The Formation of Image

lecture 6
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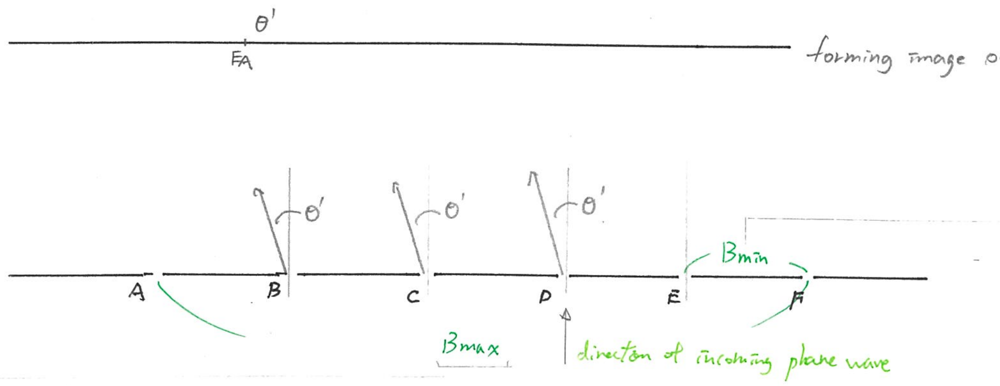
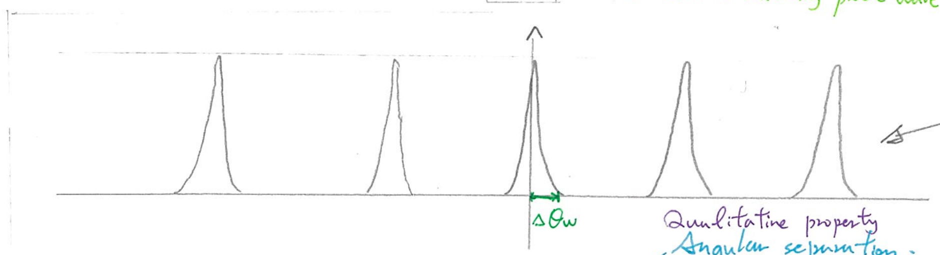
Analogy with Diffraction Gratings (c.f. Halliday, Principles of Physics, §36-5)



forming image on the infinitely distant screen (e.g. CCD)
↓
proportional to photon flux density



diffraction pattern (image)

Qualitative property

- Angular separation: $B_{min} \sin\theta = m\lambda$, $m = 0, 1, 2, 3, \ldots$
- Width of diffraction pattern: $\frac{1}{2} B_{max} \sin(\Delta\Theta_w) = \frac{1}{2}\lambda$

i.e. viewing the entire diffraction grating with width B_{max} as a single slit.

Evaluation of diffraction pattern

$$E_A = E_0 \cos(kx - \omega t + \phi_0)$$

$$\langle (E_A + E_B + E_C + E_D + \cdots + E_G)^2 \rangle = \langle E_A^2 \rangle + \langle E_B^2 \rangle + \cdots + \langle E_G^2 \rangle$$
$$+ \langle 2 E_A E_B \rangle + \langle 2 E_A E_C \rangle + 2 \langle E_A E_D \rangle + \cdots$$

$\cos\theta \cos\varphi = \frac{1}{2}\left(\cos(\theta - \varphi) + \cos(\theta + \varphi)\right)$

form of a cross-term:

$$2E_0^2 \langle \cos(kx - \omega t + \phi_0) \cos(k(x + \Delta x) - \omega t + \phi_0) \rangle$$

$$\propto \frac{1}{2} \langle \cos(k\Delta x) \rangle + \frac{1}{2} \underbrace{\langle \cos(2kx - 2\omega t + 2\phi_0 + k\Delta x) \rangle}_{= 0}$$

$\Delta x = n B_{min} \sin\theta'$

$$\simeq \frac{1}{2} \langle \cos(2\pi \frac{n B_{min}}{\lambda} \theta') \rangle$$

⇒ for any pair of slit, the maxima form at $\frac{n B_{min}}{\lambda}\theta \sim m$

⇒ $\theta \sim \frac{m\lambda}{n B_{min}}$

the angular separations between the diffraction patterns are smaller when the slits are more spatially separated.

Diffraction pattern of the diffraction gratings is a superposition of all of the cross-terms.

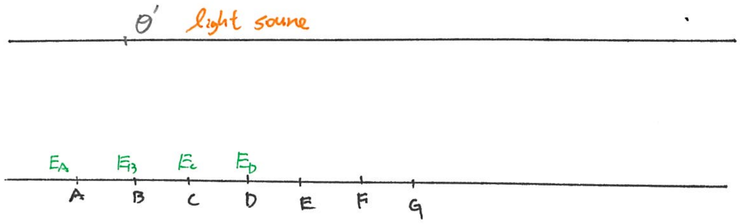
1. for any n (i.e. any pair of slits), $\theta = 0$ is always the location of a maximum (i.e. constructive interference)
2. The locations of the maxima for the $n=1$ pairs of slits are always also the locations of the maxima for any other pairs of slits.
   (e.g. the $n=1$, $m=5$ maximum is also the $n=2$, $m=10$ or $n=3$, $m=15$ maxima)
3. Between the maxima of the $n=1$ pairs of slits, the destructive interference makes the photon flux density approach 0 when $B_{max} \to \infty$

⇒ Considering any $E_i E_j$ pair, the diffraction pattern (expressed by photon flux density) has the following form

spatial separation between slits i and j

$$\frac{1}{2} \sum_{i,j} \cos(2\pi \frac{B_{ij}}{\lambda} \theta') \equiv \frac{1}{2} \sum_{k}^{M} \cos(2\pi u_k \theta') \quad \text{------ (1)}$$

Interferometric array observations towards a infinitely distant, monochromatic source



The long-term time-average of the cross-correlation measured from any two antennae is

(small angle limit) — intensity distribution of the light source

cosine correlator

$$\langle E_i E_j \rangle_{cos} \cong \left\langle \cos\left(2\pi \frac{B_{ij}}{\lambda}\theta'\right)\right\rangle = \left\langle \int \delta(\theta-\theta')\cos(2\pi u_{ij}\theta)\,d\theta \right\rangle$$

$$\langle E_i E_j \rangle_{sin} \cong \left\langle \int \delta(\theta-\theta')\sin(2\pi u_{ij}\theta)\,d\theta \right\rangle$$

$\Longrightarrow$ the complex visibility

$$V_{ij} = \int \delta(\theta-\theta')\, e^{i2\pi u_{ij}\theta}\, d\theta$$

$\downarrow$ Generalizing to any light source and baseline length:

$$V \equiv V(u) = \int A(\theta)\, e^{i2\pi u\theta}\, d\theta$$

this is the mathematical form of Fourier transform.

$\Downarrow$

When we have continuous and complete sampling of u, we can obtain $V(u)$ based on inverse Fourier transform

$$A(\theta') = \int_{-\infty}^{\infty} V(u)\, e^{-i2\pi u\theta'}\, du$$

In actual observations, we can only obtain discretized samples of u_{ij}, given by pairs of antenna, which can be described by the sampling function $S(u) = \sum_{k=1}^{M} \delta(u-u_k)$

Then the inverse F.T. of the incompletely sampled complex visibility is the image

$$\underbrace{A^D(\theta')}_{\text{dirty image}} = \int S(u)\, V(u)\, e^{-2\pi i u\theta'}\, du$$

convolution theory

$$= \underbrace{\left(\int S(u)\, e^{-2\pi i u\theta'}\, du\right)}_{\text{dirty beam} \quad f(\tau)} * \underbrace{\left(\int V(u)\, e^{-2\pi i u\theta'}\, du\right)}_{A(\theta') \quad g(x-\tau)}$$

$\parallel$ (taking real part) ← u is not directional, i.e., if we have measurement at u, there must also be a measurement at $-u$, which can cancel the imaginary part.

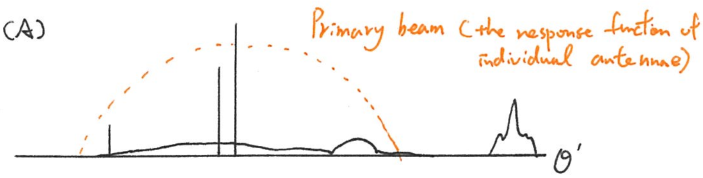
$$\sum_{k=1}^{M}\int \delta(u-u_k)\cos(2\pi u\theta')\,du = \underbrace{\sum_{k=1}^{M}\cos(2\pi u_k\theta')}_{\text{point spread function of interferometer}} \quad (2)$$

The mathematical forms of (1) and (2) are identical, i.e, the point spread function of an interferometer has the same mathematical form with the diffraction pattern of the diffraction grating.

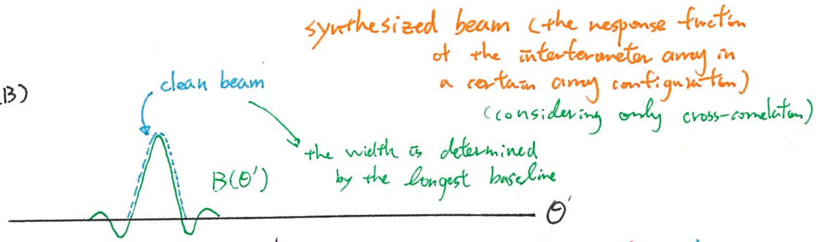
In the language of the radio interferometer community, we call the point spread function the dirty beam

The mathematical form of equation (2) is valid for arbitrary ways of placing the antennae. In the language of radio interferometry, the way to place antennae is called **array configuration**. After knowing the array configuration, we can know the sampling function $S(u)$, We can then inversely Fourier transform $S(u)$ to obtain the dirty beam. The purpose of imaging is to deduce $A(\theta)$ based on the dirty image $A^D(\theta')$ and the dirty beam.

(A)



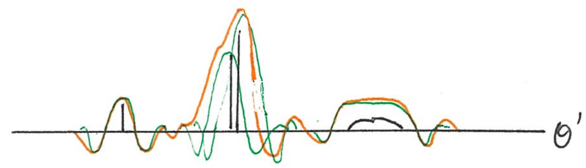
(B)



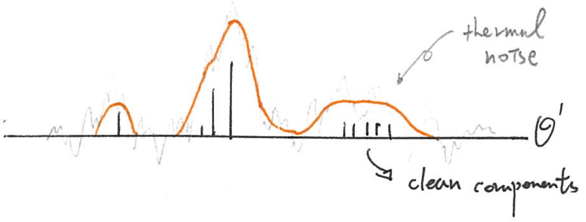
When thermal noise is present, in principle, we cannot resolve structures that are smaller than the clean beam

synthesized image (dirty image) — there are negative-intensity artifacts, which is due to that the large scale structure has been filtered out.
The structure outside of the primary beam is not detected.

(C)



(d)



The Most Commonly Adopted Deconvolution Algorithm in the Radio Interferometry Community CLEAN

Högbom algorithm

(other similar algorithms include Clark and Cotton–Schwab)

1. Looking for peak in the dirty image. (e.g. at location θ', with peak intensity I^D)
2. Subtract off $I^D \cdot \gamma \cdot (B * \delta(\theta-\theta'))$ from the dirty image
   γ → loop gain, with ≤ 1 values. normally it is chosen to be in between $0.05 \sim 0.1$
3. Record this $(I^D \cdot \gamma, \theta')$ value. This is called the **clean component**.
4. Go back to Step-1 until the peak value in the residual image becomes lower than a threshold set by the user.
5. Convolve all clean components with a clean beam (normally obtained from fitting 2D Gaussian to the dirty beam). Superimposing all convolved clean components with the residual image to obtain the restored image.

※ The Fourier transformed clean components are consistent with the directly measured visibility. Restored image is not consist with the complex visibility, although it is easier for us to get a sense about angular resolution