t=0) : x(x), y(y), t(t) {}
    Angle operator-(Angle b) const { return {x-b.x, y-b.y, t}; }
    int half() const {
        assert(x || y);
        return y < 0 || (y == 0 && x < 0);
    }
    Angle t90() const { return {-y, x, t + (half() && x >= 0)}; }
    Angle t180() const { return {-x, -y, t + half()}; }
    Angle t360() const { return {x, y, t + 1}; }
};

bool operator<(Angle a, Angle b) {
    // add a.dist2() and b.dist2() to also compare distances
    return make_tuple(a.t, a.half(), a.y * (11)b.x) <
           make_tuple(b.t, b.half(), a.x * (11)b.y);
}

// Given two points, this calculates the smallest angle between
// them, i.e., the angle that covers the defined line segment.
pair<Angle, Angle> segmentAngles(Angle a, Angle b) {
    if (b < a) swap(a, b);
    return (b < a.t180() ?
           make_pair(a, b) : make_pair(b, a.t360()));
}

Angle operator+(Angle a, Angle b) { // point a + vector b
    Angle r(a.x + b.x, a.y + b.y, a.t);
    if (a.t180() < r) r.t--;
    return r.t180() < a ? r.t360() : r;
}

Angle angleDiff(Angle a, Angle b) { // angle b - angle a
    int tu = b.t - a.t; a.t = b.t;
    return {a.x*b.x + a.y*b.y, a.x*b.y - a.y*b.x, tu - (b < a)};
}
```

## 5.2 Circles

### CircleIntersection.h

**Description:** Computes the pair of points at which two circles intersect. Returns false in case of no intersection.

```
"Point.h"
84d6d3, 11 lines

typedef Point<double> P;
bool circleInter(P a,P b,double r1,double r2,pair<P, P>* out) {
    if (a == b) { assert(r1 != r2); return false; }
    P vec = b - a;
    double d2 = vec.dist2(), sum = r1+r2, dif = r1-r2,
           p = (d2 + r1*r1 - r2*r2)/(d2*2), h2 = r1*r1 - p*p*d2;
    if (sum*sum < d2 || dif*dif > d2) return false;
    P mid = a + vec*p, per = vec.perp() * sqrt(fmax(0, h2) / d2);
    *out = {mid + per, mid - per};
    return true;
}
```

### CircleTangents.h

**Description:** Finds the external tangents of two circles, or internal if r2 is negated. Can return 0, 1, or 2 tangents – 0 if one circle contains the other (or overlaps it, in the internal case, or if the circles are the same); 1 if the circles are tangent to each other (in which case .first = .second and the tangent line is perpendicular to the line between the centers). .first and .second give the tangency points at circle 1 and 2 respectively. To find the tangents of a circle with a point set r2 to 0.

```
"Point.h"
b0153d, 13 lines

template<class P>
```

```
vector<pair<P, P>> tangents(P c1, double r1, P c2, double r2) {
    P d = c2 - c1;
    double dr = r1 - r2, d2 = d.dist2(), h2 = d2 - dr * dr;
    if (d2 == 0 || h2 < 0) return {};
    vector<pair<P, P>> out;
    for (double sign : {-1, 1}) {
        P v = (d * dr + d.perp() * sqrt(h2) * sign) / d2;
        out.push_back({c1 + v * r1, c2 + v * r2});
    }
    if (h2 == 0) out.pop_back();
    return out;
}
```

1

**CirclePolygonIntersection.h**

**Description:** Returns the area of the intersection of a circle with a ccw polygon.

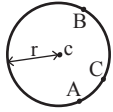
**Time:**  $\mathcal{O}(n)$

```
"../content/geometry/Point.h"
a1ee63, 19 lines

typedef Point<double> P;
#define arg(p, q) atan2(p.cross(q), p.dot(q))
double circlePoly(P c, double r, vector<P> ps) {
    auto tri = [&](P p, P q) {
        auto r2 = r * r / 2;
        P d = q - p;
        auto a = d.dot(p)/d.dist2(), b = (p.dist2()-r*r)/d.dist2();
        auto det = a * a - b;
        if (det <= 0) return arg(p, q) * r2;
        auto s = max(0., -a-sqrt(det)), t = min(1., -a+sqrt(det));
        if (t < 0 || 1 <= s) return arg(p, q) * r2;
        P u = p + d * s, v = p + d * t;
        return arg(p,u) * r2 + u.cross(v)/2 + arg(v,q) * r2;
    };
    auto sum = 0.0;
    rep(i,0,sz(ps))
        sum += tri(ps[i] - c, ps[(i + 1) % sz(ps)] - c);
    return sum;
}
```

**circumcircle.h**

**Description:**



```
"Point.h"
1caa3a, 9 lines

typedef Point<double> P;
double ccRadius(const P& A, const P& B, const P& C) {
    return (B-A).dist()*(C-B).dist()*(A-C).dist()/
           abs((B-A).cross(C-A))/2;
}

P ccCenter(const P& A, const P& B, const P& C) {
    P b = C-A, c = B-A;
    return A + (b*c.dist2()-c*b.dist2()).perp()/b.cross(c)/2;
}
```

1

**MinimumEnclosingCircle.h**

**Description:** Computes the minimum circle that encloses a set of points.

**Time:** expected  $\mathcal{O}(n)$

```
"circumcircle.h"
09dd0a, 17 lines

pair<P, double> mec(vector<P> ps) {
    shuffle(all(ps), mt19937(time(0)));
    P o = ps[0];
    double r = 0, EPS = 1 + 1e-8;
    rep(i,0,sz(ps)) if ((o - ps[i]).dist() > r * EPS) {
        o = ps[i], r = 0;
        rep(j,0,i) if ((o - ps[j]).dist() > r * EPS) {
            o = (ps[i] + ps[j]) / 2;

```

```
        r = (o - ps[i]).dist();
        rep(k,0,j) if ((o - ps[k]).dist() > r * EPS) {
            o = ccCenter(ps[i], ps[j], ps[k]);
            r = (o - ps[i]).dist();
        }
    }
    return {o, r};
}
```

1

**5.3 Polygons**

**InsidePolygon.h**

**Description:** Returns true if p lies within the polygon. If strict is true, it returns false for points on the boundary. The algorithm uses products in intermediate steps so watch out for overflow.

**Usage:** vector<P> v = {P{4,4}, P{1,2}, P{2,1}};

bool in = inPolygon(ps[i], ps[j], ps[k]);

**Time:**  $\mathcal{O}(n)$

```
"Point.h", "OnSegment.h", "SegmentDistance.h"
2bf504, 11 lines

template<class P>
bool inPolygon(vector<P> &p, P a, bool strict = true) {
    int cnt = 0, n = sz(p);
    rep(i,0,n) {
        P q = p[(i + 1) % n];
        if (onSegment(p[i], q, a)) return !strict;
        //or: if (segDist(p[i], q, a) <= eps) return !strict;
        cnt ^= ((a.y<p[i].y) - (a.y<q.y)) * a.cross(p[i], q) > 0;
    }
    return cnt;
}
```

**PolygonArea.h**

**Description:** Returns twice the signed area of a polygon. Clockwise enumeration gives negative area. Watch out for overflow if using int as T!

```
"Point.h"
f12300, 6 lines

template<class T>
T polygonArea2(vector<Point<T>&& v) {
    T a = v.back().cross(v[0]);
    rep(i,0,sz(v)-1) a += v[i].cross(v[i+1]);
    return a;
}
```

**PolygonCenter.h**

**Description:** Returns the center of mass for a polygon.

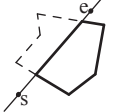
**Time:**  $\mathcal{O}(n)$

```
"Point.h"
9706dc, 9 lines

typedef Point<double> P;
P polygonCenter(const vector<P>& v) {
    P res(0, 0); double A = 0;
    for (int i = 0, j = sz(v) - 1; i < sz(v); j = i++) {
        res = res + (v[i] + v[j]) * v[j].cross(v[i]);
        A += v[j].cross(v[i]);
    }
    return res / A / 3;
}
```

**PolygonCut.h**

**Description:**



Returns a vector with the vertices of a polygon with everything to the left of the line going from s to e cut away.

**Usage:** vector<P> p = ...;

p = polygonCut(p, P(0,0), P(1,0));

```
"Point.h", "LineIntersection.h"
f2b7d4, 13 lines

typedef Point<double> P;
vector<P> polygonCut(const vector<P>& poly, P s, P e) {
    vector<P> res;
```



```
rep(i,0,sz(poly)) {
    P cur = poly[i], prev = i ? poly[i-1] : poly.back();
    bool side = s.cross(e, cur) < 0;
    if (side != (s.cross(e, prev) < 0))
        res.push_back(lineInter(s, e, cur, prev).second);
    if (side)
        res.push_back(cur);
}
return res;
}
```

ConvexHull.h

**Description:** Returns a vector of the points of the convex hull in counter-clockwise order. Points on the edge of the hull between two other points are not considered part of the hull.  
**Time:**  $\mathcal{O}(n \log n)$



```
"Point.h" 310954, 13 lines

typedef Point<ll> P;
vector<P> convexHull(vector<P> pts) {
    if (sz(pts) <= 1) return pts;
    sort(all(pts));
    vector<P> h(sz(pts)+1);
    int s = 0, t = 0;
    for (int it = 2; it--; s = --t, reverse(all(pts)))
        for (P p : pts) {
            while (t >= s + 2 && h[t-2].cross(h[t-1], p) <= 0) t--;
            h[t++] = p;
        }
    return {h.begin(), h.begin() + t - (t == 2 && h[0] == h[1])};
}
```

HullDiameter.h

**Description:** Returns the two points with max distance on a convex hull (ccw, no duplicate/collinear points).  
**Time:**  $\mathcal{O}(n)$

```
"Point.h" c571b8, 12 lines

typedef Point<ll> P;
array<P, 2> hullDiameter(vector<P> S) {
    int n = sz(S), j = n < 2 ? 0 : 1;
    pair<ll, array<P, 2>> res({0, {S[0], S[0]}});
    rep(i,0,j)
        for (; j = (j + 1) % n) {
            res = max(res, {(S[i] - S[j]).dist2(), {S[i], S[j]}});
            if ((S[(j + 1) % n] - S[j]).cross(S[i + 1] - S[i]) >= 0)
                break;
        }
    return res.second;
}
```

PointInsideHull.h

**Description:** Determine whether a point t lies inside a convex hull (CCW order, with no collinear points). Returns true if point lies within the hull. If strict is true, points on the boundary aren't included.  
**Time:**  $\mathcal{O}(\log N)$

```
"Point.h", "sideOf.h", "OnSegment.h" 71446b, 14 lines

typedef Point<ll> P;

bool inHull(const vector<P>& l, P p, bool strict = true) {
    int a = 1, b = sz(l) - 1, r = !strict;
    if (sz(l) < 3) return r && onSegment(l[0], l.back(), p);
    if (sideOf(l[0], l[a], l[b]) > 0) swap(a, b);
    if (sideOf(l[0], l[a], p) >= r || sideOf(l[0], l[b], p) <= -r)
        return false;
    while (abs(a - b) > 1) {
        int c = (a + b) / 2;
        if (sideOf(l[0], l[c], p) > 0 ? b : a) = c;
    }
}
```

```
return sgn(l[a].cross(l[b], p)) < r;
}
```

LineHullIntersection.h

**Description:** Line-convex polygon intersection. The polygon must be ccw and have no collinear points. lineHull(line, poly) returns a pair describing the intersection of a line with the polygon:  $\bullet(-1, -1)$  if no collision,  $\bullet(i, -1)$  if touching the corner  $i$ ,  $\bullet(i, i)$  if along side  $(i, i + 1)$ ,  $\bullet(i, j)$  if crossing sides  $(i, i + 1)$  and  $(j, j + 1)$ . In the last case, if a corner  $i$  is crossed, this is treated as happening on side  $(i, i + 1)$ . The points are returned in the same order as the line hits the polygon. extrVertex returns the point of a hull with the max projection onto a line.  
**Time:**  $\mathcal{O}(\log n)$

```
"Point.h" 7cf45b, 39 lines

#define cmp(i, j) sgn(dir.perp().cross(poly[(i)%n]-poly[(j)%n]))
#define extr(i) cmp(i + 1, i) >= 0 && cmp(i, i - 1 + n) < 0
template <class P> int extrVertex(vector<P>& poly, P dir) {
    int n = sz(poly), lo = 0, hi = n;
    if (extr(0)) return 0;
    while (lo + 1 < hi) {
        int m = (lo + hi) / 2;
        if (extr(m)) return m;
        int ls = cmp(lo + 1, lo), ms = cmp(m + 1, m);
        (ls < ms || (ls == ms && ls == cmp(lo, m)) ? hi : lo) = m;
    }
    return lo;
}
```

```
#define cml(i) sgn(a.cross(poly[i], b))
template <class P>
array<int, 2> lineHull(P a, P b, vector<P>& poly) {
    int endA = extrVertex(poly, (a - b).perp());
    int endB = extrVertex(poly, (b - a).perp());
    if (cml(endA) < 0 || cml(endB) > 0)
        return {-1, -1};
    array<int, 2> res;
    rep(i,0,2) {
        int lo = endB, hi = endA, n = sz(poly);
        while ((lo + 1) % n != hi) {
            int m = ((lo + hi + (lo < hi ? 0 : n)) / 2) % n;
            (cml(m) == cml(endB) ? lo : hi) = m;
        }
        res[i] = (lo + !cml(hi)) % n;
        swap(endA, endB);
    }
    if (res[0] == res[1]) return {res[0], -1};
    if (!cml(res[0]) && !cml(res[1]))
        switch ((res[0] - res[1] + sz(poly) + 1) % sz(poly)) {
            case 0: return {res[0], res[0]};
            case 2: return {res[1], res[1]};
        }
    return res;
}
```

5.4 Misc. Point Set Problems

ClosestPair.h

**Description:** Finds the closest pair of points.  
**Time:**  $\mathcal{O}(n \log n)$

```
"Point.h" ac41a6, 17 lines

typedef Point<ll> P;
pair<P, P> closest(vector<P> v) {
    assert(sz(v) > 1);
    set<P> S;
    sort(all(v), [](P a, P b) { return a.y < b.y; });
    pair<ll, pair<P, P>> ret{LLONG_MAX, {P(), P()}};
    int j = 0;
    for (P p : v) {
        P d{1 + (ll)sqrt(ret.first), 0};

```

```
while (v[j].y <= p.y - d.x) S.erase(v[j++]);
auto lo = S.lower_bound(p - d), hi = S.upper_bound(p + d);
for (; lo != hi; ++lo)
    ret = min(ret, {(lo - p).dist2(), {lo, p}});
S.insert(p);
}
return ret.second;
}
```

kdTree.h

**Description:** KD-tree (2d, can be extended to 3d)

```
"Point.h" bac5b0, 63 lines

typedef long long T;
typedef Point<T> P;
const T INF = numeric_limits<T>::max();

bool on_x(const P& a, const P& b) { return a.x < b.x; }
bool on_y(const P& a, const P& b) { return a.y < b.y; }
```

```
struct Node {
    P pt; // if this is a leaf, the single point in it
    T x0 = INF, x1 = -INF, y0 = INF, y1 = -INF; // bounds
    Node *first = 0, *second = 0;

    T distance(const P& p) { // min squared distance to a point
        T x = (p.x < x0 ? x0 : p.x > x1 ? x1 : p.x);
        T y = (p.y < y0 ? y0 : p.y > y1 ? y1 : p.y);
        return (P(x,y) - p).dist2();
    }
}
```

```
Node(vector<P>&& vp) : pt(vp[0]) {
    for (P p : vp) {
        x0 = min(x0, p.x); x1 = max(x1, p.x);
        y0 = min(y0, p.y); y1 = max(y1, p.y);
    }
    if (vp.size() > 1) {
        // split on x if width >= height (not ideal...)
        sort(all(vp), x1 - x0 >= y1 - y0 ? on_x : on_y);
        // divide by taking half the array for each child (not
        // best performance with many duplicates in the middle)
        int half = sz(vp)/2;
        first = new Node({vp.begin(), vp.begin() + half});
        second = new Node({vp.begin() + half, vp.end()});
    }
};
```

```
struct KDTree {
    Node* root;
    KDTree(const vector<P>& vp) : root(new Node({all(vp)})) {}
}
```

```
pair<T, P> search(Node *node, const P& p) {
    if (!node->first) {
        // uncomment if we should not find the point itself:
        // if (p == node->pt) return {INF, P()};
        return make_pair((p - node->pt).dist2(), node->pt);
    }
}
```

```
Node *f = node->first, *s = node->second;
T bfirst = f->distance(p), bsec = s->distance(p);
if (bfirst > bsec) swap(bsec, bfirst), swap(f, s);
```

```
// search closest side first, other side if needed
auto best = search(f, p);
if (bsec < best.first)
    best = min(best, search(s, p));
return best;
}
```



```
// find nearest point to a point, and its squared distance
// (requires an arbitrary operator< for Point)
pair<T, P> nearest(const P& p) {
    return search(root, p);
}
};
```

FastDelaunay.h

**Description:** Fast Delaunay triangulation. Each circumcircle contains none of the input points. There must be no duplicate points. If all points are on a line, no triangles will be returned. Should work for doubles as well, though there may be precision issues in 'circ'. Returns triangles in order {t[0][0], t[0][1], t[0][2], t[1][0], ... }, all counter-clockwise.  
**Time:**  $O(n \log n)$

"Point.h"	eefdf5, 88 lines
-----------	------------------

```
typedef Point<ll> P;
typedef struct Quad* Q;
typedef __int128_t ll1; // (can be ll if coords are < 2e4)
P arb(LLONG_MAX,LLONG_MAX); // not equal to any other point
```

```
struct Quad {
    Q rot, o; P p = arb; bool mark;
    P& F() { return r()->p; }
    Q& r() { return rot->rot; }
    Q prev() { return rot->o->rot; }
    Q next() { return r()->prev(); }
} *H;
```

```
bool circ(P p, P a, P b, P c) { // is p in the circumcircle?
    ll1 p2 = p.dist2(), A = a.dist2()-p2,
        B = b.dist2()-p2, C = c.dist2()-p2;
    return p.cross(a,b)*C + p.cross(b,c)*A + p.cross(c,a)*B > 0;
}
```

```
Q makeEdge(P orig, P dest) {
    Q r = H ? H : new Quad{new Quad{new Quad{new Quad{0}}}};
    H = r->o; r->r()->r() = r;
    rep(i,0,4) r = r->rot, r->p = arb, r->o = i & 1 ? r : r->r();
    r->p = orig; r->F() = dest;
    return r;
}
```

```
void splice(Q a, Q b) {
    swap(a->o->rot->o, b->o->rot->o); swap(a->o, b->o);
}
```

```
Q connect(Q a, Q b) {
    Q q = makeEdge(a->F(), b->p);
    splice(q, a->next());
    splice(q->r(), b);
    return q;
}
```

```
pair<Q,Q> rec(const vector<P>& s) {
    if (sz(s) <= 3) {
        Q a = makeEdge(s[0], s[1]), b = makeEdge(s[1], s.back());
        if (sz(s) == 2) return { a, a->r() };
        splice(a->r(), b);
        auto side = s[0].cross(s[1], s[2]);
        Q c = side ? connect(b, a) : 0;
        return {side < 0 ? c->r() : a, side < 0 ? c : b->r() };
    }
}
```

```
#define H(e) e->F(), e->p
#define valid(e) (e->F().cross(H(base)) > 0)
Q A, B, ra, rb;
int half = sz(s) / 2;
tie(ra, A) = rec({all(s) - half});
tie(B, rb) = rec({sz(s) - half + all(s)});
while ((B->p.cross(H(A)) < 0 && (A = A->next())) ||
        (A->p.cross(H(B)) > 0 && (B = B->r()->o)));
Q base = connect(B->r(), A);
```

FastDelaunay PolyhedronVolume Point3D 3dHull

```
if (A->p == ra->p) ra = base->r();
if (B->p == rb->p) rb = base;

#define DEL(e, init, dir) Q e = init->dir; if (valid(e)) \
    while (circ(e->dir->F(), H(base), e->F())) { \
        Q t = e->dir; \
        splice(e, e->prev()); \
        splice(e->r(), e->r()->prev()); \
        e->o = H; H = e; e = t; \
    }
for (;;) {
    DEL(LC, base->r(), o); DEL(RC, base, prev());
    if (!valid(LC) && !valid(RC)) break;
    if (!valid(LC) || (valid(RC) && circ(H(RC), H(LC))))
        base = connect(RC, base->r());
    else
        base = connect(base->r(), LC->r());
}
return { ra, rb };
}
```

```
vector<P> triangulate(vector<P> pts) {
    sort(all(pts)); assert(unique(all(pts)) == pts.end());
    if (sz(pts) < 2) return {};
    Q e = rec(pts).first;
    vector<Q> q = {e};
    int qi = 0;
    while (e->o->F().cross(e->F(), e->p) < 0) e = e->o;
    #define ADD { Q c = e; do { c->mark = 1; pts.push_back(c->p); \
        q.push_back(c->r()); c = c->next(); } while (c != e); }
    ADD; pts.clear();
    while (qi < sz(q)) if (!(e = q[qi++])->mark) ADD;
    return pts;
}
```

5.5 3D

PolyhedronVolume.h

**Description:** Magic formula for the volume of a polyhedron. Faces should point outwards.

	3058c3, 6 lines
--	-----------------

```
template<class V, class L>
double signedPolyVolume(const V& p, const L& trilst) {
    double v = 0;
    for (auto i : trilst) v += p[i.a].cross(p[i.b]).dot(p[i.c]);
    return v / 6;
}
```

Point3D.h

**Description:** Class to handle points in 3D space. T can be e.g. double or long long.

	8058ae, 32 lines
--	------------------

```
template<class T> struct Point3D {
    typedef Point3D P;
    typedef const P& R;
    T x, y, z;
    explicit Point3D(T x=0, T y=0, T z=0) : x(x), y(y), z(z) {}
    bool operator<(R p) const {
        return tie(x, y, z) < tie(p.x, p.y, p.z); }
    bool operator==(R p) const {
        return tie(x, y, z) == tie(p.x, p.y, p.z); }
    P operator+(R p) const { return P(x+p.x, y+p.y, z+p.z); }
    P operator-(R p) const { return P(x-p.x, y-p.y, z-p.z); }
    P operator*(T d) const { return P(x*d, y*d, z*d); }
    P operator/(T d) const { return P(x/d, y/d, z/d); }
    T dot(R p) const { return x*p.x + y*p.y + z*p.z; }
    P cross(R p) const {
        return P(y*p.z - z*p.y, z*p.x - x*p.z, x*p.y - y*p.x);
    }
    T dist2() const { return x*x + y*y + z*z; }
```

```
double dist() const { return sqrt((double)dist2()); }
//Azimuthal angle (longitude) to x-axis in interval [-pi, pi]
double phi() const { return atan2(y, x); }
//Zenith angle (latitude) to the z-axis in interval [0, pi]
double theta() const { return atan2(sqrt(x*x+y*y),z); }
P unit() const { return *this/(T)dist(); } //makes dist()==1
//returns unit vector normal to *this and p
P normal(P p) const { return cross(p).unit(); }
//returns point rotated 'angle' radians ccw around axis
P rotate(double angle, P axis) const {
    double s = sin(angle), c = cos(angle); P u = axis.unit();
    return u*dot(u)*(1-c) + (*this)*c - cross(u)*s;
}
};
```

3dHull.h

**Description:** Computes all faces of the 3-dimension hull of a point set. \*No four points must be coplanar\*, or else random results will be returned. All faces will point outwards.

**Time:**  $O(n^2)$

"Point3D.h"	5b45fc, 49 lines
-------------	------------------

```
typedef Point3D<double> P3;
```

```
struct PR {
    void ins(int x) { (a == -1 ? a : b) = x; }
    void rem(int x) { (a == x ? a : b) = -1; }
    int cnt() { return (a != -1) + (b != -1); }
    int a, b;
};
```

```
struct F { P3 q; int a, b, c; };
```

```
vector<F> hull3d(const vector<P3>& A) {
    assert(sz(A) >= 4);
    vector<vector<PR>> E(sz(A), vector<PR>(sz(A), {-1, -1}));
    #define E(x,y) E[f.x][f.y]
    vector<F> FS;
    auto mf = [&](int i, int j, int k, int l) {
        P3 q = (A[j] - A[i]).cross((A[k] - A[i]));
        if (q.dot(A[l]) > q.dot(A[i]))
            q = q * -1;
        F f{q, i, j, k};
        E(a,b).ins(k); E(a,c).ins(j); E(b,c).ins(i);
        FS.push_back(f);
    };
    rep(i,0,4) rep(j,i+1,4) rep(k,j+1,4)
        mf(i, j, k, 6 - i - j - k);
```

```
rep(i,4,sz(A)) {
    rep(j,0,sz(FS)) {
        F f = FS[j];
        if (f.q.dot(A[i]) > f.q.dot(A[f.a])) {
            E(a,b).rem(f.c);
            E(a,c).rem(f.b);
            E(b,c).rem(f.a);
            swap(FS[j--], FS.back());
            FS.pop_back();
        }
        int nw = sz(FS);
        rep(j,0,nw) {
            F f = FS[j];
        }
    }
    #define C(a, b, c) if (E(a,b).cnt() != 2) mf(f.a, f.b, i, f.c);
    C(a, b, c); C(a, c, b); C(b, c, a);
}
for (F& it : FS) if ((A[it.b] - A[it.a]).cross(
    A[it.c] - A[it.a]).dot(it.q) <= 0) swap(it.c, it.b);
return FS;
```

```
};
```

sphericalDistance.h

**Description:** Returns the shortest distance on the sphere with radius radius between the points with azimuthal angles (longitude) f1 ( $\phi_1$ ) and f2 ( $\phi_2$ ) from x axis and zenith angles (latitude) t1 ( $\theta_1$ ) and t2 ( $\theta_2$ ) from z axis (0 = north pole). All angles measured in radians. The algorithm starts by converting the spherical coordinates to cartesian coordinates so if that is what you have you can use only the two last rows. dx\*radius is then the difference between the two points in the x direction and d\*radius is the total distance between the points.

611f07, 8 lines

```
double sphericalDistance(double f1, double t1,
    double f2, double t2, double radius) {
    double dx = sin(t2)*cos(f2) - sin(t1)*cos(f1);
    double dy = sin(t2)*sin(f2) - sin(t1)*sin(f1);
    double dz = cos(t2) - cos(t1);
    double d = sqrt(dx*dx + dy*dy + dz*dz);
    return radius*2*asin(d/2);
}
```