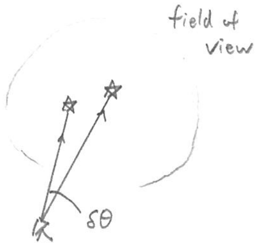
Diffraction Limit and Angular Resolution

lecture 2
呂浩宇

What is radio interferometry? Why use radio interferometry?

Radio interferometry is a sort of telescope. One of the most important parameter to characterize a telescope, is the <u>angular resolution</u>

the smallest angular offset that can be discerned



if $\delta\theta < \Delta\theta$, then we cannot tell whether there two sources or one: Rayleigh criterion

Diffraction limited angular resolution: $\Delta\theta \sim \dfrac{\lambda}{D}$ (λ → observing wavelength; D → aperture size)

※ Note that the concepts of <u>angular resolution</u> and <u>position accuracy</u> are different. Angular resolution is the smallest angular offset (e.g. between two point sources) that can be resolved by the observations. Position accuracy tells how well we can determine the location of a single target source (i.e. astrometry).

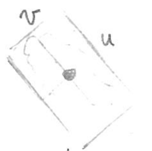
Example: if human eyes are ~1 cm aperture, then the angular resolution is 10″ (the actual human eye resolution is ~1′)

(1° = 60′ = 3600″) (′ → arcminute; ″ → arcsecond)

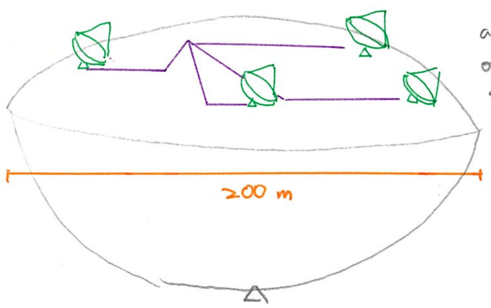
to achieve such angular resolution the required aperture size

- visible: $\lambda \sim 0.5 \cdot 10^{-6}$ m ——— 1 cm diameter
- (sub)millimeter band: $\lambda \sim 10^{-3}$ m ——— 2000 cm ~ 20m diameter
  - median-size single-dish telescope, e.g. { JCMT 15m, IRAM 30m }
- 22GHz water maser: $\lambda \sim 1$ cm ⟶ requires 200m diameter telescope ⟸ too big to be constructed
  - observing this maser has important application in the determination of black hole mass

To achieve high angular resolutions at long wavelength, we need the technique of Radio Interferometry



a nebula projected on the celestial sphere



1. Placing multiple small radio telescopes over a region that approximately has ~200 m diameter
2. Employing the radio interferometry technique to make these telescopes work effectively as a telescope that has the same angular resolution as a 200m-diameter single-dish telescope.

The directly measured results: The sparsely sampled Fourier Transform coefficient of the intensity distribution.

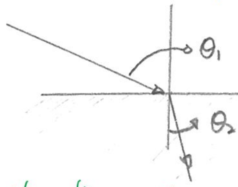
Requires calibration and inverse transformation to convert back to the original intensity distribution

吕浩宇

Principle of Interference

Interferometry is a type of optical device. Most of the aspects we need to know when understanding an optical device are

- optical device
  - principle
    - reflection, refraction (related to n) → usually only need to remember
      1. incident angle = reflected angle
      2. Snell's law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$ (refracted angle)
    - refraction interference (related to propagation & superposition)
  - layout & purpose — allows estimating
    - systematic errors
    - (thermal) noise

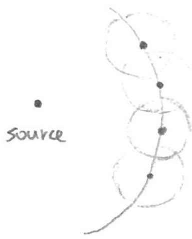


1. EM-wave at long-distance limit: plane wave: $E_0 \cos(\underbrace{k}_{\frac{2\pi}{\lambda}}x - \overbrace{\omega}^{2\pi\nu} t + \phi_0)$

   ν: frequency, ω: angular frequency, ϕ_0: phase offset.

2. The EM-wave propagation can be described by Huygen's principle
   1. every point on a wave front can be regarded as a new source
   2. The wave front is spherical symmetric with respect to the source.

3. The solutions of wave equation satisfy the principle of superposition



4. In vacuum, the speed of EM wave propagation is $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

   In the medium that the index of refraction is n:
   ① the speed of propagation is $\frac{c}{n}$
   ② frequency is not changed with n, wavelength becomes $\frac{\lambda_0}{n}$ where λ_0 is the wavelength in vacuum.

5. The flux density of photon is proportional to E_0^2 (Poynting theory)

For every principle, the exists a corresponding, simple example.

Being familiarized with the simple example helps comprehend complicated systems.

Example. Single Slit as an Optical device

Incident wave through a single slit, forming images on infinitely distant screen.

At angle $\theta=\theta'$, the optical path diffence

$$\Delta x = \frac{1}{2}D\sin\theta' \sim \frac{1}{2}D\theta' \quad (\text{small } \theta')$$

completely destructive interference:

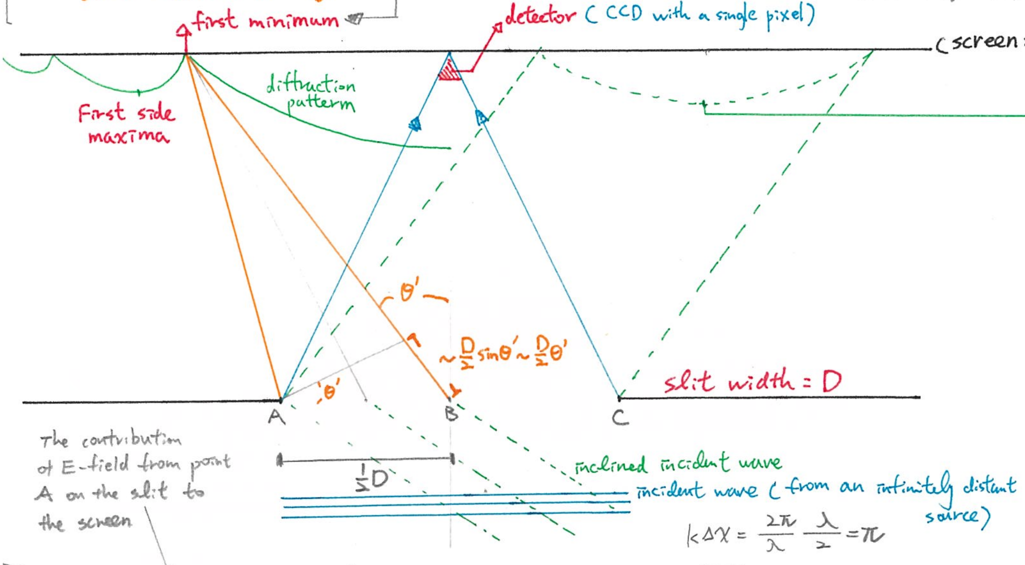
$$\frac{1}{2}D\theta' = \frac{1}{2}\lambda \Rightarrow \theta' = \frac{\lambda}{D}$$

$\theta = 0^\circ$ → points A and C at the slit have the same optical path length to $\theta=0$ position on the screen. With a long integration time, the photon number density measured by the CCD is proportional to

$$(E_A+E_C)^2 = E_0^2\left(\cos(kx-\omega t+\phi_0)+\cos(kx-\omega t+\phi_0)\right)^2$$

→ form central maxima due to the complete constructive interference

(screen: can be illustrated as a CCD) — a photon detector



For inclined incidence, the primary maxima or central maxima form at the offset position, such that the phase gradiant along the slit is compensated.

If our CCD only has one valid pixel at $\theta=0^\circ$, then when the incident wave front is not parallel to the slit, the CCD pixel will detect relatively faint signal.

In other words, this slit + CCD system has a characteristic response function $P(\theta)$, where $P(\theta)$'s form is identical to the diffraction pattern of a single slit.

The contribution of E-field from point A on the slit to the screen

$$(E_A+E_B)^2 = E_0^2\left(\cos(kx-\omega t)+\cos(k(x+\overset{\frown}{\Delta x})-\omega t)\right)^2$$

$$= E_0^2\left(\cos(kx-\omega t)+\cos(kx+\pi-\omega t)\right)^2$$

these two cosine function cancel each other completely.

The more explicit derivation: (dividing the slit into N pieces, then make $N\to\infty$)

At emergence angle θ', the photon number density is proportional to

$$\left[\sum_{n=0}^{N}\cos(k(x+\overset{\text{path length difference at the nth piece}}{\widetilde{\Delta x(n)}})-\omega t)\right]^2$$

$$= \left[\sum_{n=0}^{N}\cos\left(kx+k\frac{nD}{N}\sin\theta'-\omega t\right)\right]^2$$

when $N\to\infty$, $\frac{nD}{N}$ can be viewed as a variable of integration that is bounded by $[0,D]$

$\xrightarrow{N\to\infty}$ Electric field

using the trigonometric identity

$$\sin\theta-\sin\varphi = 2\sin\left(\frac{\theta-\varphi}{2}\right)\cos\left(\frac{\theta+\varphi}{2}\right)$$

$$\frac{\left(\sin(kx+kD\sin\theta'-\omega t)-\sin(kx-\omega t)\right)}{k\sin\theta'}$$

$$= \boxed{\frac{\sin(\frac{1}{2}kD\sin\theta')}{\frac{1}{2}k\sin\theta'}}\cos(kx-\omega t+\overbrace{\tfrac{1}{2}kD\sin\theta'}^{\text{phase modulation}})$$

amplitude modulation

- numerator: sinusoidal
- denominator: linear decay

photon number density at emergent angle θ'

$$\propto \frac{(\sin\frac{1}{2}kD\sin\theta')^2}{(k\sin\theta')^2}$$

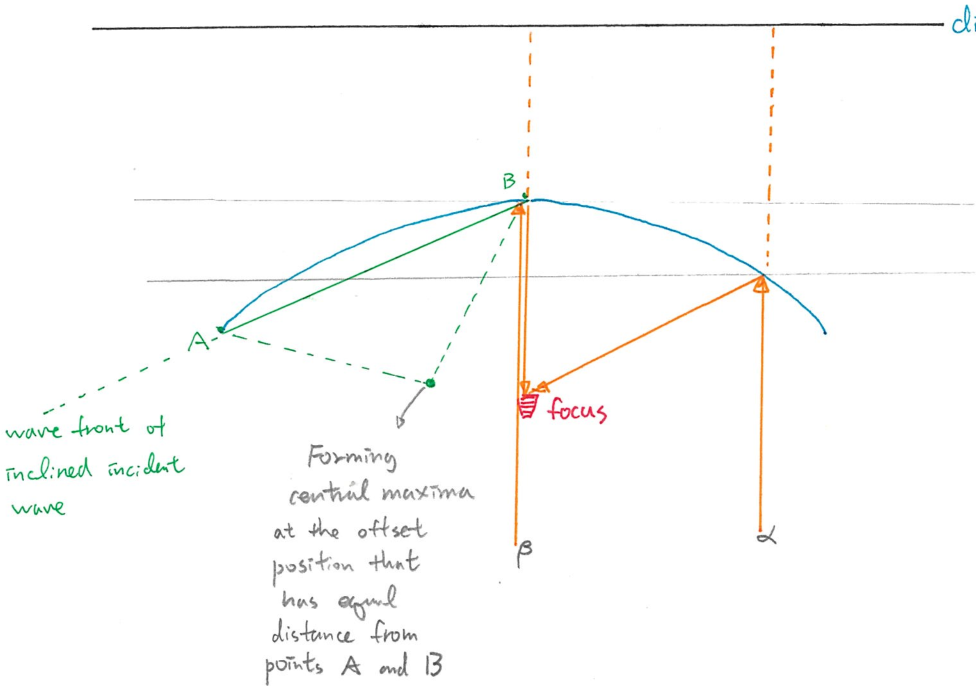
Small summary:

For an emergent angle θ', the mixed E-field has the same frequency as the incident wave. But the phase and amplitude are modulated, showing dependence on

Response of a parabolic mirror

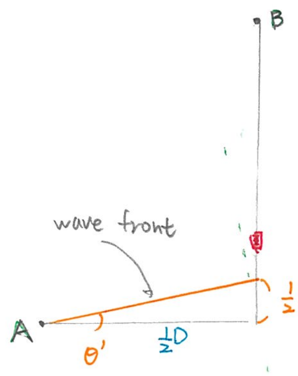
Only incident light that intersects the mirror can contribute to image

1. A parabola is defined by a directrix (準線) and a focus (焦點)
2. The vertical distance from a surface point on the parabola to the directrix is identical to the distance from that point to the focus.
3. According to the principle that the incident angle is equal to the reflected angle, any incident light beam that is normal to the directrix will be reflected to focus.



The central maximum forms at the focus. The light beams α and β have no path length difference: light beam travels a longer distance before reflection, but travel a shorter distance after reflection.

How to form minimum at the focus (small angle approximation)



$\frac{1}{2}D\theta'$: this is approximately the optical path length difference if the incident light has a small inclination.

The minimum forms at the focus when

$$\frac{1}{2}D\theta' \sim \frac{1}{2}\lambda$$

$$\Rightarrow \theta = \frac{\lambda}{D} \text{ (diffraction limited resolution)}$$

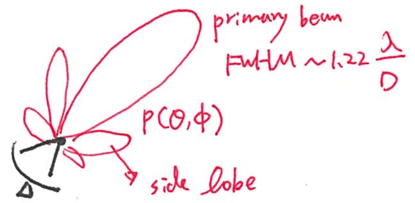
For a two dimensional mirror, this is corrected to $1.22\frac{\lambda}{D}$

The net effect of having a dish

at the focus or focal plane (similar to single slit) the electric field varies with the same frequency as the incident EM wave. But the amplitude and phase are modulated.

⇓

The bigger the mirror, the better capability in collecting light and the better angular resolution. Any reflection mirror can be comprehend in a similar way.



non-monochromatic wave can be understood as superposition of monochromatic wave. (單色光)

The Origin and Effect of Telescope Response Function

呂浩宇

All telescopes are subject to the effect of response function (or point spread function, PSF).

The mathematical form is convolution : (1-D example)

$$\underbrace{h(x)}_{\text{the image that is affected by PSF}} = \int_{-\infty}^{\infty} \overbrace{f(\tau)}^{\text{PSF}} \underbrace{g(x-\tau)}_{\text{the original image}} \, d\tau$$

e.g.

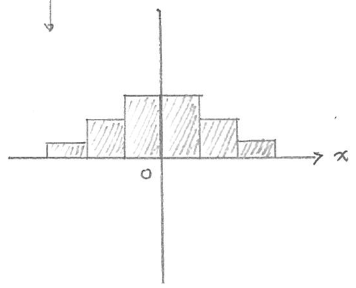
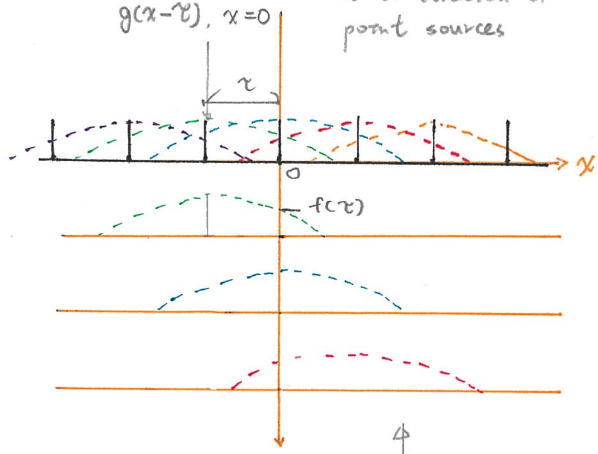


illustration: if the original image is a collection of point sources



if the image is a point source,

$$g(x) = \delta(x)$$

$$\Rightarrow h(x) = \int_{-\infty}^{\infty} f(\tau)\,\delta(x-\tau)\,d\tau = f(x)$$

↳ the shape of the image is identical to the PSF.

case with multiple point sources.

A light source with arbitrary intensity distribution can be illustrated as a superposition of multiple point sources.

↳ PSF can be originated due to

1. atmospheric perturbation
2. the properties of the optical devices. (e.g. diffraction limited resolution, refraction/diffraction, etc)

Given the measurement of $h(x)$, if the form of $f(x)$ is known, then the procedure to deduce $g(x)$ is called deconvolution. Deconvolution normally requires non-linear algorithms.

※ Convolution theorem

$$F.T.(f * g) = F.T.(f) \cdot F.T.(g)$$

$$F.T.(f \cdot g) = F.T.(f) * F.T.(g)$$