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General Physics (II) Basic Circuits

Resistance, Capacitance, and Inductance
電阻 電容 電感

※串、並聯電阻、電容、及電感在實務應用中非常實用、計算原理推導相對簡單。有心往科学或公程性質職涯發展之同学應閱讀課本修習。本普通物理課程僅介紹元件之物理原理。

本節僅討論 steady state 之情形。(電位、電流等物理性質不隨時間明顯變化)

Definition of current: for an arbitrary plane, if charge dq pass through that plane in a unit time dt, then the current through that plane is defined as:

$$i = \frac{dq}{dt}$$

SI制單位: 1 ampere = 1 A = 1 coulomb/s

When drawing a circuit, if the charge carriers carry positive charge, we draw arrows in the directions of charge carriers' motion. If the charge carriers carry negative charge (e.g. electrons), we draw arrows in the directions of charge carriors' motion.

Definition of current density: $i = \int \vec{J} \cdot d\vec{A}$ ($\vec{J}$: current density)

$$\vec{J} = (n e)\, \vec{v}_d$$

n: number density of charge; e: $1.602 \cdot 10^{-19}$ C (positive) elementary charge; $\vec{v}_d$: drift velocity (漂移速率) of charge carriers

A. Resistance, Ohm's law, and Power in Electric circuits
在電路中以 ꟿ 表示　歐姆定律

If we apply a potential difference V between two points on a conductor and detect the resulting current i, the resistance is defined as:

resistance: $R = \frac{V}{i}$ (V: potential difference; i: current)

物理: 形成單位電流所需要之電壓

R: 與外加電壓如何施加於待測導件 (e.g. 連接處之接觸面積)

SI 制單位: 1 ohm = 1 Ω = 1 volt per ampere = 1 V/A

Similarly, we can define:

resistivity: $\rho = \frac{E}{J}$ (ρ: resistivity)

物理: 形成單位電流密度所需之電場

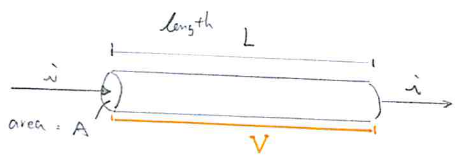
一般向量形式: $\vec{E} = \rho \vec{J}$

SI 制單位: $\frac{[E]}{[J]} = \frac{V/m}{A/m^2} = \frac{V}{A} m = \Omega \cdot m$

and we can correspondingly define **conductivity**: $\sigma = \frac{1}{\rho} \Rightarrow \vec{J} = \sigma \vec{E}$

物理: 單位電場所能造成之電流密度

The relation between these quantities.



$$E = V/L, \quad J = i/A$$

$$\rho = \frac{E}{J} = \frac{V/L}{i/A} = \frac{\overbrace{(iR)}^{V}A}{iL} \Rightarrow R = \rho \frac{L}{A}$$

給定 resistivity，長度愈長電阻愈大，截面積愈大電阻愈小

對於大部分導體，溫度愈高，ρ 值愈大。
金屬的 resistivity 滿足線性經驗公式: $\rho - \rho_0 = \rho_0 \alpha (T - T_0)$

α: temperature coefficient of resistivity

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Ohm's law: A conducting device obeys Ohm's law when the resistance of the device is independent of the $\underbrace{\text{magnitude}}_{大小}$ and $\underbrace{\text{polarity}}_{方向}$ of the applied potential difference.

注意 $R = \frac{V}{i}$ 僅為電阻之定義. Ohm's law 為 R 不為 V 的函數之非一般情形.
Ohm's law 不為 fundamental 之物理定律

B. Power in Electric Circuits

電阻造成電子在高電位與低電位間的能量差以熱耗散的方式逸散. 電容及電感則可將能量儲存於電場或磁場中.

$dq = i\,dt$

$$\underbrace{dU}_{\text{在時間經過}dt\text{，因電荷移動而造成之系統能量差}} = \underbrace{dq}_{\text{在時間經過}dt\text{，由高電位處流到低電位處之電荷量}}\, V = i\,dt\,\underbrace{V}_{電位差}$$

$\Rightarrow$ Power (功率, 即單位時間之 electric energy transfer)

$$P = \frac{dU}{dt} = iV$$

只之: 電池的正負兩極稱為電池的 term

若存在電阻, 代入電阻定義 $R = \frac{V}{i}$

$$\begin{cases} V = iR \Rightarrow P = i^2 R \\ i = \frac{V}{R} \Rightarrow P = V^2/R \end{cases}$$

電阻造成熱耗散之計算方式

C. Capacitor (電容, 用以儲存電能)

在電路圖中以 ⊣⊢

正/負極: positive/negative terminal

When a capacitor is charged, the charges $+q$ and $-q$ become spatially separated.
We refer to the charge of a capacitor as being q

In most cases, capacitors are made of conductors, for example, parallel plates (平行板). In other words, they are equipotential surfaces. When a potential difference V between two conducting surfaces is maintained when the conductor is charged, with a charge q, we can relate q and V by

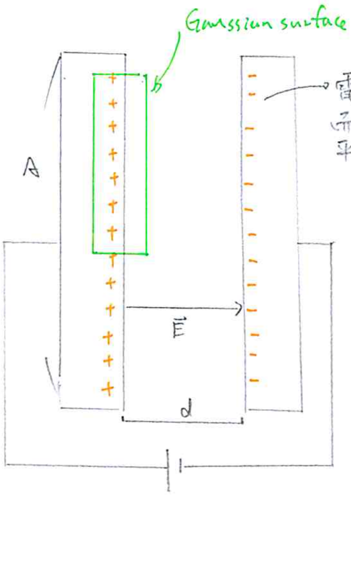
$$q = \underbrace{C}_{\text{capacitance}} V$$

物理意: 每增加一伏特的單位差可以多儲存的電量

SI 制單位: 1 farad = 1 F = 1 coulomb per volt = 1 C/V

Example: parallel-plate capacitor (平行板電容) ⟶ 將平行板近似於無窮大

(見 page 3 之推導)



電荷互相吸引, 而分布於靠近另一平行板之表面

If the surface area (one side) of the plate is A

according to Gauss's law:

$$\epsilon_0 E = \frac{q}{A} \quad \text{for any location in between the two conducting plates}$$

potential difference:

$$V = -\int_-^+ \vec{E}\cdot d\vec{s} = Ed \Rightarrow E = \frac{V}{d}$$

$$\epsilon_0 \frac{V}{d} = \frac{q}{A} \Rightarrow q = \underbrace{\frac{\epsilon_0 A}{d}}_{\text{capacitance } C} V$$

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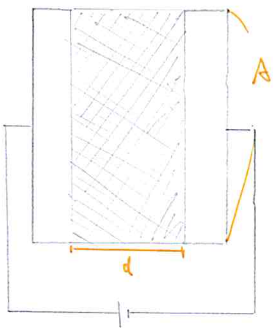
Example: capacitor with a Dielectric

(電介質 → 可被外電場偏極化之物質)

Michael Faraday (1837) = inserting certain materials in between the parallel plates can make the capacitance increased by a numerical factor $\underbrace{K}_{\text{dielectric constant}} \equiv \frac{\epsilon}{\epsilon_0}$ (見 page 5 最底下)
except that when V is greater than the breakdown potential V_{max}, the dielectric material will break down a form a conducting path.



$$C = K \underbrace{\frac{\epsilon_0 A}{d}}_{\text{沒有電介質情形之 conductance}}$$

Proof. modify the differential form of Gauss law from $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$

to $\underbrace{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon}}_{\text{見 page 5 最底下}} = \frac{\epsilon_0}{\epsilon}\frac{\rho}{\epsilon_0} = \frac{1}{K}\frac{\rho}{\epsilon_0}$

the remaining derivation of charge density distribution and potential difference is the same as the previous example. #

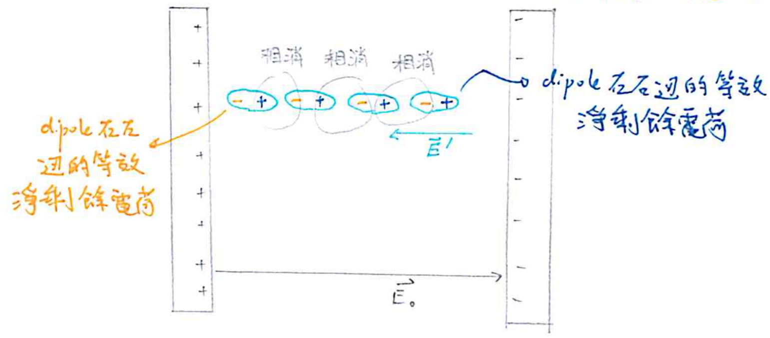
⇓

$$\vec{\nabla}\cdot(\underbrace{K\vec{E}}) = \frac{\rho}{\epsilon_0}$$

置入電介質後，電場需變為原來的 $\frac{1}{K}$ 以抵消 dielectric const 之效果

微觀理解電場因何會變為原來的 $\frac{1}{K}$

induced dipole 的電場抵消了平行板上電荷所貢獻的電場



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D. Inductor (電感, 可儲存磁能)

在電路圖中以 ℓℓℓℓ 符號表示

此處僅介紹電路學中的 inductance, 而不使用電磁學中原始的定義.
學電磁學時要注意, 原始定義與此處定義之關係才不至於混淆

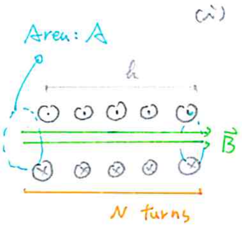
$$\underbrace{\mathcal{E}_L}_{\text{感應電動勢}} = -L \overbrace{\frac{di}{dt}}^{\text{單位時間之電流改變量}}$$

L → inductance: (負) 單位電流改變率造成的感應電動勢.

公制單位: 1 henry = 1 H = 1 $T \cdot m^2/A$

比較: (page 12)
$V = \frac{1}{C} q$: 電容可理解為單位電荷所能維持之電位差.

Example: inductance of a solenoid (→ 線圈)



(i) Faraday's law (page 10): $\mathcal{E} = -\frac{d\Phi_B}{dt}$

for a solenoid with N turns, N is large (page 9)

$$B = \mu_0 i n = \mu_0 i \frac{N}{h}$$

n → 單位長度之線圈匝數: $n = \frac{N}{h}$

the magnetic flux through each one of the N current loop:

$$BA = \frac{\mu_0 i}{h} NA$$

⇒ the emf produced by each of the N current loop:

$$\Delta\mathcal{E} = -\frac{d}{dt}(BA) = -\frac{\mu_0 N}{h} A \frac{di}{dt}$$

the overall emf:

$$\mathcal{E} = \sum \Delta\mathcal{E} = -N\frac{\mu_0 N}{h} A \frac{di}{dt} = -h\mu_0 n^2 A \frac{di}{dt}$$

⇒ the inductance: $\underline{L = \mu_0 n^2 h A}$

solenoid 電感正比於單位長度匝數之平方, 線圈長度, 及線圈截面積

↓

可重新整理為

$$L = \mu_0 i n \left(\frac{nh}{i} A\right)$$
$$= B \cdot \frac{N}{i} A = \frac{N \, \Delta\Phi_B}{i}$$

$\Delta\Phi_B$: 每一圈 current loop 造成之磁通量

表示法 $L = \frac{N \Delta\Phi_B}{i}$ 比較接近電磁學中用的 fundamental 定義

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※ 雖然大部分情形，電流因負電荷(即電子)的移動而形成，但習慣上，電路學中之電流箭頭方向仍表示正電流之流向

E. Circuits. (電路)

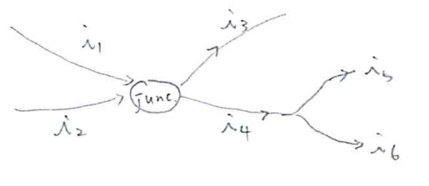
節點 交叉點

1. **Junction rule** (源自電荷守恆)：進入 junction 之電流總合等於流出 junction 之電流總合
2. **Loop rule** (源自能量守恆)：在電路中，若繞行任一迴圈，並將繞行中所遇到之電位變化加總，則加總起來的電位變化為 0

2.1：**resistance rule**：經過電阻時，順著電流方向電位變化為 $-iR$；沿相反方向則電位變化為 iR

2.2：**emf rule**：electromotive force rule：經過 emf device (如電池)時，沿 emf 方向電位增加 $\mathcal{E}$，反之則電位減少 $\mathcal{E}$.

Example 1. Junction rule



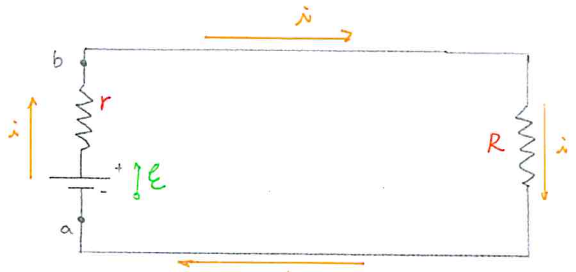
$$i_1 + i_2 = i_3 + i_4 = i_3 + i_5 + i_6$$

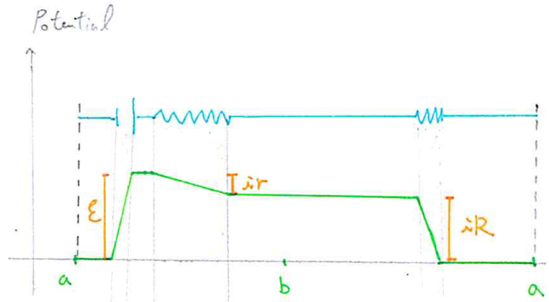
Example 2. Loop rule (僅有電池與串聯電阻之情形)

(Halliday textbook, chapter 27)

物理：以局部電位之高低來判定電流的方向。

電位之物理概念：emf device (如電池)將電荷加速，電荷獲得動能，電荷流經電阻時因碰撞而損失動能，剩餘的動能決定電荷是否應/能跨過之後的電位差
(類比被加速而得到動能的球是否能跨過重力位能差)





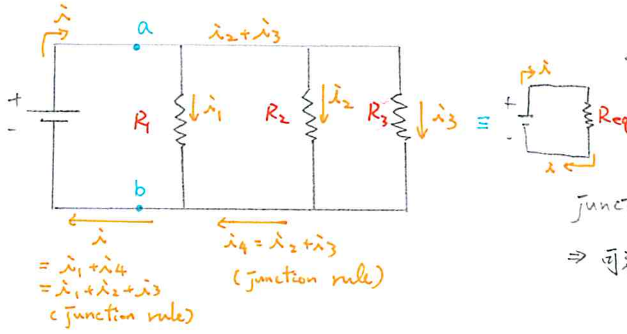
$$\text{loop rule}: \; +\mathcal{E} - ir - iR = 0$$

$$\Rightarrow i = \frac{\mathcal{E}}{R + r}$$

※ 串聯多個電阻形成之等效電阻，為各個電阻值之總和

Example 3. Junction + loop rule (電池及並聯電阻情形)

resistance in parallel



loop rule：
$$\begin{cases} \mathcal{E} - i_1 R_1 = 0 \\ \mathcal{E} - i_2 R_2 = 0 \\ \mathcal{E} - i_3 R_3 = 0 \end{cases}$$

junction rule：$i = i_1 + i_2 + i_3 = \mathcal{E}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$

$\Rightarrow$ 可視 a, b 右側為一等效電阻 $\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$

※ 並聯電阻之等效電阻之倒數為各別電阻之倒數和.

※ 碰到複雜電路時，先以此類手法簡化電路圖

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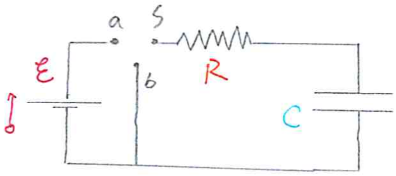
General Physics (II) Basic Circuits

(RLC例式不細請
∵ 藉電感儲存磁場之能量)

Example: RC circuits (有電阻及電容之情形)



(i) S接a不接b，電容充電 (charging the capacitor with a battery)

S: switch

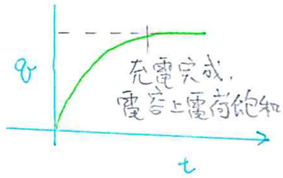
loop rule: $\mathcal{E} - iR - \dfrac{q}{C} = 0$

q: 電容上之電荷量

$q = CV$

$\Rightarrow \mathcal{E} - \dfrac{dq}{dt}R - \dfrac{1}{C}q = 0$ ：q對時間的一次常微分方程，若初始條件為電容電荷為0，則解為

$q = C\mathcal{E}(1 - e^{-t/RC})$



一階常微分方程解法：配分

$\dfrac{\mathcal{E}}{R} - \dfrac{dq}{dt} - \dfrac{1}{RC}q = 0$

$\dfrac{\mathcal{E}}{R}e^{\frac{t}{RC}} - \dfrac{dq}{dt}e^{\frac{t}{RC}} - \dfrac{1}{RC}q\,e^{\frac{t}{RC}} = 0$

$\Rightarrow \dfrac{\mathcal{E}}{R}e^{\frac{t}{RC}} - \dfrac{dq}{dt}e^{\frac{t}{RC}} - q\dfrac{d}{dt}e^{\frac{t}{RC}} = 0$

$\Rightarrow \dfrac{\mathcal{E}}{R}e^{\frac{t}{RC}} - \dfrac{d}{dt}\left(q\,e^{\frac{t}{RC}}\right) = 0$

$\Rightarrow q\,e^{\frac{t}{RC}} = \displaystyle\int \dfrac{\mathcal{E}}{R}e^{\frac{t}{RC}}dt$

$= RC\dfrac{\mathcal{E}}{R}e^{\frac{t}{RC}} + \text{const.}$

$\Rightarrow q = C\mathcal{E} + \text{const.}\,e^{-\frac{t}{RC}}$ 帶入邊界條件

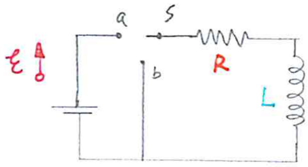
when $t = 0$, $q = 0 \Rightarrow \text{const} = -C\mathcal{E}$ #

(ii) S接b不接a，電容放電

$-iR - \dfrac{1}{C}q = 0$

$\Rightarrow q = q_0 e^{-t/RC}$ ：電容上電荷呈 exponential decay，其 decay 之特徵時間為電阻與電容值的乘積。

Example: RL circuits (有電阻及電感之情形)



(i) 由斷路情形改變為S接到a，電池放電產生電流，電感產生抵抗電池之感應電動勢。當系統慢慢趨向平衡態，電流不再隨時間變化，則電感形同不存在。

loop rule: $\mathcal{E} - iR - L\dfrac{di}{dt} = 0$ ：電流 i 之一階常微分方程，解之邊界條件為在初始時間 t=0 時電流為0，解為：

$\begin{cases} t=0 \text{ 時 } i=0 \\ t\to\infty \text{ 時 } i=\dfrac{\mathcal{E}}{R} \end{cases}$ $\longrightarrow$ $i = \dfrac{\mathcal{E}}{R}\left(1 - e^{-\frac{R}{L}t}\right)$

(ii) 在(i)描述之情形達到穩態後，開關S改接到b。因電池不再存在，電流減少，電感產生感應電動勢維持電流；

loop rule: $\underbrace{L\dfrac{di}{dt}}$ $+ iR = 0$

注意正負號，此情形之下，初始電流為順時針方向，但電感之感應電動勢為逆時針方向

i 之一階常微分方程。在時間 t=0 時電流應為 $\dfrac{\mathcal{E}}{R}$，在時間無窮大時電流為0，解為 $i = \dfrac{\mathcal{E}}{R}e^{-\frac{L}{R}t}$

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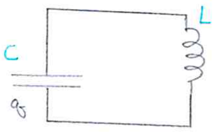
Example: LC Oscillator (LC 振盪器)

Halliday 普物課本 ch. 31. 實務上之應用重要



loop rule: $\frac{q}{C} + L\frac{di}{dt} = \frac{q}{C} + L\frac{d}{dt}\left(\frac{dq}{dt}\right) = 0$

$\Rightarrow q = Q\cos(\omega t + \phi)$

$i = \frac{dq}{dt} = -\omega Q\sin(\omega t + \phi)$

F. Energy stored in a magnetic field

先由 RL 電路看儲存在電感中之能量，再藉由推導電感中之磁場 (see page 9)

loop rule for RL circuit: $\mathcal{E} = L\frac{di}{dt} + iR$

↓ 左右同乘電流 i

$$\underbrace{\mathcal{E}i}_{\text{電池功率}} = Li\frac{di}{dt} + \underbrace{i^2R}_{\text{電阻功率}}$$

電池功率: 電池在單位時間減少之電能

電阻功率: 電阻在單位時間耗散之熱能

→ Based on energy conservation, we can see that the term $Li\frac{di}{dt}$ should be interpreted as the increase rate of the energy stored in the inductor (單位時間內電感中儲存能量之增加率)

⇓

since $Li\frac{di}{dt} = \frac{d}{dt}\left(\frac{1}{2}Li^2\right)$

we can define the (magnetic) energy stored in the inductor as

$$U_B = \frac{1}{2}Li^2$$

If the inductor has a length h and cross-section A (截面積),

then the (magnetic) energy in a unit volume is $u_B = U_B/Ah$

$$= \frac{L}{h}\frac{i^2}{2A}$$

from page 14: $L = \mu_0 n^2 hA$

$\Rightarrow u_B = \frac{1}{2}\mu_0 n^2 i^2$

from page 9: $B = \mu_0 i n$

$$\Rightarrow u_B = \frac{B^2}{2\mu_0}$$