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電波天文導論 An Introduction to Radio Interferometry

In most cases, astronomers observe light, and then deduce/testify principles based on the observations.

Light is electromagnetic (EM) wave.

A Telescope is to transform free wave to guided wave (towards receiver or detector).

A receiver/detector is to quantify the power density [energy input in unit time over unit wavelength/frequency]

1. What is EM wave

There are electric field and magnetic field in the space. The space possesses certain energy density (i.e. energy per cm^{-3}) due to this property. The energy density is related to electro/magnetic field strength. Charge is an attribute of particles (e.g. electro, proton); charges alter the electro/magnetic field, including changing the field direction and field strength. The changes are related to the location and amount of the changes (note that the total charge is a conserved physical quantity). Field is more fundamental in the theory.

When we do work to perturbe the location of charges, the EM field is perturbed too. The 1st order perturbation will propagate outward as sinusoidal wave according to the Maxwell's equations. This is the EM wave.

Electrostatic Potential Energy (high-school physics)

Electric potential at location $\vec{x}_i$:

$$\Phi(\vec{x}_i) = \frac{1}{4\pi\underbrace{\epsilon_0}_{\text{permitivity constant}}} \underbrace{\sum_{j=1}^{n-1} \frac{q_j}{|\vec{x}_j - \vec{x}_i|}}_{\text{summing over the contribution from individual charge } q_j}$$

Electric field is the spatial derivative of potential : $\vec{E} = -\vec{\nabla}\Phi$

$$\left(\text{or } E_i = -\frac{\partial}{\partial x_i}\Phi(\vec{x}_i)\right)$$

Discrete charges

Potential energy of charge q_i = W_i

$$W_i = q_i \Phi(\vec{x}_i) = \frac{q_i}{4\pi\epsilon_0}\sum_{j=1}^{n-1}\frac{q_j}{|\vec{x}_i - \vec{x}_j|}$$

Total potential energy = (summing over the potential energy of all charge)

$$W = \frac{1}{2}\sum_i\sum_j \frac{1}{4\pi\epsilon_0}\frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}$$

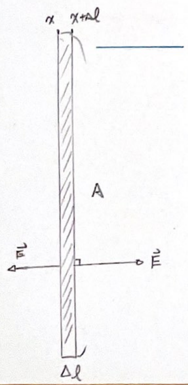
Continuous charges

$$W = \frac{1}{2}\int \underbrace{\rho(\vec{x})}_{\text{charge density}}\Phi(\vec{x})\, d^3x = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\iint \rho(\vec{x})\frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}\, d^3x\, d^3x'$$

How we can express this by the more fundamental quantity, electric field ?

Gauss's law : $\epsilon_0 \oint \vec{E}\cdot d\vec{A} = q_{enc}$

Integral form of Coulomb's law



Thin slab with constant charge density ρ

Enclosed charge : $\rho A \Delta l$

Gauss law :

$$q_{enc} = \rho A \Delta l \xrightarrow[\Delta l \to 0]{} 2\epsilon_0 A E \Rightarrow \rho = 2\frac{E}{\Delta l}\epsilon_0$$

$$\vec{E}(x) = -\vec{E}(x+\Delta l)$$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} \cong \frac{2E}{\Delta l}$$

$$\Rightarrow \rho = \epsilon_0 \vec{\nabla}\cdot\vec{E}$$

charge density is proportional to the spatial derivative of electric field

Electrostatic potential energy

$$W = \frac{1}{2}\int \rho(\vec{x})\,\Phi(\vec{x})\,d^3x = \frac{\epsilon_0}{2}\int (\vec{\nabla}\cdot\vec{E})\,\Phi(\vec{x})\,d^3x$$

integration by part

$$= \frac{\epsilon_0}{2}\int \vec{E}\cdot\underbrace{(-\vec{\nabla}\Phi(\vec{x}))}_{\vec{E}=-\vec{\nabla}\Phi}\,d^3x = \frac{\epsilon_0}{2}\int |E|^2 d^3x$$

magnetic field energy has a similar form due to the symmetry of Maxwell's equations

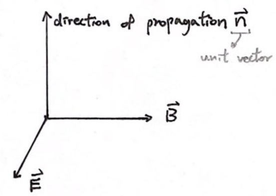
$$W = \frac{1}{2\mu_0}\int |B|^2 d^3x$$

μ_0 → magnetic permeability

※ Energy density in the space is proportional to field strength square.

energy density is transferred with the propagation of EM wave.

→ Energy is a capability to do work. ⟹ Electric field accelerates electrons, i.e., the accelerated electron gain energy, the field loses energy. The capability of doing work propagates with the EM wave. In the astronomical observations, the electrons in the receiver/detector eventually extract energy from the EM waves, which is then converted to heat (bolometer), photo-electrons' kinetic energy, or the energy carried by the current.



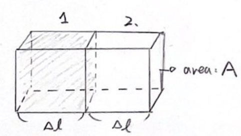
According to the Maxwell's equation, the ① direction of EM wave propagation is the same as $\vec{E}\times\vec{B}$

② in vacuum, the speed of EM wave propagation $c = \frac{1}{\sqrt{\epsilon_0\mu_0}}$

(energy propagation rate per unit time per unit area)

(Energy) flux density : Poynting vector $\vec{S} = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}|E|^2\vec{n}$

→ How to comprehend and remember the energy transfer rate ? (handwavy derivation)



Supposedly, a wave packet is propagating from box 1 to box 2 in time $\Delta t = \frac{\Delta l}{c}$

initial total energy in box 1 : $\left(\frac{\epsilon_0}{2}E^2 A\Delta l\right) \equiv W$

$$\boxed{\text{flux density}} = \frac{W}{A\,\Delta t} = \frac{cW}{A\Delta l} = \frac{1}{2}\epsilon_0 c E^2 = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}|E|^2 \quad \left(c = \frac{1}{\sqrt{\epsilon_0\mu_0}}\right)$$

→ To convert to photon flux density, we can divide the photon energy $h\nu$

※ To quantify energy flux density
1. optical observations directly measure counts of photons in a certain area in a unit of time.
2. Radio detectors measure E

2. Flux density : F_ν

ν : frequency of EM wave, in unit of Hz

In astrophysical literature, flux density is usually expressed as F. When the wave is not monochromatic, then we use F_ν to denote that it is a function of frequency ν.

The SI unit of F_ν is $\mathrm{Joul\ s^{-1}\,m^{-2}\,Hz^{-1}} = \mathrm{W\,m^{-2}\,Hz^{-1}}$

Instead of SI unit, most of the astrophysical literature use the unit

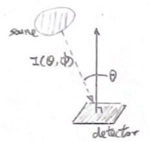
$$1\ \mathrm{Jy} = 10^{-26}\ \mathrm{W\,m^{-2}\,Hz^{-1}}$$

If the detector is a 2D surface, eg. a CCD, when the incoming flux density is F_ν,
the overall energy we receive is $dE = F_\nu \, dA \, d\nu \, dt$, where dA is the area of the detector.
The incoming power, by definition, is $dP = dE/dt = F_\nu \, dA \, d\nu$

3. Intensity and brightness temperature

※ solid angle = $d\Omega = \sin\theta \, d\theta \, d\phi$

The incoming flux density can be contributed by multiple celestial sources that are located in various directions. To describe how they contribute to the incoming flux density, we define the quantity, intensity.



include angle (夹角) between the incoming light direction and the normal vector of the detector surface

Intensity $\equiv I_\nu(\theta, \phi)$

$$F_\nu = \int I_\nu(\theta, \phi) \cos\theta \, d\Omega$$

The SI unit of intensity is Joul $s^{-1} \, m^{-2} \, Hz^{-1} \, sr^{-1}$

In astronomical observations, we can in principle use the unit of $Jy \, sr^{-1}$

However, since in radio observations, our calibrition sources can often be nearly ideal black bodies, therefore, it is also common to refer to intensity simply with the temperature of the black body that contributes to the same intensity as our target source at the observing frequency ν.

The intensity of an ideal black body emitter is $B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$ ($B_\nu(T)$ and I_ν have the same dimension)

Planck const.

Boltzmann const.

temperature of the black body

In the Rayleigh-Jean limit (i.e. long wavelength limit, $h\nu \ll kT$)

$B_\nu^{RJ}(T) \equiv \frac{2\nu^2}{c^2} kT$ [This approximation is very commonly adopted in radio interferometry]

※ When $I_\nu(\theta, \phi) = B_\nu(T)$, we say the brightness temperature in the direction (θ, ϕ)

$T_B = T$

→ In the Rayleigh-Jeans limit

$$I_\nu = \frac{2\nu^2}{c^2} k T_B$$

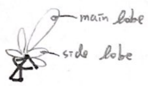
$$\Rightarrow T_B = \frac{c^2}{2k\nu^2} I_\nu = \frac{\lambda^2}{2k} I_\nu$$

(※ this is the formula to convert from intensity to brightness temperature)

4. Angular Resolution and $\mathrm{Jy\,beam^{-1}}$ unit

In reality, we never achieve infinitely fine angular resolution, i.e. any element in our detector/receiver always integrate the incoming intensity over a finite solid angle. That is, only the flux density over a finite incoming solid angle $d\Omega$ is a directly measurable quantity. Intensity $I_\nu(\theta,\phi)$ is not directly measurable, although it is more fundamental than F_ν.

Practically, for a certain telescope, the smallest $d\Omega$ we can use is the solid angle of the main lobe of the response function of our telescope or interferometry array.



In the case of single-dish telescope, we call this main lobe the primary beam.
In the case of interferometry, we call it synthesized beam.

a function $P(\theta,\phi)$ that is normalized to have peak $=1.0$
In radio astronomy, $P(\theta,\phi)$ often can be approximated by a 2D Gaussian in the main lobe.

※ The beam solid angle Ω_B is defined as $\Omega_B = \int_0^{4\pi} P(\theta,\phi)\,d\Omega \longrightarrow$ in the interferometry studies, we also often only integrate over the main lobe

The power received by an antenna with effective cross-section A_e (single polarization)

$$P_\nu = \frac{1}{2} A_e \int I_\nu(\theta,\phi)\,P(\theta,\phi)\,d\Omega \equiv k\underbrace{T_A}$$

this $\frac{1}{2}$ factor is to take into account polarization.

definition of antenna temperature,
The temperature of an equivalent resistor

If we consider dual polarization receivers, the

$$P_\nu = A_e \int I_\nu(\theta,\phi)\,P(\theta,\phi)\,d\Omega$$

$$\underbrace{\frac{P_\nu}{A_e}}_{\text{flux density}} = \int I_\nu(\theta,\phi)\,P(\theta,\phi)\,d\Omega$$

if $\frac{P_\nu}{A_e} = 1$ Jy and I_ν is a constant over the beam.

then $\quad I_\nu(\theta,\phi)\int P(\theta,\phi)\,d\Omega = I_\nu\,\Omega_B = 1$ Jy

$$\Rightarrow I_\nu = \frac{1\ \mathrm{Jy}}{\Omega_B}$$ pronounced as 1 Jansky per beam

※ This is a very practical intensity unit in radio interferometry. The intensity that is N times this value is called

N Jansky per beam

Example: Elliptical Gaussian beam with major and minor axis full width at half maxima (FWHM) θ_a and θ_b (small angle approximation)

※ Gaussian FWHM $= 2\sqrt{2\ln 2}\,\sigma$

$$P = \exp\left(\frac{-\xi^2}{2\sigma_a^2} + \frac{-\eta^2}{2\sigma_b^2}\right)$$

$$= \exp\left[4\ln(2)\left(\frac{-\xi^2}{(2\sqrt{2\ln 2}\,\sigma_a)^2} + \frac{-\eta^2}{(2\sqrt{2\ln 2}\,\sigma_b)^2}\right)\right] = \exp\left[4\ln(2)\left(\frac{-\xi^2}{\theta_a^2} + \frac{-\eta^2}{\theta_b^2}\right)\right]$$

$$\Rightarrow \Omega_B = \int \exp\left[-4\ln(2)\left(\frac{\xi^2}{\theta_a^2} + \frac{\eta^2}{\theta_b^2}\right)\right]d\xi\,d\eta = \frac{\pi\theta_a\theta_b}{4\ln 2}$$

Gaussian integration: $\int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$

For Gaussian beam, the conversion between Jy beam^{-1} unit and brightness temperature is

intensity in SI unit

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(see page 3)

$$T_B = \frac{\lambda^2}{2k} I_\nu = \frac{\lambda^2}{2k}\left(\frac{F_\nu}{\Omega_B}\right)$$

intensity in Kelvin unit, i.e., brightness temperature

$$= \frac{\lambda^2}{2k}\left(\frac{F_\nu}{beam}\right)\left(\frac{beam}{\Omega_B}\right) = \frac{\lambda^2}{2k} I_\nu^{Jy/beam} \frac{beam}{\Omega_B}$$

intensity in Jy beam^{-1} unit

$$= \frac{\lambda^2}{2 \cdot 1.38 \cdot 10^{-23}} I_\nu^{Jy/beam} \frac{4 \ln 2}{\pi \theta_a \theta_b}$$

$$= 3.2 \cdot 10^{22} \frac{\lambda}{\theta_a \theta_b} I_\nu^{Jy/beam}$$

beam FWHM in the major and minor axes.

intensity in Jy beam^{-1} unit.

The key information when we look up data archive

target source name/coordinates, ranges of observing frequency, frequency resolution, angular resolution ($\theta_a \times \theta_b$; P.A.). the lowest (1σ) T_B or $I_\nu^{Jy/beam}$ can be detected,

polarization (X, Y, R, L,)

In the interferometric observations, the maximum recoverable angular scale is also important. (to be introduced later)