An Introduction to Radio Interferometry

2-2 Diffraction pattern of a single slit

You can find relevant material
on my personal webpage

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Lecture Unit 1-1

Radio Interferometer is an optical device

1. Light is EM-wave.
2. **EM-wave at long-distance limit: plane wave $\boldsymbol{E} = \boldsymbol{E_0}\cos(\boldsymbol{kx} - \boldsymbol{wt} + \boldsymbol{\phi_0})$**
   ϕ_0: phase offset; $w = 2\pi\nu$: angular frequency; ν: frequency
3. The EM-wave propagation can be described by Huygens' principle
   (i) every point on a wave front can be regarded as a new source
   (ii) the wave front is spherically symmetric with respect to the source
4. The solutions of EM-wave equation satisfy the principle of superposition.
5. In vacuum, the speed of EM wave propagation is $c = \frac{1}{\sqrt{\epsilon_0\mu_0}}$; in the medium that the index of refraction is n, the speed of EM-wave propagation is c/n, and the wavelength becomes ${}^{\lambda_0}/_{n}$, where λ_0 is the wavelength in vacuum.

1. EM-wave at long-distance limit: plane wave $E = E_0 \cos(kx - wt + \phi_0)$

Lecture Unit 1-1

1. **EM-wave at long-distance limit: plane wave $E = E_0 \cos(kx - wt + \phi_0)$**

Lecture Unit 1-2

Flux density

Energy is proportional to the square of electric or magnetic field strength.
The **energy flux density**, which is the amount of energy passing a unit area in a unit of frequency and time,
Is also proportional to the **square of electric and/or magnetic field strength**.

Electric field: $\vec{E} = -\nabla\Phi$

Total electric potential energy : $W_E = \frac{1}{2}\int \rho(\vec{x})\Phi(\vec{x})d^3x = \frac{\epsilon_0}{2}\int (\nabla\cdot\vec{E})\Phi(\vec{x})d^3x$

$$= \frac{\epsilon_0}{2}\int \vec{E}\cdot\left(-\nabla\Phi(\vec{x})\right)d^3x = \frac{\epsilon_0}{2}\int |E|^2 d^3x$$

In EM wave

From the symmetry of Maxwell's equations we expect magnetic field energy to have a similar form $W_B = \frac{1}{2\mu_0}\int |B|^2 d^3x$

$$\frac{E_m}{B_m} = \frac{1}{\mu_0\epsilon_0 c}$$

1. EM-wave at long-distance limit: plane wave $E = E_0 \cos(kx - wt + \phi_0)$
2. Energy flux density $\propto E_0^2$

Lecture Unit 1-2

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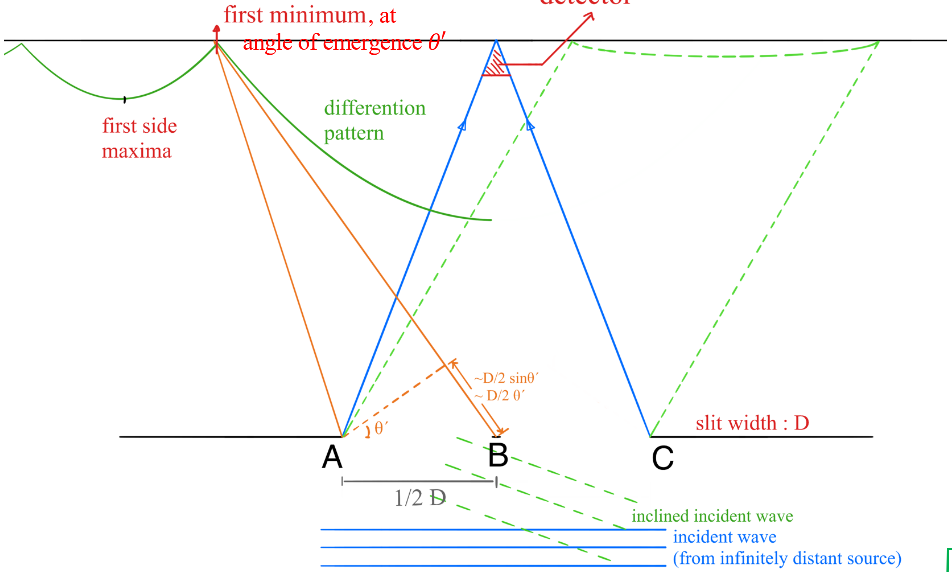

Path length difference

$$(E_A + E_B)^2 = E_0^2[\cos(kx - \omega t) + \cos(k(x + \underbrace{\Delta x}) - \omega t)]^2$$

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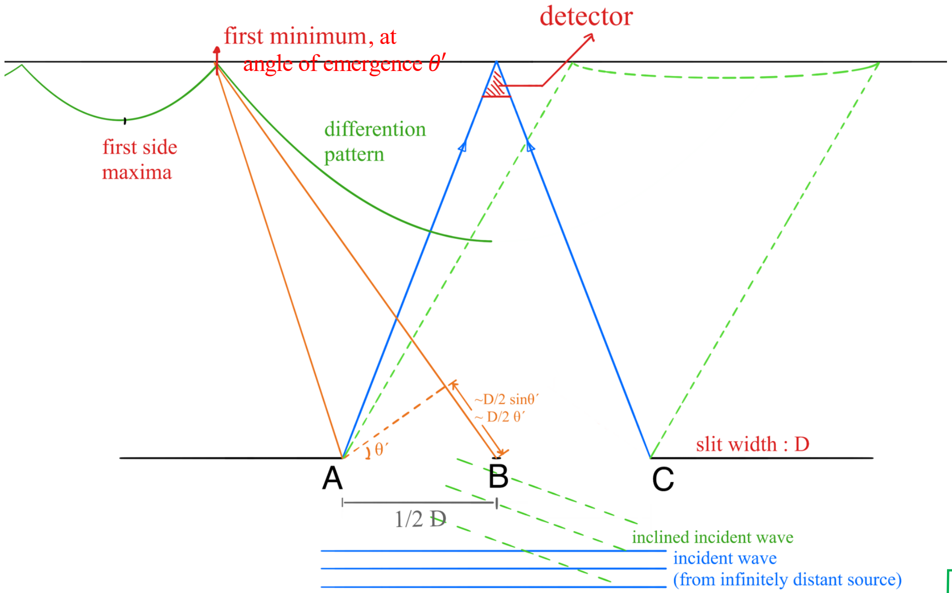

$$k\Delta x = \frac{2\pi}{\lambda}\frac{\lambda}{2} = \pi \Longrightarrow \frac{D}{2}\sin\theta' \sim \frac{D}{2}\theta' = \frac{\lambda}{2} \Longrightarrow \theta' \sim {}^{\lambda}/_{D}$$

$$(E_A + E_B)^2 = E_0^2[\cos(kx - \omega t) + \cos(k(x + \Delta x) - \omega t)]^2$$
$$= E_0^2[\underbrace{\cos(kx - \omega t) + \cos(kx + \pi - \omega t)}]^2$$

Two cosine functions that cancel each other completely.

1. <u>EM-wave at long-distance limit: plane wave $E = E_0 \cos(kx - wt + \phi_0)$</u>
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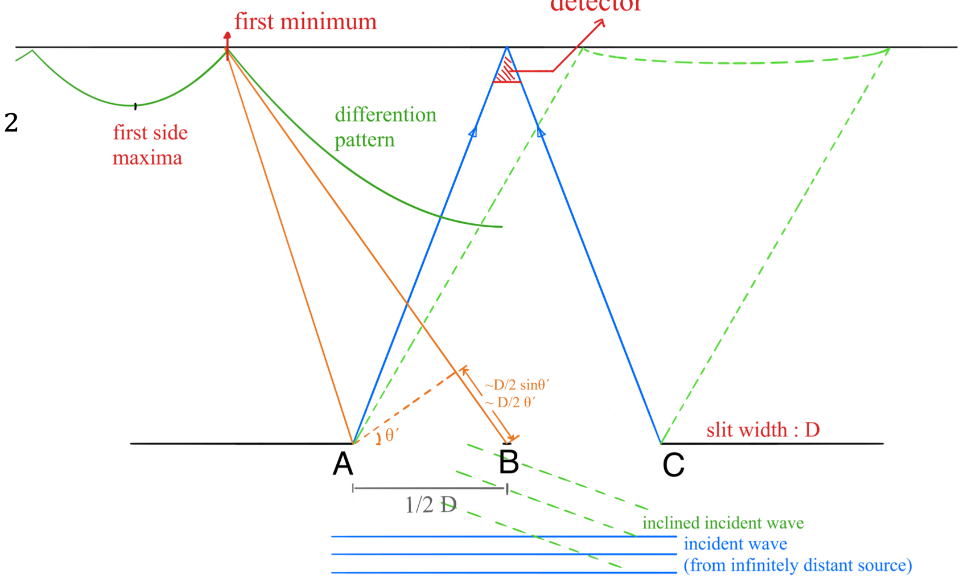
$$\left[\sum_{n=0}^{N-1} \cos(k(x + \Delta x_n) - \omega t)\right]^2 = \left[\sum_{n=0}^{N-1} \cos(kx + k\frac{nD}{N}\sin\theta' - \omega t)\right]^2$$



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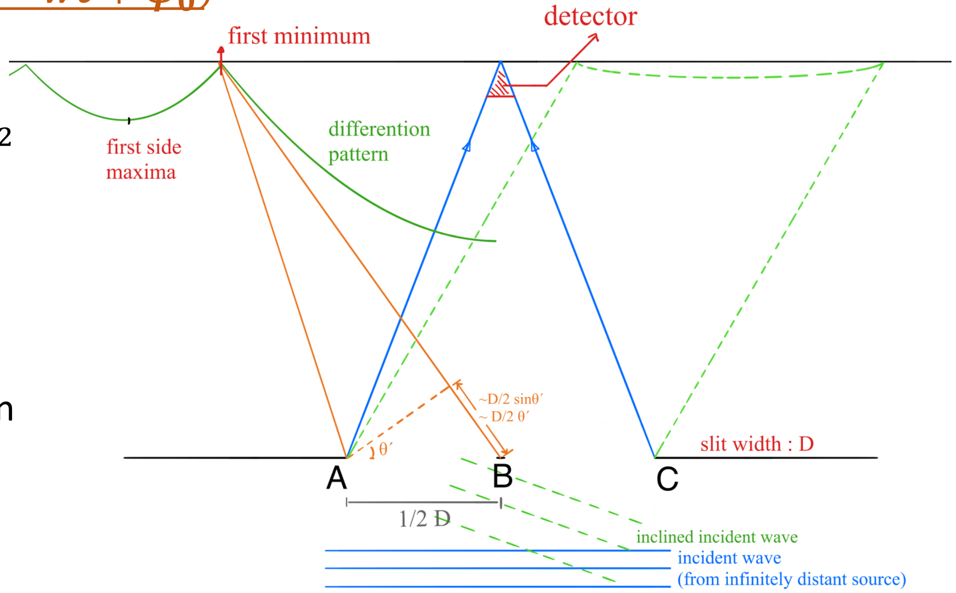
In the limit of $N \to \infty$, the summation becomes integration



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$$\sin\theta - \sin\varphi = 2\sin\left(\frac{\theta - \varphi}{2}\right)\cos\left(\frac{\theta + \varphi}{2}\right)$$

$$-\frac{[\sin(kx + kD\sin\theta' - \omega t) - \sin(kx - \omega t)]}{k\sin\theta'}$$

$$= -\frac{\sin\left(\frac{1}{2}kD\sin\theta'\right)\cos\left(kx - \omega t + \frac{1}{2}kD\sin\theta'\right)}{\frac{1}{2}k\sin\theta'}$$

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<u>Diffraction pattern</u>: $\tilde{P}(\theta)$

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Diffraction pattern: $\tilde{P}(\theta)$

First zero: $\frac{1}{2}kD\theta' = \pi \Longrightarrow \frac{1}{2}\frac{2\pi}{\lambda}D\theta' = \pi \Longrightarrow \theta' = \frac{\lambda}{D}$

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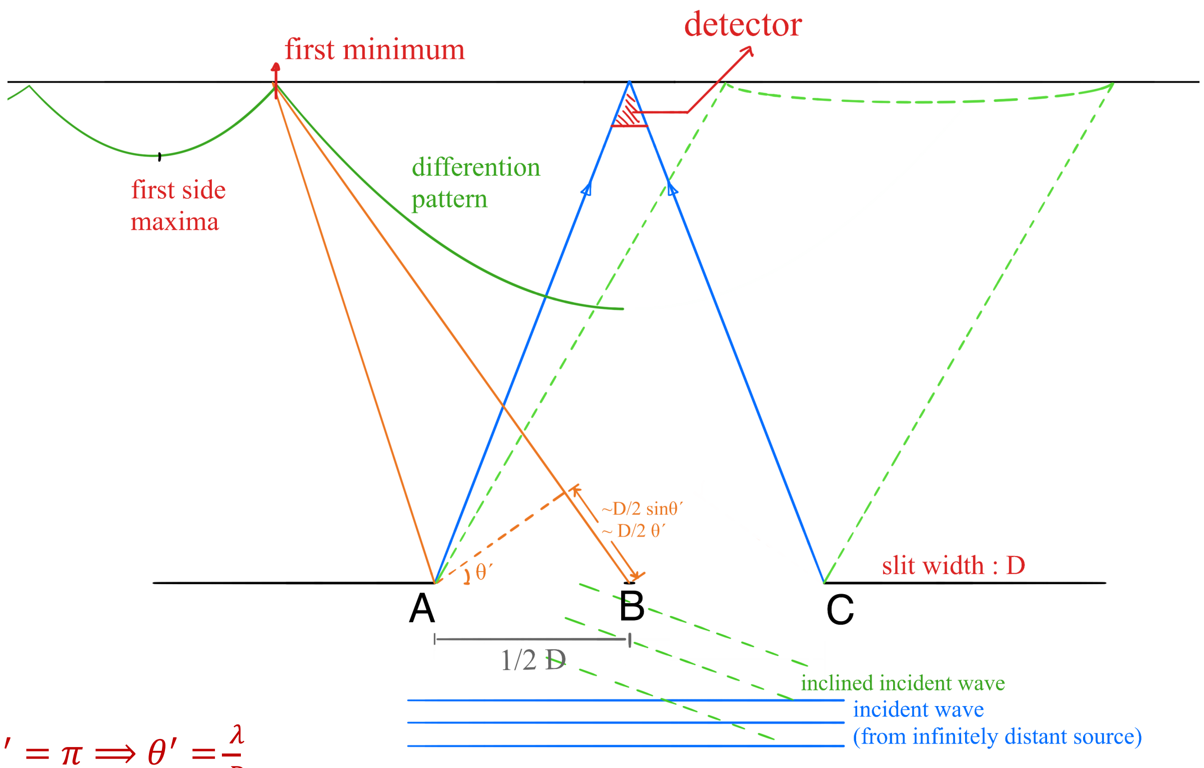

$$\sqrt{\tilde{P}(\theta)} = \frac{\sin\left(\frac{1}{2}kD\sin\theta\prime\right)}{\frac{1}{2}k\sin\theta\prime} \sim \frac{\sin\left(\frac{1}{2}kD\theta\prime\right)}{\frac{1}{2}k\theta\prime}$$

<u>Diffraction pattern</u>: $\tilde{P}(\theta)$

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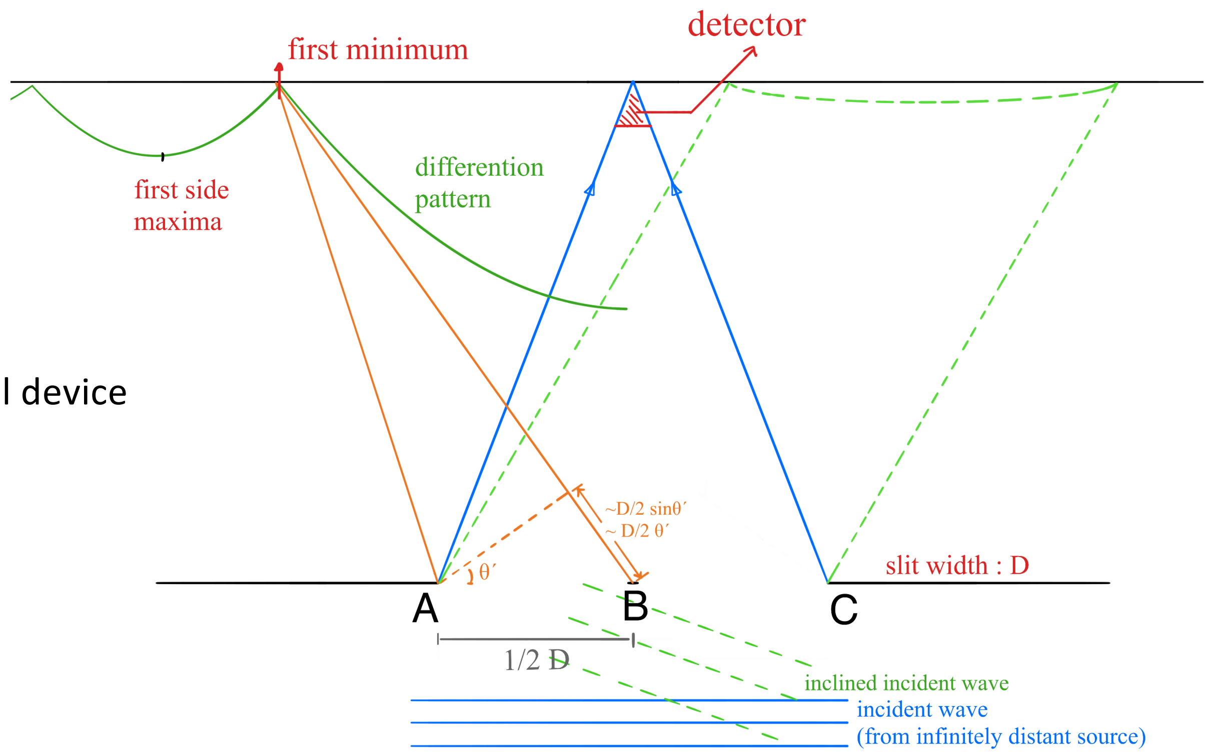

The slit and the detector make an observational device that the response function is the same as the diffraction pattern of the single slit.

1. The diffraction pattern of a single slit: $\tilde{P}(\theta)$, $\sqrt{\tilde{p}(\theta)} \sim \frac{\sin\left(\frac{1}{2}kD\theta\right)}{\frac{1}{2}k\theta}$

2. The first zero of the diffraction pattern is at the angle of emergence $\theta' = \frac{\lambda}{D}$