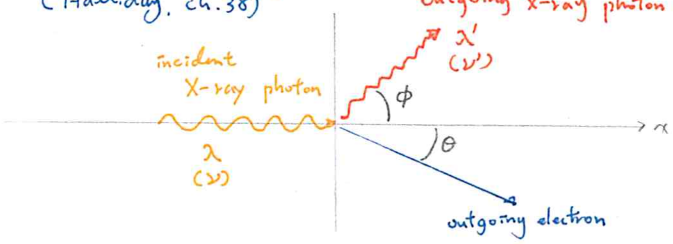
呂治宇

General Physics (II) Quantum Physics

Compton scattering (Arthur Compton 1922)
(Halliday, ch.38)



He also verified Maxwell's prediction that light is EM wave.

discovered by Heinrich Hertz (1887)

$\boxed{\text{photoelectric effect experimentally implies that the energy of a photon is } h\nu}$

Planck constant

Assumption made by Max Planck in 1890s in the derivation of the formula to describe black body radiation.

energy conservation: (rest energy of the electron has been subtracted from the l.h.s. & r.h.s. of the equation)

$$\underbrace{h\nu}_{\text{energy of incoming photon}} = \overbrace{h\nu'}^{\text{energy of outgoing photon}} + \underbrace{mc^2(\gamma - 1)}_{\text{energy of outgoing electron}}, \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

v: speed of outgoing electron.

momentum conservation:

$$\text{horizontal: } \frac{h\nu}{c} = \frac{h\nu'}{c}\cos\phi + \gamma m v \cos\theta$$

$$\text{vertical: } 0 = \frac{h\nu'}{c}\sin\phi - \gamma m v \sin\theta$$

There are 3 equations for us to elliminate 3 of the 5 independent variables $(\nu, \nu', v, \phi, \theta)$

Defining $\Delta\lambda \equiv \lambda' - \lambda$, with a little (unimportant) algebra, we got:

$$\underbrace{\overset{\text{compton shift}}{\Delta\lambda} = \frac{h}{mc}(1 - \cos\phi)}$$

If we place detectors at various outgoing angle ϕ, this theory predict that we should detect the scattered light at different wavelength.
Note that this theory is only true when the hypothesis of photons is true, which is distinct from the prediction of the classical EM wave propagation

photons behave like particles. This is a problem!
Can particles behave like wave ?

de Broglie (1923) : $\underset{\text{(wavelength)}}{\lambda} = \dfrac{h}{|\vec{p}|}$ ⟶ verified by

- L.H. Germer & C. J. Davisson (1927)
- G.P. Thomson

利用 X-ray Bragg diffraction 測量 crystal 晶面間距，再用電子來驗證電子的 diffraction pattern

粒子：侷限於小的空間中，具有一定的動量與能量，
其位置與速度隨時間之變化由 equation of motion 決定.

波：瀰漫於空間中之物理量，由波函數 $\psi(\vec{r}, t)$ 描述.
波函數隨時間的演化由 wave equation 決定.

General Physics (II) Quantum Physics

Experiments that show photons behave like particles.

1. Compton scattering (Arthur Compton 1922)

2. Blackbody radiation (the earliest development about QM, since 1859)
   even before we know the light is EM wave.

   (i) Before 1900, Gustav Kirchhoff

   Argued based on thermal physics, that the intensity of blackbody radiation only depends on the temperature T and frequency ν.

   (ii) mid-1880s, { Ludwig Boltzmann, Josef Stefan

   Argued that the energy density of blackbody emission u [Joul cm^{-3}] is proportional to intensity. In addition, its value is

   $$\underbrace{u = \sigma T^4}_{\text{Stefan-Boltzmann constant } (\sigma)}$$ — Stefan-Boltzmann law.

   (iii) Wilhelm Wien

   Attempted to derive the blackbody intensity based on statistical physics, assuming that the photons are similar to classical gas particles.

   He obtained the Wien's law which is consistent with observations at short wavelength

   $$I_{wien}(\lambda, T) = b\,\lambda^{-5} \exp(-a/\lambda T)$$

   b, a: constants to be determined experimentally.

   (iv) Early 1900, Max Planck.

   Assumed the walls in a blackbody chamber can only emit and absorb EM wave of frequency ν by energy in unit of $h\nu$

   such energy units were named "quanta" by Planck

   Based on this assumption, Planck obtained the correct form of blackbody intensity by applying methods of statistical physics.

   $$I(\lambda, T) = \frac{b}{\lambda^5} \frac{1}{\exp(a/\lambda T) - 1}$$

   詮釋量子效應為h為有之結果，若$h \to 0$，即光子能量單位可為無窮小，則blackbody radiation formula趨近於Rayleigh-Jeans推導出古典型式，在短波長處積分発散

   (v) Einstein (1905).

   At a certain frequency, the energy of EM wave can only be integer times $h\nu$. This predicts the photoelectric effect (光電效應) that can be tested by further experiments.

3. Photoelectric effect (Heinrich Hertz, then Robert Millikan in 1904-1913)
   電子僅能吸收單一光子

   classical expectation {
   1. Accumulating the incoming radiative energy with time can eventually excite photoelectron.
   2. There can be a time lag between illumination and the ejection of photoelectron.

   actual case: {
   1. Capability of exciting photoelectron only relates to frequency, not intensity. Kinetic energy of photoelectron depends on frequency and the surface work-function.
   2. Ejection of photoelectron is instantaneous.

古二百寸

General Physics (II) Quantum Physics.

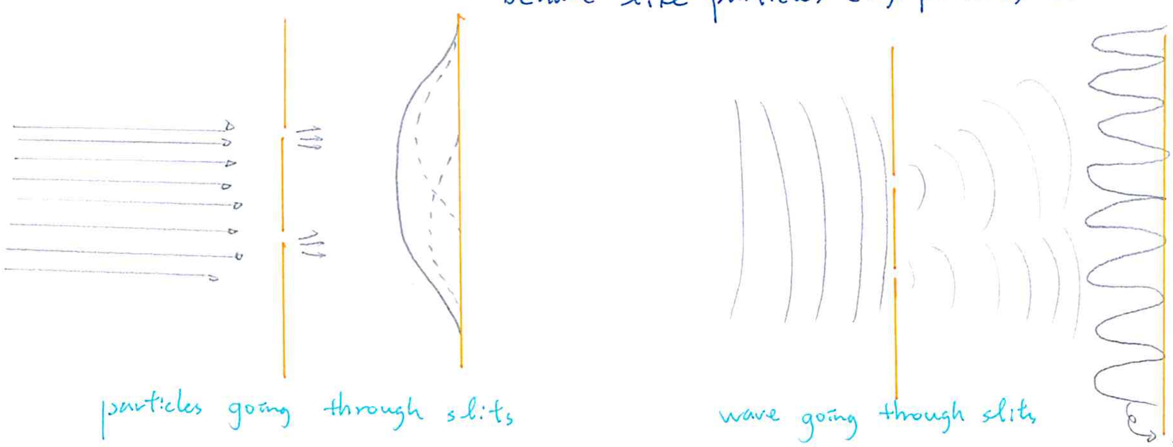
(此處講一下光的雙狹縫實驗, Halliday ch. 35、36)

Problem: if photon behave like particles, then how we understand the inteference of light in the double slit experiments?

(i) the interference pattern emerge even when there is only one photon at a time

(ii) when we try to change the aparatus such that we can know which slit the photon is going through, the interference pattern disappear.

The way we measure somehow "decide" whether light behave like particles (e.g. photons) or waves.



particles going through slits

wave going through slits

interference pattern

Question: can particle behaves like wave?

Louis de Broglie: matter has wave-like property.

The wavelength is

$$\lambda = \frac{h}{|\vec{p}|} \quad \text{de Broglie wavelength.}$$

(i) C.J. Davisson & L.H. Germer, G.P. Thomson — observing the interference pattern of electrons scattered by planes in crystals.

(ii) slit experiments (see textbook) Tonomura et al. (1989)

Max Born's hypothesis: each photon is associated a wave ψ, called probability amplitude (机率幅)

電場正比於電子出現之機率幅.
机率正比於机率幅平方

In the EM wave, the $|E|^2$ determines the probability for a photon to appear.

Can electrons be described by another sort of probability wave?

General Physics (II) — Quantum Physics

Double-slit interference of EM wave & photon(s)

電場之其中一個空間分量，解為正/餘弦函數

1. 回顧課程開始時所說，電磁波中之電場滿足波方程 $\left(\sum_i \frac{\partial^2}{\partial x_i^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right)\psi = 0$
2. 對於任意一小面積（如一個 CCD pixel），單位時間內入射之能量正比於電場平方

期中考後會詳細介紹。
目前先看電場為向量而能量為純量，電場平方亦為純量。此外亦可藉由因次分析討論

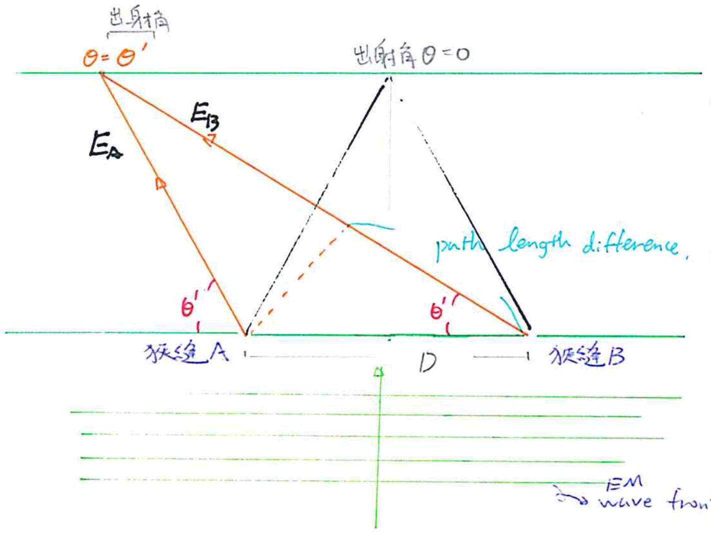
Energy of electric field

$$W = \frac{1}{2}\int \underbrace{\rho(\vec{x})}_{\text{charge density}}\ \overbrace{\Phi(\vec{x})}^{\text{electric potential}}\, d^3x = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\iint \rho(\vec{x})\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x\, d^3x'$$

$$\overset{\text{(Gauss's law)}}{=} \frac{1}{2}\int (\epsilon_0 \vec{\nabla}\cdot\vec{E})\,\Phi(\vec{x})\, d^3x$$

$$\overset{\text{integration by part}}{=} \frac{\epsilon_0}{2}\int \vec{E}\cdot \overbrace{(-\vec{\nabla}\Phi(\vec{x}))}^{\text{電位的 gradient，即為電場：} \vec{E} = -\vec{\nabla}\Phi} = \frac{\epsilon_0}{2}\int |E|^2 d^3x$$

energy flux density（能量在單位時間之通量）正比於空間中之電場能量乘以光速



Screen (CCD)

假設屏幕位於無窮遠處，經過一段時間的曝光之後，CCD相當於測量單位時間單位面積通過之光子數之期望值

path length difference, $\sim D\sin\theta' \equiv \Delta x$

when the incident light is bright (when there are lots of incoming photons) for any angle of emergence θ', the photon number flux density is proportional to energy flux density

because $E = h\nu$

$$\propto \overbrace{\langle (E_A + E_B)^2 \rangle}^{\text{長時間平均}} = \left\langle \left(\cos(k\overbrace{x}^{\text{行經之光程}} - \omega t + \phi_0) + \cos(k(x + \overbrace{\Delta x}^{\text{光程差}}) - \omega t + \phi_0) \right)^2 \right\rangle$$

$$= \underbrace{\langle \cos^2(kx - \omega t + \phi_0)\rangle + \langle \cos^2(k(x+\Delta x) - \omega t + \phi_0)\rangle}_{\text{auto-correlation, each contribute } \frac{1}{2}} + 2\overbrace{\langle \cos(kx - \omega t + \phi_0)\cos(k(x+\Delta x) - \omega t + \phi_0)\rangle}^{\text{cross-correlation}}$$

利用 trigonometric identity
$\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$

$$\langle \cos(k\Delta x)\rangle + \underbrace{\langle \cos(2kx - 2\omega t + 2\phi_0 + k\Delta x)\rangle}_{\text{zero}}$$

$$\langle \cos(k\Delta x)\rangle = \langle \cos(kD\sin\theta')\rangle$$

the interference patterns have $\begin{cases}\text{minimum} \\ \text{maximum}\end{cases}$ when $\cos(k\Delta x) = \begin{cases} -1 \\ +1 \end{cases}$

$\Rightarrow$ thus the minimum appears at $\frac{2\pi}{\lambda} D\sin\theta' = m\pi$, $m = 1, 3, 5, \cdots$

for small θ', $\theta' \sim \frac{1}{2} m \frac{\lambda}{D}$

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General Physics (II) Quantum Physics

※ What happens if we turn down the intensity of light to make it so faint, such that there can be only one photon passing the double-slit at a time over a long period of time?

The experimental result is that there will still be the interference pattern!

Max Born's hypothesis

無法將一個光子分割成具有更小能量之粒子

1. A photon has energy $h\nu$, and it is indivisible.
2. When the incident light is very faint, the incoming photons are casually disconnet with each other. Before a photon hit the screen, it cannot know what have happened to the other photons. Therefore, it cannot (and does not need to) interfere with the other photon.
3. A single photon may be described by the one-photon electric field $\vec{e}(\vec{r},t)$

$$\vec{e}(\vec{r},t) = \overbrace{\vec{e}_A(\vec{r},t)}^{\text{field coming through slit A}} + \underbrace{\vec{e}_B(\vec{r},t)}_{\text{field coming through slit B}}$$

4. The probability for this photon to be found at location $\vec{r}$ and time t is proportional to $[\vec{e}(\vec{r},t)]^2 d^3\vec{r}$. Therefor, the interference pattern can still be developed. $\vec{e}$ must obey a linear equation such that the interference pattern appears like how we can expect classically.
5. When many photons are involved, the individual one-photon field somehow combine to create the classical electric field $\vec{E}$

※ Backing to the problem of electron.

Schrödinger's (1926) approach

1. Making an analogy to the EM wave problem.
2. Electron has locality, and has to bounce like billiard ball (classica
   This does not sound like waves. But waves in fact also have some aspects that can be analogous to these aspects., e.g.,
   the light pulse has locality, and a light ray can reflect in a way that is similar to billiard ball.

It is not impossible to describe electron with wave function (波函數)

physical Hilbert space : The space of functions that can be either normalized to unity, or to the Dirac delta function.

Postulate = (Max Born, Erwin Schrödinger 1926)

a function in the physical Hilbert space, s.t. there is a concept of probability conservation

Each electron is associated with a wave function $\psi(\vec{r}, t)$ that obeys some linear equation. (i.e., Schrödinger equation)

机率幅

可對比於光波問題中之電場，但$\psi(\vec{r},t)$無相對應之古典場

linearity is a key to allow interference

ψ can be a complex function, with a complex conjugate ψ^*.

The probability of finding a electron at time t in a infinitesimal space $d^3\vec{r}$ is $\underbrace{\overbrace{|\psi(\vec{r},t)|^2}^{\text{机率密度}} d^3\vec{r}}_{\text{机率}}$

The electron must be somewhere in the space. So

$$\int_{\text{all space}} |\psi(\vec{r},t)|^2 d^3\vec{r} = 1$$

Particle may be described as a ψ wave pulse that has a finite extension in space.

making partial t derivative on both side

$$\Rightarrow \int_{\text{all space}} \left(\frac{\partial \psi^*(\vec{r},t)}{\partial t} \psi(\vec{r},t) + \psi^*(\vec{r},t) \frac{\partial \psi(\vec{r},t)}{\partial t} \right) d^3\vec{r} = 0$$

a way to satisfy this equation at all time : requiring $\frac{\partial \psi}{\partial t}$ to be related to ψ in some way. should still be linear.

related by the "Governing equation" which is the Schrödinger Eq. we are contructing.

建構 Ansatz : ψ 必須以類似電磁波的 sin, cos wave 型式干涉，且的時間微分要為ψ的 linear operation

先依據光波干涉的型式猜解，再由解的型式推論 equation of motion

最自然的猜法

先猜ψ的型式為 $\cos(kx-\omega t)$, $\sin(kx-\omega t)$ 的線性组合

$$\psi(x,t) = A[\cos(kx-\omega t) + i\sin(kx-\omega t)] = Ae^{i(kx-\omega t)}$$

角頻率，為頻率的2π倍

$$\frac{\partial \psi}{\partial t} = -i\omega\psi(x,t)$$

(what are k and ω ?)

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General Physics (II) Quantum Physics

free electron (自由電子)
(non-relativistic)

plane wave : $k = \frac{2\pi}{\lambda}$ (wave number) $\Rightarrow k = 2\pi\left(\frac{p}{h}\right) = \frac{p}{\hbar}$, $\hbar \equiv h/2\pi$

de Broglie wave length $\lambda = \frac{h}{p}$

annlogous to light (EM) wave, we expect the energy of the electron to be related to frequency ν by $E = h\nu = \frac{h}{2\pi}(2\pi\nu) = \hbar\omega \Rightarrow \omega = \frac{E}{\hbar}$

$\Longrightarrow$ plane-wave form of the electron wave function

$$\psi(x,t) = \underbrace{A}_{\text{normalization constant}} e^{\frac{i(px - Et)}{\hbar}}$$

For a non-relativistic electron, we expect $E = \frac{1}{2}\frac{p^2}{m}$

$$\Rightarrow \frac{\partial\psi}{\partial t} = -i\omega\psi = -i\frac{E}{\hbar}\psi = -i\frac{p^2}{\hbar\, 2m}\psi$$

$i\hbar\frac{\partial\psi}{\partial t} = E\psi$

$-\frac{p^2}{2m}\psi = E\psi$ (time independent Schrödinger equ.)

Observing that $\frac{\partial^2\psi}{\partial x^2} = -k^2\psi = -\frac{p^2}{\hbar^2}\psi = -\frac{i}{\hbar}2m\left(-i\frac{p^2}{\hbar\, 2m}\right)\psi = -i\frac{2m}{\hbar}\frac{\partial\psi}{\partial t}$

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi$$

time-indedepdent Schrödinger eqn. for free-electron.

electron in potential

公設：波長 (de Broglie wavelength) 仍由動量給出

$$\underbrace{(E - V)}_{\text{動能}} = \frac{p^2}{2m}$$

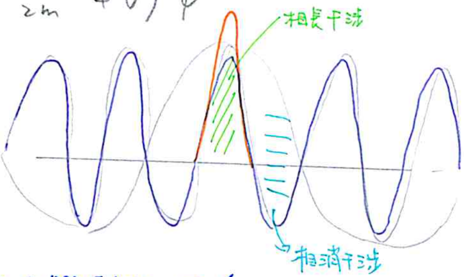
總能

modifying from $E = \frac{1}{2}\frac{p^2}{m}$ to $E = \frac{p^2}{2m} + V$

following the derivation as the free-electron case

$$\frac{\partial\psi}{\partial t} = i\omega\psi = -i\frac{E}{\hbar}\psi = -i\frac{1}{\hbar}\left(\frac{p^2}{2m} + V\right)\psi$$

$$\Rightarrow i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + V\psi$$



A localized particle.

A sin/cos wave does not has locality. Nevertheless, we can superimpose sin/cos waves of various length to make a pulse. By observing the wave function of a free election, we see that this is saying a localized electron is in a state that is mixing the probabilities of being at various different momentum. The more localized is the electron, the probability distribution in the momentum space is wider.

位置之確定性愈高

動量之不確定性愈高

uncertainty principle : $\underbrace{\Delta x}_{\text{位置之標準差}} \cdot \overbrace{\Delta p}^{\text{動量之標準差}} \geq \hbar$

呂浩宇

General Physics (II) — Quantum Physics

Problem with energy eigen wave function (能量本徵態，即具有特定能量之電子)

解題方法：列出以下方程式解題
1. 要求波函數為連續
2. 要求波函數之空間微分連續
3. localized particle 問題要求波函數可被 normal

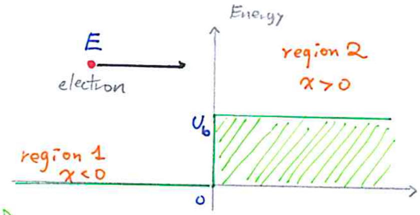
A. Reflection / transmission of electron from a potential step

注意此處解的只是電子 energy eigen wave function 的行為而不是 localized 的，具有一定動量机率分布的 wave pocket. 然而結果是 qualitatively similar，可看出重要的量子效應.

Case：
Electron with energy E is approaching the step potential of height U_b.
Considering $E > U_b$

- classical (deterministic) physics：electron either go straightly or bounce back.
- quantum physics：electron has the probability of transmitting and bouncing back.

Question：what is the probability of being {transmitted / reflected} ?



(i) Wave function at $x<0$ (Region 1)   (ψ is basically the free particle's wave function)

$$i\hbar\frac{\partial\psi}{\partial t} = E\psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi \qquad \text{Eq. (1)}$$

the general solution for the time independent part of ψ is

$$\psi_1(x) = \underbrace{Ae^{ikx}}_{\text{incident beam}} + \underbrace{Be^{-ikx}}_{\text{reflected wave}} \qquad \text{Eq. (2)}$$

Including the time dependent part, wave $\begin{cases} e^{i(kx-\omega t)} \\ e^{i(-kx-\omega t)} \end{cases}$ is traveling toward the $\begin{cases} \text{right} \\ \text{left} \end{cases}$.

Substituting (2) into (1), we get $E = +\frac{\hbar^2}{2m}k^2$ $\hbar = \frac{h}{2\pi}$

$$\Rightarrow k = \frac{2\pi}{\lambda} = \frac{\sqrt{2mE}}{\hbar} = \frac{2\pi\sqrt{2mE}}{h}$$

(ii) wave function at $x>0$ (region 2)

$$i\hbar\frac{\partial\psi}{\partial t} = E\psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + V\psi \qquad \text{Eq. (3)}$$

the general solution for the time independent part of ψ is

$$\psi_2(x) = \underbrace{Ce^{ik_bx}}_{\text{transmitted wave}} + De^{-ik_bx} \qquad \text{Eq. (4)}$$

substituting (4) into (3), we get $E = -\frac{\hbar^2}{2m}k_b^2 + V \Rightarrow k_b = \frac{\sqrt{2m(E-U_b)}}{\hbar}$

(iii) Initial and boundary condition

1. There is no incident electron from the right. Therefore, $D=0$
2. Wave function needs to be continuous at $x=0$ ⟶ 代入 Eq.(2) 及 (4)，令 $D=0$ 並令 Eq.(2) 及 (4) 相等 $\Rightarrow Ae^0 + Be^0 = C \Rightarrow A+B=C$
3. $\frac{\partial\psi}{\partial x}$ needs to be continuous at $x=0$ $\Rightarrow Ake^0 - Bke^0 = Ck_be^0 \Rightarrow Ak - Bk = Ck_b$

$$Ak + Bk = Ck$$
$$Ak - Bk = Ck_b$$

上下式相加 $\Rightarrow 2kA = (k+k_b)C \Rightarrow A = \frac{k+k_b}{2k}C$

上下式相減 $\Longrightarrow B = \frac{k-k_b}{2k}C$

reflection coefficient R: ratio of the probability density in the reflection to the probability in the incident beam.

$$R = \frac{|B|^2}{|A|^2}$$

transmission coefficient T:

$$T = 1-R$$

記得在講原子結構時介紹
α-decay
β-decay

B. Tunneling through a potential barrier (Quantum tunneling)

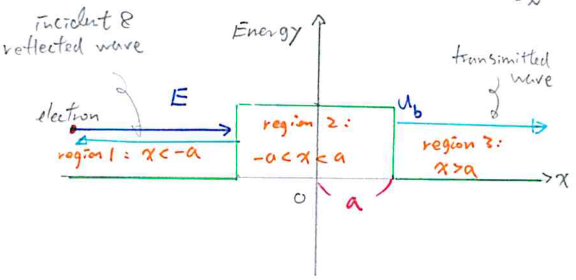
應用: Scanning Tunneling Microscope (STM): 掃描式穿隧顯微鏡

Case: Electron with energy E is approaching a potential barrier $U_b > E$
The width of the barrier is $2a$

classical (deterministic) physics: electron can only bounce back.
quantum physics: electron has some probability to tunnel through the barrier.

first noting the when $E < U_b$, assuming $\psi = e^{ikx}$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + V\psi = +\frac{\hbar^2}{2m}k^2\psi + U_b\psi = E\psi$$



$$\Rightarrow \frac{\hbar^2}{2m}k^2 = (E-U_b) < 0$$

Therefore, ψ $\begin{cases}\text{decrease}\\ \text{increase}\end{cases}$ exponentially in such region.

總是可以先選擇 incident wave amplitude 為 1, 之後有需要再做 normalization

(i) general solution in region:

共四個獨立變數 R, A, B, T

$$\begin{cases} 1: \ \psi(x) = e^{ikx} + Re^{-ikx}, & k = \frac{\sqrt{2mE}}{\hbar} \\ 2: \ \psi(x) = Ae^{iqx} + Be^{-iqx}, & q^2 = \frac{2m}{\hbar^2}(E-U_b) \\ 3: \ \psi(x) = Te^{ikx}, & k = \frac{\sqrt{2mE}}{\hbar} \end{cases}$$

對於 $E > U_b$ 情形 解題方法亦同

共四個獨立的 equation

(ii) continuity of ψ:

at $x=-a$: $e^{-ika} + Re^{ika} = Ae^{-iqa} + Be^{iqa}$

at $x=a$: $Ae^{iqa} + Be^{-iqa} = Te^{ika}$

(iii) continuity of $\frac{\partial}{\partial x}\psi$:

at $x=-a$: $ike^{-ika} - ikRe^{ika} = iqAe^{-iqa} - iqBe^{iqa}$

at $x=a$: $iqAe^{iqa} - iqBe^{-iqa} = ikTe^{ika}$

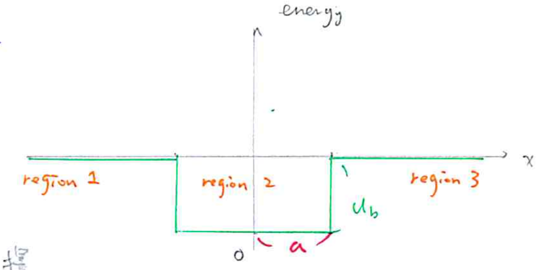
solution (with a little algebra)

$$T^{-1} = 1 + \frac{U_b^2}{4E(U_b-E)}\sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(U_b-E)}\right)$$

性質: 若 barrier 愈寬 (a愈大) 或 愈高 (U_b愈大), 穿隧机率愈低

呂浩宇

C. Finite square potential well (有限位能井)



解題要列的 equations 基本與 barrier 問題相同，
只是把 U_0 改成負數.

bound state: $E<0$ 之能量狀態

↳ 此處為相對於 free-electron 所受位能之選擇

→ 在 region 1 及 3 能選擇的 solution 只有
$$\begin{cases} \text{region 1: } Be^{\kappa x} \\ \text{region 3: } Ae^{\kappa x} \end{cases}, \quad \kappa = \frac{2m|E|}{\hbar^2}$$

region 2 的 general solution 為
$$\psi(x) = C\underbrace{\cos(qx)}_{\text{相對 } x=0 \text{ 對稱解}} + D\overbrace{\sin(qx)}^{\text{相對 } x=0 \text{ 反對稱解}}, \quad q^2 = \frac{2m}{\hbar^2}U_0 - \kappa^2$$

同樣解法為要求 $\psi(x)$ 在 $x=\pm a$ 處為連續，並且 $\frac{\partial\psi}{\partial x}(x)$ 亦連續.

bound state 的解，E 必為 discretized (能階量子化)

↓

特例: infinite square well
(假設 potential 在 $x>a$ 與 $x<-a$ 處為無窮大，
在 $-a<x<a$ 之間為 0)

解為滿足 free-particle Schrödinger eq. 且在 $x=\pm a$ 處為0之波函數

free particle:
$$\begin{cases} k = \frac{\sqrt{2mE}}{\hbar} = \frac{2\pi}{\lambda} \Rightarrow E = \frac{\hbar^2}{2m}k^2 \\ -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \end{cases}$$

even solution

$$\cos ka = 0 \Rightarrow k = \frac{(n+\frac{1}{2})\pi}{a}, \quad n = 0, 1, 2, 3 \ldots$$
$$= \frac{(2n+1)\pi}{2a}$$

性質: 最低能階為 even function.

odd solution

$$\sin ka = 0 \Rightarrow k = \frac{2m\pi}{2a}, \quad m = 1, 2, 3 \ldots$$

注意不可選 $m=0$, 否則波函數隨處為 0, 無意義

→ all solutions: $k = \frac{n\pi}{2a}$, $n = 1, 2, 3, 4, 5, 6, \ldots$

All possible energy levels:
$$E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{2a}\right)^2$$
$$= \frac{n^2\pi^2\hbar^2}{2m(2a)^2}$$

性質: 位能井愈寬 (a 愈大), 最小能階愈低. 能階能量正比於 n^2.
半古典情形: n^2 大, 即 k 很大之情形 (λ很小). 粒子看起來在 box 中各處被找到的機率都差不多. 不侷限於少數波峰.