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# General Physics (II) Electromagnetic theory

## Static Electric Field (Halliday & Resnick ed. 11, chapter 21 – 24)

charge is quantized : $q = ne$, $n = \pm 1, \pm 2, \pm 3$

elementary charge : $e = 1.602 \times 10^{-19}$ C

coulomb (SI 制之電荷量單位)

$1C = (1A)(1s)$

。 – 庫侖之電荷量為 – 安培之電流在一秒間所通過的電荷數

charge is conserved (不可隨意產生或湮滅消電荷)

charges can move on the (semi/super…) conductors.

傾向在導体表面重新分布而使得電荷隨處為 0

## Coulomb's law (Charles-Augustin de Coulomb, 1785)

清乾隆年間

describing the electrostatic force between two charge particles

for charged particles that have charge $q_1$ and $q_2$, $\vec{r} \equiv \vec{x}_1 - \vec{x}_2$, $\hat{r} \equiv \frac{\vec{r}}{|\vec{r}|}$ 方向向量

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}|^2} \hat{r}$$

charged particle 2 施加給 charged particle 1 之靜電力

electrostatic constant, or coulomb's constant.

$$\frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^9 \ N \cdot m^2/C^2$$

permittivity constant : $\epsilon_0 = 8.85 \cdot 10^{-12} \ C^2/N \cdot m^2$

superposition of force : $\vec{F}_{1,net} = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \vec{F}_{15} + \cdots + \vec{F}_{1n}$

(靜電力之疊加)

SI 制單位: N/C

electric field : $\vec{E} \equiv \frac{\vec{F}}{q_0}$

單位電荷受的 electric force

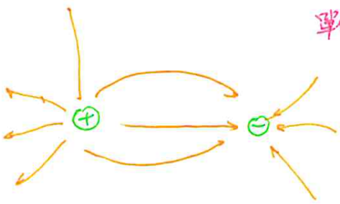
$\vec{F}$: 該無窮小電荷所受之總電磁力

due to this relation, electric field can be superposed in the same way.

concept introduced by Michael Faraday in the 19th century

$q_0$: 某無窮小電荷之 test charge

因無窮小, 故其電荷產生之電磁力不改變空間中之電荷分布



### Electric field lines :

Extend away from positive charge toward negative charge.

(i) at any point, the electric field vector must be tangent to the electric field line.

(ii) In the plane perpendicular to the field lines, the relative density of lines represents the relative magnitude of the field.

### Electric field due to a point charge

at location $\vec{x}_2$ (point charge); at location $\vec{x}_1$ (field)

$$\vec{E} = \frac{\vec{F}}{q_0} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

$\hat{r} = \frac{\vec{r}}{|\vec{r}|}$, $r = |\vec{x}_2 - \vec{x}_1|$

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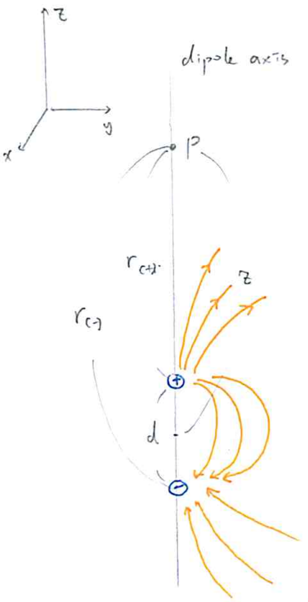
# General Physics (II) Electromagnetic theory

## Electric field due to a dipole

(因大部分情形考慮電中性物体，故時常須面對 dipole field)
如極性分子的性質




At point P, the electric field in the z direction set up by the dipole 為

$$E_z = E_{z(+)} - E_{z(-)}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{r_{(+)}^2} - \frac{1}{4\pi\epsilon_0}\frac{q}{r_{(-)}^2} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{(z-\frac{1}{2}d)^2} - \frac{1}{(z+\frac{1}{2}d)^2}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\frac{2zd}{\left((z-\frac{1}{2}d)(z+\frac{1}{2}d)\right)^2} \xrightarrow[d \ll z]{} \frac{q}{4\pi\epsilon_0}\frac{2zd}{z^4} = \frac{1}{2\pi\epsilon_0}\frac{qd}{z^3}$$

$$= \frac{1}{2\pi\epsilon_0}\frac{p}{z^3}$$

(electric dipole moment: $\vec{p} = q\vec{d}$)
$\vec{d}$ 方向為由負極指向正極

一般的應用之中我們僅知道 $\vec{p}$ 而不知道 q 及 d. dipole field 之電場與 $r^3$ 成反比
dipole field 隨距離 decay 得比 monopole 快. 長距離情形只有 monopole 重要

$r \gg d$，確 z 軸測量之電場 (dipole field) 一般形式

$$E_r = \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}2\cos\theta, \quad E_\theta = \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\sin\theta, \quad E_\phi = 0$$

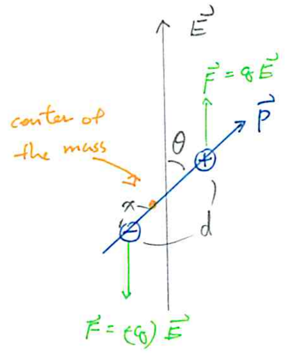
## Electric field due to a continuous distribution of charge

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \underbrace{\rho(\vec{x}')}_{\text{在位置 }\vec{x}'\text{ 處單位空間內之電荷數}} \underbrace{\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}}_{\hat{r}/r^2} d^3x'$$

課本 §22-4, §22-5 中舉 line charge 及 disk of charge 等例
不清楚積分如何處理者可參考.
實作中多為較複雜情形，可/須靠數值積分幫助解決

## A dipole in an electric field

在大多數的應用中，可視外電場為均勻電場，此時 dipole 的行為僅與 $\vec{E}$ 與 $\vec{p}$ 有關




torque (about the c.o.M)

$$\tau = q\overset{\text{均勻外場之場強}}{E}\left((d-x)\sin\theta\right) - (-q)Ex\sin\theta$$

$$= E(qd)\sin\theta = Ep\sin\theta \longrightarrow \text{力矩方向為穿出紙面}$$

$$\Rightarrow \underline{\underline{\vec{\tau} = \vec{p}\times\vec{E}}}$$

(類似地，我們也可以定義 magnetic dipole moment $\vec{\tau} = \vec{\mu}\times\vec{B}$ 在磁場 $\vec{B}$ 中受到力矩 $\vec{\tau}$ 之物体具有 magnetic dipole moment $\vec{\mu}$)

$$\text{位能 } U = \int(-\tau)\,d\theta = \int pE\sin\theta\,d\theta = \underbrace{-pE\cos\theta}_{\text{隨}\theta\text{變化之部分}} + \overbrace{\text{const.}}^{\text{可自行定義之位能零點}}$$

以與電磁力相反方向的力作功而使系統儲存的能量

$$\Longrightarrow \underline{\underline{U = -\vec{p}\cdot\vec{E}}}$$

(magnetic dipole 在磁場的能量 $U = -\vec{\mu}\cdot\vec{B}$)

※此處 $\tau$ 與 U 為重要之一般性質，應該記憶。物質電中性情形多，dipole 對電場貢獻重要

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# General Physics (II) Electomagnetic theory

## Gauss law:

(※對於大學以上物理為非常常用之定理。有需要學習物化者或應在学習向量微積分後研讀)
此處跳過證明

Relating electric field on a closed surface to the net charge enclosed by that surface

↳ Application
- ① evaluate $\vec{E}$ when the charge distribution is known.
- ② evaluating the charge distribution when $\vec{E}$ is known.

積分型式

$$\oint \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0}$$

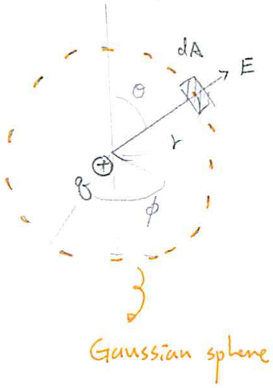
enclosed charge density

同一定理之微分型式：

$$\oint \vec{E}\cdot d\vec{A} = \underbrace{\int (\vec{\nabla}\cdot\vec{E})\,dV}_{\text{体積積分}} \Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

電荷密度

### Example 1: electric field on the sphere that is centered on a point charge




$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$$

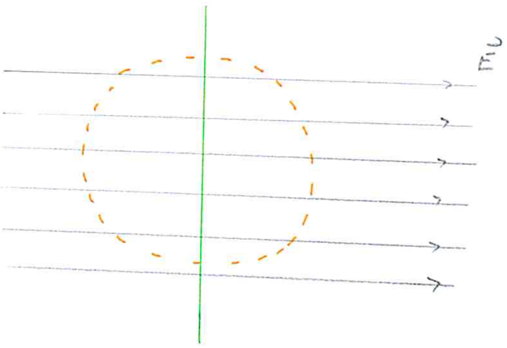
$$dA = (r\sin\theta\, d\phi)(r\,d\theta) = r^2\sin\theta\, d\theta\, d\phi$$

$$\boxed{\oint \vec{E}\cdot d\vec{A}} = \int_0^{2\pi}\!\!\int_0^{\pi} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} r^2 \sin\theta\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\!\!\int_0^{\pi} \frac{q}{4\pi\epsilon_0}(d\cos\theta)\,d\phi$$

$$= \frac{q}{2\epsilon_0}\int_0^{\pi}(d\cos\theta) = \boxed{\frac{q}{\epsilon_0}}$$

### Example 2: charge density when there is an uniform electric field




from the symmety of the system

( $\int \vec{E}\cdot dA$ 在綠線左半边與右右半边之貢獻相消)

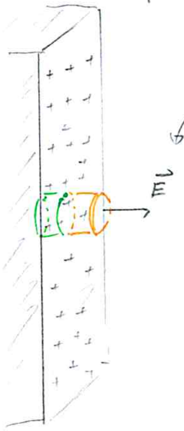
$\oint \vec{E}\cdot d\vec{A} = 0$ no matter how big/small is the sphere

⇓

there is no net local charge density

### Example 3: charge on a flat and large conductor

(廣泛應用於在實驗之中建立均匀電磁場)
或應用與制作電容)




由於對稱性電場垂直於導体平面

- Inside the conductor: $\int \vec{E}\cdot d\vec{A} = 0$ ($\vec{E}$ 隨處為 0)
- outside the conductor: $\int \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0} \Rightarrow E = \frac{1}{\epsilon_0}\frac{q_{enc}}{dA} = \frac{\sigma}{\epsilon_0}$

導体表面之電荷密度

* 導体外電場強度與電荷密度有簡單關係.

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# General Physics (II) Electromagnetic theory

electric force is conservative $\longrightarrow$ has an associated potential energy（定義之方式與 gravitational potential energy 相同）

真正的動力學問題用 field，平衡狀態用 potential.

計算中用 potential 而非 electric field 的好處

1. 計算上純量場較向量場好處理（適用 stationary 問題）系統狀態不隨時間改變，如處於平衡狀態
2. 能量守恆之概念

**electric potential**: the amount of the electric potential energy per unit charge when a positive test charge is brought in from infinity

SI制單位：$1$ volt $= 1$ joul per coulomb（當我們將一帶有正電荷之 test charge 由無窮遠處移入電場時，單位正電荷所帶有之電位能）

極小，對系統不造成干擾之電荷

$$V_f - V_i = -\int_i^f \vec{E}\cdot d\vec{s}$$

（積分路徑可任取；順著電場的方向向下走是往 potential 低的地方去）

**electric potential energy**: $U = qV$

potential energy — $U$；potential — $V$

若將電場單位與 volt 關連：$1\ N/C = \left(1\frac{N}{C}\right)\left(\frac{1V}{1\frac{J}{C}}\right)\left(\frac{1J}{1N\cdot m}\right) = 1\ V/m$

一庫倫電荷在1伏特電位中的能量，相當於以一牛頓的力推行一米所做的功

**energy conservation**: $\underbrace{U_i}_{\text{initial potential energy}} + \overbrace{K_i}^{\text{initial mechanical energy}} = U_f + K_f - \underbrace{W_{app}}_{\text{外界對系統做的功}}$

**electric field**: $\vec{E} = -\vec{\nabla}V$ (i.e., $E_x = -\frac{\partial V}{\partial x}$, $E_y = -\frac{\partial V}{\partial y}$, $E_z = \frac{\partial V}{\partial z}$) $\Rightarrow \vec{\nabla}\times\vec{E} = 0$

Example: Potential due to a charged particle(s)

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

$\rightarrow$ setting $V_f = 0$ (at $\infty$), $V_i = V$ (at $|\vec{r}| = R$)

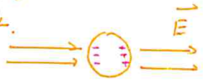
we get

$$0 - V = \frac{-q}{4\pi\epsilon_0}\int_R^\infty \frac{1}{r^2}\hat{r}\cdot d\vec{r}$$

$d\vec{r}$ 為往半徑增大方向

$$= -\frac{1}{4\pi\epsilon_0}\frac{q}{R} \Rightarrow V = \frac{1}{4\pi\epsilon_0}\frac{q}{R}$$

※ 平衡狀態時，導體內部電場為 0（isolated 的）
（即無電荷因受 electric force 作用而運動）
依本節計算，導體內部隨處為等電位




↓

電荷移動，使得導體內部電荷造成的電場剛好 cancel 外電場

↓

由於需要一定數量之電荷來 cancel 外電場（或與其它部分電荷之電場達到平衡）
當導體表面尖銳時，電荷密度高
（造成局部電場高，易發生尖端放電）

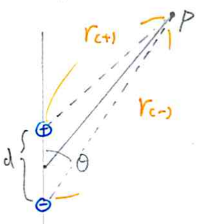
when there are multiple charged particles, due to the superposition of electric field

$$V = \sum_{i=1}^{n} V_i = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i}$$

with continuous charge distribution

$$V = \int dV = \frac{1}{4\pi\epsilon_0}\int\frac{dq}{r}$$

Example: Potential due to an electric dipole




$$V = \sum_{i=1}^{2} V_i = V_{(+)} + V_{(-)} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_{(+)}} + \frac{-q}{r_{(-)}}\right) = \frac{q}{4\pi\epsilon_0}\frac{r_{(-)} - r_{(+)}}{r_{(+)}r_{(-)}}$$

遠距離近似 $\begin{cases} r_{(-)} - r_{(+)} \approx d\cos\theta \\ r_{(-)}r_{(+)} \approx r^2 \end{cases}$

$p = qd$

$$V = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

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# General Physics (II) Electromagnetic theory

## Energy carried by electric field. (in vacuum)

Potential energy of charge $q_i$ : $W_i$

$$W_i = q_i V(\vec{x}_i) = \frac{q_i}{4\pi\epsilon_0}\sum_{j=1}^{n-1}\frac{q_j}{|\vec{x}_i - \vec{x}_j|}$$

Total potential energy (note the extra factor ½ to avoid duplicating the summation)

$$W = \frac{1}{2}\sum_i\sum_j \frac{1}{4\pi\epsilon_0}\frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}$$

With continuous charge distribution

Gauss law 的微分型

$\rho = \epsilon_0 \vec{\nabla}\cdot\vec{E}$

$$W = \frac{1}{2}\int \rho(\vec{x})\, V(\vec{x})\, d^3x = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\iint \rho(\vec{x})\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x\, d^3x'$$

$$= \frac{\epsilon_0}{2}\int (\vec{\nabla}\cdot\vec{E})V(\vec{x})\, d^3x$$

(integration by part) $= \frac{\epsilon_0}{2}\int \vec{E}\cdot\underbrace{(-\vec{\nabla}V(\vec{x}))}_{\text{by definition } \vec{E} = -\vec{\nabla}V} d^3x = \frac{\epsilon_0}{2}\int \vec{E}\cdot\vec{E}\, d^3x$

$$\Rightarrow \quad W_E = \frac{1}{2}\epsilon_0\int |\vec{E}|^2 d^3x$$

依類似計算過程可得 energy carried by magnetic field.

$$W_B = \frac{1}{2\mu_0}\int |\vec{B}|^2 d^3x$$

## Dielectricity

若物質可被電場偏極化: $\vec{P} = \epsilon_0 \chi_e \vec{E}$ $\chi_e$ : electric susceptibility.

$\vec{P}$ : electric dipole in a unit volume ≡ electric polarization

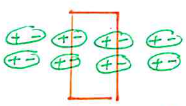
electric potential caused by charge density and electric polarization

$$V(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x'\left[\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} + \frac{\vec{P}(\vec{x}')\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\right] = \frac{1}{4\pi\epsilon_0}\int d^3x'\left[\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} + \vec{P}(\vec{x}')\cdot\vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\right]$$

第二項 integration by part,

$$= \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|}\left[\rho(\vec{x}') - \vec{\nabla}'\cdot\vec{P}(\vec{x}')\right]$$

對於 potential, 此項的作用即為沒有 electric polarization 情形之 charge 項



當 polarization 有梯度時, locally, 相當於有電荷積聚

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}[\rho - \vec{\nabla}\cdot\vec{P}]$$

define electric displacement $\vec{D} = \epsilon_0\vec{E} + \vec{P}$

$$\Rightarrow \vec{\nabla}\cdot\vec{D} = \rho$$ (可用 Gauss' law 解 $\vec{D}$)

若物質可被偏極化: $\vec{D} = \epsilon_0\vec{E} + \vec{P}$

$= \epsilon_0\vec{E} + \epsilon_0\chi_e\vec{E} = \epsilon_0(1+\chi_e)\vec{E}$

定義: electric permitivity $\epsilon = \epsilon_0(1+\chi_e)$

vacuum permitivity : $\epsilon_0$

dielectric constant: $\epsilon/\epsilon_0 = 1 + \chi_e \Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon}$

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# General Physics (II) Electromagnetic theory

definition of current: $i = \frac{dq}{dt}$ (單位時間通過的電荷數)

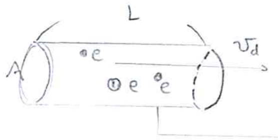
SI 制單位: 1 ampere $= 1\,A = 1$ coulomb per second.

$$\Rightarrow q = \int dq = \int_0^t i\,dt \quad \text{(累積通過的電荷數)}$$

defining current density: $i = \int \underbrace{\vec{J}}_{\text{單位投影面積之電流}} \cdot d\vec{A}$

投影於與 current 垂直之方向

relation between $\vec{J}$ and the number density of charge carrier $n$, the charge per carrier $e$; and the drift velocity of the charge carrier $v_d$




total charge in this volume: $q = (AL)\cdot n \cdot e$

time for all charge to leave this volume: $t = \frac{L}{v_d}$

$$\Rightarrow i = \frac{q}{t} = \frac{ALne}{L/v_d} = nAev_d$$

$$\Rightarrow \frac{i}{A} = J = \underbrace{(ne)\,v_d}_{\text{電流密度等於電荷密度乘以 drift velocity}} \Longrightarrow v_d = \frac{J}{ne}$$

microscopic view of the drift velocity.

conduction electrons that are free to move have high speed ($10^6$ m/s) due to Pauli exclusion principle. They collide with the atoms frequently, with average time $\tau$. They are also accelerated by the external field $E$.

加速度

$$\text{drift velocity } v_d = a\tau = \left(\frac{eE}{m}\right)\tau$$

$m$: mass of the charge carrier

$$\Rightarrow \frac{J}{ne} = \frac{eE\tau}{m}$$

$$\Rightarrow J = \frac{e^2 n \tau}{m} E$$

resistivity $\rho$

conductivity $\sigma \equiv \frac{1}{\rho} = \frac{e^2 n\tau}{m}$

$= \sigma E$ 電流密度等於電場乘以導電率

$J = \frac{i}{A}$, $E = \frac{V}{L}$

$V$: 導体兩端的電位差

$L$: 導体長度

$$\Rightarrow \rho = \frac{E}{J} = \frac{V/L}{i/A} \Rightarrow \underbrace{\frac{V}{i} \equiv R}_{} = \rho\frac{L}{A}$$

$R$: resistance

$V = iR$ omic law

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# General Physics (II) Electromagnetic theory

## Magnetic Field.

produced by $\begin{cases} \text{(1) moving charge} \\ \text{(2) spin magnetic moment} \end{cases}$ (在光譜中與 Zeeman splitting 有關)

Magnetic field is <u>defined</u> by the force law

$$\underbrace{\vec{F}_B}_{} = q\vec{v} \times \vec{B}$$

( 若 $\vec{v}$ 與磁場的夾角為 $\phi$, 則 $|\vec{F}_B| = |q||\vec{v}||\vec{B}|\sin\phi$ ) 電荷所受之力垂直於電荷之速度方向及磁場方向

因此形式, 不受其它外力之電荷繞磁場進行圓周運動

若一電荷為 $q$, 速度為 $\vec{v}$ 的粒子受到的磁力為 $\vec{F}_B$, 則磁場為 $\vec{B}$

SI 制單位: $1 \text{ tesla} = 1\,T = 1 \dfrac{\text{newton}}{(\text{coulomb})(m/s)}$

$$= 1 \frac{\text{newton}}{\underbrace{(\text{coulomb}/s)}_{\text{電流單位: 安培}}(m)} = 1 \cdot \frac{N}{A \cdot m}$$

$$1 \text{ gauss} = 10^{-4} \text{ tesla}$$

There is no magnetic monopole. Magnetic field lines are connecting from magnetic <u>north pole</u> to magnetic <u>south pole</u>.

## Magnetic force on a current-carrying wire (如長度為 L)

$$q = i\,t = i\frac{L}{v} \Longrightarrow F_B = qvB\sin\phi$$

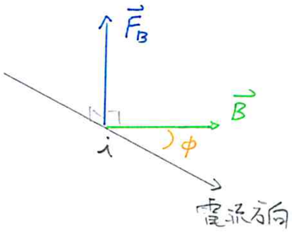
電流 ($i$)

時間 $t$ 通過的總電荷數 ($q$)

經過時間 ($t$)

$$= i\frac{L}{v}vB\sin\phi$$

$$\Rightarrow \vec{F}_B = i\vec{L} \times \vec{B}$$




對於一段無窮短之電纜: $d\vec{F}_B = i(d\vec{L}) \times \vec{B}$

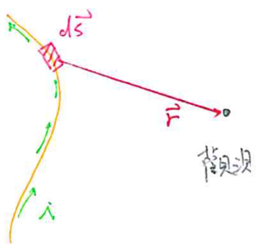
## Magnetic field due to a current: Biot & Savart law

導線 $$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i\,d\vec{s} \times \hat{r}}{r^2}$$

$d\vec{s}$: 一小段 wire 之方向向量

$\hat{r}$: 由 wire 位置指到觀測點之方向向量

$r$: 由一小段 wire 到觀測點之距離




general form for a current density $\vec{J}(\vec{x})$.

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}') \times \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\,d^3x' \qquad \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = -\nabla\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$$

$$= \frac{\mu_0}{4\pi}\vec{\nabla} \times \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x' \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{B} = 0$$

(對應於電場問題的 $\vec{\nabla} \times \vec{E} = 0$)

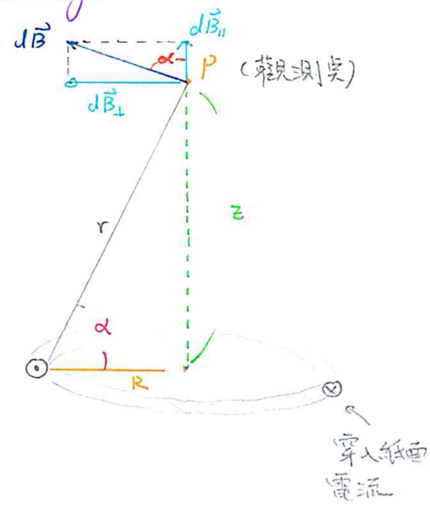
呂浩宇

# General Physics (II) Electromagnetic theory

## Example: magnetic field of a coil

根據 Biot & Savart law，在 z-軸上本就無 $\phi$ 方向磁場
由於系統之對稱性，磁場只有在 z 方向而無在 $\hat{r}$ 方向

※ 解題時的座標選擇：有柱對稱性的問題，一般使用柱座標較方便



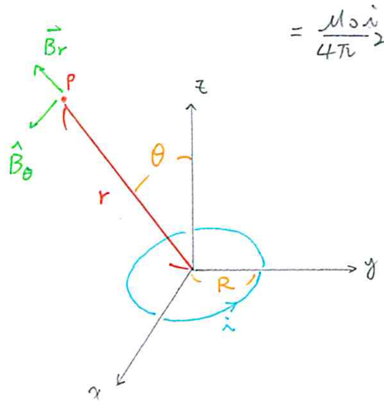

z 方向磁場 $B_\parallel = \int dB_\parallel$

$$= \int \frac{\mu_0}{4\pi}\frac{i\,ds\cos\alpha}{r^2} \qquad \begin{cases} r = (R^2+z^2)^{\frac{1}{2}} \\ \cos\alpha = \dfrac{R}{r} \end{cases}$$

$$= \frac{\mu_0 i}{4\pi}2\pi R\,\frac{R}{r^3} = \frac{\mu_0 i R^2}{2(R^2+z^2)^{\frac{3}{2}}} \xrightarrow[z \gg R]{} \frac{\mu_0 i}{2\pi z^3}(\pi R^2) = \frac{\mu_0}{2\pi}\frac{iA}{z^3}$$

記憶：不在 z-軸上測磁場時，z 代換為 r，$B_r/B_\theta$ 分子掛 $\cos\theta / \frac{1}{2}\sin\theta$

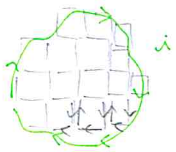
比較 page 2 electric dipole 型式，可看出此處磁場為 magnetic dipole，magnetic dipole moment 為 $iA$




r>>R，在觀測點 P 之電場一般形式

$$\begin{cases} B_r = \dfrac{\mu_0}{2\pi}\dfrac{\cos\theta}{r^3}(i\pi R^2) \\ B_\theta = \dfrac{\mu_0}{4\pi}\dfrac{\sin\theta}{r^3}\underbrace{(i\pi R^2)}_{iA} \end{cases}$$

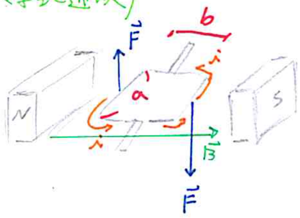
dipole field!




## Magnetic torque on a current loop (適用於馬達)

(先考慮方形線圈，則任意形狀之線圈皆可以以方形無窮好地近似)

定義線圈截面垂直磁場為傾角 $\theta = 0$ 位置)




$F = iaB$

$$\text{torque} = 2\cdot\left(F\cdot\frac{1}{2}b\sin\theta\right)$$

$$= 2\left(iaB\cdot\frac{1}{2}b\sin\theta\right)$$

$$= iabB\sin\theta = i\underbrace{A}_{\text{方形線圈之截面積}}B\sin\theta$$

$$\Rightarrow \vec{\tau} = i\vec{A}\times\vec{B}$$

$\vec{A}$：以右手定則定義之線圈截面積法向量


無論以其產生的磁場來看，或是以其在外磁場中受到的力矩看，線圈之行為皆為 magnetic dipole，其 magnetic dipole moment 為 $i\vec{A}$

※ 此處環形電流產生之磁場亦為重要之一般性質。
一般會推算電荷之角動量，再得出角動量與磁矩之關係。
原子/分子中帶有角動量之電荷產生之磁場會反應於能階中的 fine structure (spin-orbital coupling) 及 hyper-fine structure (coupling between electron and nuclear spin)

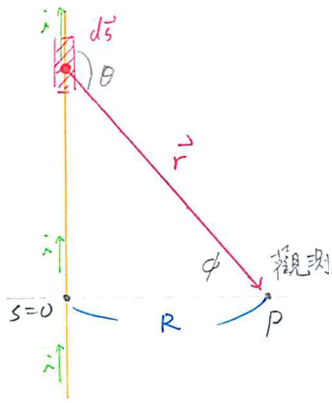
呂喆宇

# General Physics (II)  Electromagnetic theory

## Example: magnetic field due to a current in a long straight wire

由於無窮長直導線問題之對稱性，在P点，磁場方向為穿入紙面

magnetic field contributed by a small sector of wire




$$dB = \frac{\mu_0}{4\pi}\frac{i\,ds\sin\theta}{r^2}$$

$$B = 2\int_{s=0}^{s=\infty} dB = 2\int_{s=0}^{s=\infty}\frac{\mu_0 i}{4\pi}\frac{\sin\theta\, ds}{r^2}$$

$r = (s^2+R^2)^{\frac{1}{2}}$

$\sin\theta = \frac{R}{\sqrt{s^2+R^2}}$

$$= \frac{\mu_0 i}{2\pi}\int_0^{\infty}\frac{R\,ds}{(s^2+R^2)^{3/2}} = \frac{\mu_0 i}{2\pi R}$$


$r = \frac{R}{\cos\phi}$, $s = R\tan\phi$

$\Rightarrow \frac{ds}{d\phi} = R\frac{d\tan\phi}{d\phi}$

$= R\sec^2\phi = \frac{R}{\cos^2\phi}$

$\Rightarrow ds = \frac{R}{\cos^2\phi}d\phi$

$$dB = \frac{\mu_0}{4\pi}\frac{i\cos\phi}{r^2}\frac{R}{\cos^2\phi}d\phi$$

$$= \frac{\mu_0 i}{4\pi R}\cos\phi\, d\phi$$

$$\Rightarrow B = 2\int_{\phi=0}^{\phi=90^\circ}\frac{\mu_0 i}{4\pi R}\cos\phi\, d\phi$$

$$\frac{\mu_0 i}{2\pi R}(\sin 90^\circ - \sin 0^\circ) = \frac{\mu_0 i}{2\pi R}$$

## Ampère's law

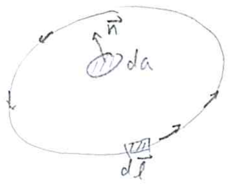
(推導利用 identity: $\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -\vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$ 及 $\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\delta(\vec{x}-\vec{x}')$)

brute force

differential form: $\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\vec{\nabla}\times\vec{\nabla}\times\int\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x' = \mu_0\vec{J}$ $\Rightarrow \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

Integrate form (for an open surface $S$ bounded by a closed contour $C$)

Stokes' theorem: $\int_S(\vec{\nabla}\times\vec{A})\cdot\vec{n}\,da = \oint_C\vec{A}\cdot d\vec{l}$




$$\int_S(\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \int_S(\vec{\nabla}\times\vec{B})\cdot\vec{n}\,da$$

$$\oint_C\vec{B}\cdot d\vec{l} = \mu_0\int_S\vec{J}\cdot\vec{n}\,da \Rightarrow \oint_C\vec{B}\cdot d\vec{l} = \mu_0 i$$

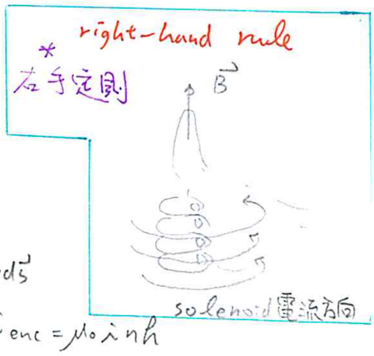
total current passing through the closed contour C

## Example: magnetic field due to a current in a long straight wire

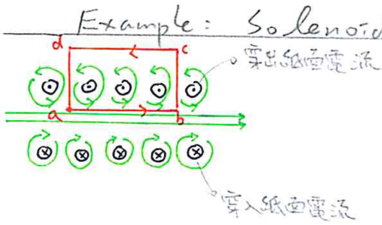
Ampère's law

$$\oint\vec{B}\cdot ds = B\oint ds = B(2\pi R) = \mu_0 i$$

$$\Rightarrow B = \frac{\mu_0 i}{2\pi R}$$




## Example: Solenoids (線圈，應用於產生實驗室中均勻磁場)




$$\oint\vec{B}\cdot d\vec{s} = \mu_0 i_{enc} = \int_a^b\vec{B}\cdot d\vec{s} + \int_b^c\vec{B}\cdot d\vec{s} + \int_c^d\vec{B}\cdot d\vec{s} + \int_d^a\vec{B}\cdot d\vec{s}$$

令a到b的長度為h，單位長度之線圈數為n $\Rightarrow \mu_0 i_{enc} = \mu_0 i n h$

理想線圈只有$\int_a^b\vec{B}\cdot d\vec{s}$部分有積分貢獻

$\Rightarrow Bh = \mu_0 i n h \Rightarrow B = \mu_0 i n$ 磁場強度(約)等於$\mu_0$乘以電流及單位長度之線圈數