Beam Splitter for Spin Waves in Quantum Spin Network

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We theoretically design and analytically study a controllable beam splitter for the spin wave propagating in a star-shaped (e.g., a Y-shaped beam) spin network. Such a solid state beam splitter can display quantum interference and quantum entanglement by the well-aimed controls of interaction on nodes. It will enable an elementary interferometric device for scalable quantum information processing based on the solid system.

I.   INTRODUCTION

Beam splitters are the elementary optical devices frequently used in classical and quantum optics [1], which can even work well in the level of single photon quanta [2] and are applied to generate quantum entanglement [3]. For matter waves, an early beam splitter can be referred to the experiments of neutron interference based on a perfect crystal interferometer with wavefront and amplitude division [4]; and now an atomic beam splitter has been experimentally implemented on the atom chip [5]. The theoretical protocols have been suggested to realize the beam splitter for the Bose-Einstein condensate [6].

In this paper, we propose and study the implementation of beam splitter for the spin wave propagations in the star-shaped spin networks (SSSNs) with $m + 1$ weighted legs (see the Fig. 1a), where each leg is a one-dimensional (1-D) spin chain with XY couplings. This investigation is mostly motivated by our recent researches on the perfect transfer of quantum states along a single quantum spin chain [7] and for a 1-D Bloch electron system [8, 9]. The similar quantum networks have been considered for a coupled many harmonic oscillator system [10] and for quantum cloning via spin networks [11].

A basic SSSN is a Y-shaped network or called Y-beam [5] for $m = 2$, which can be regarded as an elementary block, in principle, to the architecture of complicated networks (such as a solid state interferometer) for quantum information processing. It can function to transfer quantum state coherently in multi-channel and to generate entanglement between two spins which are a long distance apart. Furthermore, we will show that the quantum coherence of spin waves propagating in two legs can be well controlled by adjusting the coupling strengths only at the node; and then a controllable solid state interferometer is built based on this crucial function.

The basic element of an arbitrary spin network is the coupling between spins, which is usually described by the XY Hamiltonianin

$$H^{XY} = J_{ij} \sum_{i,j} (S_i^+ S_j^- + H.c.), \qquad (1)$$

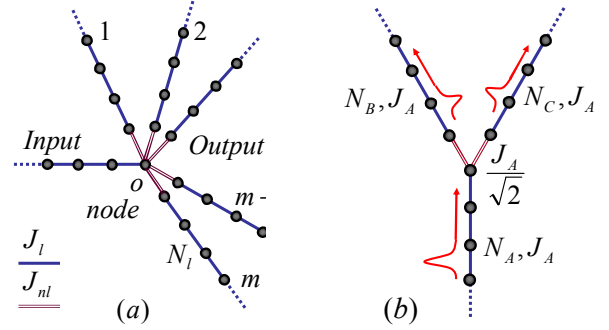where $S_i^\pm$ are the Pauli spin operators acting on the in-

FIG. 1: *(Color on line) (a) The star-shaped spin network with an input spin chain A and m output spin chain. (b) Y-shaped network or called Y-beam, a special star-shaped spin network, serves as the fundamental block for the architecture of complicated quantum spin networks*

ternal space of electron on the ith site. One can apply magnetic field B_z beforehand and then switch off it, to prepare a polarized initial states with all spins down for the quantum network. The dynamics of the lower excitations (magnons) from this polarized state is attractive because of their relevance to quantum information applications. In the low-temperature and low magnon density limit, the magnon can be regarded as boson by the Holstein-Primakoff transformation $S_j^+ \simeq b_j^\dagger$. Then one can translates a XY spin network into the bosonic system with the Hamiltonian $b_{a,j}^\dagger b_{a,j+1} + h.c$, approximately. Note that such map is exact for the case of single magnon.

II.   STAR-SHAPED BEAM SPLITTER AND ITS REDUCTION

We consider a spin network of a star shape (we call the SSSN) as shown in Fig. 1a. Under the Holstein-Primakoff transformation the Hamiltonian of leg l consisting of N_l spins with XY interactions can be written as

$$H_l = H_l(J_l, N_l) \stackrel{\circ}{=} J_l \sum_{j=1}^{N_l-1} (b_{l,j}^\dagger b_{l,j+1} + H.c.), \qquad (2)$$

where $b_{l,j}^\dagger, b_{l,j}$ are the boson operators on the jth site of the lth leg. Here we have assumed that the couplings J_l are the same for a given bosonic chain l. We uniquely denote the Hamiltonian by $H_l(J_l, N_l)$ for the bosonic chain hereafter. The SSSN is constructed by linking the m output bosonic chains to the one end (or the node) O of the input leg A by the couplings J_{nl}. The Hamiltonian of an SSSN is of the form as the same as Eq. (2) except for the part around the node O.

We will show that, due to the quantum interference mechanism, by some constrain for the coupling constants J_l and J_{nl}, an SSSN can be reduced into an imaginary linear bosonic chain with homogeneous coupling plus a smaller complicated network if only the single-magnon case is concerned. The fact that the input chain A is a part of this virtual linear chain implies that the bosonic wave packet can perfectly propagate in this virtual linear chain without the reflection by the node. This indicates that there is a coherent split of the input bosonic wave packet because the magnon excitation in this virtual chain actually is just a superposition of magnon excitations in the m bosonic chains.

To sketch the central idea, we first consider a special SSSN, which consists of m identical "output" chains $B_1, B_2, \cdots, B_m$ with homogeneous coupling $J_l = J$, $J_{nl} = J_n$ and the same length N, while the length of chain A is M. The Hamiltonian

$$H = \sum_{p=1}^{m} H_{B_p}(J, N) + H_A(J, M) + H_{nod} \quad (3)$$

can be explicitly written in terms of the leg Hamiltonians $H_{B_p}(J, N)$ and $H_A(J, M)$ defined by Eq. (1) and the interactions around the node O

$$H_{nod} = -J_n(b_{A,M}^\dagger \sum_{p=1}^{m} b_{B_p,1} + H.c.). \quad (4)$$

Now we construct the virtual bosonic chain a of length $M + N$ with the boson operators $b_{a,j}^\dagger = b_{A,j}^\dagger$ ($j = 1, 2, \cdots, M$) in the real chain A and the collective operator

$$b_{a,M+j}^\dagger = \frac{1}{\sqrt{m}} \sum_{p=1}^{m} b_{B_p,j}^\dagger \quad (5)$$

for the virtual part, where $j = 1, 2, \cdots, N$. There exist $m-1$ complementary linear bosonic chains with the collective operators

$$b_{b_q,j}^\dagger = (1/\sqrt{m}) \sum_{p=1}^{m} \exp(-i2\pi pq/m) b_{B_p,j}^\dagger \quad (6)$$

where $p = 1, 2, \cdots, m$, $q = 1, 2, \cdots, m-1$. It can be checked that, together with $b_{a,j}^\dagger$, the above defined collective operators $b_{a,M+j}^\dagger$ and $b_{b_q,j}^\dagger$, ($q = 1, 2, \cdots, m-1$) and their conjugates also satisfy the commutative relations of boson operators.

Using operators $b_{a,M+j}^\dagger$ and $b_{b_q,j}^\dagger$, we divide the total Hamiltonian into two commutative parts

$$H_b = \sum_{q=1}^{m-1} H_{b_q}(J, N) \quad (7)$$

and

$$H_a = H_a(J, M + N) + H_{vn}, \quad (8)$$

where

$$H_{vn} = (J - \sqrt{m}J_n)b_{a,M}^\dagger b_{a,M+1} + H.c. \quad (9)$$

The first Hamiltonian H_b describes $m-1$ independent virtual bosonic chains without input from H_A while the second one describes a linear bosonic chain with an impurity at the Mth site. In usual it can reflect the bosonic wave packet from the input leg.

Only when the coupling matching conation $J_n = J/\sqrt{m}$ is satisfied, the virtual bosonic chain described by H_a is just a standard bosonic chain since $H_{vn} = 0$. In this case no reflection occurs at the node. With this matched node coupling, an ideal beam splitter can be realized with m coherent outputs since each operator $b_{a,M+j}^\dagger$ is a linear combination of $b_{B_p,j}^\dagger$. Then it can create a superposition from the vacuum state with bosons excitation. Each component of this superposition represents a magnon or boson excitation in a leg. The detailed analysis will be done with the special SSSN of $m = 2$ in the next section.

III. Y-SHAPED BEAM SPLITTER DECOUPLING

Actually, we need not require the two output legs to be identical. To be convenient, we consider the asymmetric Y-beam consisting of three legs A, B and C with three hopping integrals J_F for $F = A, B, C$ and the node interactions J_{nF} for $F = B, C$. The total Hamiltonian reads

$$H = \sum_{F=A,B,C} H_F - \sum_{F=B,C} (J_{nF}b_{A,M}^\dagger b_{F,1} + H.c.) \quad (10)$$

where $H_F = H_F(J_F, N_F)$ and $N_A = M$, $N_B = N_C = N$.

In order to decouple this Y-beam as two virtual linear bosonic chains, we need to optimize the asymmetric couplings so that the perfect transmission can occur in the decoupled linear bosonic chains. To this end we introduce two sets of operators by

$$b_{a,M+j}^\dagger = \cos\theta b_{B,j}^\dagger + \sin\theta b_{C,j}^\dagger;$$
$$b_{b,j}^\dagger = \sin\theta b_{B,j}^\dagger - \cos\theta b_{C,j}^\dagger, \quad (11)$$

for $j = 1, 2, \cdots, N$. A straight forward calculation shows that the two sets of operator act as boson operators and
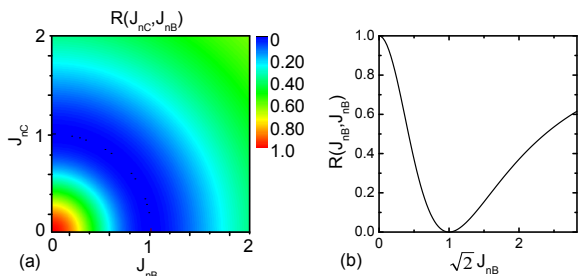
FIG. 2: *(Color on line) (a) The contour map of the reflection factor $R(J_{nC}, J_{nB})$ as a function of J_{nC}, J_{nB} for the GWP with $\alpha = 0.3$ and momentum $\pi/2$ in a finite system with $N_A = N_B = N_C = 50$. It shows that around the matching condition, i.e, the circle $J_{nC}^2 + J_{nB}^2 = J_A^2$, the reflection factor approaches zero. (b) The profile of $R(J_{nC}, J_{nB})$ along $J_{nC} = J_{nB}$.*

FIG. 3: *(Color on line) (a) The contour map of maximal concurrence of two GWPs at two legs A and B, $C_{\max}(J_{nC}, J_{nB})$ for the same setup as that in Fig. 2. It is found that two GWPs yield the maximal entanglement at the point $J_{nC} = J_{nB} = J_A/\sqrt{2}$. (b) The profile of $C_{\max}(J_{nC}, J_{nB})$ along $J_{nC} = J_{nB}$.*

commutative. Here, the mixing angle θ is to be determined as follows by the optimization for quantum information transmission. In comparison with the optical beam splitter, the above equation can be regarded as a fundamental issue for the boson beam splitter.

Together with the original boson operator $b_{a,j}^\dagger = b_{A,j}^\dagger$ for the input leg, the set with $b_{a,M+j}^\dagger$ defines a new linear chain a with the effective couplings $J_{aj} = J_A$ ($j \in [1, M-1]$), $J_{aM} = J_{nB}\cos\theta + J_{nC}\sin\theta$ and $J_{a,M+j} = J_B\cos^2\theta + J_C\sin^2\theta$, for $j \in [1, N-1]$. Another virtual linear chain b is defined by $b_{b,j}^\dagger$ with homogeneous couplings $J_{bj} = J_B\sin^2\theta + J_C\cos^2\theta$ for $j \in [1, N-1]$.

In general, these two linear chains do not decouple with each other since there exists a connection interaction around the node

$$
\begin{aligned}
H_{con} &= g(b_{b,j}^\dagger b_{a,M+j+1} + b_{a,M+j}^\dagger b_{b,j+1} + H.c) \\
&\quad - J_{AB}(b_{a,M}^\dagger b_{b,1} + H.c.)
\end{aligned}
\tag{12}
$$

where $g = (J_B - J_C)\sin 2\theta/2$ and $J_{AB} = J_{nB}\sin\theta - J_{nC}\cos\theta$. Fortunately, the two bosonic chains decouple with each other when we optimize the mixing angle θ and the inter-chain coupling by setting them as $\tan\theta = J_{nC}/J_{nB}$, $J_B = J_C$ and then $J_{aM} = J_{nB}/\cos\theta$. Thus when we set $J_{nB} = J_A\cos\theta$, the coupling matching condition

$$
J_A = \sqrt{J_{nC}^2 + J_{nB}^2} = J_B = J_C
\tag{13}
$$

holds. Especially, the virtual bosonic chain a becomes homogenous when condition (13) is satisfied. Then it can be employed to transfer the quantum state without reflection on the node in the transformed picture. By transforming back to the original picture, the quantum state transfer is shown to be a perfect beam splitting. Similar to the point of view of linear optics, such beam splitting process can generate entanglement. We will show that the values of J_{nB} and J_{nC} can determine the amplitudes of the bosonic wave packet on legs B and C.
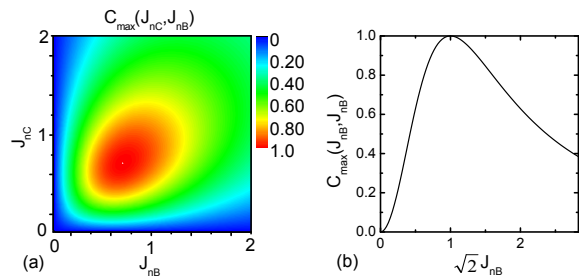
Now we apply the beam splitter to a special spin wave packet, a Gaussian wave packet (GWP) with momentum $\pi/2$, which has the form

$$
\left|\psi_{A\frac{\pi}{2}}(N_0)\right\rangle = \frac{1}{\sqrt{\Omega_1}} \sum_j e^{-\frac{\alpha^2}{2}(j-N_0)^2} e^{i\frac{\pi}{2}j} |j\rangle
\tag{14}
$$

at $t = 0$, where Ω_1 is the normalization factor and N_0 is the initial central position of the GWP at the input chain A. The single excitation basis vector $|j\rangle = S_{A,j}^+|d\rangle$ is defined by the polarized state $|d\rangle$ with all spins aligned down. As mentioned in the introduction, the conclusion we obtained for bosonic system is exact for the single-magnon case. It is known from the previous work [9] that such GWP can approximately propagate along a homogenous chain without spreading. Then at a certain time t, such GWP evolves into

$$
|\Phi(t)\rangle = \cos\theta \left|\psi_{B\frac{\pi}{2}}(N_t)\right\rangle + \sin\theta \left|\psi_{C\frac{\pi}{2}}(N_t)\right\rangle
\tag{15}
$$

where $N_t = N_0 + 2tJ_A - M$, i.e., the beam splitter can split the GWP into two cloned GWPs completely.

In order to verify the above analysis, a numerical simulation is performed for a GWP with $\alpha = 0.3$ in a finite system with $N_A = N_B = N_C = 50$. Let $|\Phi(0)\rangle$ be a normalized initial state. Then the reflection factor at time t can be defined as

$$
R(J_{nC}, J_{nB}, t) = \sum_{j \in D'} \left| \langle j| e^{-iHt} |\Phi(0)\rangle \right|^2
\tag{16}
$$

to depict the reflection at the node where $D' = [1, M-1]$. At an appropriate instant t_0, $R(J_{nC}, J_{nB}) = R(J_{nC}, J_{nB}, t_0)$ as a function of J_{nC} and J_{nB} is plotted in Fig. 2. It shows that around the coupling matching condition (13), the reflection factor vanishes, which is just in agreement with our analytical result.

IV. DYNAMIC OF BEAM SPLITTER AS ENTANGLER

Now we consider how the SSSN can behave as an entangler to produce spin entanglement with the Y-beam

as an illustration. Let the input state $|\phi(0)\rangle$ to be a single magnon excitation state in the leg A (e.g., $= S_{A,j}^+|d\rangle$ or $|\psi_{A\frac{\pi}{2}}(N_0)\rangle$ introduced by E.q. (14)). It can propagate into the legs B and C through the node with some reflection. On the other hand, the spin wave can be regarded as being transferred along the virtual legs a and b. Once we manipulate the coupling constants to satisfy the coupling matching condition, the spin wave can only enter the leg a rather than b without any reflection. Then the final state is of the magnon excitation only in the leg a. As an illustration the single-magnon excitation can be described by the state $|\phi(t)\rangle = S_{a,M+j}^+|d\rangle$ or

$$\langle d_A|\phi(t)\rangle = \cos\theta|u_{jB}\rangle \otimes |d_C\rangle + \sin\theta|d_B\rangle \otimes |u_{jC}\rangle \quad (17)$$

where $|u_{jF}\rangle = S_{F,j}^+|d_F\rangle$ $(F = B, C)$ represents one-magnon excitation from the polarized state $|d_F\rangle$ of the chain F with all spin down. This is just an entangled state and then the Y-beam acts as an entangler similar to that in quantum optical systems.

To quantitatively characterize entanglement of two separated waves $|\psi_{B,C\frac{\pi}{2}}(N_t)\rangle$ obtained by the beam splitter, the total concurrence with respect to the two wave packets located at the ends of legs B and C can be calculated as

$$C(t) = \sum_{i \in D} \left| \langle\Phi(t)| (S_{B,i}^+ S_{C,i}^- + S_{B,i}^- S_{C,i}^+) |\Phi(t)\rangle \right| \quad (18)$$

according to refs.[17, 18]. Here the domain $D = [N-W, N]$ and $W = 4\sqrt{\ln 2}/\alpha$ is the width of the wave packet. On the other hand, the concurrence is also the function of J_{nC} and J_{nB}. Numerical simulation is performed for a GWP with $\alpha = 0.3$ and momentum $\pi/2$ in a finite system with $N_A = 50$, $N_B = 50$, and $N_C = 50$. The maximal concurrence $C_{\max}(J_{nC}, J_{nB}) = \max\{C(t)\}$ as a function of J_{nC} and J_{nB} is plotted in Fig. 3. It shows that two split wave packets yield the maximal entanglement just at the coupling matching point $J_{nC} = J_{nB} = J_A/\sqrt{2}$.

V. QUANTUM INTERFEROMETER FOR SPIN WAVE

Finally, we consider in details a more complicated SSSN than the Y-beam, the quantum interferometer for spin wave, which consists of two Y-beams (see Fig. 4a). Similar to the optical interferometer, where the polarization of photon is utilized to encode information, the SSSN uses the spin down and up to encode the quantum information.

We still use the evolution of GWP to demonstrate the physical mechanism of such setup. Firstly, we consider the simplest case with the path difference (defined in Fig. 4a) $\Delta = 0$. It is easy to show that such network is equivalent to two independent virtual chains with lengths
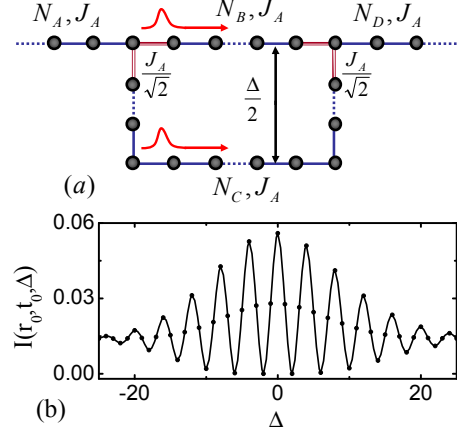

FIG. 4: (Color on line) (a) The interferometric network with an input chain A and output chain D, which consists of two Y-beams. Δ is the "optical path difference" which determines the interference pattern of output spin wave. (b) The interference pattern of output wave in the leg D ($r_0 = 50$, $t_0 = 100/J_A$) for the GWP with $\alpha = 0.3$ in the interferometric network with $N_A = N_B = N_D = 50$, $N_C = N_B + \Delta$.

will be transmitted into the leg D without any reflection. This fact can be understood according to the interference of two split GWPs. It means that the nonzero Δ should affect the shape of the pattern of output wave.

Actually, from the above analysis about the GWP propagating in the Y-beam, we note that the conclusion can be extended to the Y-beam consisting of two different length legs $N_B \neq N_C$. It is due to the locality of the GWP and the fact that the speed of the GWP only depends on the coupling constant. Thus the interference pattern at site r_0 and time t_0 in leg D can be presented as $I(r_0, t_0, \Delta) = |\langle r_0| \exp(-iHt_0) |\Phi(0)\rangle|^2$. Numerical simulation of $I(r_0, t_0, \Delta)$ for the input GWP in the interferometric network with $N_A = N_B = N_D = 50$, $N_C = N_B + \Delta$ is performed. For $r_0 = 50$, $t_0 = 100/J_A$, a perfect interference phenomenon by $I(r_0, t_0, \Delta)$ is observed for the range $\Delta \in [-25, 25]$ in Fig. 4b.

In summary, we point out that our coherent quantum network for spin wave can be implemented by an array of quantum dots or other artificial atoms. It will enable an elementary quantum device for scalable quantum computation, which can coherently transfer quantum information among the qubits to be integrated. The observable effects for spin wave interference may be discovered in the dynamics of magnetic domain in some artificial quantum material.

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