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# General Physics (II) Electromagnetic theory

definition of current: $i = \frac{dq}{dt}$ (單位時間通過的電荷數)

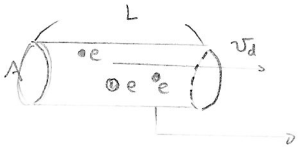
SI 制單位: 1 ampere = 1 A = 1 coulomb per second.

$\Rightarrow q = \int dq = \int_0^t i\,dt$ (累積通過的電荷數)

defining current density: $i = \int \vec{J} \cdot d\vec{A}$ → 單位投影面積之電流

投影於與 current 垂直之方向

relation between $\vec{J}$ and the number density of charge carrier $n$, the charge per carrier $e$, and the drift velocity of the charge carrier $v_d$



→ total charge in this volume: $q = (AL) \cdot n \cdot e$

time for all charge to leave this volume: $t = \frac{L}{v_d}$

$$\Rightarrow i = \frac{q}{t} = \frac{ALne}{L/v_d} = nAev_d$$

$$\Rightarrow \frac{i}{A} = J = \underbrace{(ne)v_d}_{} \Longrightarrow v_d = \frac{J}{ne}$$

電流密度等於電荷密度乘以 drift velocity

microscopic view of the drift velocity.

conduction electrons that are free to move have high speed ($10^6$ m/s) due to Pauli exclusion principle. They collide with the atoms frequently, with average time $\tau$. They are also accelerated by the external field $E$.

加速度

$$\text{drift velocity } v_d = a\tau = \left(\frac{eE}{m}\right)\tau$$

mass of the charge carrier.

$$\Rightarrow \frac{J}{ne} = \frac{eE\tau}{m}$$

$$\Rightarrow J = \frac{e^2 n\tau}{m} E$$

resistivity $\rho$

conductivity $\sigma \equiv \frac{1}{\rho} = \frac{e^2 n \tau}{m}$

$= \sigma E$ 電流密度等於電場乘以導電率

$J = \frac{i}{A}$, $E = \frac{V}{L}$ 導体兩端的電位差；導体長度

resistance

$$\Rightarrow \rho = \frac{E}{J} = \frac{V/L}{i/A} \Rightarrow \underbrace{\frac{V}{i} \equiv R}_{} = \rho\frac{L}{A}$$

$V = iR$ omic law

若外加電場(電位差)不影響 charge carrior 的密度 n 及平均碰撞間隔時間 $\tau$，則電阻、電壓、及電流間關係滿足歐姆定律

大部分導体皆在某特定小電壓範圍内滿足歐姆定律

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## Magnetic Field.

produced by $\begin{cases} (1) \text{ moving charge} \\ (2) \text{ spin magnetic moment} \end{cases}$ (在光譜中與 Zeeman splitting 有關)

Magnetic field is defined by the force law

因此形式，不受其它外力之電荷繞磁場進行圓周運動

$$\vec{F_B} = q\vec{v} \times \vec{B}$$

(若 $\vec{v}$ 與磁場的夾角為 $\phi$，則 $|\vec{F_B}| = |q||\vec{v}||\vec{B}|\sin\phi$) 電荷所受之力垂直於電荷之速度方向及磁場方向

若一電荷為 $q$，速度為 $\vec{v}$ 的粒子受到的磁力為 $\vec{F_B}$，則磁場為 $\vec{B}$

SI 制單位：$1 \text{ tesla} = 1\,T = 1 \dfrac{\text{newton}}{(\text{coulomb})(m/s)}$

$$= 1 \frac{\text{newton}}{(\text{coulomb}/s)(m)} = 1 \cdot \frac{N}{A \cdot m}$$

電流單位：安培

$$1 \text{ gauss} = 10^{-4} \text{ tesla}$$

There is no magnetic monopole. Magnetic field lines are connecting from magnetic north pole to magnetic south pole.

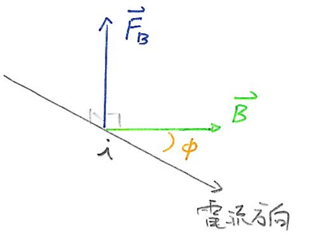
## Magnetic force on a current-carrying wire (如長度為 L)

$$q = i\,t = i\frac{L}{v} \implies F_B = qvB\sin\phi$$

時間 t 通過的總電荷數 ; 電流 ; 經過時間

$$= i\frac{L}{v} vB\sin\phi$$

$$\Rightarrow \vec{F_B} = i\vec{L} \times \vec{B}$$




對於一段無窮短之電纜：$d\vec{F_B} = i(d\vec{L}) \times \vec{B}$

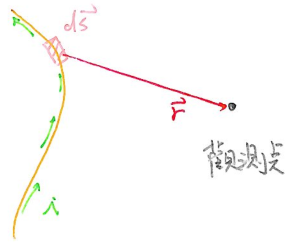
## Magnetic field due to a current : Biot & Savart law

導線 $$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i\,d\vec{s} \times \hat{r}}{r^2}$$

$d\vec{s}$：一小段 wire 之方向向量

$\hat{r}$：由 wire 位置指到觀測點之方向向量

$r$：由一小段 wire 到觀測點之距離




general form for a current density $\vec{J}(\vec{x})$.

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}') \times \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} d^3x' \qquad \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = -\nabla\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$$

$$= \frac{\mu_0}{4\pi}\nabla \times \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' \Rightarrow \nabla \cdot \vec{B} = 0$$

(對應於电場問題的 $\nabla \times \vec{E} = 0$)

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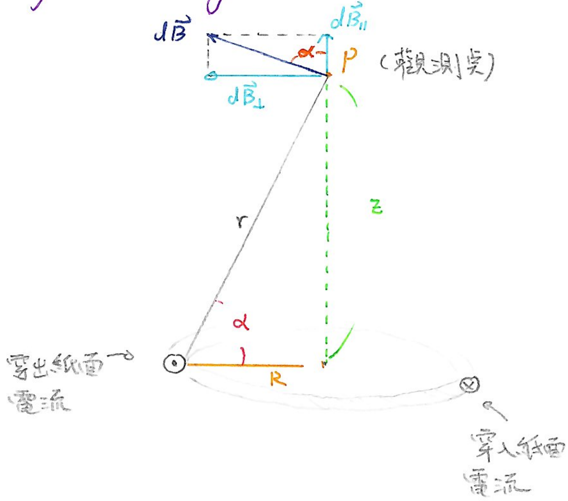
# General Physics (II) Electromagnetic theory

## Example: magnetic field of a coil

根據 Biot & Savart law，在 z-軸上本就無 $\phi$ 方向磁場
由於系統之對稱性，磁場 B 只有在 $\hat{z}$ 方向而無在 $\hat{r}$ 方向

※ 解題時的座標選擇：有柱對稱性的問題，
一般使用柱座標較方便



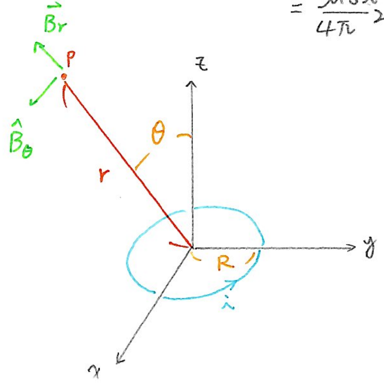

z 方向磁場 $B_{\parallel} = \int dB_{\parallel}$

$$= \int \frac{\mu_0}{4\pi}\frac{i\,ds\cos\alpha}{r^2} \qquad \begin{cases} r = (R^2+z^2)^{\frac{1}{2}} \\ \cos\alpha = \frac{R}{r} \end{cases}$$

$$= \frac{\mu_0 i}{4\pi}2\pi R\,\frac{R}{r^3} = \frac{\mu_0 i R^2}{2(R^2+z^2)^{\frac{3}{2}}} \xrightarrow[z\gg R]{} \frac{\mu_0 i}{2\pi z^3}(\pi R^2) = \underline{\frac{\mu_0}{2\pi}\frac{iA}{z^3}}$$

記憶：不在 z-軸上測磁場時，z 代換為 r，$B_r/B_\theta$ 分子掛 $\cos\theta / \frac{1}{2}\sin\theta$

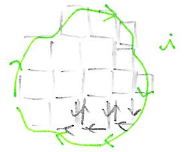
比較 page 2 electric dipole 型式，可看出此處磁場為 magnetic dipole，magnetic dipole moment 為 $iA$




r >> R，在觀測點 p 之電場一般形式

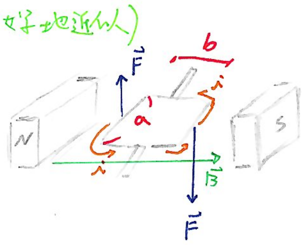
$$\begin{cases} B_r = \frac{\mu_0}{2\pi}\frac{\cos\theta}{r^3}(i\pi R^2) \\ B_\theta = \frac{\mu_0}{4\pi}\frac{\sin\theta}{r^3}\underbrace{(i\pi R^2)}_{iA} \end{cases}$$

dipole field!




## Magnetic torque on a current loop (適用於馬達)

(先考慮方形線圈，則任意形狀之線圈皆可以以方形無窮好地近似)



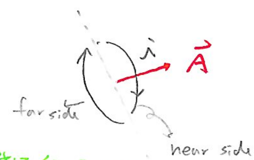

定義線圈截面垂直磁場為傾角 $\theta = 0$ 位置)

$F = iaB$

$$\text{torque} = 2\cdot\left(F\cdot\frac{1}{2}b\sin\theta\right)$$

$$= 2\left(iaB\cdot\frac{1}{2}b\sin\theta\right)$$

$$= iabB\sin\theta = i\overset{\text{方形線圈之截面積}}{A}B\sin\theta$$




$$\Rightarrow \vec{\tau} = i\vec{A}\times\vec{B}$$

$\vec{A}$：以右手定則定義之線圈截面積法向量

無論以其產生的磁場來看，或是以其在外磁場中受到的力矩，線圈之行為皆為 magnetic dipole，其 magnetic dipole moment 為 $i\vec{A}$

※ 此處環形電流產生之磁場亦為重要之一般性質，
一般會推算電荷之角動量，再得出角動量與磁矩之關係。
原子/分子中帶有角動量之電荷產生之磁場會反應於能階中的 fine structure (spin-orbital coupling)
及 hyper-fine structure (coupling between electron and nuclear spin)

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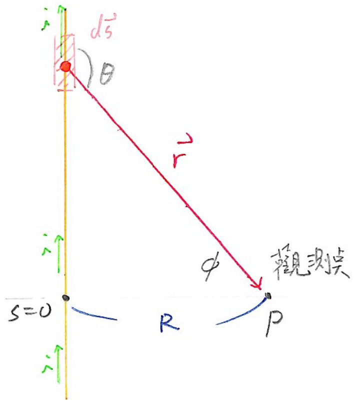
# General Physics (II) Electromagnetic theory

## Example: magnetic field due to a current in a long straight wire

由於無窮長直導線問題之對稱性，在P點，磁場方向為穿入紙面

magnetic field contributed by a small sector of wire




$$dB = \frac{\mu_0}{4\pi}\frac{i\,ds\sin\theta}{r^2}$$

$$B = 2\int_{s=0}^{s=\infty} dB = 2\int_{s=0}^{s=\infty}\frac{\mu_0 i}{4\pi}\frac{\sin\theta\,ds}{r^2}$$

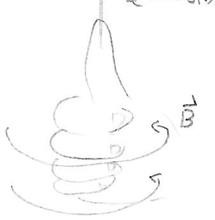
$r = (s^2+R^2)^{\frac{1}{2}}$

$\sin\theta = \frac{R}{\sqrt{s^2+R^2}}$

$$= \frac{\mu_0 i}{2\pi}\int_0^\infty \frac{R\,ds}{(s^2+R^2)^{3/2}} = \frac{\mu_0 i}{2\pi R}$$

* right-hand rule

右手定則




$r = \frac{R}{\cos\phi}$, $s = R\tan\phi$

$\Rightarrow \frac{ds}{d\phi} = R\frac{d\tan\phi}{d\phi}$

$= R\sec^2\phi = \frac{R}{\cos^2\phi}$

$\Rightarrow ds = \frac{R}{\cos^2\phi}d\phi$

$$dB = \frac{\mu_0}{4\pi}\frac{i\cos\phi}{r^2}\frac{R}{\cos^2\phi}d\phi$$

$$= \frac{\mu_0 i}{4\pi R}\cos\phi\,d\phi$$

$$\Rightarrow B = 2\int_{\phi=0}^{\phi=90^\circ}\frac{\mu_0 i}{4\pi R}\cos\phi\,d\phi$$

$$\frac{\mu_0 i}{2\pi R}(\sin 90^\circ - \sin 0^\circ) = \frac{\mu_0 i}{2\pi R}$$

## Ampère's law

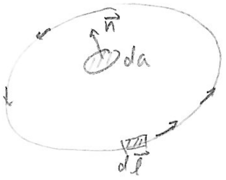
(推導利用 identity: $\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -\vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$ 及 $\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\delta(\vec{x}-\vec{x}')$)

brute force

differential form: $\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\vec{\nabla}\times\vec{\nabla}\times\int\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x' = \mu_0\vec{J}$ $\Rightarrow \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

Integrate form (for an open surface S bounded by a closed contour C)

Stokes' theorem: $\int_S (\vec{\nabla}\times\vec{A})\cdot\vec{n}\,da = \oint_C \vec{A}\cdot d\vec{l}$




$$\underbrace{\int_S (\vec{\nabla}\times\vec{B})\cdot d\vec{a}}_{\oint_C \vec{B}\cdot d\vec{l}} = \int_S (\vec{\nabla}\times\vec{B})\cdot\vec{n}\,da$$

$$= \mu_0\int_S \vec{J}\cdot\vec{n}\,da \Rightarrow \oint_C \vec{B}\cdot d\vec{l} = \mu_0 i$$

total current passing through the closed contour C

## Example: magnetic field due to a current in a long straight wire

Ampère's law

$$\oint \vec{B}\cdot d\vec{s} = B\oint ds = B(2\pi R) = \mu_0 i$$

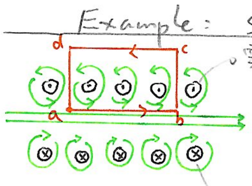
$$\Rightarrow B = \frac{\mu_0 i}{2\pi R}$$

* right-hand rule

右手定則


solenoid電流方向

## Example: Solenoids (線圈，應用於產生實驗室中均勻電場)




$\oint\vec{B}\cdot d\vec{s} = \mu_0 i_{enc} = \int_a^b\vec{B}\cdot d\vec{s} + \int_b^c\vec{B}\cdot d\vec{s} + \int_c^d\vec{B}\cdot d\vec{s} + \int_d^a\vec{B}\cdot d\vec{s}$

令 a 到 b 的長度為 h，單位長度之線圈數為 n $\Rightarrow \mu_0 i_{enc} = \mu_0 i n h$

理想線圈只有 $\int_a^b\vec{B}\cdot d\vec{s}$ 部分有積分貢獻

$\Rightarrow Bh = \mu_0 i n h \Rightarrow B = \mu_0 i n$ 磁場強度(約)等於 $\mu_0$ 乘以電流及單位長度之線圈數

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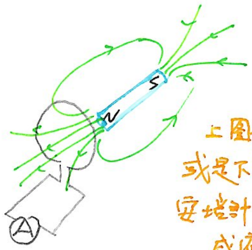
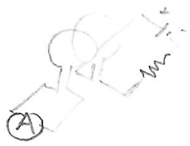
# General Physics (II) Electromagnetic theory

## Faraday's law of induction & Lenz's law

↳原始觀察：線圈中磁通量改變時，觀察到線圈中有感應電流，可詮釋為受感應電場驅動而產生感應電流 ⟶ 磁場隨時間改變促生了感應電場 (fundamental)




上圖磁鐵移動，或是下圖電路接通時，安培計 Ⓐ 測到感應電流




magnetic flux（磁通量）: $\Phi_B = \int \vec{B} \cdot d\vec{A}$（$d\vec{A}$：面積）

Faraday's law : $\underbrace{\mathcal{E}}_{\text{electromotive force (emf)}} = -\dfrac{d\Phi_B}{dt}$ 感應電動勢正比於負的電通量改變率

electromotive force (emf) $= \oint \vec{E} \cdot d\vec{l}$ 對單位電荷的作功

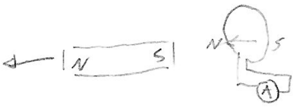
↳ 當有 N 匝線圈時可寫為 $\mathcal{E} = -N\dfrac{d\Phi_B}{dt}$

Lenz's law : an induced current has a direction such that the magnetic field due to the current opposes the change in the magnetic flux that induces the current.

(感應電流的效果為補償線圈中增/減的磁通量)

另一種觀點：由於 Faraday's law，若要改變線圈的磁通量，則外力必須作功

↓ e.g.




若將左圖磁鐵拉離線圈，則感應電流造成如圖示之 magnetic dipole 吸引磁鐵，為對抗此磁吸引力，外力必須作功

能量守恆：
- 外力功率：$P = F v$（F：力；v：磁棒移動速率）
- 線圈熱功耗：$P = i^2 R$（i：电流；R：電阻）

⊳ 兩者必須相等

## Energy density of a Magnetic field (Halliday & Resnick Ed. 11, §30-8)

magnetic energy density : $u_B = \dfrac{B^2}{2\mu_0}$ 形式與 page 5 中介紹之空間中電場能量密度形式相似 ⟵ 電與磁場在理論中具有對稱性

（$u_B$：單位空間中因存在磁場，而具有之能量）