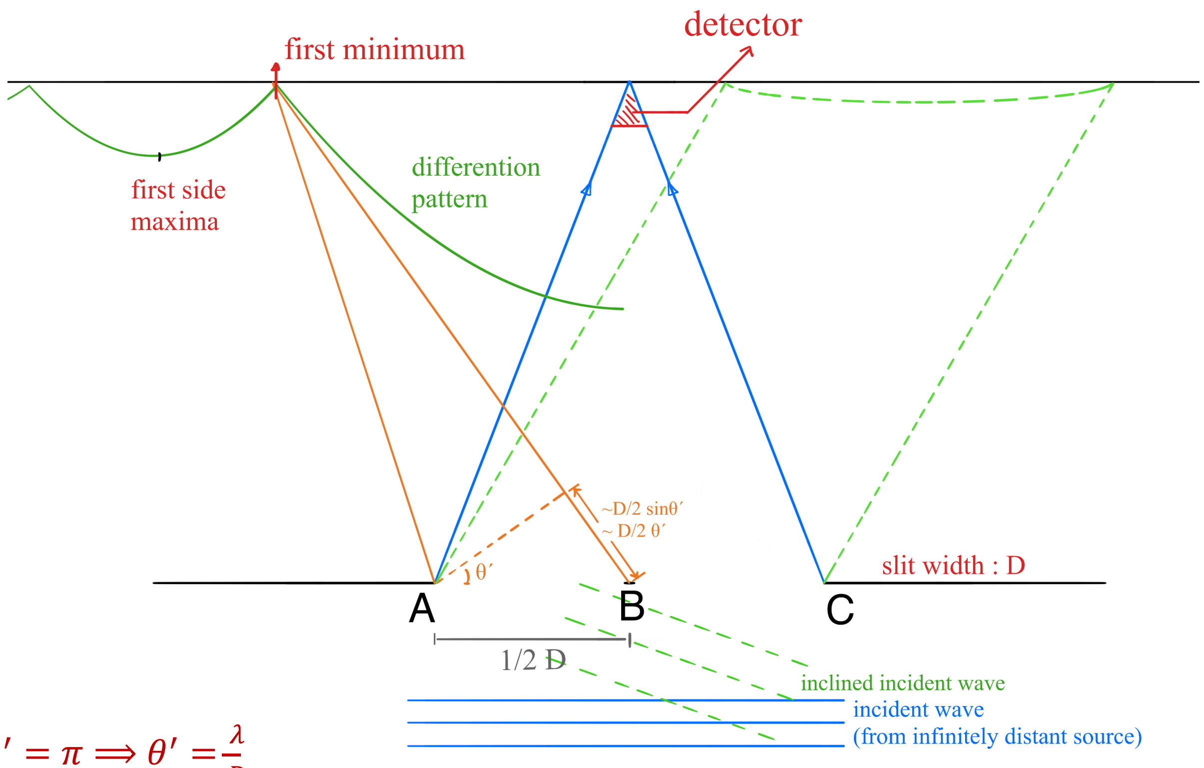
An Introduction to Radio Interferometry

4-1 Diffraction grating

You can find relevant material
on my personal webpage

1. EM-wave at long-distance limit: plane wave $E = E_0 \cos(kx - wt + \phi_0)$
2. Energy flux density $\propto E_0^2$



$$\sqrt{\tilde{P}(\theta)} = \frac{\sin\left(\frac{1}{2}kD\sin\theta\prime\right)}{\frac{1}{2}k\sin\theta\prime} \sim \frac{\sin\left(\frac{1}{2}kD\theta\prime\right)}{\frac{1}{2}k\theta\prime}$$

Diffraction pattern: $\tilde{P}(\theta)$

First zero: $\frac{1}{2}kD\theta' = \pi \Longrightarrow \frac{1}{2}\frac{2\pi}{\lambda}D\theta' = \pi \Longrightarrow \theta' = \frac{\lambda}{D}$

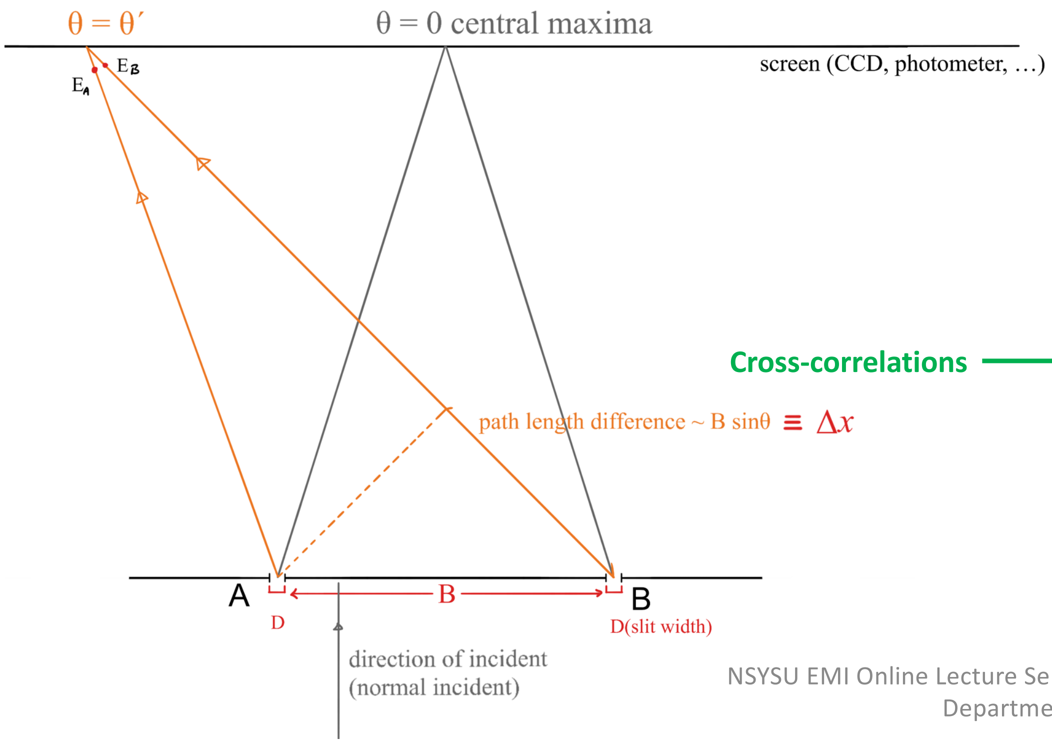
1. EM-wave at long-distance limit: plane wave $E = E_0\cos(kx - wt + \phi_0)$
2. Energy flux density $\propto E_0^2$

Lecture Unit 1-2

Lecture Unit 2-2

Single slit field $= -\dfrac{\sin\left(\frac{1}{2}kD\sin\theta'\right)}{\frac{1}{2}k\sin\theta'}\cos\left(kx-\omega t+\frac{1}{2}kD\sin\theta'\right)$

$$\equiv \underbrace{\sqrt{\tilde{P}(\theta)}}_{\textbf{Amplitude modulation}}\cos(kx-\omega t+\underbrace{\phi_s}_{\textbf{Phase modulation}})$$



Number of incoming photons in a unit of time and a unit area (at the angle of emergence $\theta = \theta'$)

$$\propto \langle [E_A + E_B]^2 \rangle$$

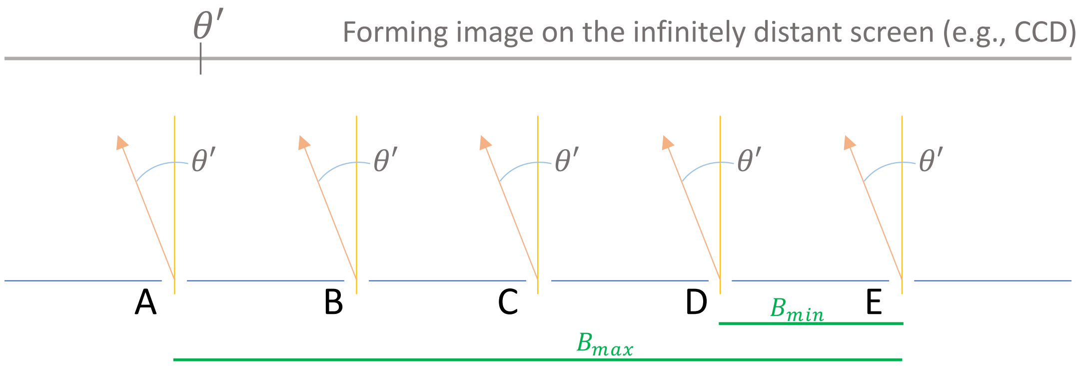
$$= \langle [\cos(kx-\omega t+\phi_0) + \cos(k(x+\Delta x)-\omega t+\phi_0)]^2 \rangle$$

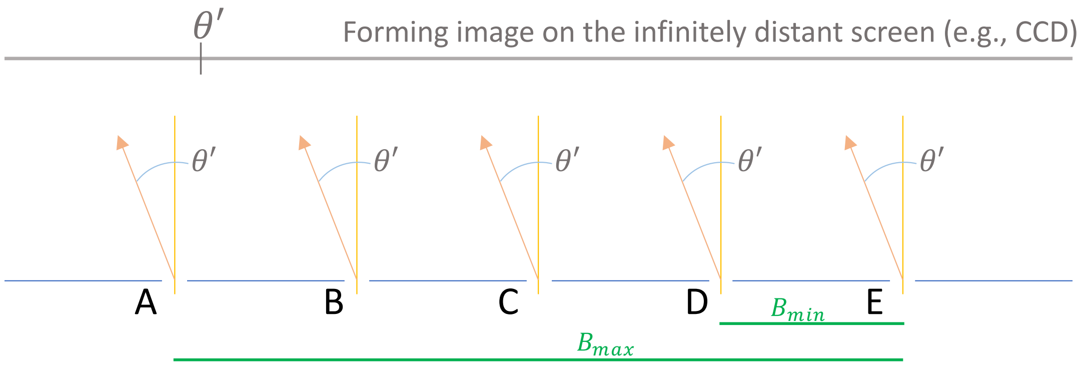
$$= \left.\begin{matrix} 1/2 \\ +\ 1/2 \end{matrix}\right] \textbf{Auto-correlations}$$

Cross-correlations $\longrightarrow$ $+ \langle \boxed{2\underbrace{\cos(kx-\omega t+\phi_0)}_{\varphi}\ \underbrace{\cos(k(x+\Delta x)-\omega t+\phi_0)}_{\psi}} \rangle$

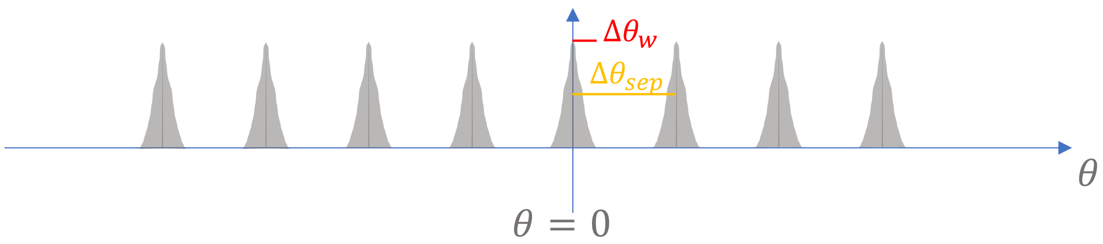
$$\langle [E_A + E_B]^2 \rangle = \tilde{P}(\theta)\left[1+\cos\left(\frac{2\pi}{\lambda}B\sin\theta\right)\right]$$

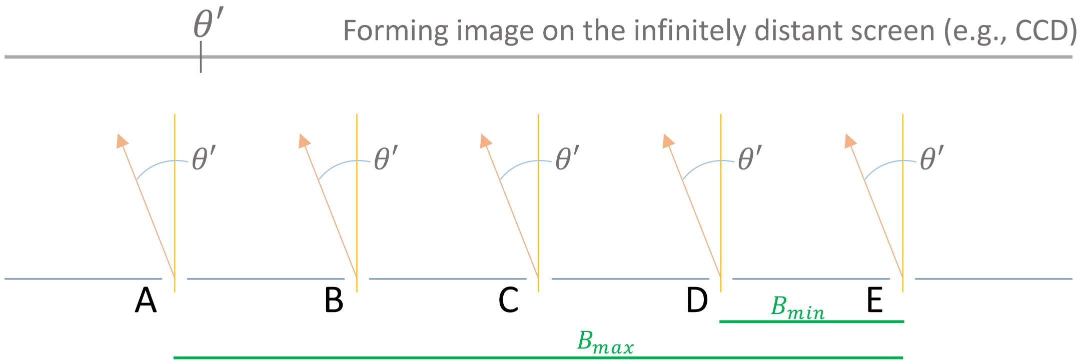
$$\sim \tilde{P}(\theta)\left[\underbrace{1}_{\textbf{Auto-correlations}} + \underbrace{\cos\left(2\pi\frac{B}{\lambda}\theta\right)}_{\textbf{Cross-correlations}}\right]$$


θ′
Forming image on the infinitely distant screen (e.g., CCD)
θ′
θ′
θ′
θ′
θ′
A
B
C
D
E
B_{min}
B_{max}



Diffraction pattern
(proportional to the number of incoming photons
in a unit of time and a unit of area)





Diffraction pattern

Location of maxima: $B_{min}\sin(\theta) = m\lambda, \quad m = 0,1,2,3 \ldots$

(same as the case of a double slit)

Diffraction pattern
(proportional to the number of incoming photons
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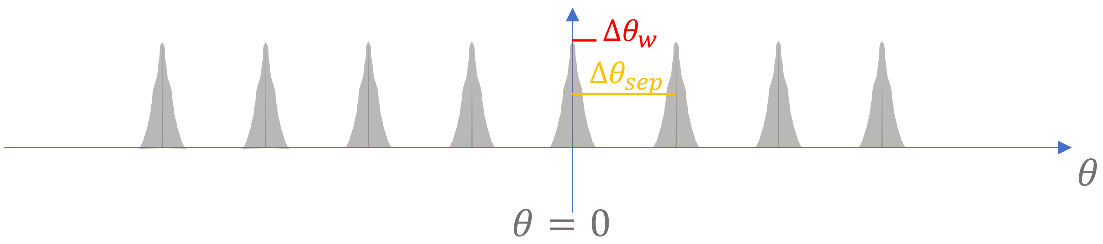

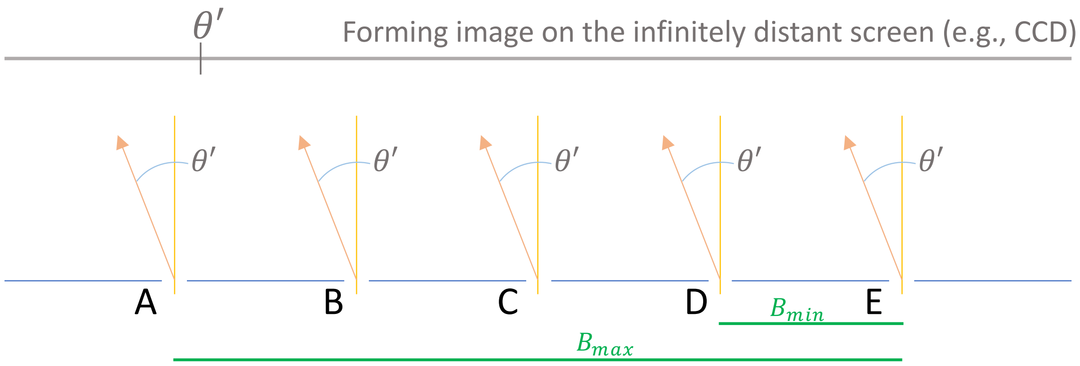

Diffraction pattern

Location of maxima: $B_{min}\sin(\theta) = m\lambda, \qquad m = 0,1,2,3\ \ldots$

Width of pattern: $\frac{1}{2}B_{max}\sin(\Delta\theta_w) = \frac{1}{2}\lambda$

(same as the case of a single slit)

Diffraction pattern
(proportional to the number of incoming photons
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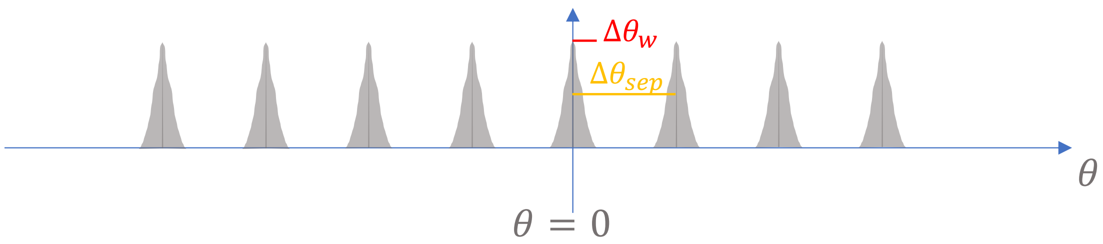
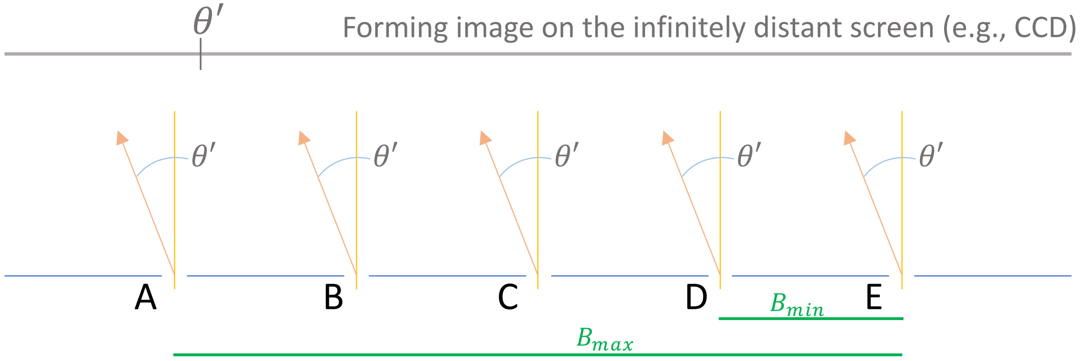


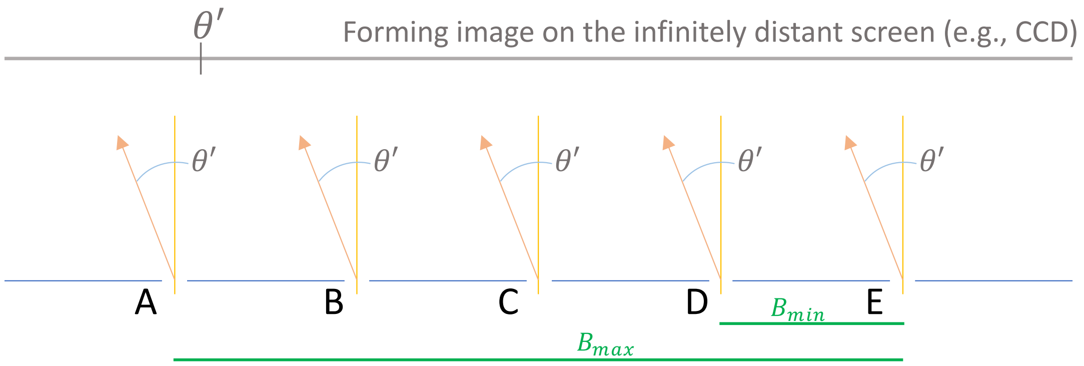


Evaluation of diffraction pattern

(proportional to the number of incoming photons in a unit of time and a unit of area)

$$E_A = E_0 \cos(kx - \omega t + \phi_0)$$

$$< (E_A + E_B + E_C + E_D + E_E + \cdots)^2 >$$



Evaluation of diffraction pattern

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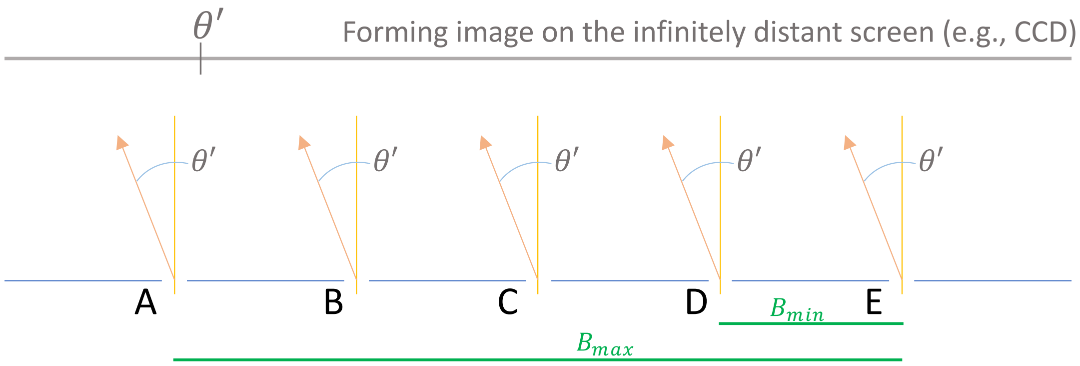
$$\begin{aligned} < (E_A + E_B + E_C + E_D + E_E + \cdots)^2 > =& < {E_A}^2 > + < {E_B}^2 > + < {E_C}^2 > + < {E_D}^2 > + < {E_E}^2 > \\ &+ < 2E_AE_B > + < 2E_AE_C > + < 2E_AE_D > + < 2E_AE_E > \\ &+ < 2E_BE_C > + < 2E_BE_D > + < 2E_BE_E > \\ &+ < 2E_CE_D > + < 2E_CE_E > \\ &+ < 2E_DE_E > \end{aligned}$$



Evaluation of diffraction pattern

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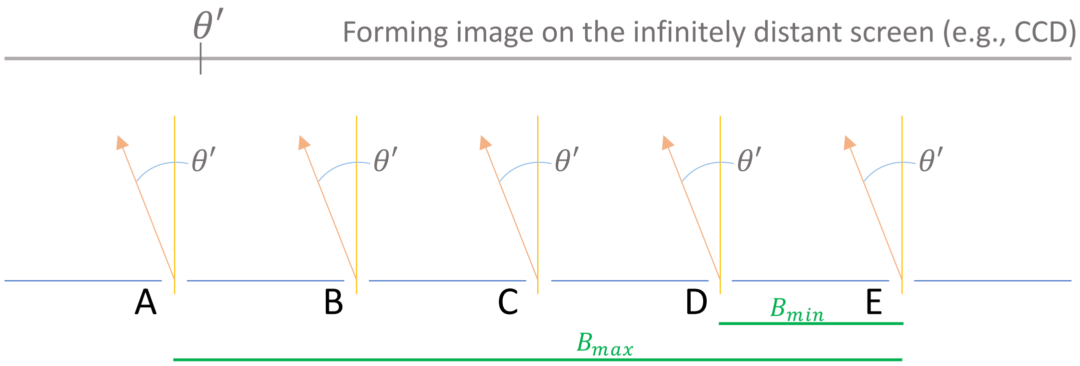
$$E_A = E_0 \cos(kx - \omega t + \phi_0)$$

$$< (E_A + E_B + E_C + E_D + E_E + \cdots)^2 > = \overbrace{< {E_A}^2 > + < {E_B}^2 > + < {E_C}^2 > + < {E_D}^2 > + < {E_E}^2 >}^{\text{Auto-correlation}}$$

$$\text{Cross-correlation}\left\{\begin{array}{l} + < 2E_AE_B > + < 2E_AE_C > + < 2E_AE_D > + < 2E_AE_E > \\ + < 2E_BE_C > + < 2E_BE_D > + < 2E_BE_E > \\ + < 2E_CE_D > + < 2E_CE_E > \\ + < 2E_DE_E > \end{array}\right.$$



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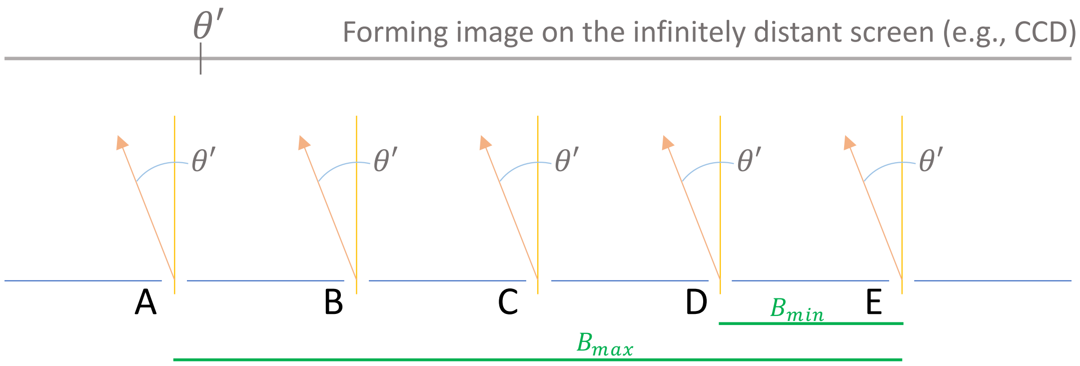
$$\text{Cross-correlation} \left\{ \begin{array}{l} + \boxed{< 2E_AE_B >} + < 2E_AE_C > + < 2E_AE_D > + < 2E_AE_E > \\ + \boxed{< 2E_BE_C >} + < 2E_BE_D > + < 2E_BE_E > \\ + \boxed{< 2E_CE_D >} + < 2E_CE_E > \\ + \boxed{< 2E_DE_E >} \end{array} \right.$$

4 slit-pairs at shortest separation



Evaluation of diffraction pattern

(proportional to the number of incoming photons in a unit of time and a unit of area)

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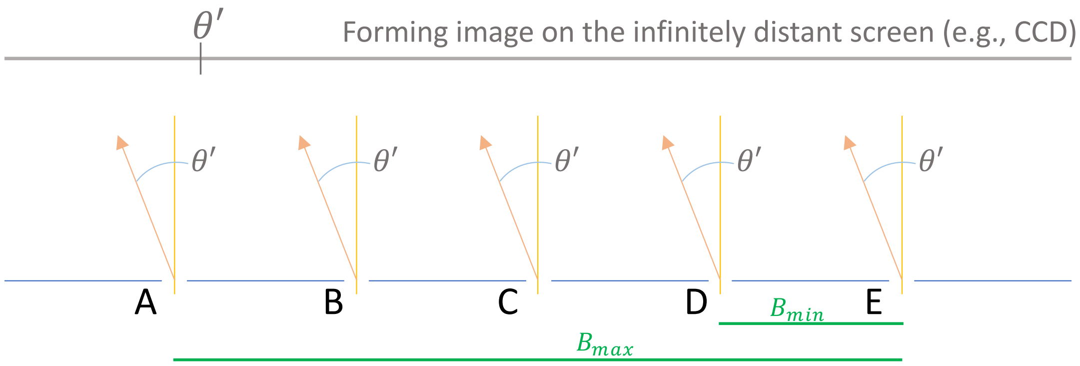
3 slit-pairs at the 2nd shortest separation



Evaluation of diffraction pattern

(proportional to the number of incoming photons in a unit of time and a unit of area)

$$E_A = E_0 \cos(kx - \omega t + \phi_0)$$

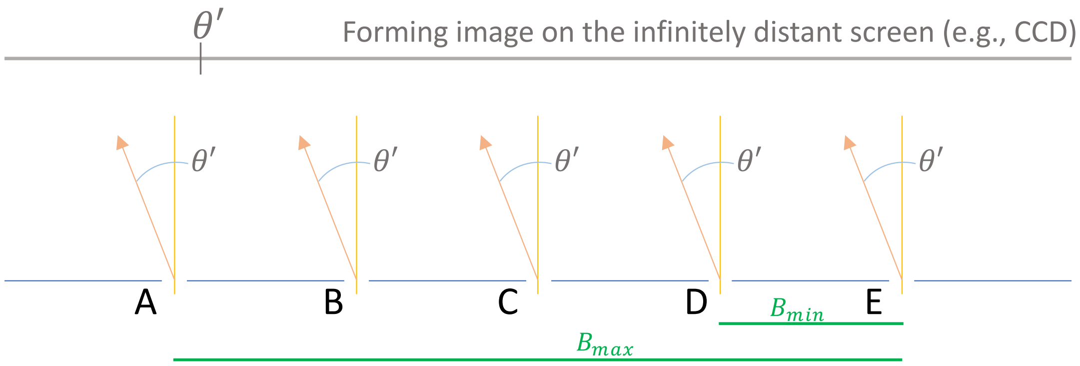
Auto-correlation

$$< (E_A + E_B + E_C + E_D + E_E + \cdots)^2 > = \overbrace{< {E_A}^2 > + < {E_B}^2 > + < {E_C}^2 > + < {E_D}^2 > + < {E_E}^2 >}^{\text{Auto-correlation}}$$

Cross-correlation

$$\begin{aligned}
&+ < 2E_AE_B > + < 2E_AE_C > + \boxed{< 2E_AE_D >} + < 2E_AE_E > \\
&+ < 2E_BE_C > + < 2E_BE_D > + \boxed{< 2E_BE_E >} \\
&+ < 2E_CE_D > + < 2E_CE_E > \\
&+ < 2E_DE_E >
\end{aligned}$$

2 slit-pairs at the 3rd shortest separation



Evaluation of diffraction pattern

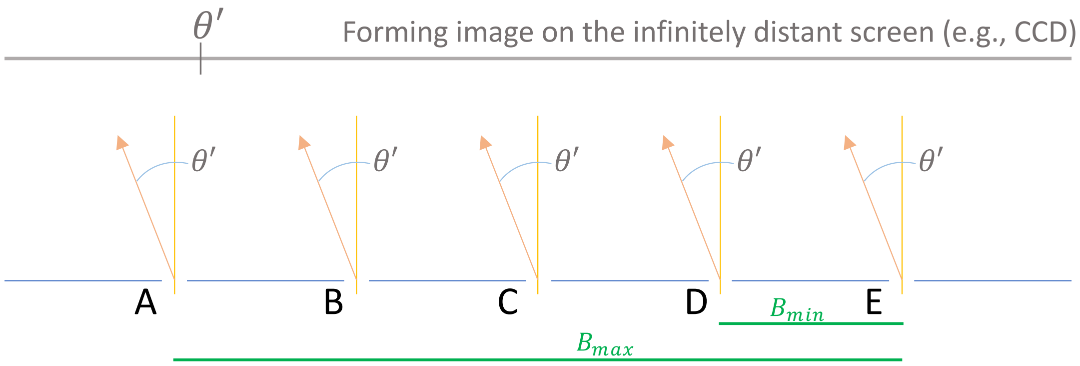
(proportional to the number of incoming photons in a unit of time and a unit of area)

$$E_A = E_0 \cos(kx - \omega t + \phi_0)$$

$$< (E_A + E_B + E_C + E_D + E_E + \cdots)^2 > = \overbrace{< {E_A}^2 > + < {E_B}^2 > + < {E_C}^2 > + < {E_D}^2 > + < {E_E}^2 >}^{\text{Auto-correlation}}$$

$$\text{Cross-correlation} \left\{ \begin{array}{l} + < 2E_A E_B > + < 2E_A E_C > + < 2E_A E_D > + \boxed{< 2E_A E_E >} \\ + < 2E_B E_C > + < 2E_B E_D > + < 2E_B E_E > \\ + < 2E_C E_D > + < 2E_C E_E > \\ + < 2E_D E_E > \end{array} \right.$$

1 slit-pair at longest separation



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$$\left\{\begin{array}{l} + < 2E_AE_B > + < 2E_AE_C > + < 2E_AE_D > + \boxed{< 2E_AE_E >} \\ + < 2E_BE_C > + < 2E_BE_D > + < 2E_BE_E > \\ + < 2E_CE_D > + < 2E_CE_E > \\ + < 2E_DE_E > \end{array}\right.$$

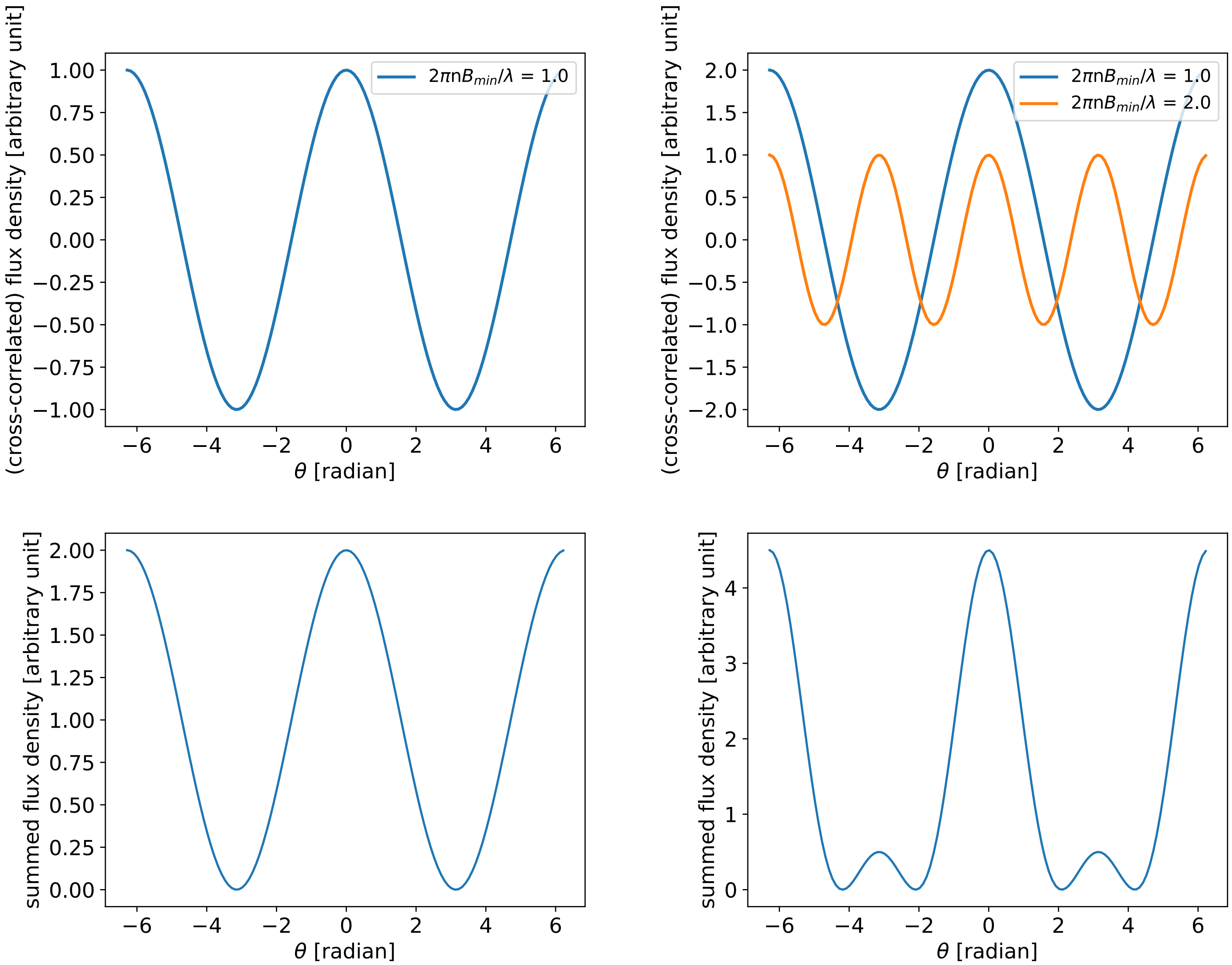
1 slit-pair at longest separation

Form of each cross-term

$$< 2E_0{}^2 \cos(kx - \omega t + \phi_0)\cos(k(x + \Delta x) - \omega t + \phi_0) >$$
$$= < \cos(k\Delta x) > + < \cos(2kx - 2\omega t + 2\phi_0 + k\Delta x) >, \qquad \Delta x = nB_{min}\sin\theta'$$
$$= \cos\left(\frac{2\pi}{\lambda} nB_{min}\sin\theta'\right) \sim \cos\left(\frac{2\pi}{\lambda} nB_{min}\theta'\right)$$

Trigonometric identity:
$\cos\psi\cos\varphi$
$= \frac{1}{2}[\cos(\psi - \varphi) + \cos(\psi + \varphi)]$

Assuming a diffraction grating with 2 and 3 slits



Flux density:

$\cos(k\theta) = \cos\left(2\pi \frac{nB_{min}}{\lambda}\theta\right)$,

$k = 1, 2, 3, 4, 5, \ldots$, (N-1)

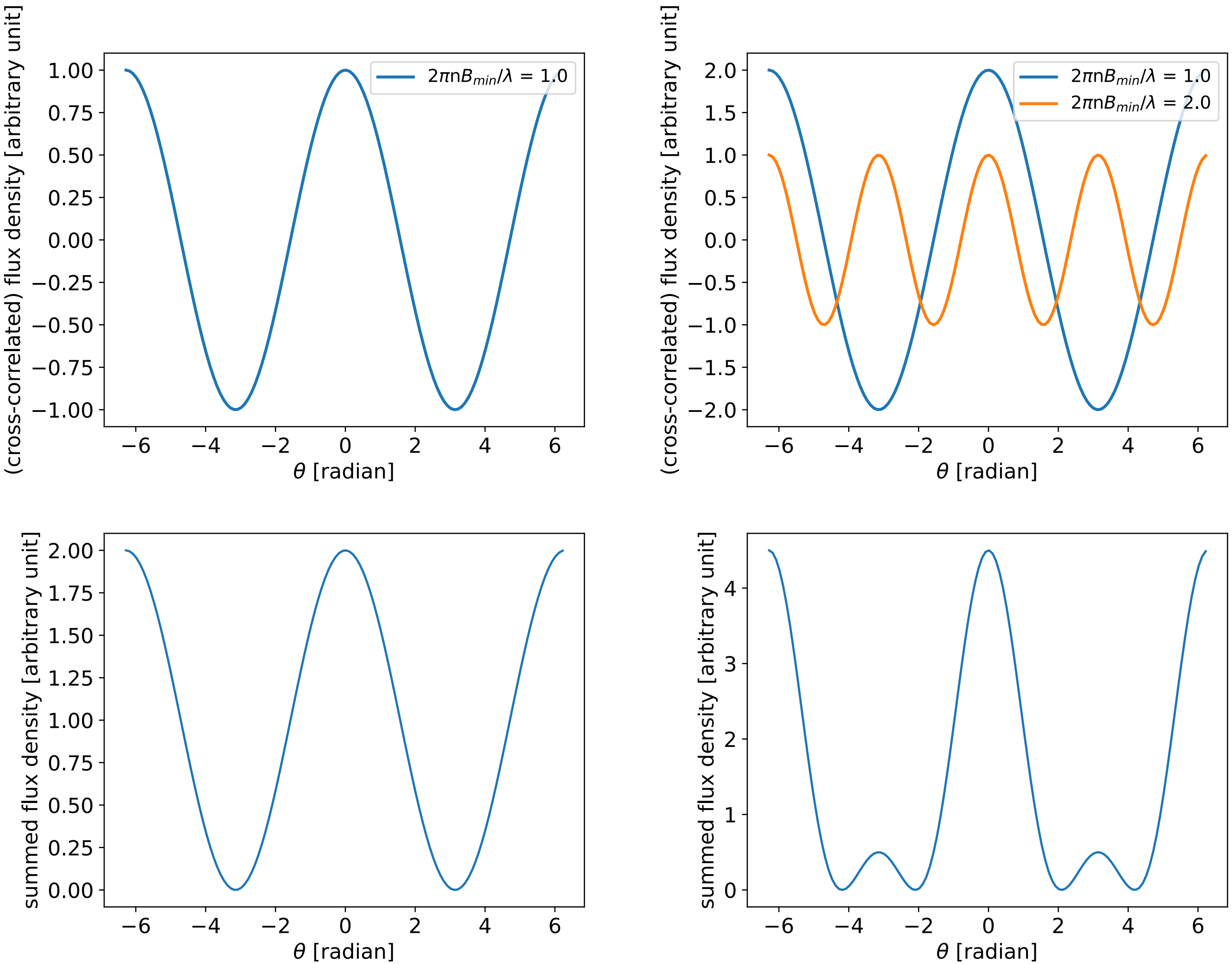
Summed flux density:

$\sum_k (N-k)\cos(k\theta)$ + auto-corr.

Properties:

1. For any n, $\theta = 0$ is always the location of the central maximum (i.e., constructive interference)

Assuming a diffraction grating with 2 and 3 slits



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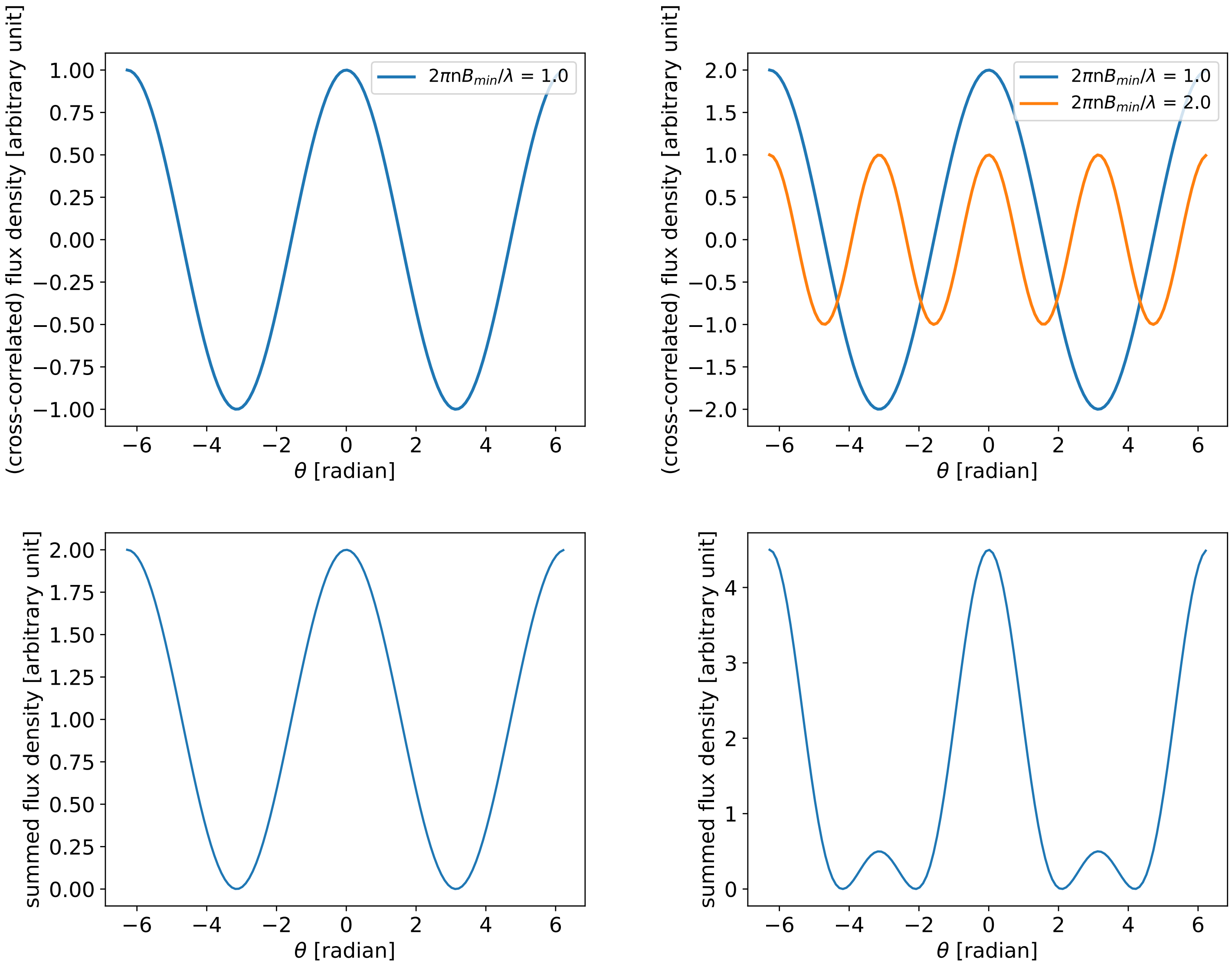
2. The locations of the maxima for the $n = 1$ slit-pairs are always the locations of the maxima for any other pairs of slits.

Assuming a diffraction grating with 2 and 3 slits



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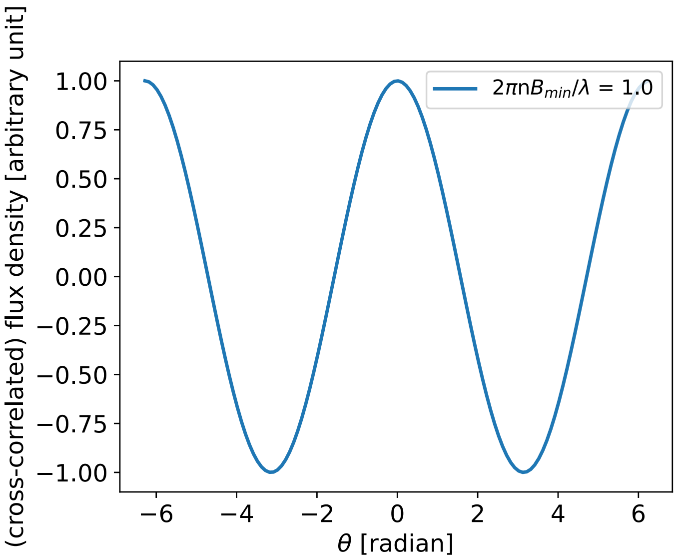
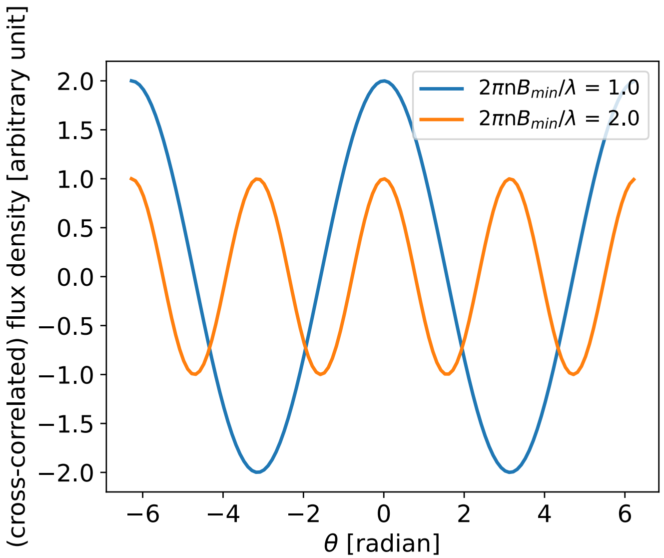
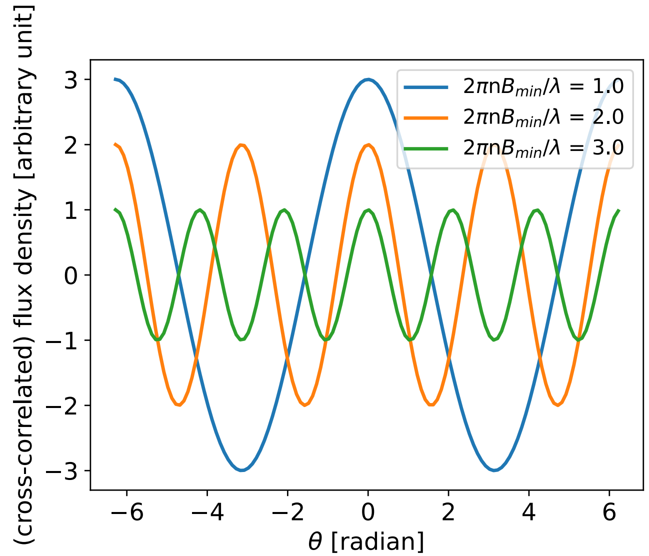
Properties:

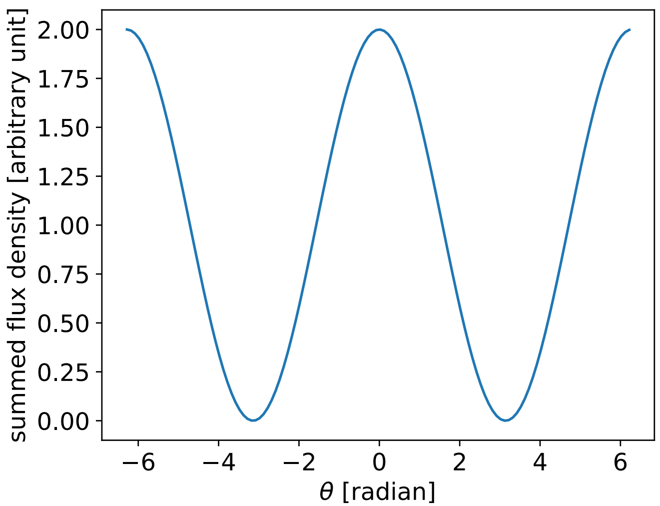
1. For any n, $\theta = 0$ is always the location of the central maximum (i.e., constructive interference)
2. The locations of the maxima for the $n = 1$ slit-pairs are always the locations of the maxima for any other pairs of slits.
3. Between the maxima of the n=1 pairs of slits, the destructive interference makes the photon flux density approach 0. (i.e., the cross-correlation terms cancel each other).

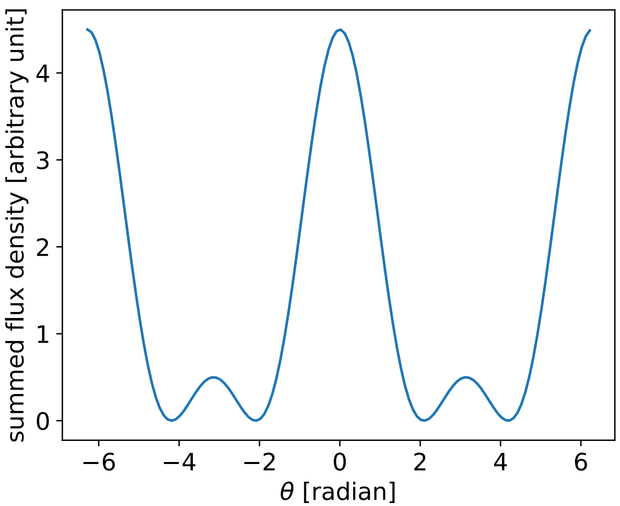
Assuming a diffraction grating with 2, 3, and 4 slits

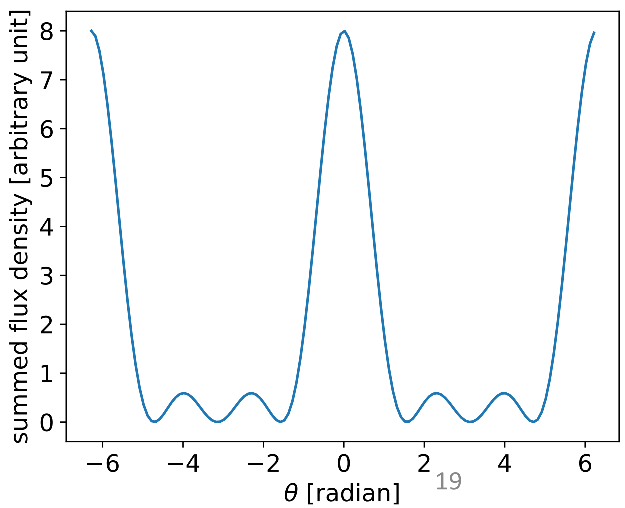




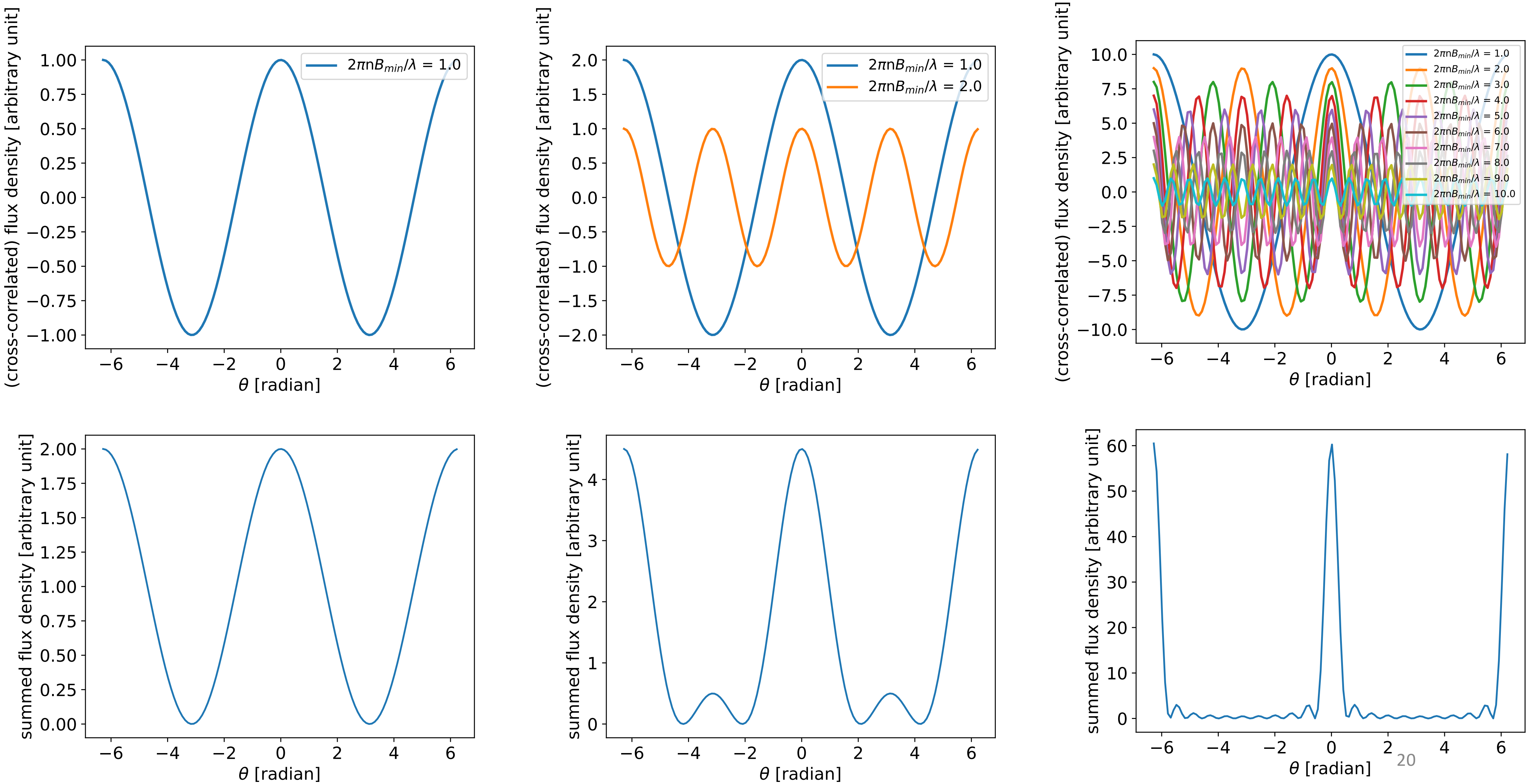








Assuming a diffraction grating with 2, 3, and 10 slits



Assuming a diffraction grating with 2, 3, and 10 slits

Properties:

1. For any n, $\theta = 0$ is always the location of the central maximum (i.e., constructive interference)
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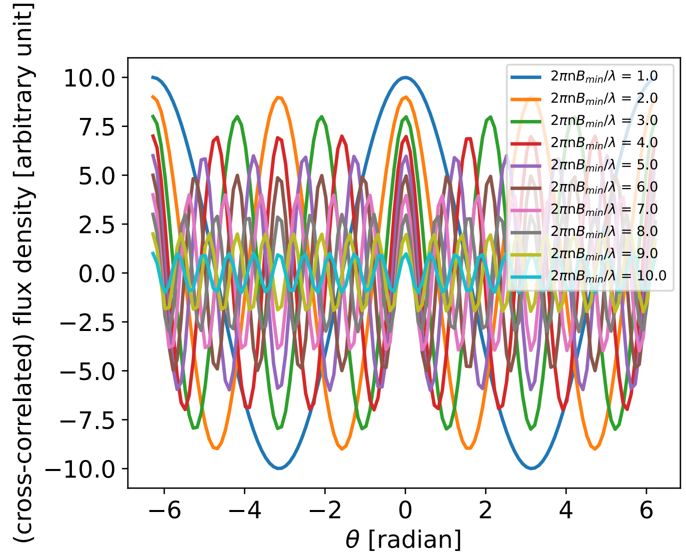
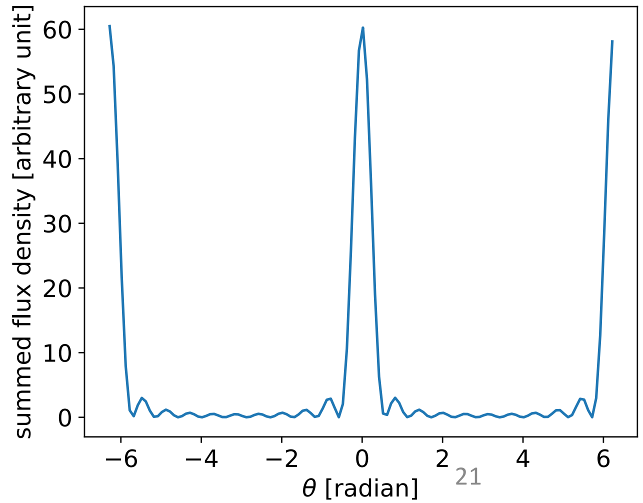
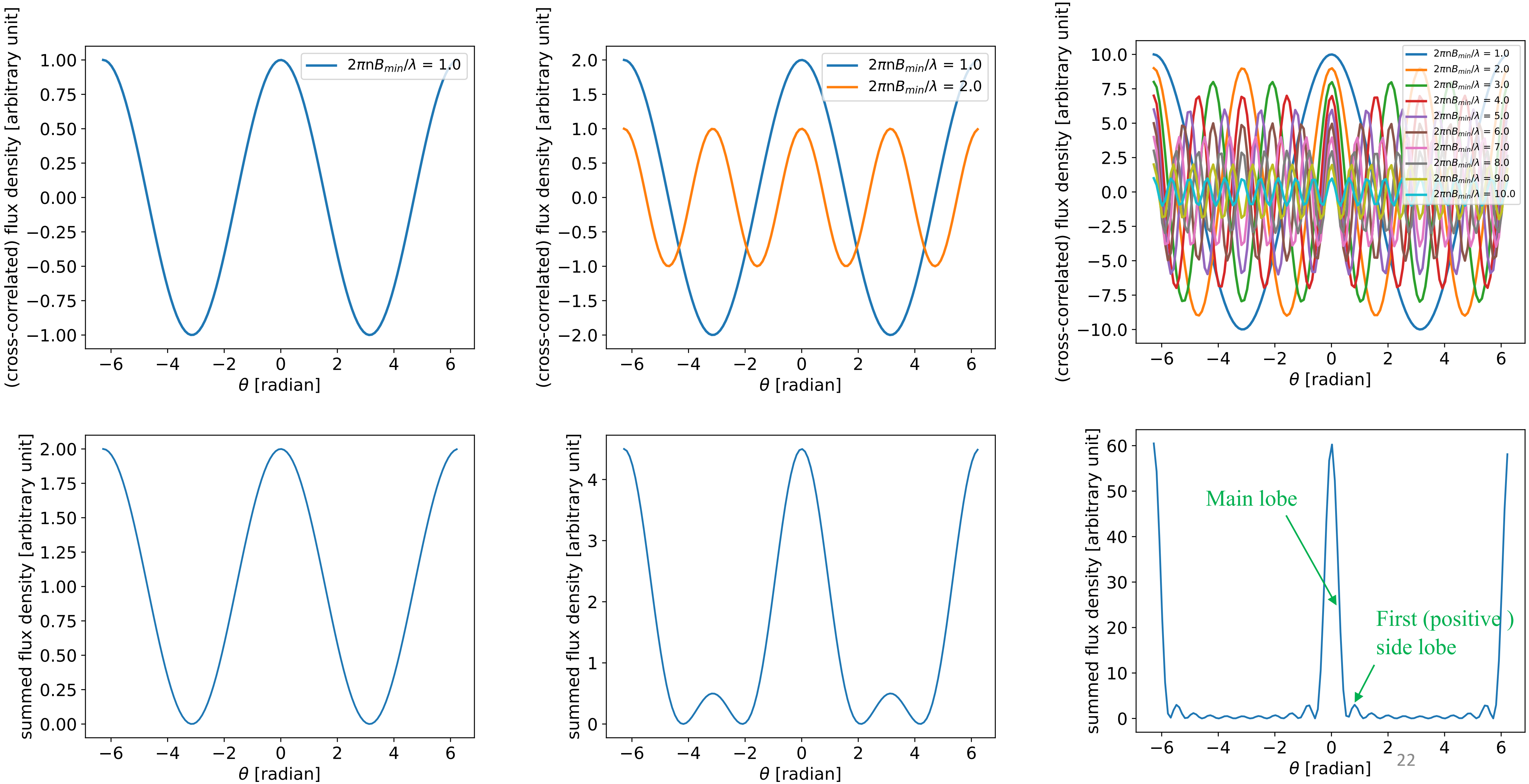
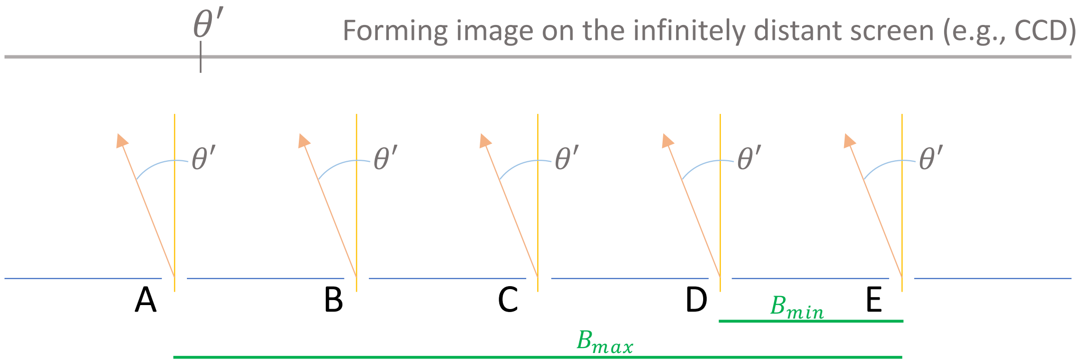




Assuming a diffraction grating with 2, 3, and 10 slits





Diffraction pattern

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