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# General Physics (II) Maxwell Equations

## A summary of the basic equations introduced in the previous pages

庫倫定律 → Gauss law for electric field : $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$ (page 3) 電場與 charge density distribution 之關係

Gauss law for magnetic field : $\vec{\nabla}\cdot\vec{B} = 0$ (page 7) 沒有磁單極 — 注意此為經驗公式，而非無條件地限制 magnetic monopole 不許存在．反之，若 monopole 存在，則量子力學允許證明電荷之量化：Dirac quantization condition

故尋找 monopole 至今仍為重要的實驗物理課題

Faraday's law : $\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$ (page 10) 磁生電

Biot & Savart 定律 — Ampère's law : $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ (page 9) 電流與磁場之關係

積分形式

$$
\begin{cases}
\oint \vec{E}\cdot d\vec{A} = \frac{1}{\epsilon_0} q_{enc} \\
\oint \vec{B}\cdot d\vec{A} = 0 \\
\oint \vec{E}\cdot d\vec{l} + \frac{\partial \Phi_B}{\partial t} = 0 \\
\oint \vec{B}\cdot d\vec{l} - \mu_0 i = 0
\end{cases}
$$

Divergence theorem

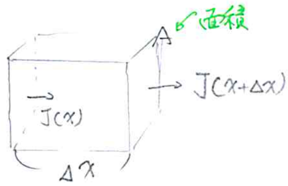
$$\left(\int \vec{\nabla}\cdot\vec{E}\, d^3x = \int \vec{E}\cdot d\vec{A}\right)$$

Stokes' theorem

$$\left(\oint \vec{E}\cdot d\vec{l} = \int (\vec{\nabla}\times\vec{E})\cdot d\vec{A}\right)$$

## J. C. Maxwell (1865) : the Ampère's law is incomplete from the point of view of charge conservation

電荷守恆 : $\boxed{\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0}$




由差分形式理解 : (一維情形)

單位時間，在小 box 之中，總電荷之增加量

$$\underbrace{J(x)A}_{\text{單位時間流入之電荷}} - \underbrace{J(x+\Delta x)A}_{\text{單位時間流出之電荷}} = \Delta Q/\Delta t$$

$\Rightarrow$ 電荷密度增加量 $\frac{\Delta Q/\Delta t}{\text{volume}} = \frac{J(x)A - J(x+\Delta x)A}{A\,\Delta x} = \frac{J(x) - J(x+\Delta x)}{\Delta x} \longrightarrow -\frac{\partial J}{\partial x}$

$$\frac{\Delta Q/\Delta t}{\text{volume}} = \frac{\Delta(Q/\text{volume})}{\Delta t} \longrightarrow \frac{\partial\rho}{\partial t}$$

$$\Rightarrow \frac{\partial\rho}{\partial t} + \frac{\partial J}{\partial x} = 0$$

若完整考量三維情形 : $\frac{\partial\rho}{\partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}$

$$= \frac{\partial\rho}{\partial t} + \overbrace{\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)}^{\vec{\nabla}}\cdot\underbrace{(J_x\hat{x} + J_y\hat{y} + J_z\hat{z})}_{\vec{J}}$$

$$= \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

**Ampère's law 之問題** : $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0$ 對於任意 vector field $\vec{B}$ 皆成立

(Ampère's law) $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ $\Rightarrow \mu_0\vec{\nabla}\cdot\vec{J} = 0$ $\Rightarrow$ $\frac{\partial\rho}{\partial t} = 0$，直接不允許空間中電荷密度改變，必有錯誤 which is too strict when comparing with the general law of charge conservation

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# General Physics (II) Maxwell equations

the correction proposed by J. C. Maxwell

$$\vec\nabla\times\vec B = \mu_0\vec J + \mu_0\left(\overbrace{\epsilon_0\frac{\partial\vec E}{\partial t}}\right)$$

此項與電流密度具有同樣的因次
稱為 displacement current density
此項為完成電磁波波動方程之關鍵

在電荷密度
不隨時間改變的平衡態,
此式之效果與原本的
Ampère's law 完全相同

$$\Rightarrow \vec\nabla\cdot(\vec\nabla\times\vec B) = 0 = \mu_0\left(\vec\nabla\cdot\vec J + \epsilon_0\frac{\partial}{\partial t}(\nabla\cdot\vec E)\right)$$

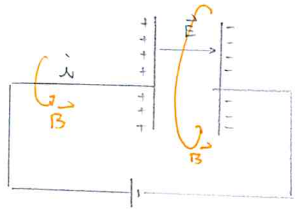
Gauss's law

$$= \mu_0\left(\vec\nabla\cdot\vec J + \epsilon_0\frac{\partial}{\partial t}\frac{\rho}{\epsilon_0}\right)$$

$$= \mu_0\left(\vec\nabla\cdot\vec J + \frac{\partial\rho}{\partial t}\right) = 0$$

displacement current: $i_d \equiv \epsilon_0\frac{d\Phi_E}{dt} = \epsilon_0 \underbrace{\frac{\partial}{\partial t}\int \vec E\cdot d\vec A}$

電場通量隨時間的改變率




displacement current 項之效果:
若電池使平行板電容隨時間充電, 則電流產生磁場,
平行板間亦因電通量改變而產生磁場.

## ※ Maxwell equations (真空情形)

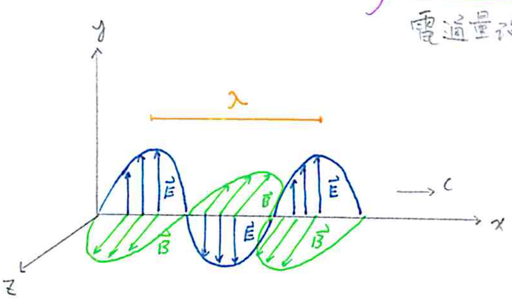
$$\vec\nabla\cdot\vec E = \frac{\rho}{\epsilon_0} \qquad \vec\nabla\times\vec B = \mu_0\vec J + \mu_0\left(\epsilon_0\frac{\partial\vec E}{\partial t}\right)$$

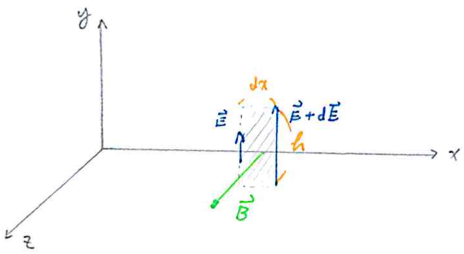
$$\vec\nabla\cdot\vec B = 0 \qquad \vec\nabla\times\vec E + \frac{\partial\vec B}{\partial t} = 0$$

Halliday ed. 11 ch 32 後半磁性之由來部分對於了解原子能階重要,
建議閱讀

## Electromagnetic Waves.

電通量改變使磁場生成, 磁通量改變使電場生成, 而成為可傳遞之電磁波.
其傳遞之方向為 $\vec E\times\vec B$, 波速為 c




Faraday's law: $\oint \vec E\cdot d\vec s = -\frac{d\Phi_B}{dt}$

$$\Rightarrow (E+dE)h - Eh = h\,dE = -\frac{d}{dt}(B\,h\,dx)$$

$$= -h\,dx\frac{dB}{dt}$$

$$\Rightarrow \boxed{\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}}$$

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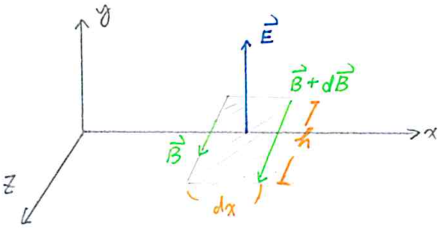
# General Physics (II) Maxwell equations




Maxwell's law → the Ampère's law modified by Maxwell (assuming no current)

$$\oint \vec{B}\cdot d\vec{s} = \mu_0\epsilon_0 \frac{d\Phi_E}{dt}$$

$$\Rightarrow -(B+dB)h + Bh = -h\,dB = \mu_0\epsilon_0 h\,dx\frac{dE}{dt}$$

$$\Downarrow$$

$$\boxed{-\frac{\partial B}{\partial x} = \mu_0\epsilon_0\frac{\partial E}{\partial t}}$$ 與 page 19 底下式子比較

回顧狹義相對論.

電磁波在真空中即可傳遞. 不需要以太作為介質, 並且波速總是為 c

↓

迫使物理學對 Galileo transformation 做出修正, 而成為 Lorentz transformation.

$$\Rightarrow \frac{\partial^2 E}{\partial x^2} = -\frac{\partial}{\partial x}\frac{\partial}{\partial t}B = -\frac{\partial}{\partial t}\frac{\partial}{\partial x}B = \frac{\partial}{\partial t}\left(-\frac{\partial}{\partial x}B\right) = \frac{\partial}{\partial t}\left(\mu_0\epsilon_0\frac{\partial E}{\partial t}\right)$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\right)E = 0$$ 通解為 sinusoidal wave

$E = E_m\sin(kx-\omega t)$, 波速為 $\omega/k$.

$$\Rightarrow k^2 - \mu_0\epsilon_0\omega^2 = 0 \Rightarrow \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0\epsilon_0}} = c$$

$$B = B_m\sin(kx-\omega t)$$

$$\frac{E_m}{B_m} = \frac{1}{\mu_0\epsilon_0 c}$$

## Energy transport and the Poynting vector

John Henry Poynting

Poynting vector: $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$

SI 制單位: $\frac{J}{m^2 s} = \frac{W}{m^2}$

energy transport by a EM wave across a unit area in a unit time.

單位時間單位面積通過的能量

↓

單位時間因吸收或反射光而造成的動量改變為力之因次. 除以面積則成為壓力之因次 (光壓)

可利用光子動量與能量之關係 $E = pc$ 轉換

↑

回憶狹義相對論 $E^2 = p^2c^2 + m^2c^4$, 光子質量 $m = 0$