An Introduction to Radio Interferometry

3-2 Response of a 2-element interferometer

You can find relevant material
on my personal webpage

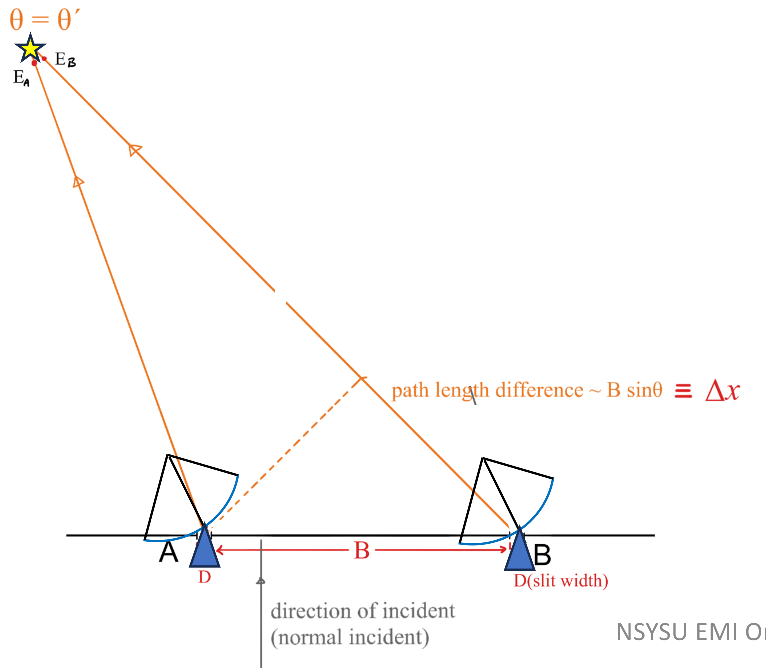
1. EM-wave at long-distance limit: plane wave $E = E_0 \cos(kx - wt + \phi_0)$
2. Energy flux density $\propto E_0^2$

Lecture Unit 1-2

Single slit field $= -\dfrac{\sin\left(\frac{1}{2}kD\sin\theta'\right)}{\frac{1}{2}k\sin\theta'}\cos\left(kx - \omega t + \frac{1}{2}kD\sin\theta'\right)$

$$\equiv \underbrace{\sqrt{\tilde{P}(\theta)}}_{\text{Amplitude modulation}}\cos(kx - \omega t + \underbrace{\phi_s}_{\text{Phase modulation}})$$

Lecture Unit 2-2



Number of incoming photons in a unit of time and a unit area (at the angle of emergence $\theta = \theta'$)

$$\begin{aligned}
&\propto [E_A + E_B]^2 \\
&= \tilde{P}(\theta)[\cos(kx - \omega t + \phi_0) + \cos(k(x + \Delta x) - \omega t + \phi_0)]^2 \\
&= \tilde{P}(\theta)\{[\cos(kx - \omega t + \phi_0)]^2 \\
&\quad + [\cos(k(x + \Delta x) - \omega t + \phi_0)]^2 \\
&\quad + 2\cos(kx - \omega t + \phi_0)\cos(k(x + \Delta x) - \omega t + \phi_0)\}
\end{aligned}$$

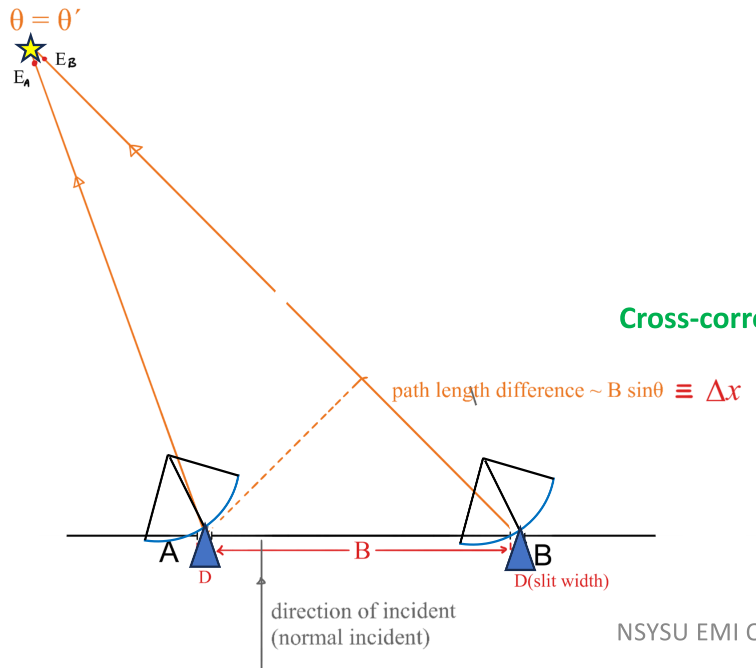
1. <u>EM-wave at long-distance limit: plane wave $E = E_0 \cos(kx - wt + \phi_0)$</u>
2. Energy flux density $\propto E_0^2$

Lecture Unit 1-2

Single slit field $= -\dfrac{\sin\left(\frac{1}{2}kD \sin\theta'\right)}{\frac{1}{2}k \sin\theta'} \cos\left(kx - \omega t + \frac{1}{2}kD \sin\theta'\right)$

Lecture Unit 2-2

$$\equiv \underbrace{\sqrt{\tilde{P}(\theta)}}_{\text{Amplitude modulation}} \cos(kx - \omega t + \underbrace{\phi_s}_{\text{Phase modulation}})$$



Number of incoming photons in a unit of time and a unit area (at the angle of emergence $\theta = \theta'$)

$$\propto < [E_A + E_B]^2 >$$

$$\propto < [\cos(kx - \omega t + \phi_0) + \cos(k(x + \Delta x) - \omega t + \phi_0)]^2 >$$

$$= < [\cos(kx - \omega t + \phi_0)]^2 >$$

$$+ < [\cos(k(x + \Delta x) - \omega t + \phi_0)]^2 >$$

Auto-correlations

Cross-correlations $\longrightarrow$ $+ < 2\cos(kx - \omega t + \phi_0)\cos(k(x + \Delta x) - \omega t + \phi_0) >$

$$< \cdot > \equiv \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \int_{-\frac{1}{2}\Delta t}^{\frac{1}{2}\Delta t} \cdot \, dt$$

1. EM-wave at long-distance limit: plane wave $E = E_0 \cos(kx - wt + \phi_0)$
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Lecture Unit 1-2

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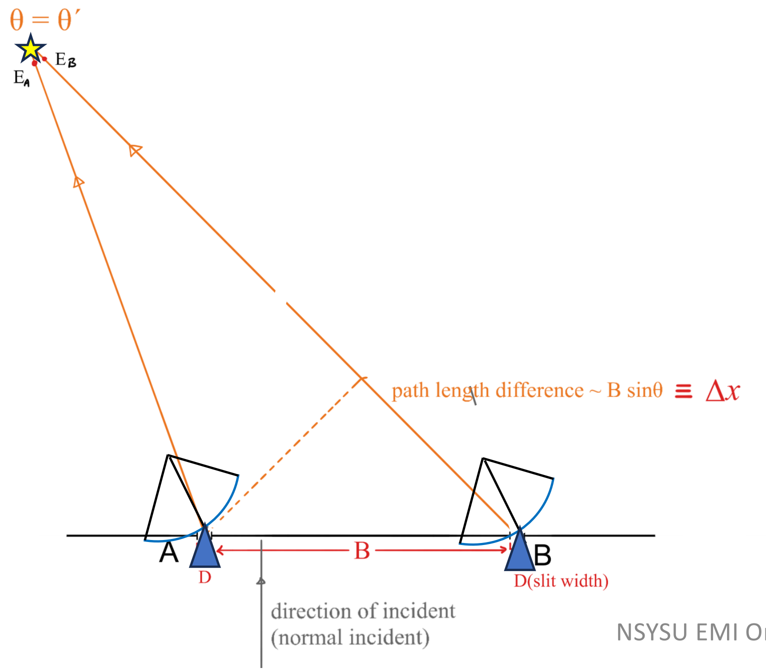
Lecture Unit 2-2



Number of incoming photons in a unit of time and a unit area (at the angle of emergence $\theta = \theta'$)

$$\propto \langle [E_A + E_B]^2 \rangle$$

$$= \langle [\cos(kx - \omega t + \phi_0) + \cos(k(x + \Delta x) - \omega t + \phi_0)]^2 \rangle$$

$$= \left.\begin{matrix} 1/2 \\ + 1/2 \end{matrix}\right\} \text{Auto-correlations}$$

$$\underbrace{+ \langle \boxed{2\cos(kx - \omega t + \phi_0)\cos(k(x + \Delta x) - \omega t + \phi_0)} \rangle}_{\text{Cross-correlations}}$$

$$\langle [E_A + E_B]^2 \rangle = \tilde{P}(\theta)\left[1 + \cos\left(\frac{2\pi}{\lambda}B\sin\theta\right)\right]$$

$$\sim \tilde{P}(\theta)\left[1 + \cos\left(2\pi\frac{B}{\lambda}\theta\right)\right]$$

1. EM-wave at long-distance limit: plane wave $E = E_0 \cos(kx - wt + \phi_0)$
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Lecture Unit 1-2

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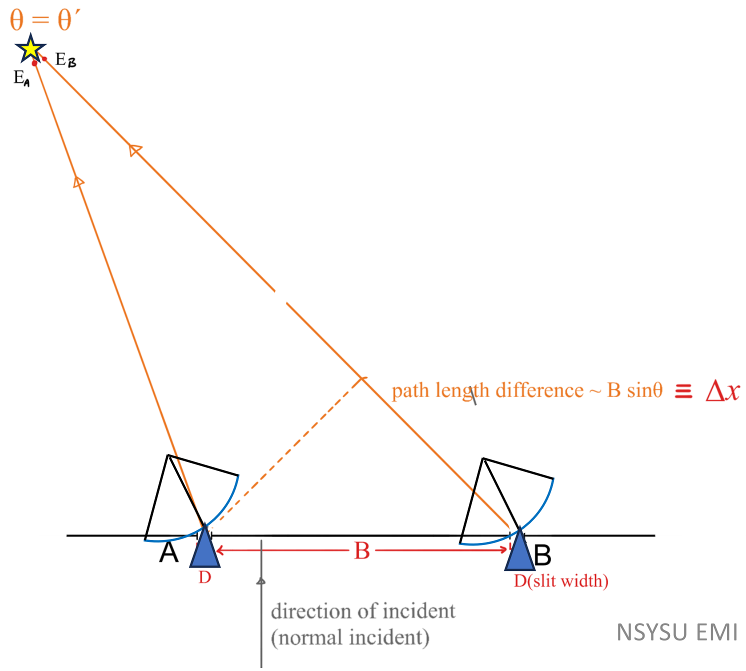
Lecture Unit 2-2



Number of incoming photons in a unit of time and a unit area (at the angle of emergence $\theta = \theta'$)

$\propto < [E_A + E_B]^2 >$

$= < [\cos(kx - \omega t + \phi_0) + \cos(k(x + \Delta x) - \omega t + \phi_0)]^2 >$

$= \left.\begin{array}{l} 1/2 \\ + 1/2 \end{array}\right]$ **Auto-correlations**

$+ \underbrace{< \boxed{2\cos(kx - \omega t + \phi_0)\cos(k(x + \Delta x) - \omega t + \phi_0)} >}_{\text{Cross-correlations}}$

$< [E_A + E_B]^2 > \sim \tilde{P}(\theta)[\underbrace{1}_{\text{Auto-correlations}} + \cos(2\pi u\theta)]$

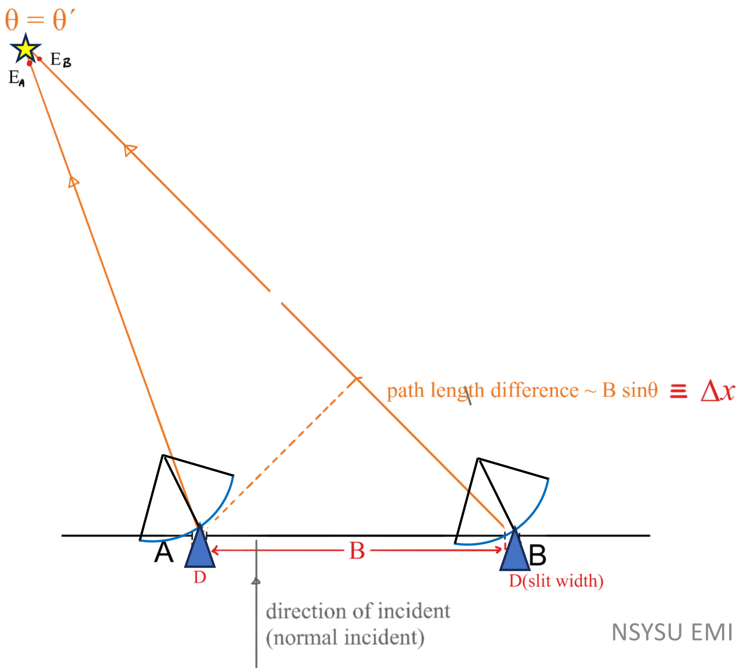
1. <u>**EM-wave at long-distance limit: plane wave $E = E_0 \cos(kx - wt + \phi_0)$**</u>
2. **Energy flux density $\propto E_0^2$**

Lecture Unit 1-2

Single slit field $= -\dfrac{\sin\left(\frac{1}{2}kD\sin\theta'\right)}{\frac{1}{2}k\sin\theta'}\cos\left(kx - \omega t + \frac{1}{2}kD\sin\theta'\right)$

Lecture Unit 2-2

$$\equiv \underbrace{\sqrt{\tilde{P}(\theta)}}_{\text{Amplitude modulation}} \cos(kx - \omega t + \underbrace{\phi_s}_{\text{Phase modulation}})$$



Number of incoming photons in a unit of time and a unit area (at the angle of emergence $\theta = \theta'$)

$$\propto < [E_A + E_B]^2 >$$

$$= < [\cos(kx - \omega t + \phi_0) + \cos(k(x + \Delta x) - \omega t + \phi_0)]^2 >$$

$$= \left.\begin{array}{l} 1/2 \\ + 1/2 \end{array}\right\} \text{Auto-correlations}$$

$$+ \underbrace{< \boxed{2\cos(kx - \omega t + \phi_0)\cos(k(x + \Delta x) - \omega t + \phi_0)} >}_{\text{Cross-correlations}}$$

$$< [E_A + E_B]^2 > \sim \tilde{P}(\theta)[1 + \underbrace{\cos(2\pi u\theta)}_{\text{Cross-correlations}}]$$

1. EM-wave at long-distance limit: plane wave $E = E_0 \cos(kx - wt + \phi_0)$
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Lecture Unit 1-2

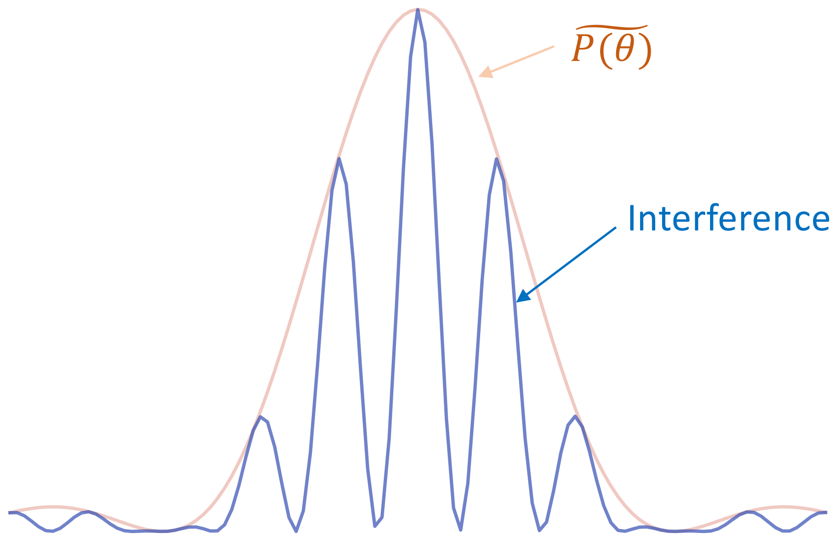
Lecture Unit 2-2

Single slit field $= -\dfrac{\sin\left(\frac{1}{2}kD \sin\theta\prime\right)}{\frac{1}{2}k \sin\theta\prime} \cos\left(kx - \omega t + \frac{1}{2}kD\sin\theta'\right)$

$$\equiv \underbrace{\sqrt{\tilde{P}(\theta)}}_{\text{Amplitude modulation}} \cos(kx - \omega t + \underbrace{\phi_s}_{\text{Phase modulation}})$$

Amplitude modulation

Phase modulation



Number of incoming photons in a unit of time and a unit area (at the angle of emergence $\theta = \theta'$)

$$\propto < [E_A + E_B]^2 >$$

$$= < [\cos(kx - \omega t + \phi_0) + \cos(k(x + \Delta x) - \omega t + \phi_0)]^2 >$$

$$= \left.\begin{matrix} 1/2 \\ + 1/2 \end{matrix}\right] \text{Auto-correlations}$$

$$+ \underbrace{< \boxed{2\cos(kx - \omega t + \phi_0)\cos(k(x + \Delta x) - \omega t + \phi_0)} >}_{\text{Cross-correlations}}$$

$$< [E_A + E_B]^2 > \sim \underbrace{\tilde{P}(\theta)}_{\text{Primary beam: response function of each reflector}}[1 + \cos(2\pi u\theta)]$$

1. EM-wave at long-distance limit: plane wave $E = E_0 \cos(kx - wt + \phi_0)$
2. Energy flux density $\propto E_0^2$

Lecture Unit 1-2

Single slit field $= -\dfrac{\sin\left(\frac{1}{2}kD\sin\theta\prime\right)}{\frac{1}{2}k\sin\theta\prime}\cos\left(kx - \omega t + \frac{1}{2}kD\sin\theta'\right)$

$$\equiv \underbrace{\sqrt{\tilde{P}(\theta)}}_{\text{Amplitude modulation}}\cos(kx - \omega t + \underbrace{\phi_s}_{\text{Phase modulation}})$$

Lecture Unit 2-2

Cosine correlation: $< [E_A + E_B]^2 >_{cos} \sim \tilde{P}(\theta)[1 + \cos(2\pi u\theta)]$

Sine correlation: $\propto < [E_A + E'_B]^2 >$

$$= < \left[\cos(kx - \omega t + \phi_0) + \cos(k(x + \Delta x) - \omega t + \phi_0 + \frac{\pi}{2})\right]^2 >$$

$$= {}^1/_2$$
$$+ {}^1/_2$$

Auto-correlations

$$+ < \underbrace{2\cos(kx - \omega t + \phi_0)\cos\left(k(x + \Delta x) - \omega t + \phi_0 + \frac{\pi}{2}\right)} >$$

$$2\cos(kx - \omega t + \phi_0)\left[\cos(k(x + \Delta x) - \omega t + \phi_0)\underbrace{\cos\frac{\pi}{2}}_{0} - \sin(k(x + \Delta x) - \omega t + \phi_0)\underbrace{\sin\frac{\pi}{2}}_{1}\right]$$

1. EM-wave at long-distance limit: plane wave $E = E_0 \cos(kx - wt + \phi_0)$
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Single slit field $= -\dfrac{\sin\left(\frac{1}{2}kD\sin\theta'\right)}{\frac{1}{2}k\sin\theta'}\cos\left(kx - \omega t + \frac{1}{2}kD\sin\theta'\right)$

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Lecture Unit 2-2

Cosine correlation: $< [E_A + E_B]^2 >_{cos} \sim \tilde{P}(\theta)[1 + \cos(2\pi u\theta)]$

Sine correlation: $\propto < [E_A + E'_B]^2 >$

$$= < \left[\cos(kx - \omega t + \phi_0) + \cos\left(k(x + \Delta x) - \omega t + \phi_0 + \frac{\pi}{2}\right)\right]^2 >$$

$$= \left.\begin{matrix} {}^1/_2 \\ + {}^1/_2 \end{matrix}\right] \text{Auto-correlations}$$

$$+ < \underbrace{2\cos(kx - \omega t + \phi_0)\cos\left(k(x + \Delta x) - \omega t + \phi_0 + \frac{\pi}{2}\right)}_{-2\cos(kx - \omega t + \phi_0)\sin(k(x + \Delta x) - \omega t + \phi_0)} >$$

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$= < \left[\cos(kx - \omega t + \phi_0) + \cos(k(x + \Delta x) - \omega t + \phi_0 + \frac{\pi}{2})\right]^2 >$

$= {}^1/_2$

$+ {}^1/_2$ Auto-correlations

$+ < \underbrace{2\cos(kx - \omega t + \phi_0)\cos\left(k(x + \Delta x) - \omega t + \phi_0 + \frac{\pi}{2}\right)}_{\underbrace{-2\cos(kx - \omega t + \phi_0)\sin(k(x + \Delta x) - \omega t + \phi_0)}_{-[\sin(2kx - 2\omega t + 2\phi_0 + k\Delta x) - \sin(k\Delta x)]}} >$

Trigonometric identity:
$\cos\psi \sin\varphi$
$= \frac{1}{2}[\sin(\psi + \varphi) - \sin(\psi - \varphi)]$

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Lecture Unit 2-2

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$$= < \left[\cos(kx - \omega t + \phi_0) + \cos\left(k(x + \Delta x) - \omega t + \phi_0 + \frac{\pi}{2}\right)\right]^2 >$$

$$= \left.\begin{array}{l} {}^1/_2 \\ + {}^1/_2 \end{array}\right] \text{Auto-correlations}$$

$$+ < \underbrace{\underbrace{2\cos(kx - \omega t + \phi_0)\cos\left(k(x + \Delta x) - \omega t + \phi_0 + \frac{\pi}{2}\right)}_{-2\cos(kx - \omega t + \phi_0)\sin(k(x+\Delta x) - \omega t + \phi_0)}}_{-[\cancel{\sin(2kx - 2\omega t + 2\phi_0 + k\Delta x)} - \sin(k\Delta x)]} >$$

Zero in the <> bracket

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$$= {}^1/_2 \quad + {}^1/_2 \quad \text{Auto-correlations}$$

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Lecture Unit 2-2

Cosine correlation: $< [E_A + E_B]^2 >_{cos} \sim \tilde{P}(\theta)[1 + \cos(2\pi u\theta)]$

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Lecture Unit 2-2

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Cosine correlation: $< [E_A + E_B]^2 >_{cos} \sim \tilde{P}(\theta)[1 + \cos(2\pi u\theta)]$

Sine correlation: $< [E_A + E_B]^2 >_{sin} \sim \tilde{P}(\theta)[1 + \sin(2\pi u\theta)]$

Cosine correlation and sine correlation are the direct measurement of a two-element radio interferometer

Cosine correlation: $< [E_A + E_B]^2 >_{cos} \sim \tilde{P}(\theta)[1 + \cos(2\pi u\theta)]$

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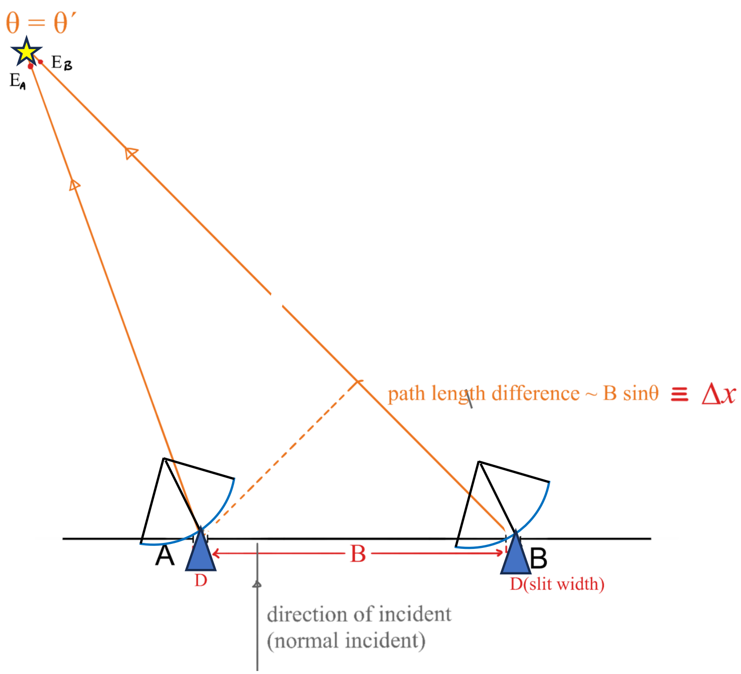

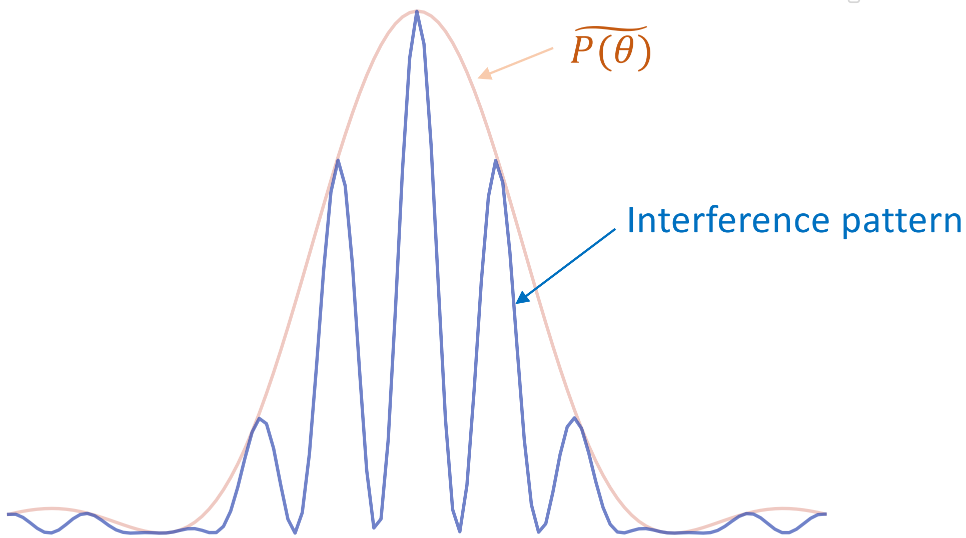

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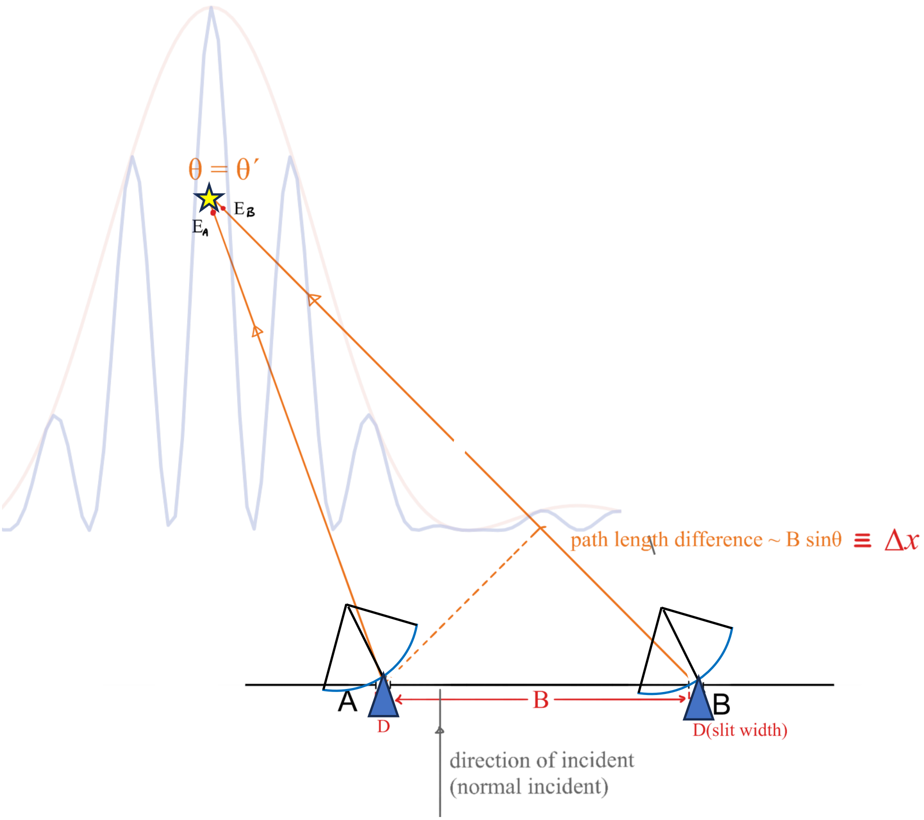

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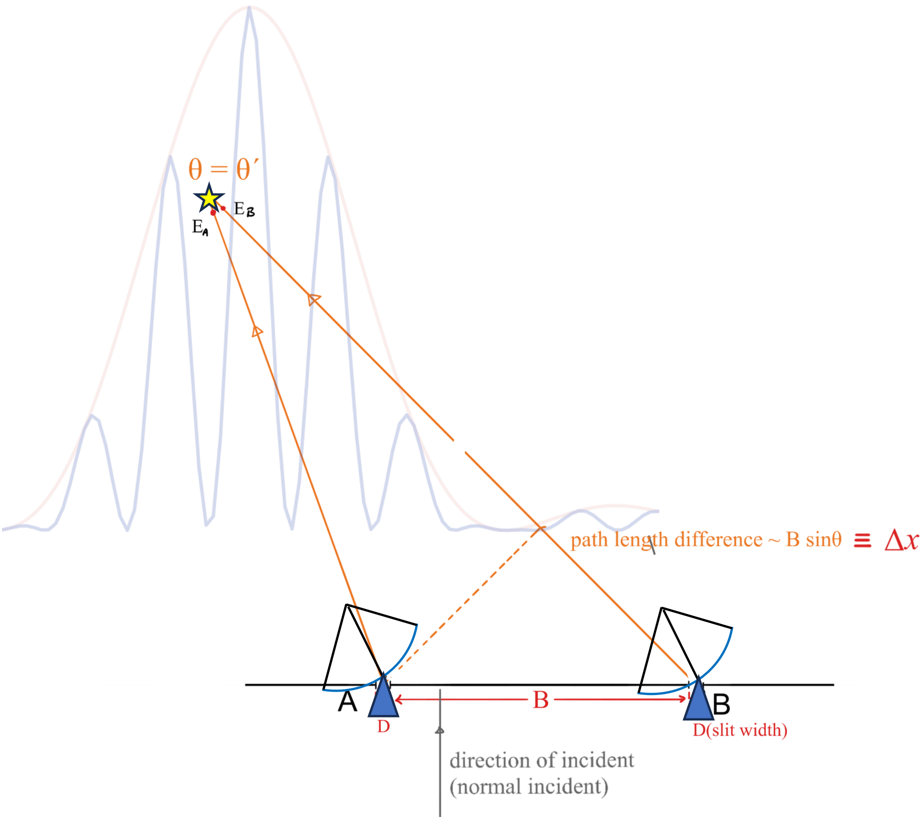

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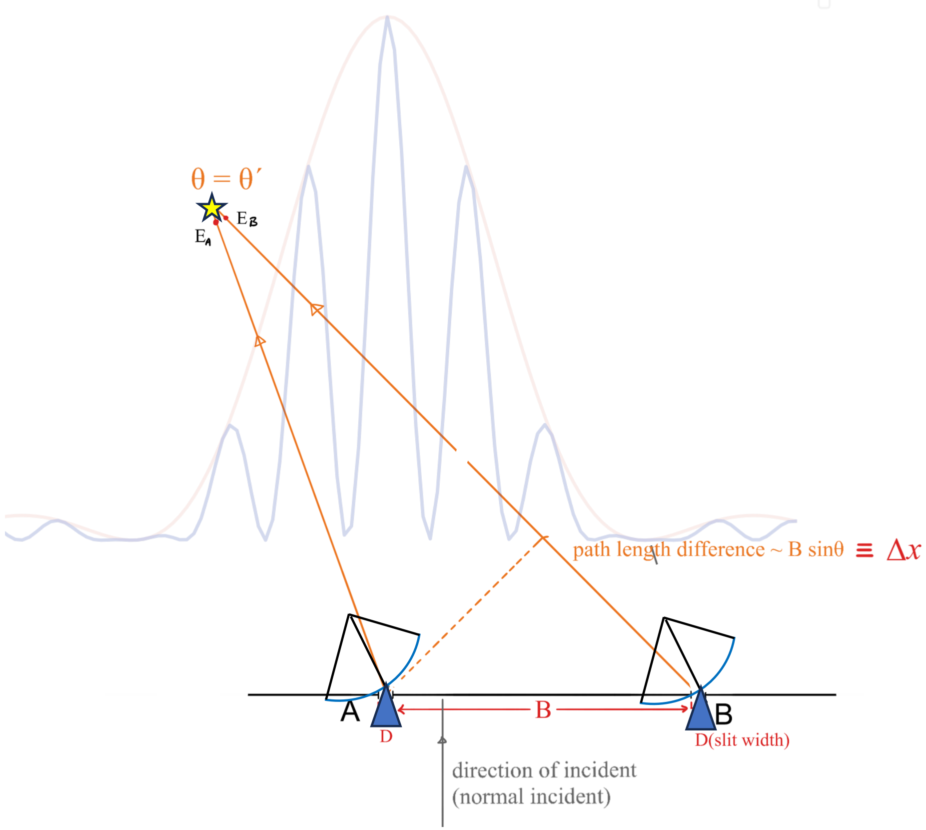

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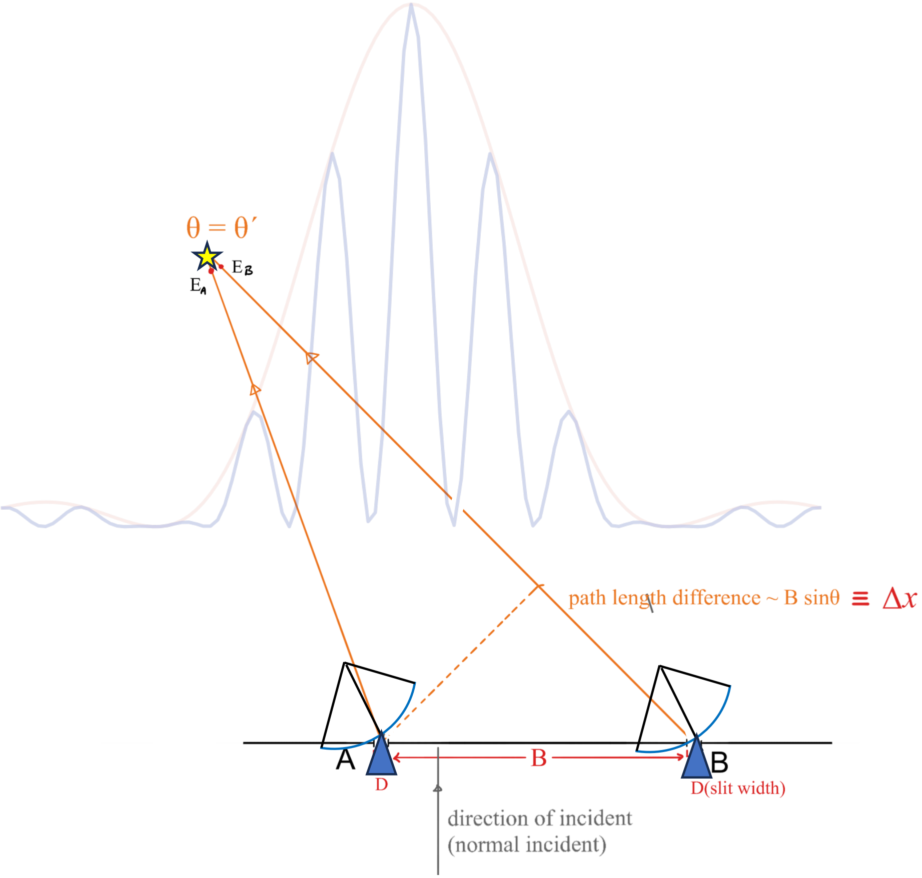

This operation: $< \cdot > \equiv \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \int_{-\frac{1}{2}\Delta t}^{\frac{1}{2}\Delta t} \cdot \; dt$

is filtering out (i.e., averaging out) some information.

The cosine correlation filters out the information that is asymmetric with respective to $\theta = 0$. In other words, cosine correlation probes the component of intensity distribution that is symmetric with respect to $\theta = 0$. On the other hand, sine correlation probes the component of intensity distribution that is asymmetric with respect to $\theta = 0$.

$\theta = 0$ at a chosen phase referencing center, which is defined by the delay line of an interferometer. For more details, see:
Synthesis Imaging in Radio Astronomy II

Cosine correlation: $< [E_A + E_B]^2 >_{cos} \sim \tilde{P}(\theta)[1 + \cos(2\pi u\theta)]$

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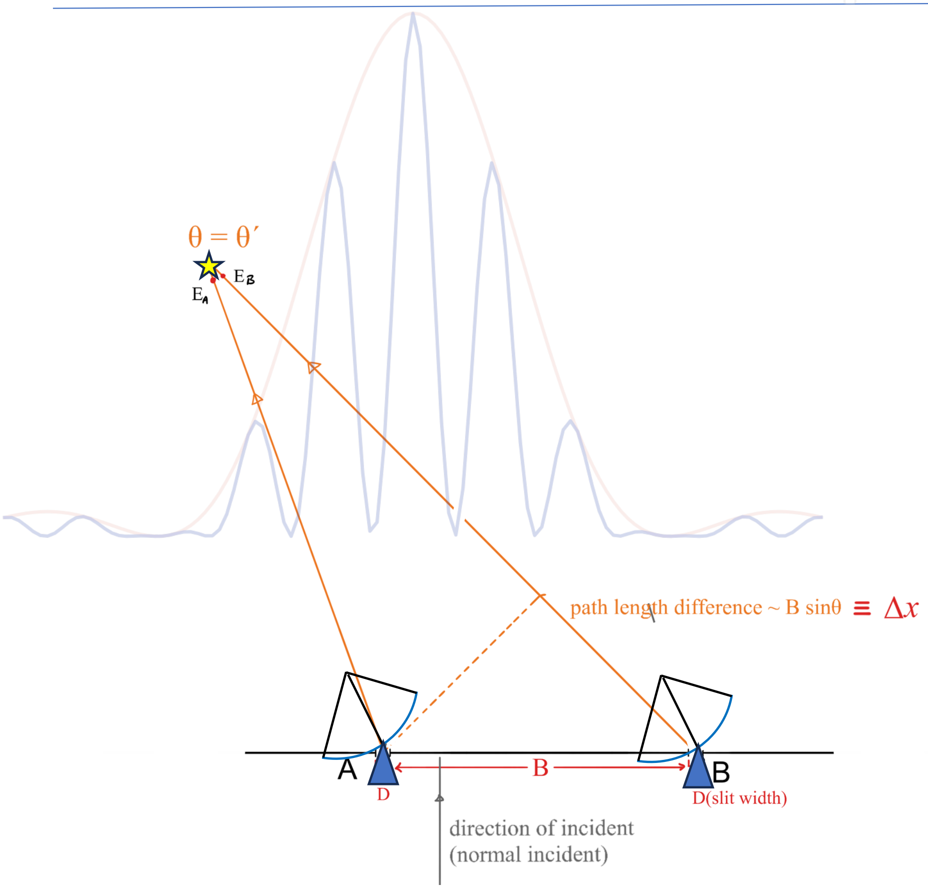

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By construction [from $f(x)$]:

$h(x) = \frac{1}{2}[f(x) + f(-x)]$, $h(-x) = h(x)$ symmetric function

$g(x) = \frac{1}{2}[f(x) - f(-x)]$, $g(-x) = -g(x)$ asymmetric function

$h(x) + g(x) = f(x)$

1. The response function of a 1D, two-element interferometer is very similar to the interference pattern of a double-slit $\tilde{P}(\theta)\left[1+\cos\left(2\pi\frac{B}{\lambda}\theta\right)\right] \sim \tilde{P}(\theta)[1+\cos(2\pi u\theta)]$
2. The correlator products often include sine and cosine correlations:

Cosine correlation: $<[E_A + E_B]^2>_{cos} \sim \tilde{P}(\theta)[1+\cos(2\pi u\theta)]$

Sine correlation: $<[E_A + E_B]^2>_{sin} \sim \tilde{P}(\theta)[1+\sin(2\pi u\theta)]$