Entanglement and non-Markovianity of Quantum Evolutions

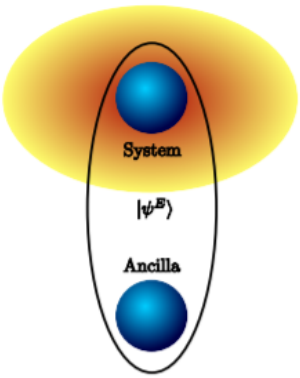
ABHISHEK MITTAL[1], ALAPAN CHAUDHURI[2]

CONTENTS

1 INTRODUCTION

During our study of Quantum Systems, mostly we have encountered Markovian evolutions (which are supposed to be the ideal case). This arises from weak coupling with an environment that acts as a memoryless reservoir.

This works very well from quantum optics, but when dealing with interacting many-body systems. Why does it not work for multi-body systems? Because, subsystem's coupling strength may be comparable to the coupling to the bath in case of interacting many-body systems.



In this report, we discuss Markovianity and non-Markovianity for quantum systems and how we can quantify them[1].

[1] *Abhishek Mittal, SPCRC, IIITH*, 2019102004
[2] *Alapan Chaudhuri, CSTAR & CQST, IIITH*, 2019111023

2 PREREQUISITES

2.1 Partial Transpose

Partial transpose as mentioned above is described as follows.

$$\rho_{AB}^{T_B} = (\mathbb{I} \otimes T)\rho_{AB}$$

2.2 Choi-Jamiolkowski Isomorphism

Let $\mathcal{E} : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$ be a linear map. Then, the following are equivalent:

1. $\mathcal{E}$ is n-positive (i.e. $\mathcal{E}(A) \in \mathbb{C}^{m\times m}$ is positive whenever $A \in \mathbb{C}^{n\times n}$ is positive).
2. The matrix $\Gamma_{\mathcal{E}}$, sometimes called the Choi matrix of $\mathcal{E}$, is positive. Here, ϕ is a maximally entangled state in the suited dimension.

$$\Gamma_{\mathcal{E}} = (\mathrm{id}_n \otimes \mathcal{E})\left(|\phi\rangle\langle\phi|\right)$$

3. $\mathcal{E}$ is completely positive.

The Choi-Jamiolkowski isomorphism allows to map many statements about states to statements about channels and vice versa. For example, a channel is completely positive exactly if the Choi state is positive, a channel is entanglement breaking exactly if the Choi state is separable, and so further.

Clearly, the isomorphism is very straightforward, and thus, one could equally well transfer any proof from channels to states and vice versa; however, often it is much more intuitive to work with one or the other, and to transfer the results later on.

2.3 Norms

Norm is a distance metric such that

1. $|cv| = |c|.|v|$
2. $|v + w| \leqslant |v| + |w|$
3. $|v| = 0$ iff $v = 0$

Examples of norms are as follows

- Manhattan (L_1), corresponds to probability distributions
- Euclidean (L_2), corresponds to pure quantum states
- L_p norm $\to |v|_{L_p} = (\sum v_i^p)^{1/p}$
- $L_\infty = \max_i |v_i|$

Similarly, schatten p-norm : $|M|_{S_p} = |\sigma(M)|_{S_p}$ where $\sigma(M)$ is a vector of singular values of a matrix, say M.

$$S_1 = \sum \text{singular values}$$

$$S_2 = \sqrt{\sum \text{singular values}^2}$$

$$S_\infty = \max_i |\text{singular values}|$$

In case of density matrices (in general to positive semi-definite Hermitian operators), we have singular values = eigenvalues thus, $\sigma(M) = \lambda(M)$.

$$|X|_{S_p} = |\sigma(X)|_{L_p} = (\sum_i \sigma_i^p)^{1/p}$$

If $X \geqslant 0$, then $\sigma(X) = \lambda(X)$ then, $|X|_{S_1} = trX \implies |\rho|_{S_1} = tr(\rho)$. Thus, ρ is a density matrix $\iff$ $\rho \geqslant 0$ and $|\rho|_{S_1} = 1$.

3 CLASSICAL AND QUANTUM MARKOV PROCESSES

Any ideal open quantum system will undergo Markovian dynamics provided that its evolution satisfies a Master equation.

$$\frac{d\rho}{dt} = \mathcal{L}(\rho) = -i[H, \rho] + \sum_k \gamma_k (V_k \rho V_k^\dagger - \frac{1}{2}\{V_k^\dagger V_k, \rho\})$$

This Markovian quantum system given by TP maps $(\mathcal{E})$ that define one-parameter semigroup of CP maps such that $\mathcal{E}_{r_1+r_0} = \mathcal{E}_{r_1}\mathcal{E}_{r_0}$.

- $\mathcal{E}_{t_2,t_0} = \mathcal{E}_{t_2,t_1}\mathcal{E}_{t_1,t_0}$ where $\mathcal{E}_{t_j,t_i} = \mathcal{T}e^{\int_{t_i}^{t_j} d\mathcal{T}} \longrightarrow$
- This is similar to classical Chapman-Kolmogorov equation (given below in functional and matrix form).

$$p_{i_1,\ldots,i_{n-1}}(f_1,\ldots,f_{n-1}) = \int_{-\infty}^{\infty} p_{i_1,\ldots,i_n}(f_1,\ldots,f_n)\, df_n$$

$$P(t+s) = P(t)P(s)$$

At the level of one-point probabilities, divisible and Markovian processes are equivalent. It is required to known the complete hierarchy of time-conditional probabilities to make any distinction.

$$\mathbb{P}(x_1,t_1) = \sum_{x_0 \in \mathcal{X}} T(x_1,t_1|x_0,t_0)\mathbb{P}(x_0,t_0)$$

3.1 Relations between classical and quantum Markovian systems

Consider a Markov process $\{X(t), t \in I\}$. Given any two time instants t_1 and t_2 we have $T(x_2,t_2|x_1,t_1) = P(x_2,t_2|x_1,t_1)$.

Let us consider a system in a quantum state given by some (non-degenerate) density matrix ρ, the spectral decomposition yields

$$\rho = \sum_x p(x)|\psi(x)\rangle\langle\psi(x)|$$

Here the eigenvalues $p(x)$ form a classical probability distribution, which may be interpreted as the probabilities for the system to be in the corresponding eigenstate $|\psi(x)\rangle$, $\mathbb{P}(|\psi(x)\rangle) = p(x)$.

Consider now some time evolution of the quantum system such that the spectral decomposition of the initial state is preserved, $\rho(t_0) = \sum_x p(x,t_0)|\psi(x)\rangle\langle\psi(x)|$,

$$\rho(t) = \sum_x p(x,t)|\psi(x)\rangle\langle\psi(x)| \in \mathcal{S}$$

where $\mathcal{S}$ denotes the set of quantum states with the same eigenvectors as $\rho(t_0)$.

Since this process can be seen as a classical stochastic process on the variable x, which labels the eigenstates $|\psi(x)\rangle$, we consider it to be divisible if the evolution of $p(x,t)$ satisfies the classical definition of divisibility. In such a case, there are transition matrices $T(x_1,t_1|x_0,t_0)$, such that

$$p(x_1,y_1) = \sum_{x_0\in\mathcal{X}} T(x_1,t_1|x_0,t_0)p(x_0,t_0)$$

can be written in terms of density matrices as $\rho(t_1) = \mathcal{E}_{(t_1,t_0)}[\rho(t_0)]$.

Here, $\mathcal{E}_{(t_1,t_0)}$ is a dynamical map that preserves the spectral decomposition of $\rho(t_0)$ and satisfies the following equation.

$$\begin{aligned}\mathcal{E}(t_1,t_0)\rho(t_0) &= \sum_{x_0\in X} p(x_0,t_0)\mathcal{E}(t_1,t_0)[|\psi(x)\rangle\langle\psi(x)|] \\ &= \sum_{x_0\in X} T(x_1,t_1|x_0,t_0)p(x_0,t_0)[|\psi(x)\rangle\langle\psi(x)|]\end{aligned}$$

Furthermore, $\mathcal{E}(t_2,t_1)$ preserves positivity and trace of any state in $\mathcal{S}$ and obeys the composition law.

$$\mathcal{E}(t_3,t_1) = \mathcal{E}(t_3,t_2)\mathcal{E}(t_2,t_1)$$

	Classical	Quantum
Normalization	$\sum_{x_2\in\mathcal{X}} T(x_2,t_2\|x_1,t_1) = 1$	$\mathcal{E}_{(t_2,t_1)}$ trace-preserving
Positivity	$T(x_2,t_2\|x_1,t_1) \geq 0$	$\mathcal{E}_{(t_2,t_1)}$ completely positive
Composition Law	$T(x_3,t_3\|x_1,t_1) = \sum_{x_2\in\mathcal{X}} T(x_3,t_3\|x_2,t_2)T(x_2,t_2\|x_1,t_1)$	$\mathcal{E}_{(t_3,t_1)} = \mathcal{E}_{(t_3,t_2)}\mathcal{E}_{(t_2,t_1)}$

Figure 1: Comparision between Classical and Quantum Markovianity

4 UNDERSTANDING MARKOVIANITY

We can understand Markovianity under serveral paradigms as follows.

1. Information flow: In case of a Markovian evolution, information goes out of the system, to the environment and does not come back. However, in case of non-Markovian evolution, there is influx of information from environment to system, at some point of time.

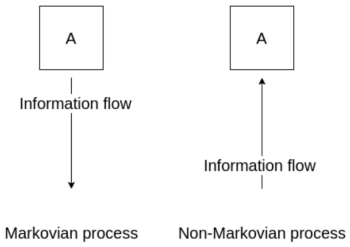


Figure 2: Information Flow

2. Trace distance: In general, for open quantum systems (under Markovian dynamics) you can never increase the distance of two states, no matter what. So,

basically when you wanna protect against noise all we are doing is slowing down the rate of noise (indistinguishable nonsense).

$$T(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leqslant T(\rho, \sigma)$$

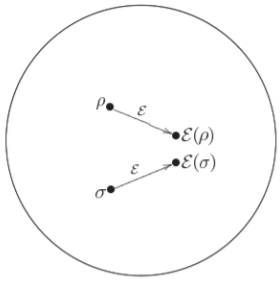


Figure 3: Trace distance

Definition of trace distance given below. [2]

$$T(\rho, \sigma) = \frac{1}{2}|\rho - \sigma|_{S_1} = \frac{1}{2}\,\mathrm{tr}\,|\rho - \sigma|$$

However, if the distance between them increases then they become more distinguishable. This implies that information has entered the system from the environment, further implying non-Markovianity.

3. Coupling: Weak coupling of system to a memoryless reservoir results in Markovianity, whereas strong coupling mostly leads to non-Markovianity.
4. Entanglement: For Markovian evolutions, the decay of the entanglement with an ancillary system (in environment) will be monotonically decreasing.

5 MEASURES OF QUANTUM NON-MARKOVIANITY

5.1 Optimization problem

The measure of non-Markovianity can be formulated as an optimization problem. Let $\mathcal{E}$ be the physical dynamical map that is unknown to us, then we are interested in finding the map from the family of Markovian Maps $\mathcal{E}^{M}_{t_0+\epsilon,t_0}$ such that its S_1 norm with $\mathcal{E}$ is as low as possible. At the same time, we also want to find the maximum non-Markovianity over the entire duration, hence we treat ϵ as a variable and choose the one that gives us the largest non-Markovianity. This leads to a min-max problem. The major disadvantage of this formulation is that it requires the knowledge of $\mathcal{E}$ beforehand, which we definitely don't know about.

$$\max_{\epsilon>0} \min_{\mathcal{E}^M} \|\mathcal{E}_{(t_0+\epsilon,t_0)} - \mathcal{E}^{M}_{(t_0+\epsilon,t_0)}\|$$

5.2 Entanglement measure

We know that decay of entanglement with an ancillary system will be monotonically decreasing for Markovian evolutions.

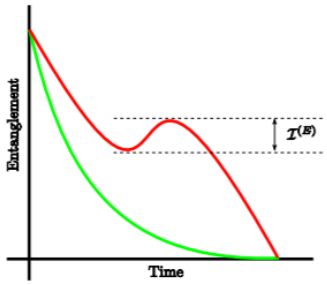


Figure 4: Entanglement vs Time

Thus, we can construct some measure for non-Markovianity of an unknown quantum evolution by computing the amount of entanglement between system and ancilla at different times within a selected interval $[t_0, t_{max}]$ and check for strict monotonic decrease of the quantum correlations.

So how do we compute entanglement? We use measures such as:

- Logarithmic negativity: $\mathbf{E}(\rho_{SE}) = \log_2 \|\rho_{SE}^{T_S}\|_1$
- Concurrence: $\mathbf{E}(\rho_{SE}) = \max(0, \sqrt{\lambda}_1\sqrt{\lambda}_2\sqrt{\lambda}_3\sqrt{\lambda}_4)$

where $\mathbf{E}(.)$ denotes measure of entanglement and S stands for system whereas E denotes environment and ρ_{SE} denotes a quantum state entangled with an ancillary system from the environment.

Now, we define $\mathcal{I}^{(E)}$ as:

$$\mathcal{I}^{(E)} = \int_{t_0}^{t_{max}} |\dot{\mathbf{E}}[\rho_{SA}(t)]|dt - \Delta\mathbf{E}$$

$$= \int_{t_0}^{t_{max}} |\dot{\mathbf{E}}[\rho_{SA}(t)]|dt - \int_{t_0}^{t_{max}} d\mathbf{E}[\rho_{SA}(t)]$$

where,

$$\Delta\mathbf{E} = \mathbf{E}[\rho_{SA}(t_0)] - \mathbf{E}[\rho_{SA}(t_{max})]$$

and,

$$\rho_{SA}(t_0) = |\phi\rangle\langle\phi|, \quad \phi = \frac{1}{d}\sum_{i=0}^{d-1} |i\rangle \otimes |i\rangle$$

Hence we obtain the sufficient condition to show that an evolution is non-Markovian as:

1. If $I(E) = 0$ then, the integral term is equal to $\Delta\mathbf{E}$. Which means there was no increase in entanglement at any point.
2. If $\mathcal{I}(E) > 0$, we conclude that the evolution is non-Markovian. However, there can be however non-Markovian quantum evolutions that remain undetected by the proposed measure.

6 STRONGER CONDITION FOR MARKOVIANITY

In the previous case (when using entanglement measure), it is worth noticing that that we never knew what the exact dynamical map was. We just kept applying it locally on our system and measuring the entanglement. Moreover, it wasn't a necessary condition for distinguishing Markovianity and non-Markovianity.

However, if we can reconstruct the dynamical map, then we can achieve a necessary and sufficient condition for non-Markovianity. This reconstruction can be done using tomography.

Now, this allows us to further construct a **necessary as well as sufficient condition** for Markovianity. Givn that we are dealing with dynamical map $\mathcal{E}_t$, we can say that if the map is CPTP, $\forall t$ then the evolution is Markovian and the following property shall hold.

$$\mathcal{E}_{t+\epsilon,0} = \mathcal{E}_{t+\epsilon,t}\mathcal{E}_{t,0}$$

We use the Choi-Jamiolkowski isomorphism to check whether $\mathcal{E}_{t+\epsilon,t}$ is CP, in other wordss we have to check if the following holds true.

$$(\mathbb{I} \otimes \mathcal{E}_{t+\epsilon,t})|\phi\rangle\langle\phi| \geqslant 0$$

Given the trace preserving property, we have the following as a measure of non-markovianity.

$$f_{NCP}(t+\epsilon,t) = \|(\mathbb{I} \otimes \mathcal{E}_{t+\epsilon,t})(|\phi\rangle\langle\phi|)\|_1$$

Therefore, $\mathcal{E}(t+\epsilon,t)$ is CP if and only if $f_{NCP}(t+\epsilon,t) = 1$, otherwise $f_{NCP}(t+\epsilon,t) > 1$ which we shall use to arrive at a measure of non-markovianity.

$$g(t) = \lim_{\epsilon\to 0^+} \frac{f_{NCP}(t+\epsilon,t)-1}{\epsilon} = \begin{cases} g(t) = 0, & \text{iff } \mathcal{E}_{t+\epsilon,t} \text{ is CP (markovian)} \\ g(t) > 0, & \text{otherwise (non-markovian)} \end{cases}$$

Further, we define $\mathcal{I}$ as follows.

$$\mathcal{I} = \int_0^\infty g(t)dt$$

Now, $\mathcal{I}$ can be taken as a measure of non-Markovianity, and as long as $g(t)$ decreases fast enough and still is finite.

We can further obtained a normalized version of the above measure given by $\mathcal{D}_{NM}$.

$$\mathcal{D}_{NM} = \frac{\mathcal{I}}{\mathcal{I}+1} = \begin{cases} \mathcal{D}_{NM} = 0 \text{ when } \mathcal{I} = 0 \text{ (markovian)} \\ \mathcal{D}_{NM} \to 1 \text{ when } \mathcal{I} \to \infty \text{ (non-markovian)} \end{cases}$$

7 EXAMPLES

7.1 Single Damped Harmonic Oscillator

Single Damped Harmonic Oscillator: In this case, in order to visulaize the sensitivity of the proposed measure $\mathcal{I}^{(E)}$, two different spectral densities of the bath have been considered, along with several initial temperatures.

Now, we have

$$J(\omega) = \alpha\omega^k e^{-\omega/\omega_c}$$

where $k = 1$ for ohmic spectral density and $k = 3$ for super-ohmic spectral density.

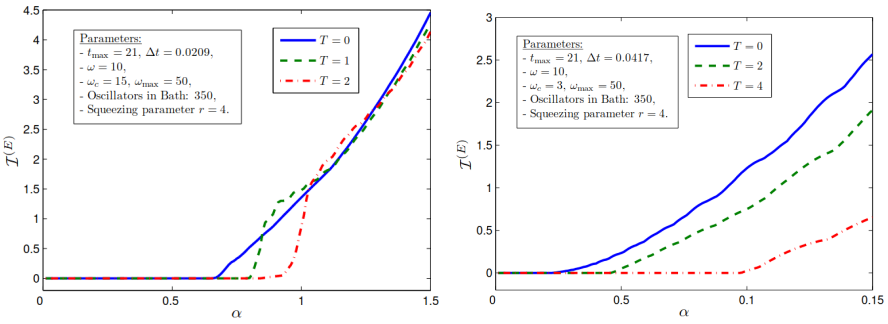


Figure 5: Ohmic and super-Ohmic spectral density

Non-markovian character increases as α increases. As $\alpha \to 0$, the evolution approaches markovian character.

7.2 Pure Dephasing

The dynamics of pure dephasing is given by the following Lindblad equation.

$$\frac{d\rho}{dt} = \gamma(t)(zz)$$

From the above we obtain,

$$g(t) = \begin{cases} 0, & \gamma(t) \geqslant 0 \\ -2\gamma(t), & \gamma(t) < 0 \end{cases}$$

$$\mathcal{I} = -2 \int_{\gamma(t)<0} \gamma(t) dt$$

Thus, if $\gamma(t) \geqslant 0$ we have Markovian evolution, else if its negative (< 0) then we have non-Markovian evolution.

8 CONCLUSION

We explored some basic concepts related to what non-Markovian quantum evolution is and how it can be measured. We derived the necessary and sufficient conditions to indicate whether a dynamical map is Markovian or non-Markovian or not. We attempted to demonstrate our ideas by also using a classical analogue in thermodynamics as well.

REFERENCES

[1] Susana F. Huelga Angel Rivas and Martin B. Plenio. Entanglement and Non-Markovianity of Quantum Evolutions. July 29,2010.

[2] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*. 1889.