INDEX

No.	Question No.	Page No.	Total Marks	Marks Obt.	Teacher Signature
1.	Question No 1	1–4	10		
2.	Question No 2	5	10		
3.	Question No 3	6	10		
4.	Question No 4	7–9	10		
5	Question No 5	10	10		
6.	Question No 6	11–15	10		
7.	Question No 7	16–18	20		
8.	Question No 8	19–21	20		

SUBJECT NAME: PROBABILITY THEORY

SUBJECT CODE: MST-003

Question No1 State whether the following statements are True or False. and also give the reason in support of your answer.

(a) Sample space of a (i) random experiment tossing two coins simultaneously and (ii) one coin two times is the same.

Solution :- TRUE

If a coin is tossed twice or if two coins are tossed simultaneously, the sample space is equal on both occasions.

$$SS = \{HH, HT, TH, TT\}$$

(b) Standard deviation of a random variable X may take any real value, i.e its value lies in the interval $(-\infty, \infty)$

Solution:- False

To see why, think about the numerator and denominator inside the radical:.

$$SD = \sqrt{\frac{\sum (x-\bar{x})^2}{n}}$$

Notice how n is always positive. It's the number of data points and we can't have a negative number of data points.

Also, notice that $\sum |x-\bar{x}|^2$ involves a quantity getting squared. whenever we square something, we get a non-negative number.

Since both the denominator and numerator are positive, the entire expression must be positive too.

(c) If events $E_1, E_2, E_3, E_4, \ldots, E_n$ are mutually exclusive and exhaustive then $P(E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n)$ will be greater than $1/2$ but less than 1.

Solution:- $\boxed{\text{FALSE}}$

Cases are said to be mutually exclusive if the happening of any one of them prevents the happening of all others in a single experiment.

here $P(E_1 \cup E_2 \cup \ldots \cup E_n) = 1$.

(d) If S is sample space of a random experiment and E is an event defined on this sample space then $P(S|E) = 1$.

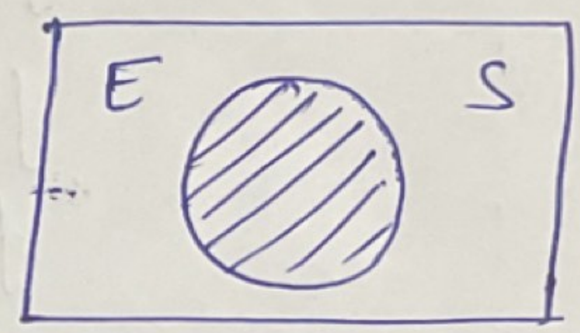
Solution:- $\boxed{\text{TRUE}}$

If S is a sample space of a random experiment and E is an event defined on this sample space then

$$P(S|E) = \frac{P(S \cap E)}{P(E)}$$

$$= \frac{P(E)}{P(E)}$$

$$= 1$$



$\because E \subset S$ so $E \cap S = E$

(e) If X is a random variable having range set $\{0,1,2,3\}$ then the set $\{x \in S : X(x) = 0\}$ is an event having at least one outcome of the random experiment.

Solution:- $\boxed{\text{TRUE}}$

If X is a random variable having range set $\{0,1,2,3\}$ $\quad X(S) = \{0,1,2,3\}$

Suppose we want to calculate the probability

of having atleast two tails in the coin flipping experiment. this calculation corresponds with the event $A = \{(HTT), (THT), (TTH), (TTT)\} \subseteq S$

we express the prob. as $P(A) = P(X \geq 2)$

The value of $P(X \geq 2)$ is computed with a probability function f that assign every value.

$$f(0) = 1/8$$
$$f(1) = 3/8$$
$$f(2) = 3/8$$
$$f(3) = 1/8$$

Then $P(A) = P(X \geq 2)$

$$= P(X \in \{2,3\})$$
$$= f(2) + f(3) = 1/2$$

i.e the set having atleast one outcome of random experiment.

Question NO2 There are 4 black, 3 blue and 8 red balls in an urn. Three balls are selected one by one without replacement. What is the probability that

(i) First ball drawn is black, second one is red and third one is blue.

(ii) All the three balls are of same colour.

Solution:- Given that

no. of Black Balls $n(E_1) = 4$

no. of Blue Balls $n(E_2) = 3$

no. of Red Balls $n(E_3) = 8$

(i) Prob that first ball drawn is black, second is red and third one is blue

$$= \frac{4}{15} \times \frac{8}{14} \times \frac{3}{13}$$

$$= \frac{16}{455} = 0.035$$

(ii) All the three balls are of same colour.

(a.) Prob. of 3 black balls $= \frac{4}{15} \times \frac{3}{14} \times \frac{2}{13} = \frac{4}{455}$

(b.) Prob. of 3 blue balls $= \frac{3}{15} \times \frac{2}{14} \times \frac{1}{13} = \frac{1}{455}$

(c.) Prob. of 3 Red balls $= \frac{8}{15} \times \frac{7}{14} \times \frac{6}{13} = \frac{56}{455}$.

Prob. same colour $= 61/455 = 0.134$

Question No3 A random 5-card poker hand is dealt from a standard deck of cards. Find the probability (in terms of binomial coefficients) of getting a flush (all 5 cards being of the same suit: do not count a royal flush, which is a flush with an ace, King, queen, jack and 10).

Solution:-

No. of Sample Space $= {}^{52}C_5$

Now pick one of four suits and fix it at the moment. There exist ${}^{13}C_5$ possible hands of that suit.

But, we have to exclude royal flush hand in this suit, so there exist ${}^{13}C_5$ possible wanted hands.

Doing that for all suits, we get.

$$P(E) = \frac{\text{no. of favourable cases}}{\text{no. of exhaustive cases.}}$$

$$P(E) = \frac{4({}^{13}C_5 - 1)}{{}^{52}C_5}$$

$$P(E) = \frac{643}{324870} = P(E) = 1.97 \times 10^{-3}$$

$$\approx 0.0020$$

Question No 4 Show that $f(x) = \left(\frac{1}{2}\right)^{x+1}, x = 0,1,2,3,4,5$ ... is a valid PMF for a discrete random variable. Also, find out its CDF.

Solution :→ Probability Mass Function

Let X be a random variable which takes the values $x_1, x_2, \ldots, x_n$ and let $P[X = x_i] = p(x_i)$. This function $p(x_i), i = 1, 2 \ldots$ defined for the values $x_1, x_2, \ldots$ $p(x_i) \geq 0$ and $\sum_i p(x_i) = 1$

Consider $\sum_i p(x_i) = \sum \left(\frac{1}{2}\right)^{x+1} : x = 0,1,2,3$

$$= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots$$

$$= \frac{1/2}{1 - 1/2} \qquad \left\{ S_\infty = \frac{a}{1-r} \right\}$$

$$= \frac{1/2}{1/2} = 1$$

$$\boxed{\sum_i p(x_i) = 1}$$

which means $f(x) = \left[\frac{1}{2}\right]^{x+1}, x = 0,1,2$ is a valid PMF

Cumulative Density function

$$f(x) = \left[\frac{1}{2}\right]^{x+1}, \; x = 0, 1, 2$$

The C.D.F is

for $x < 0$ $\quad F_X(x) = 0$

$0 \le x \le 1$ $\quad F_X(x) = P_X(0) = \frac{1}{2}$

$1 \le x < 2$ $\quad F_X(x) = P_X(0) + P_X(1) = \frac{1}{2} + \frac{1}{4}$

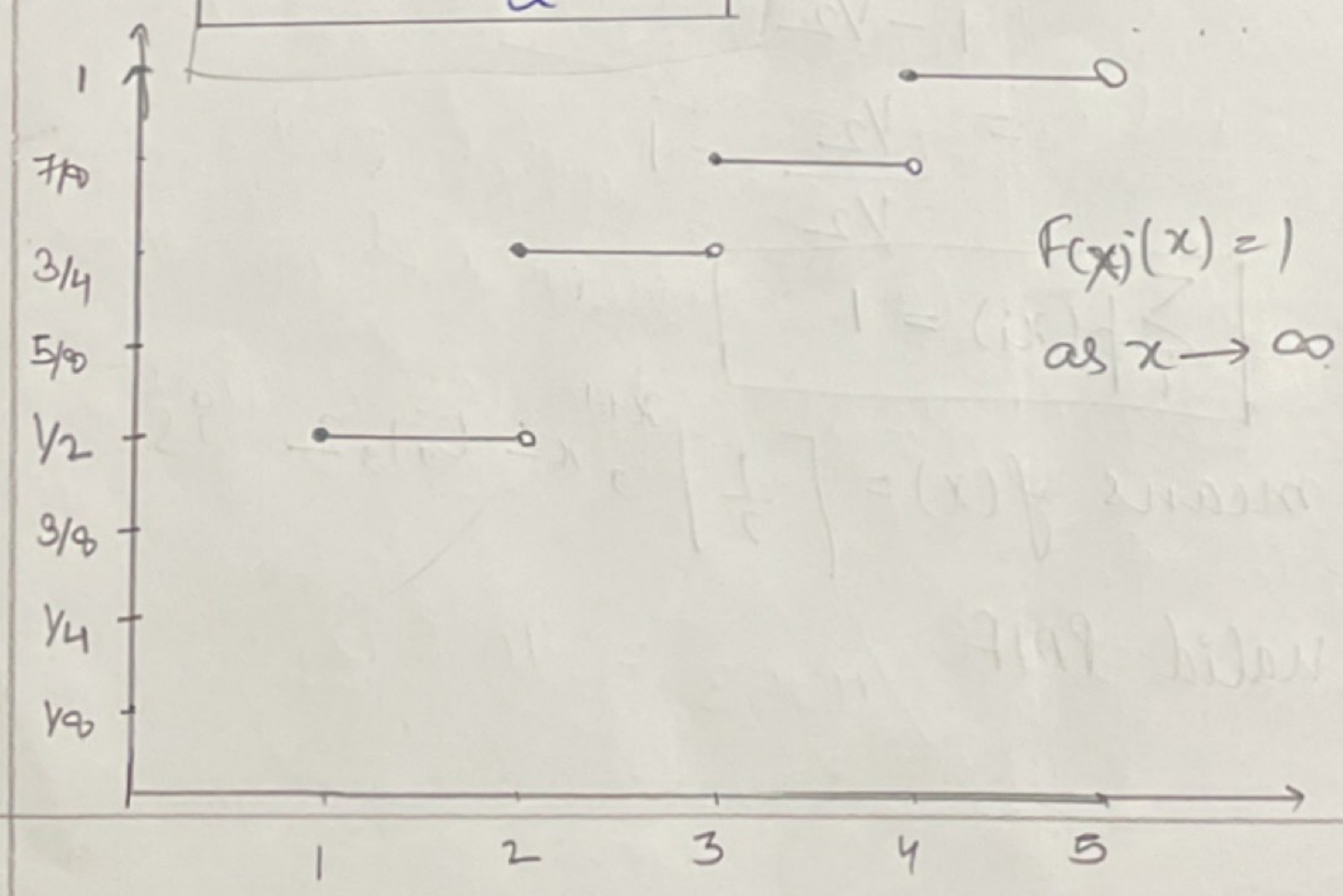
$$= \frac{3}{4}$$

In general we have, for $0 < k \le x < k+1$

$$F_X(x) = P_X(0) + P_X(1) + P_X(2) + \dots + P_X(k)$$

$$= 0 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^{k+1}}$$

$$\boxed{F_X(x) = \frac{2^{k+1} - 1}{2^{k+1}}}$$



It is easy to obtain that

$$E(I_{1,2}) = P(I_{1,2}=1) = \frac{1}{365}$$

Finally $E\left(\sum_{j,k} I_{j,k}\right) = {}^{100}C_2 \times \frac{1}{365}$

Now define following random variables

$$D_l = \begin{cases} 1, & \text{at } l\text{th day in the year there exist birthday match} \\ 0, & \text{otherwise} \end{cases} \qquad l = 1, 2, \ldots, 365$$

Similarly we have

$$E(\Sigma_l D_l) = \Sigma_l E(D_l) = 365\, E(D_1)$$

$$E(D_1) = P(D_1 = 1) = 1 - P(D_1 = 0)$$

$$= 1 - \left[\left(\frac{364}{365}\right)^{100} + 100 \cdot \frac{1}{365} \times \left(\frac{364}{365}\right)^{99}\right]$$

The event that there exist no match on the first day of the year means that no people were born at the first day of the year or that one and only one people was born at the first day of the year.

Hence $E\left(\sum_l D_l\right) = 365\left[1 - \left\{\left(\frac{364}{368}\right)^{100} + \frac{100}{365}\left(\frac{364}{365}\right)^{99}\right\}\right]$

Question No 5 A group of 100 people are comparing their birthdays (as usual, assume their birthdays are independent and not on February 29, etc) Find the expected number of pairs of people with the same birthday, and the expected number of days in the year on which at least two of these people were born.

Solution:→ Number of people with numbers 1, 2, ..., 99, 100. Defining following random variable.

$$I_{j,k} = \begin{cases} 1, & j^{th} \text{ and } k^{th} \text{ person share the birthday} \\ 0, & \text{otherwise} \end{cases} \qquad j \neq k \in \{1, 2, \ldots 100\}$$

We want to find

$$E\left[\sum_{j,k} I_{j,k}\right] = \sum_{j,k} E(I_{j,k})$$

$$= {}^{100}C_2 \, E(I_{1,2})$$

[we have used the fact that there exist ${}^{100}C_2$ possible pairs of people, linearity of expectation and the symmetry].

Question No 6 Random variable X follows Beta distribution with parameter $a=3$, $b=2$ and has pdf.

$$f(x) = \begin{cases} 12x^2(1-x), & 0<x<1 \\ 0, & \text{otherwise.} \end{cases}$$

Find (i) CDF of X (ii) $P[0<X<1/2]$ (iii) mean and variance of X without using direct formula for mean and variance.

Solution :→ Given function

$$f(x) = \begin{cases} 12x^2(1-x) & : 0<x<1 \\ 0 & : \text{otherwise} \end{cases}$$

(i) CDF of X

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{x} 12x^2(1-x)\,dx$$

$$= 12\int_0^x (x^2 - x^3)\,dx$$

$$= 12\left[\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^x\right]$$

$$= 12\left[\frac{x^3}{3} - \frac{x^4}{4} - 0 + 0\right] = 4x^3 - 3x^4$$

$$\boxed{\text{C.D.F of } X: \quad F(x) = \begin{cases} 4x^3 - 3x^4 & \text{for } 0<x<1 \\ 0, & \text{otherwise.} \end{cases}}$$

(ii) $P\left[0 < x < \frac{1}{2}\right] = \int_0^{1/2} f(x)\,dx$

$$= \int_0^{1/2} 12x^2(1-x)\,dx$$

$$= 12\int_0^{1/2} (x^2 - x^3)\,dx$$

$$= 12\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^{1/2}$$

$$= 12\left[\frac{1}{24} - \frac{1}{64} - 0 + 0\right]$$

$$= 12 \times \frac{5}{192} = \frac{5}{16}$$

$$\boxed{P\left[0 < x < \frac{1}{2}\right] = 0.3125}$$

(iii) Mean and variance of X

Mean of X $E(X) = \int_0^1 x \cdot f(x)\,dx$

$$= \int_0^1 x \cdot 12x^2(1-x)\,dx$$

$$= 12\int_0^1 (x^3 - x^4)\,dx$$

$$= 12\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1$$

$$= 12\left[\frac{1}{4} - \frac{1}{5} - 0 + 0\right]$$

$$\boxed{E(X) = \frac{3}{5}}$$

(iv) Variance of X

$$E(X^2) = \int_0^1 x^2 \cdot f(x)\,dx = \int_0^1 x^2 \cdot 12x^2(1-x)\,dx$$

$$= 12\int_0^1 x^4(1-x)\,dx$$

$$= 12 \int_0^1 (x^4 - x^5)\, dx$$

$$= 12 \left[\frac{x^5}{6} - \frac{x^6}{6} \right]_0^1$$

$$= 12 \left[\frac{1}{5} - \frac{1}{6} - 0 + 0 \right]$$

$$\boxed{E(x^2) = 2/5}$$

Now, $Var(x) = E(x^2) - [E(x)]^2$

$$= \frac{2}{5} - \left(\frac{3}{5}\right)^2$$

$$= \frac{2}{5} - \frac{9}{25}$$

$$\boxed{Var(x) = \frac{1}{25}}$$

(b) Calculate the 75% quantile for the marginal distribution of Y.

Y	1	2	3
P(Y)	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{1}{4}$

Here $n=1$

$$Q_3 = \left[\frac{3(n+1)}{4}\right]^{th} \text{ value of obs.}$$

$$= \left(\frac{3-2}{4}\right)^{th} \text{ value of observation}$$

$$= 0.25^{th} \text{ value of observation.}$$

x	Frequency (f)	cf
1	0.1667	0.1667
2	0.5833	0.75
3	0.25	1
...	...	...
	n=1	...

from above table 0.25^{th} value is 1.

75% Quantile for the marginal distribution of Y=1

Question No 7 Consider the joint PDF for the type of customer service X (0 = telephonic hotline, 1 = Email) and of satisfaction score Y (1 = unsatisfied, 2 = satisfied, 3 = very satisfied).

	Y		
X	1	2	3
0	0	1/2	1/4
1	1/6	1/12	0

(a) Determine and interpret the marginal distributions of both X and Y.

Solution

	Y			p(x)
X	1	2	3	
0	0	1/2	1/4	3/4
1	1/6	1/12	0	1/4
p(y)	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{1}{4}$	

Marginal distribution of (X)

X	0	1
P(X)	$\frac{3}{4}$	$\frac{1}{4}$

Marginal distribution of (Y)

Y	1	2	3
P(Y)	1/6	7/12	1/4

(c) Determine and interpret the conditional distribution of satisfaction level for $X=1$.

$$P[Y=1 \mid X=1] = \frac{P[Y=1, X=1]}{P[X=1]} = \frac{\frac{1}{6}}{\frac{1}{4}} = \frac{2}{3}$$

$$P[Y=2 \mid X=1] = \frac{P[Y=2, X=1]}{P[X=1]} = \frac{\frac{1}{12}}{\frac{1}{4}} = \frac{1}{3}$$

$$P[Y=3 \mid X=1] = \frac{P[Y=3, X=1]}{P[X=1]} = \frac{0}{1/4} = 0$$

$\therefore$ The conditional distribution of Y given $X=1$ is.

Y	1	2	3
$P[Y=y \mid X=]$	$\frac{2}{3}$	$\frac{1}{3}$	0

(d) Are the two variables independent?

$P[X=1] = 1/4$ and $P[Y=1] = 1/6$

$P[X=1, Y=1] = 1/6$

Here $P[X=1, Y=1] = 1/6$ & $P[X=1] \cdot P[Y=1] = \frac{1}{4} \times \frac{1}{6} = \frac{1}{24}$

i.e $P[X=1, Y=1] \neq P[X=1] \cdot P[Y=1]$

Hence X & Y are not independent variables.

(e) Calculate and interpret the covariance of X & Y.

$$E(X) = \sum_{x=0}^{1} x \cdot p(x) = 0 \cdot \frac{3}{4} + 1 \cdot \frac{1}{4} = \frac{1}{4}$$

$$E(Y) = \sum_{y=1}^{3} y \cdot p(y) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{7}{12} + 3 \cdot \frac{1}{4} = \frac{25}{12}$$

$$E(XY) = (0 \times 1 \times 0) + \left(0 \times 2 \times \frac{1}{2}\right) + \left(0 \times 3 \times \frac{1}{4}\right) + \left(1 \times 1 \times \frac{1}{6}\right) + \left(1 \times 2 \times \frac{1}{12}\right) + (1 \times 3 \times 0)$$

$$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

So, $Cov(X, Y) = E(XY) - E(X) \cdot E(Y)$

$$= \frac{1}{3} - \left(\frac{1}{4} \times \frac{25}{12}\right)$$

$$= -3/16$$

$Cov(X, Y) = -3/16$

Question 8:→ State Monty Hall problem and solve it.

Solution:→

MONTY HALL PROBLEM STATEMENT

Suppose your on a game show and you are given the choice of three doors. Behind one door is a car, behind the others goats. You pick a door, say No 1 and the host who knows what's behind the doors, opens another door, say No 3, which has a goat. He then says to you. "Do you want to pick door No. 2?" Is it your advantage to switch your choice?

USING BAYE'S THEORAM TO SOLVE THE MONTY HALL PROBLEM

A more rigorous solution can be found using Baye's Theorem. Credit goes to Christopher Long for this interesting solution. I'm going to assume that you are familiar with Baye's Theorem which is a way to figure out conditional probability. (if event A happens, what's the probability of event B will happen?)

The basis for the solution is the same as in the above scenario. There are three doors, one has the car behind it. You pick a door, then Monty opens one of the other doors to reveal a goat.

Let's assume you pick door 1 and then Monty shows you the goat behind door 2. In order to use Baye's Theorem we need to first assign an event to A and B.

- Let event A be that the car is behind door number 1.
- Let event B be that Monty opens up door 2 to show the goat.

Here's the Baye's solution-

$$Pr(A|B) = \frac{Pr(B|A) \times Pr(A)}{Pr(B)}$$

$$= \frac{1/2 \times 1/3}{1/3 \times 1/2 + 1/3 \times 0 + 1/3 \times 1}$$

Pr(A) is pretty simple to figure out. There is a 1/3 chance that the car is behind door 1. There are two doors left, and each has a 1/2 chance of being chosen- which gives us Pr(B|A), or the probability of event B, given A.

Pr(B) in the denominator, is a little trickier to figure out. Consider that:

1. You choose door 1. Monty shows you a goat behind door 2.
2. If the car is behind door 1, Monty will not choose it. He'll open door 2 and show a goat 1/2 of the time.
3. If the car is behind door 2, Monty will always open door 3, as he never reveals the car.
4. If the car is behind door 3, Monty will open door 2 100% of the time.

As Monty has opened door 2, you know the car is either behind door 1 (your choice) or door 3. The probability of the car being behind door 1 is 1/3. This means that the probability of the car being behind door 3 is $1 - (1/3) = 2/3$. And that is why you switch.