McCOMBS SCHOOL OF BUSINESS
Salem Center for Policy

Probability

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Outline

The basics and conditional probability

Independence

Paradoxes, mixtures, and the rule of total probability

Random variables, distributions, and simulation

What is probability?

- A measure of uncertainty
- Answering the question: "How likely is a given event?"
- As with any mathematical concept, there are a set of axioms setting the "ground rules"
- Separately, there are different ways to interpret probability ...

  (i) **frequentist**: limit of relative frequency after repeating an experiment an infinite number of times (coin flip!)

  (ii) **Bayesian**: subjective belief about the likelihood of an event occurence

Probability basics

If A denotes some event, then $P(A)$ is the probability that this event occurs:

- $P(\text{coin lands heads}) = 0.5$
- $P(\text{rainy day in Ireland}) = 0.85$
- $P(\text{cold day in Hell}) = 0.0000001$

And so on...

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Some probabilities are estimated from direct experience over the long run:

- $P(\text{newborn baby is a boy}) = \frac{106}{206}$
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Others are synthesized from our best judgments about unique events:

- $P(\text{Apple stock goes up after next earnings call}) = 0.54$
- $P(\text{Djokovic wins next US Open}) = 0.4$ (6 to 4 odds)
- etc.

Probability basics: conditioning

A conditional probability is the chance that one thing happens, given that some other thing has already happened.

A great example is a weather forecast: if you look outside this morning and see gathering clouds, you might assume that rain is likely and carry an umbrella.

We express this judgment as a conditional probability: e.g. "the conditional probability of rain this afternoon, given clouds this morning, is 60%."

Probability basics: conditioning

In statistics, we write this a bit more compactly:

- $P(\text{rain this afternoon} \mid \text{clouds this morning}) = 0.6$
- That vertical bar means "given" or "conditional upon."
- The thing on the left of the bar is the event we're interested in.
- The thing on the right of the bar is our knowledge, also called the "conditioning event" or "conditioning variable": what we believe or assume to be true.

$P(A \mid B)$: "the probability of A, given that B occurs."

Probability basics: conditioning

Conditional probabilities are how we express judgments in a way that reflects our partial knowledge.

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A really important fact is that conditional probabilities are **not symmetric**:

$$P(A \mid B) \neq P(B \mid A)$$

As a quick counter-example, let the events A and B be as follows:

- A: "you can dribble a basketball"
- B: "you play in the NBA"

Probability basics: conditioning

- A: "you can dribble a basketball"
- B: "you play in the NBA"

Clearly $P(A \mid B) = 1$: every NBA player can dribble a basketball!

Probability basics: conditioning

- A: "you can dribble a basketball"
- B: "you play in the NBA"

But $P(B \mid A)$ is nearly zero!

Uncertain outcomes and probability models

An uncertain outcome (more formally called a “random process”) has two key properties:

1. The set of possible outcomes, called the sample space, *is known* beforehand.

2. The particular outcome that occurs is *not known* beforehand.

We denote the **sample space** as Ω, and some particular element of the sample space as $\omega \in \Omega$

Uncertain outcomes and probability models

Examples:

1. NBA finals, Golden State vs. Toronto:

$$\Omega = \{\text{4-0, 4-1, 4-2, 4-3, 3-4, 2-4, 1-4, 0-4}\}$$

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3. Number of no-shows on an AA flight from Austin to DFW:

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3. Number of no-shows on an AA flight from Austin to DFW:

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4. Poker hand

$$\Omega = \text{all possible five-card deals from a 52-card deck}$$

An **event** is a *subset of the sample space*, i.e. $A \subset \Omega$. For example:

1. NBA finals, Golden State vs. Toronto. Let A be the event "Toronto wins". Then

$$A = \{\text{3-4, 2-4, 1-4, 0-4}\} \subset \Omega$$

2. Austin weather. Let A be the event "cooler than 90 degrees". Then

$$A = [10, 90) \subset [10, 115]$$

3. Flight no-shows. Let A be "more than 5 no shows":

$$A = \{6, 7, 8, \ldots, N_{\text{seats}}\}$$

Some set theory concepts

We need some basic set-theory concepts to make sense of probability, since the sample space Ω is a set, and since "events" are subsets of Ω.

Union: $A \cup B = \{\omega : \omega \in A \text{ or } \omega \in B\}$

Intersection: $A \cap B = \{\omega : \omega \in A \text{ and } \omega \in B\}$

Complement: $A^C = \tilde{A} = \{\omega : \omega \notin A\}$

Difference: $A \setminus B = \{\omega : \omega \in A, \omega \notin B\}$

Disjointness: A and B are disjoint if $A \cap B = \emptyset$ (the empty set).

Axioms of probability (Kolmogorov)

These are the ground rules!

Consider an uncertain outcome with sample space Ω. "Probability" $P(\cdot)$ is a set function that maps Ω to the real numbers, such that:

1. Non-negativity: For any event $A \subset \Omega$, $P(A) \geq 0$.
2. Normalization: $P(\Omega) = 1$ and $P(\emptyset) = 0$.
3. Finite additivity: If A and B are disjoint, then $P(A \cup B) = P(A) + P(B)$.

3a. Finite additivity (general): For any sets A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
(bonus: prove this with set theory!)

Not that intuitive! Notice no mention of frequencies...

Summary of terms

- Uncertain outcome/"random process": we know the possibilities ahead of time, just not the specific one that occurs
- Sample space: the set of possible outcomes
- Event: a subset of the sample space
- Probability: a function that maps events to real numbers and that obeys Kolmogorov's axioms

OK, so how do we actually *calculate* probabilities?

Calculation

Now that we have an understanding of the axioms, notation, and interpretation, how do we calculate probabilities?

Counting!

(review 6.1.3-6.1.4 in the QSS book … ways to count objects in structured sets are discussed)

Suppose our sample space Ω is a finite set consisting of N elements $\omega_1, \ldots, \omega_N$.

Suppose further that $P(\omega_i) = 1/N$: each outcome is equally likely, i.e. we have a discrete uniform distribution over possible outcomes.

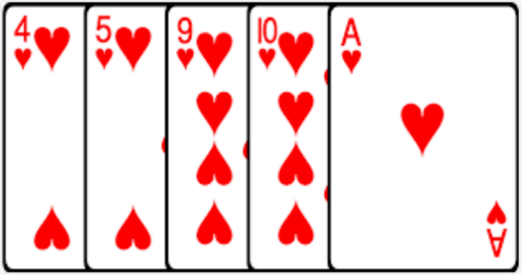
Then for each set $A \subset \Omega$,

$$P(A) = \frac{|A|}{N} = \frac{\text{Number of elements in } A}{\text{Number of elements in } \Omega}$$

That is, to compute $P(A)$, **we just need to count how many elements are in** A.

Counting example

Someone deals you a five-card poker hand from a 52-card deck. What is the probability of a flush (all five cards the same suit)?



Note: this is a very historically accurate illustration of probability, given its origins among bored French aristocrats!

Counting example

- Our sample space has $N = \binom{52}{5} = 2{,}598{,}960$ possible poker hands, each one equally likely.

- How many possible flushes are there? Let's start with hearts:
  $\rightarrow$ There are 13 hearts

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  $\rightarrow$ The same argument works for all four suits, so there are $4 \times 1287 = 5{,}148$ flushes. Thus:

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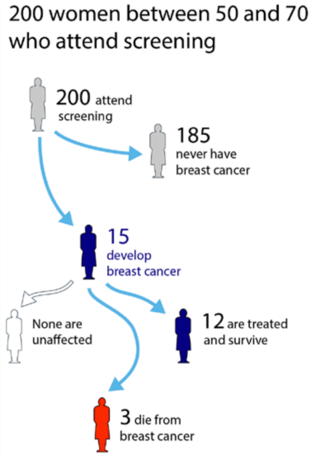
  $\rightarrow$ To make a flush with hearts, you need any 5 of these 13 cards.
  $\rightarrow$ Thus there are $\binom{13}{5} = 1287$ possible flushes with hearts.
  $\rightarrow$ The same argument works for all four suits, so there are 4 $\times$ 1287 = 5,148 flushes. Thus:

$$P(\text{flush}) = \frac{|A|}{|\Omega|} = \frac{5148}{2598960} = 0.00198079$$

So we know how to count, but what about conditioning?

Probability trees are very useful for this task! This involves counting at different levels of the tree.

Conditioning example: mammograms



- $P(\text{cancer}) = \frac{15}{200}$
- $P(\text{die,cancer}) = \frac{3}{200}$
- $P(\text{die} \mid \text{cancer}) = \frac{3}{15}$

– In general, we can estimate the conditional probability as:

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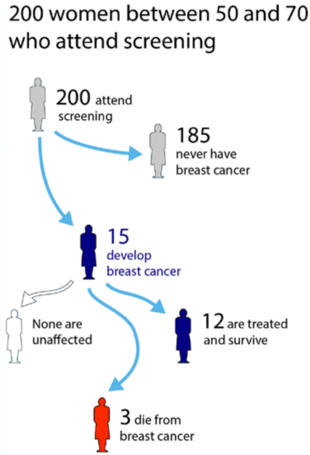

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  - In general, we can estimate the conditional probability as:

$$P(A \mid B) = \frac{\text{Frequency of } A \text{ and } B \text{ both happening}}{\text{Frequency of } B \text{ happening}}$$

This is actually a new axiom

The multiplication rule – it is an axiom since it can't be derived from the original axioms.

$$P(A \mid B) = \frac{P(A, B)}{P(B)}$$

Alternate version

We can also use this alternative version if we want to go in reverse, from a conditional probability to a joint probability.

It says the same thing with the terms rearranged.

$$P(A, B) = P(A \mid B) \cdot P(B)$$