呂語宇

General Physics (II) Electromagnetic theory

Static Electric Field (Halliday & Resnick ed. 11, chapter 21-24)

charge is quantized : $q = ne$, $n = \pm 1, \pm 2, \pm 3$

elementary charge : $e = 1.602 \times 10^{-19}$ C

C : coulomb (SI 制之電荷量單位)

$1C = (1A)(1s)$ 。一庫倫之電荷量為一安培之電流在一秒間所通過的電荷數

charge is conserved (不可隨意產生或湮滅淨電荷)

charges can move on the (semi/super...) conductors. 傾向在導体表面重新分布而使得電場趨近為 0

Coulomb's law (Charles-Augustin de Coulomb, 1785) 清乾隆年間

describing the electro static force between two charge particles

for charged particles that have charge q_1 and q_2, $\vec{r} = \vec{x}_1 - \vec{x}_2$, $\hat{r} = \dfrac{\vec{r}}{|\vec{r}|}$ 方向向量

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}|^2} \hat{r}$$

$\vec{F}_{12}$: charged particle 2 施加給 charged particle 1 之靜電力

$\frac{1}{4\pi\epsilon_0}$: electrostatic constant, or coulomb's constant.

$$\frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^{9} \ N \cdot m^2/C^2$$

permittivity constant : $\epsilon_0 = 8.85 \cdot 10^{-12} \ C^2/N \cdot m^2$

superposition of force : $\vec{F}_{1,net} = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \vec{F}_{15} + \cdots + \vec{F}_{1n}$
(靜電力之疊加)

SI 制單位 : N/C

electric field : $\vec{E} = \dfrac{\vec{F}}{q_0}$

單位電荷受的 electric force

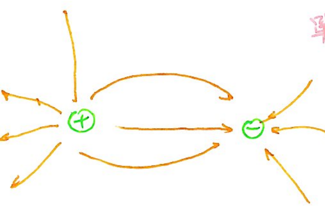
concept introduced by Michael Faraday in the 19th century

$\vec{F}$: 該無窮小電荷所受之總電磁力

q_0 : 某無窮小電荷之 test charge

因無窮小，故其電荷產生之電磁力不改變空間中之電荷分布

due to this relation, electric field can be superposed in the same way.



Electric field lines :

Extend away from positive charge toward negative charge.

(i) at any point, the electric field vector must be tangent to the electric field line.

(ii) In the plane perpendicular to the field lines, the relative density of lines represents the relative magnitude of the field.

Electric field due to a point charge

(point charge at location $\vec{x}_2$; field at location $\vec{x}_1$)

$$\vec{E} = \frac{\vec{F}}{q_0} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

$\hat{r} = \dfrac{\vec{r}}{|\vec{r}|}$, $r = |\vec{x}_1 - \vec{x}_2|$

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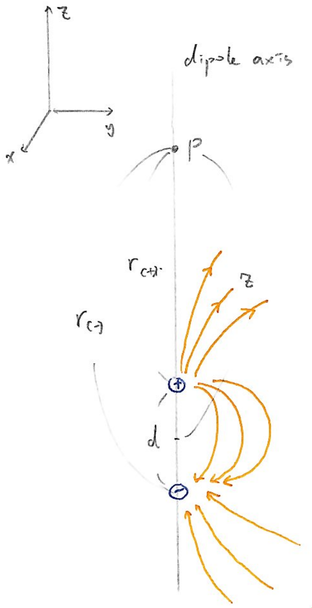
Electric field due to a dipole

(因大部分情形考慮電中性物體，故時常須面對 dipole field)

如極性分子的性質



At point P, the electric field in the z direction set up by the dipole is

$$E_z = E_{z(+)} - E_{z(-)}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{r_{(+)}^2} - \frac{1}{4\pi\epsilon_0}\frac{q}{r_{(-)}^2} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{(z-\frac{1}{2}d)^2} - \frac{1}{(z+\frac{1}{2}d)^2}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\frac{2zd}{\left((z-\frac{1}{2}d)(z+\frac{1}{2}d)\right)^2} \xrightarrow[d \ll z]{} \frac{q}{4\pi\epsilon_0}\frac{2zd}{z^4} = \frac{1}{2\pi\epsilon_0}\frac{qd}{z^3}$$

$$= \frac{1}{2\pi\epsilon_0}\frac{p}{z^3}$$ (electric dipole moment: $\vec{p} = q\vec{d}$)

$\vec{d}$ 方向為由負極指向正極

一般的應用之中我們僅知道 $\vec{p}$ 而不知道 q 及 d. dipole field 之電場與 r^3 成反比

dipole field 隨距離 decay 得比 monopole 快. 長距離情形只有 monopole 重要

$r \gg d$, 在 z 軸測量之電場 (dipole field) 一般形式

$$E_r = \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}2\cos\theta, \quad E_\theta = \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\sin\theta, \quad E_\phi = 0$$

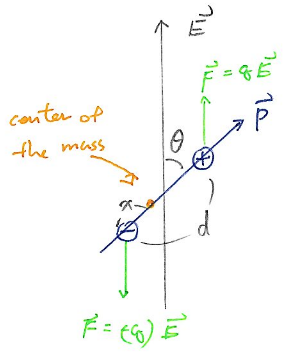
Electric field due to a continuous distribution of charge

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \underbrace{\rho(\vec{x}')}_{\text{在位置 } \vec{x}' \text{ 處單位空間內之電荷數}} \underbrace{\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}}_{\hat{r}/r^2} d^3x'$$

課本 §22-4, §22-5 中舉 line charge 及 disk of charge 為例. 不清楚積分如何處理者可參考. 實作中多為較複雜情形, 可/須靠數值積分解決

A dipole in an electric field

在大多數的應用中, 可視外電場為均勻電場. 此時 dipole 的行為僅與 $\vec{E}$ 與 $\vec{p}$ 有關



torque (about the c.o.m)

均勻外場之場強

$$\tau = qE\left((d-x)\sin\theta\right) - (-q)Ex\sin\theta$$

$$= E(qd)\sin\theta = Ep\sin\theta \longrightarrow \text{力矩方向為穿出紙面}$$

$$\Rightarrow \boxed{\vec{\tau} = \vec{p}\times\vec{E}}$$

(類似地, 我們也可以定義 magnetic dipole moment $\vec{\mu}$, $\vec{\tau} = \vec{\mu}\times\vec{B}$ 在磁場中受到力矩 $\vec{\tau}$ 之物體具有 magnetic dipole moment $\vec{\mu}$)

位能 U (以與電磁力相反方向的力作功而使系統儲存的能量)

$$U = \int(-\tau)\,d\theta = \int pE\sin\theta\,d\theta = \underbrace{-pE\cos\theta}_{\text{隨}\theta\text{變化之部分}} + \overbrace{\text{const.}}^{\text{可自行定義之位能零點}}$$

$$\Longrightarrow \boxed{U = -\vec{p}\cdot\vec{E}}$$

(magnetic dipole 在磁場中的能量 $U = -\vec{\mu}\cdot\vec{B}$)

※此處 $\vec{\tau}$ 與 U 為重要之一般性質, 應該記憶. 物質電中性情形多, dipole 對電場貢獻重要

General Physics (II) Electromagnetic theory

Gauss law:

(※ 對於大學以上物理為非常常用之定理。有需要學習物化者或應在学習向量微積分後研讀) 此處跳過證明

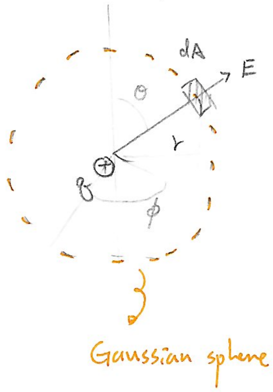
Relating electric field on a closed surface to the net charge enclosed by that surface

↳ Application
- ① evaluate $\vec{E}$ when the charge distribution is known.
- ② evaluating the charge distribution when $\vec{E}$ is known.

積分型式 $$\oint \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0}$$ (enclosed charge density)

同一定理之微分型式: $$\oint \vec{E}\cdot d\vec{A} = \int (\vec{\nabla}\cdot\vec{E})\,dV \Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$ (体積積分; 電荷密度)

Example 1: electric field on the sphere that is centered on a point charge



$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$$

$$dA = (r\sin\theta\, d\phi)(r\, d\theta) = r^2\sin\theta\, d\theta\, d\phi$$

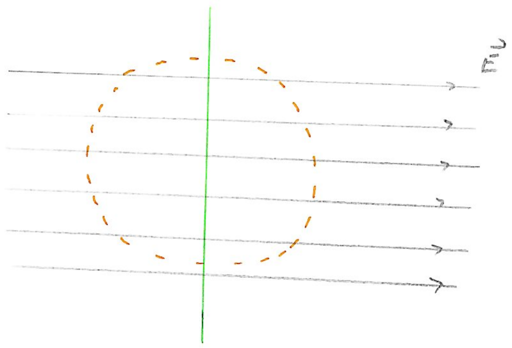
$$\boxed{\oint \vec{E}\cdot d\vec{A}} = \int_0^{2\pi}\int_0^{\pi}\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}r^2\sin\theta\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\frac{q}{4\pi\epsilon_0}(d\cos\theta)\,d\phi$$

$$= \frac{q}{2\epsilon_0}\int_0^{\pi}(d\cos\theta) = \boxed{\frac{q}{\epsilon_0}}$$

可把此處之Gauss's law視為是比上頁中的平方反比定律更為fundamental之對電場描述 (效果equivalent, 但形式更為根本)

Example 2: charge density when there is an uniform electric field



from the symmetry of the system
($\int \vec{E}\cdot dA$ 在綠線左半边與右右半边之貢獻相消)

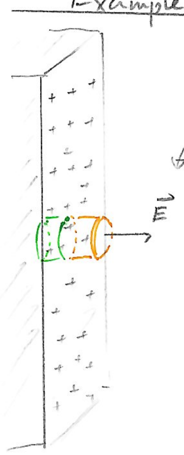
$\oint \vec{E}\cdot d\vec{A} = 0$ no matter how big/small is the sphere

⇓

there is no net local charge density

Example 3: charge on a flat and large conductor

(廣泛應用於在實驗之中建立均勻電磁場)
或應用與製作電容)



由於對稱性, 電場垂直於導体平面

- Inside the conductor: $\int \vec{E}\cdot d\vec{A} = 0$ ($\vec{E}$ 隨處為 0)
- outside the conductor: $\int \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0} \Rightarrow E = \frac{1}{\epsilon_0}\frac{q_{enc}}{dA} = \frac{\sigma}{\epsilon_0}$ (導体表面之電荷密度)

→ 導体外電場強度與電荷密度有簡單關係.

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General Physics (II) Electromagnetic theory

定義之方式與 gravitational potential energy 相同

electric force is conservative ⟶ has an associated potential energy

真正的動力學問題用 field,
平衡狀態用 potential.

計算中用 potential 而非 electric field 的好處
1. 計算上純量場較向量場好處理 (適用 stationary 問題)
2. 能量守恆之概念

系統狀態不隨時間改變,
如處於平衡狀態

注意只有電荷產生的電場適用此處 potential 定義. 因磁場改變生成之感應電場不適用

electric potential: the amount of the electric potential energy per unit charge when a positive test charge is brought in from infinity

SI 制單位:
1 volt = 1 joul per coulomb

(當我們將一帶有正電荷之 test charge 由無窮遠處移入電場時, 單位正電荷所帶有之電位能)

極小, 對系統不造成干擾之電荷

$$V_f - V_i = -\int_i^f \vec{E}\cdot d\vec{s}$$ (積分路徑可任取; 順著電場的方向向下走是往 potential 低的地方去)

electric potential energy: $\underbrace{U}_{\text{potential energy}} = q\underbrace{V}_{\text{potential}}$

若將電場單位與 volt 關連: $1\ N/C = \left(1\frac{N}{C}\right)\left(\frac{1\,V}{1\,\frac{J}{C}}\right)\left(\frac{1\,J}{1\,N\cdot m}\right) = 1\ V/m$

一庫倫電荷在1伏特電位中的能量, 相當於以一牛頓的力推行一米所做的功

energy conservation: $\underbrace{U_i}_{\text{initial potential energy}} + \overbrace{K_i}^{\text{initial mechanical energy}} = U_f + K_f - \underbrace{W_{app}}_{\text{外界對系統做的功}}$

electric field: $\vec{E} = -\vec{\nabla}V$ (i.e., $E_x = -\frac{\partial V}{\partial x}$, $E_y = -\frac{\partial V}{\partial y}$, $E_z = \frac{\partial V}{\partial z}$) $\Rightarrow \vec{\nabla}\times\vec{E} = 0$

Example: Potential due to a charged particle(s)

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

setting $V_f = 0$ (at ∞), $V_i = V$ (at $|\vec{r}| = R$)

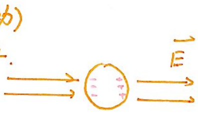
we get

$$0 - V = \frac{-q}{4\pi\epsilon_0}\int_R^\infty \frac{1}{r^2}\hat{r}\cdot d\vec{r}$$

$d\vec{r}$ 為往半徑增大方向

$$= -\frac{1}{4\pi\epsilon_0}\frac{q}{R} \Rightarrow V = \frac{1}{4\pi\epsilon_0}\frac{q}{R}$$

※ 平衡狀態時, isolated的導体內部電場為 0
(即無電荷因受 electric force 作用而運動)
依本節計算, 導体內部隨處為等電位.



↓

電荷移動, 使得導体內部電荷
造成的電場剛好 cancel 外電場

↓

由於需要一定數量之電荷來 cancel 外電場, (或與其它部分电荷之电場達到平衡)
當導体表面尖銳時, 電荷密度高
(造成局部電場高, 易發生尖端放電)

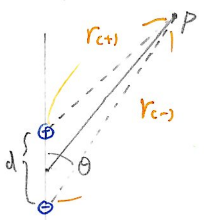
when there are multiple charged particles, due to the superposition of electric field

$$V = \sum_{i=1}^{n} V_i = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i}$$

with continuous charge distribution

$$V = \int dV = \frac{1}{4\pi\epsilon_0}\int\frac{dq}{r}$$

Example: Potential due to an electric dipole



$$V = \sum_{i=1}^{2} V_i = V_{(+)} + V_{(-)} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_{(+)}} + \frac{-q}{r_{(-)}}\right) = \frac{q}{4\pi\epsilon_0}\frac{r_{(-)} - r_{(+)}}{r_{(+)}r_{(-)}}$$

遠距離近似 $\begin{cases} r_{(-)} - r_{(+)} \approx d\cos\theta \\ r_{(-)}r_{(+)} \approx r^2 \end{cases}$

$p = qd$

$$V = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

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General Physics (II) Electromagnetic theory

Energy carried by electric field. (in vacuum)

Potential energy of charge q_i : W_i

$$W_i = q_i V(\vec{x}_i) = \frac{q_i}{4\pi\epsilon_0}\sum_{j=1}^{n-1}\frac{q_j}{|\vec{x}_i - \vec{x}_j|}$$

Total potential energy (note the extra factor 1/2 to avoid duplicating the summation)

$$W = \frac{1}{2}\sum_i\sum_j \frac{1}{4\pi\epsilon_0}\frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}$$

With continuous charge distribution

Gauss law 的微分型 $\rho = \epsilon_0 \vec{\nabla}\cdot\vec{E}$

$$W = \frac{1}{2}\int \rho(\vec{x}) V(\vec{x}) d^3x = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\iint \rho(\vec{x}) \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x\, d^3x'$$

$$= \frac{\epsilon_0}{2}\int (\vec{\nabla}\cdot\vec{E}) V(\vec{x})\, d^3x$$

(integration by part) $$= \frac{\epsilon_0}{2}\int \vec{E}\cdot\underbrace{(-\vec{\nabla}V(\vec{x}))}_{\text{by definition } \vec{E} = -\vec{\nabla}V} d^3x = \frac{\epsilon_0}{2}\int \vec{E}\cdot\vec{E}\, d^3x$$

$$\Rightarrow \underline{W_E = \frac{1}{2}\epsilon_0 \int |\vec{E}|^2 d^3x}$$

依類似計算過程可得 energy carried by magnetic field.

$$W_B = \frac{1}{2\mu_0}\int |\vec{B}|^2 d^3x$$

Dielectricity

若物質可被電場偏極化: $\vec{P} = \epsilon_0 \chi_e \vec{E}$ $\quad \chi_e$: electric susceptibility.

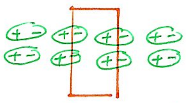
$\vec{P}$: electric dipole in a unit volume ≡ electric polarization

electric potential caused by charge density and electric polarization

$$V(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x' \left[\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} + \frac{\vec{P}(\vec{x}')\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\right] = \frac{1}{4\pi\epsilon_0}\int d^3x' \left[\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} + \vec{P}(\vec{x}')\cdot\vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\right]$$

第二項 integration by part,

$$= \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|}\left[\rho(\vec{x}') - \vec{\nabla}'\cdot\vec{P}(\vec{x}')\right]$$



當 polarization 有梯度時, locally, 相當於有電荷積聚

對於 potential, 此項的作用即為沒有 electric polarization 情形之 charge 項

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}[\rho - \vec{\nabla}\cdot\vec{P}]$$

define electric displacement $\vec{D} \equiv \epsilon_0\vec{E} + \vec{P}$

$$\Rightarrow \vec{\nabla}\cdot\vec{D} = \rho \quad \text{(可用 Gauss' law 解 } \vec{D}\text{)}$$

若物質可被偏極化: $\vec{D} = \epsilon_0\vec{E} + \vec{P}$

$$= \epsilon_0\vec{E} + \epsilon_0\chi_e\vec{E} = \epsilon_0(1+\chi_e)\vec{E}$$

定義: electric permitivity $\epsilon = \epsilon_0(1+\chi_e)$

vacuum permitivity : ϵ_0

dielectric constant: $\epsilon/\epsilon_0 = 1 + \chi_e \Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon}$