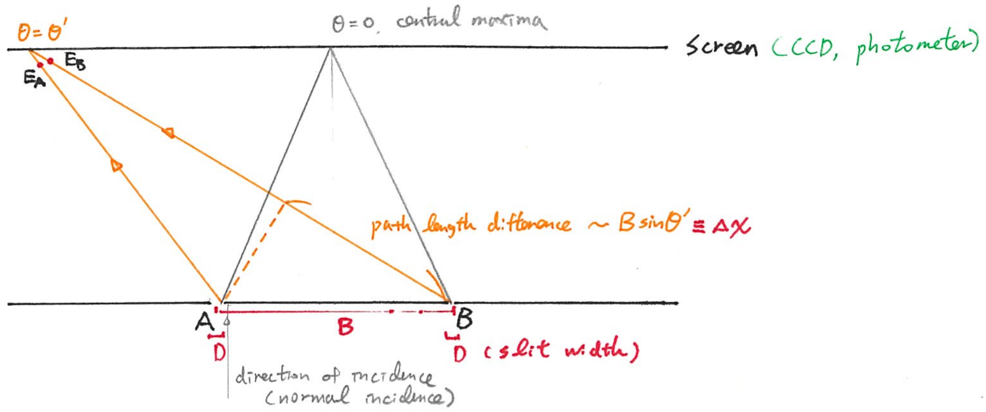
From double-slit to radio interferometer

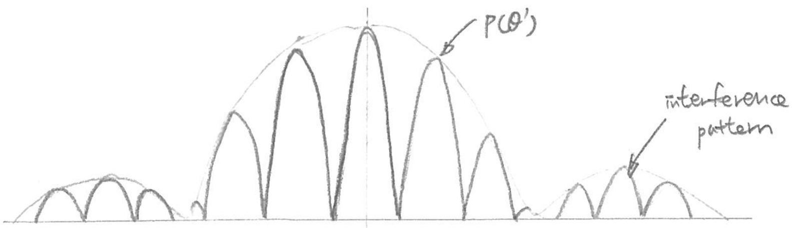


(i) if $D \sim 0$, for normal incident light with wavelength λ (making long-term exposure)
light sources A and B can be viewed as monochromatic light sources that have identical initial phase.

photon number flux density at $\theta = \theta'$ on the screen

$$\propto \Big(\underbrace{\cos(kx - \omega t + \phi_0)}_{E_A:\ \text{electric field contributed by light source A}} + \underbrace{\cos(k(x+\Delta x) - \omega t + \phi_0)}_{E_B} \Big)^2$$

$$= \overbrace{\cos^2(kx - \omega t + \phi_0)}^{\text{auto-correlation}} + \overbrace{\cos^2(k(x+\Delta x) - \omega t + \phi_0)}^{\text{auto-correlation (long-term average over time is } \frac{1}{2})} + 2\underbrace{\cos(kx - \omega t + \phi_0)\cos(k(x+\Delta x) - \omega t + \phi_0)}_{\text{cross-correlation}}$$



using trigonometric identity:
$\cos\theta\cos\varphi = \frac{1}{2}\left(\cos(\theta - \varphi) + \cos(\theta + \varphi)\right)$

$$\cos(k\Delta x) + \underbrace{\cos(2kx - 2\omega t + 2\phi_0 + k\Delta x)}_{\text{long-term time average is } 0}$$

Amplitude equal to 1. At the minimum $\cos(k\Delta x) = -1$
this cross-correlation term (together with the factor 2)
can cancel the contribution of the auto-correlation
terms.

$\frac{2\pi}{\lambda} \cdot B\sin\theta = m\pi$,
$m = 1, 3, 5, \ldots$
$\theta' \sim \frac{m}{2}\frac{\lambda}{D}$

When D is not negligibly small, the amplitude of the electric field contributed by each of the two slits is modulated by the factor $\dfrac{\sin(\frac{1}{2}kD\sin\theta')}{\frac{1}{2}k\sin\theta'}$, and carries an extra phase offset $\frac{1}{2}kD\sin\theta'$. Since the phase offsets are identical, the calculation of the interference pattern is not affected. The amplitude modulation can be expressed as a envelope factor $P(\theta)$ in the final interference pattern:

(photon number flux density)
Interference pattern can be expressed by

$$\frac{I(\theta')}{A} = P(\theta')\Big[\underbrace{1}_{\text{contribution of auto-correlation terms}} + \overbrace{\cos\left(\tfrac{2\pi}{\lambda}B\theta'\right)}^{\text{contribution of cross-correlation term}}\Big] = P(\theta')\left[1 + \cos(2\pi u\theta')\right]$$

A: a coefficient that describes how bright is each of the light source

$u \equiv \frac{B}{\lambda}$: slit separation measured in unit of wavelength

As compared to cross-correlation term auto-correlation terms are less sensitive to angle of emergence θ

when $P(\theta')$ and u are known, making two independent measurements of $I(\theta')$ (i.e., at two different θ' positions) will allow unambiguously determining θ' and A

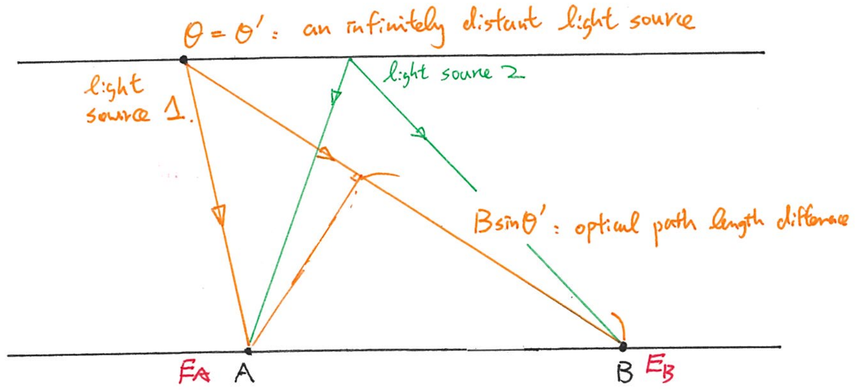
Radio detector technique

Directly measuring the electric field strength $E_0 \cos(kx - \omega t + \phi_0)$ instead of photon number density, i.e., directly measuring $\begin{cases} E_A = \cos(kx - \omega t + \phi_0) \\ E_B = \cos(k(x+\Delta x) - \omega t + \phi_0) \end{cases}$

then use correlator to evaluate

$\langle E_A \rangle^2$, $\langle E_B \rangle^2$, $\langle E_A E_B \rangle^2$ → denotes long-term time average.

Radio Interferometry



(i) when there is only one infinitely distant, monochromatic light source, the $\langle E_A E_B \rangle$ is exactly the same with the case of double slit

(ii) when there are two light sources, denoted as light sources 1 and 2, then $\begin{cases} E_A = E_A^1 + E_A^2 \\ E_B = E_B^1 + E_B^2 \end{cases}$

$$\begin{aligned} \langle (E_A + E_B) \rangle^2 = &\langle E_A^1 \rangle^2 + \langle E_B^1 \rangle^2 + \langle 2E_A^1 E_B^1 \rangle \text{ — interference pattern of source 1 alone} \\ &+ \langle E_A^2 \rangle^2 + \langle E_B^2 \rangle^2 + \langle 2E_A^2 E_B^2 \rangle \text{ — interference pattern of source 2 alone} \\ &+ \underbrace{\langle 2E_A^1 E_A^2 \rangle + \langle 2E_A^1 E_B^2 \rangle + \langle 2E_B^1 E_A^2 \rangle + \langle 2E_B^1 E_B^2 \rangle} \end{aligned}$$

if the two sources are not correlated, then the phase of these cross-term will walk randomly over time. In this case, the time average of these cross-term over time is zero

↓

Acoherent source

thermal noise is also acoherent source, of which the expectation value is 0, but the root-mean-square is positively definite.

Radio Interferometry : generalized to N acoherent sources

If we only look at the cross-correlation terms.

$$\langle E_A E_B \rangle \propto \sum_{i=0}^{N} P(\theta_i') \overbrace{A_i}^{\text{a coefficient to describe how bright is the source}} \cos(2\pi \underbrace{u}_{\frac{B}{\lambda}} \theta')$$

generalizing from descrete sources to continuous sources.

F.T. coefficient

$$\int \underbrace{P(\theta)A(\theta)}_{\text{the original intensity distributon, modulated by the primary beam response function.}} \overbrace{\cos(2\pi u \theta)\, d\theta}^{\text{Fourier transform}} \longrightarrow \text{output of cosine correlator (sensitive to symetric } P(\theta)A(\theta) \text{ function)}$$

If there is a way to manually add an $\frac{1}{2}\pi$ phase to E_B. (denoting phase shifted case as E_B')

$$(E_A + E_B')^2 = \left(\cos(kx - \omega t + \phi_0) + \cos\left(k(x+\Delta x) - \omega t + \phi_0 + \frac{\pi}{2}\right)\right)^2$$

$$= \left(\cos(kx - \omega t + \phi_0) + \cos(k(x+\Delta x) - \omega t + \phi_0) \underbrace{\cos\frac{\pi}{2}}_{=0} - \sin(k(x+\Delta x) - \omega t + \phi_0) \underbrace{\sin\frac{\pi}{2}}_{=1} \right)^2$$

$$= \cos^2(kx - \omega t + \phi_0) + \sin^2(k(x+\Delta x) - \omega t + \phi_0) \underbrace{- 2\cos(\overbrace{kx - \omega t + \phi_0}^{\theta}) \sin(\overbrace{k(x+\Delta x) - \omega t + \phi_0}^{\varphi})}$$

using trigonometric identity

$$\cos\theta \sin\varphi = \frac{1}{2}\left(\sin(\theta+\varphi) - \sin(\theta - \varphi)\right)$$

$$\underbrace{\sin(2kx - 2\omega t + 2\phi_0 + k\Delta x)}_{\text{long-term average is } 0} - \underbrace{\sin(k\Delta x)}_{\text{sine correlation}}$$

generalizing from descrete sources to continuous sources

$$\int P(\theta)A(\theta) \sin(2\pi u \theta)\, d\theta \longrightarrow \text{output of sine correlator (sensitive to asymmetric } P(\theta)A(\theta) \text{ function)}$$

we can combine these measurements to yield the <u>complex visibility</u>

The outputs of cosine/sine correlators are the real/imaginary parts of the complex visibility. A complex visibility includes two independent measurements. The flux density and position of a single point-source can be determined by one complex visibility.

※ Imaging: the inversion from $\langle E_A E_B \rangle$ to $P(\theta)A(\theta)$

If we can vary the value of u continously (i.e., changing the separation of the two radio detectors), then we can make a complete fourier transform of $P(\theta)A(\theta)$. In this case, imaging can be a simple inverse Fourier transform. Otherwis, we obtain incomplete samplings that requires a more complicate

Example of complex visibility: point source

$$A(\theta) = \delta(\theta - \theta')$$

⇒ output of cosine correlator

complex gain
contributed by atmospheric and instrumental effects.

$$g \int P(\theta)\,\delta(\theta-\theta')\cos(2\pi u\theta)\,d\theta$$

$$= g\,P(\theta')\cos(2\pi u \theta') \quad \text{for any } u$$

similarly, the output of sine correlator is $g\,P(\theta')\sin(2\pi u\theta')$

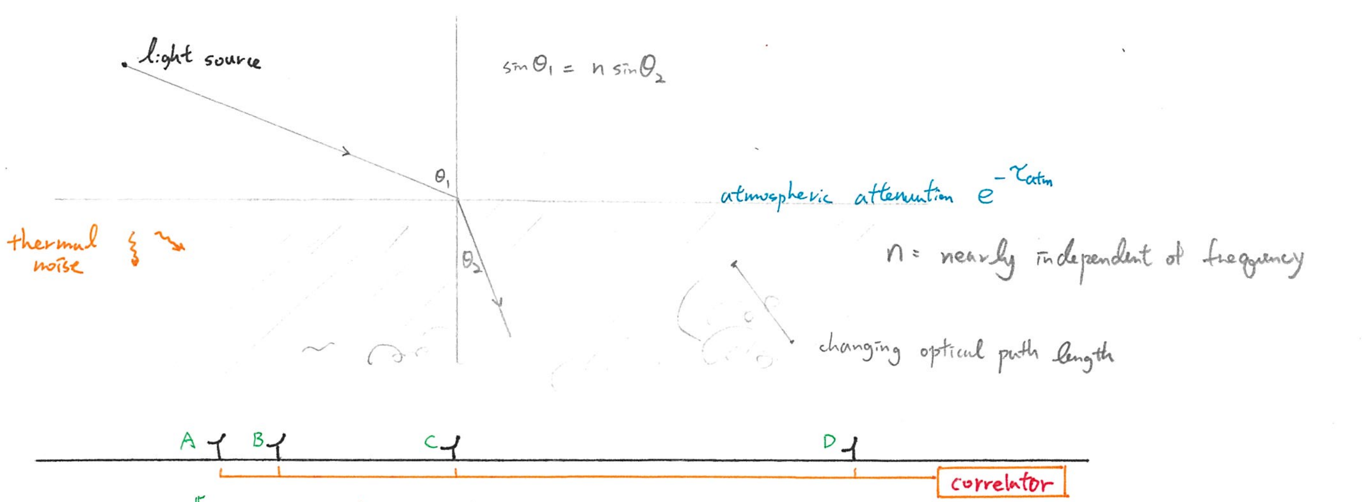
complex visibility: $g\,P(\theta')\left(\cos(2\pi u\theta') + i\sin(2\pi u\theta')\right)$

$$= \underbrace{g\,P(\theta')}_{\text{uncalibrated visibility amplitude}}\, e^{i\overbrace{2\pi u\theta'}^{\text{phase}}}$$

(e.g., in unit of volts)

Summary: for a point source, visibility amplitude does not vary with $|u|$.
For a fixed source location θ', phase depends linearly on u.
We can regard $2\pi u$ as the rate for the phase to vary with θ'.
With a longer baseline length $|u|$, phase will vary more rapidly with θ'

1D linear radio interferometer



u_{AB}, u_{AC}, u_{AD}, u_{BC}, u_{BD}, u_{CD}, ...

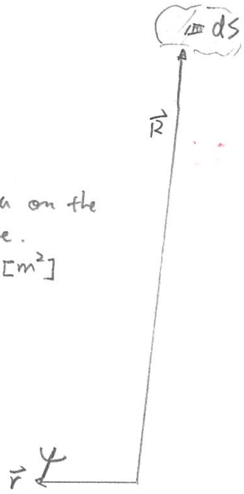
The total number of independent baseline is $\frac{1}{2}N(N-1)$

When designing an array, the less redundant is the sampling u_{ij} (i.e., less repeats of the measurements at the same baseline length), the better for the image reconstruction.

※ if you cannot understand the derivation, you can read the 1-D case again, and take the conclusion for 2D as granted for a moment)

Generalization to a 2D array observations towards acoherent celestial sources (quasi-monochromatic)

(ignoring polarization for this moment)



Electric field contributed from infinitely distant sources:

$$E_\nu(\vec{r}) = \int \mathcal{E}_\nu(\vec{R}) \frac{e^{ik|\vec{R}-\vec{r}|}}{|\vec{R}-\vec{r}|} dA = \int \mathcal{E}_\nu(\vec{R}) \frac{e^{i\frac{2\pi}{c/\nu}|\vec{R}-\vec{r}|}}{|\vec{R}-\vec{r}|} dA$$

$E_\nu(\vec{r})$: the electric field detected at the station that is located at $\vec{r}$

$\mathcal{E}_\nu(\vec{R})$: field strength at the source location

$\frac{1}{|\vec{R}-\vec{r}|}$: Huygens' Principle: dilution of spherical wave field-strength during the propagation

dA → a surface area on the celestial sphere. Dimension is $[m^2]$

$dA = |\vec{R}|^2 d\Omega$

if we correlated the signal taken at two different locations

$$V_\nu(\vec{r}_1, \vec{r}_2) = \langle E_\nu(\vec{r}_1) E_\nu^*(\vec{r}_2) \rangle = \left\langle \iint \mathcal{E}_\nu(\vec{R}_1) \mathcal{E}_\nu^*(\vec{R}_2) \frac{e^{ik|\vec{R}_1-\vec{r}_1|}}{|\vec{R}_1-\vec{r}_1|} \frac{e^{-ik|\vec{R}_2-\vec{r}_2|}}{|\vec{R}_2-\vec{r}_2|} dA_1 dA_2 \right\rangle$$

$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)$

for acoherent source,

$\langle \mathcal{E}_\nu(\vec{R}_1) \mathcal{E}_\nu^*(\vec{R}_2) \rangle = 0$ for $\vec{R}_1 \neq \vec{R}_2$

(i.e. this can be expressed as delta function)

※ 1D:

$$\frac{1}{|R-r|} = \frac{1}{((R-r)^2)^{\frac{1}{2}}} \xrightarrow[\text{1st order Taylor expansion}]{R \gg r} \frac{1}{R(1-2\frac{r}{R})^{\frac{1}{2}}} \simeq \frac{1}{R(1-\frac{r}{R})} \simeq \frac{1}{R}(1+\frac{r}{R})$$

3D generalization ($|\vec{R}| \gg |\vec{r}|$)

$$|\vec{R}-\vec{r}| \simeq |\vec{R}| - \frac{\vec{R}\cdot\vec{r}}{|\vec{R}|}$$

$$V_\nu(\vec{r}_1, \vec{r}_2) \simeq \left\langle \iint \mathcal{E}_\nu(\vec{R}_1) \mathcal{E}_\nu^*(\vec{R}_2) \frac{e^{ikR} e^{-ik\vec{s}_1\cdot\vec{r}_1}}{|R_1|} \frac{e^{-ikR} e^{ik\vec{s}_2\cdot\vec{r}_2}}{|R_2|} |\vec{R}_1|^2 d\Omega_1 |\vec{R}_2|^2 d\Omega_2 \right\rangle$$

$s_1 \equiv \frac{\vec{R}_1}{|\vec{R}_1|}$: unit vector

intensity : $I_\nu(\vec{s}) \equiv |\vec{R}|^2 \langle |\mathcal{E}_\nu(\vec{s})|^2 \rangle$, $s \equiv \frac{\vec{R}}{|\vec{R}|}$

$$\Rightarrow V_\nu(\vec{r}_1, \vec{r}_2) \approx \int I_\nu(\vec{s}) e^{-ik\vec{s}\cdot(\vec{r}_1-\vec{r}_2)} d\Omega$$

$$= \int I_\nu(\vec{s}) e^{-2\pi i \vec{s}\cdot\frac{(\vec{r}_1-\vec{r}_2)}{\lambda}} d\Omega$$

① for a coplanar array, we can always select the Cartesian coordinate system such that $\vec{r}_1 - \vec{r}_2$ can be explicitly expressed as $(\lambda\vec{u}, \lambda\vec{v}, \lambda\vec{w})$ — baseline length measured in units of wavelength

② the unit direction vector $\vec{s}$ can be expressed in a cartesian coordinate system as $(l, m, \sqrt{1-l^2-m^2})$

$$V_\nu(u, v, w\equiv 0) = \iint I_\nu(l, m) \frac{e^{-2\pi i (ul + vm + w\sqrt{1-l^2-m^2})}}{\sqrt{1-l^2-m^2}} dl\, dm$$

For target source with very small angular scale

$\vec{s} \equiv \vec{s}_0 + \vec{\sigma}$ — $\vec{\sigma}$: angular offset from the center

$\vec{s}_0$: unit vector in the line-of-sight direction towards the center of the source → we call $\vec{s}_0$ the phase tracking center

choosing a Cartesian coordinate system such that $\vec{s}_0 = (0, 0, 1)$

$$\Rightarrow V'_\nu(u, v, w) = e^{-2\pi i w} \iint I_\nu(l, m) e^{-2\pi i (ul + vm)} dl\, dm$$

if we redefine $V_\nu = e^{2\pi i w} V'_\nu$, then

$$V_\nu(u, v) = \iint I_\nu(l, m) e^{-2\pi i (vl + um)} dl\, dm$$

The relation between V_ν and I_ν is a (inverse) Fourier transform